Hyper-power series and analytic generalized smooth functions

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Definition ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$

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The ring of Robinson-Colombeau defined by the index set $\mathbb{N} \times I$ and ordered as $(n, \varepsilon) \leq (m, e)$ if and only if $\varepsilon \leq e$, is denoted as ${}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ i.e. $\exists Q \in \mathbb{N} \forall^0 \varepsilon \forall n \in \mathbb{N} : \ |a_{n\varepsilon}| \leq \rho_{\varepsilon}^{-Q}$ and $\forall q \in \mathbb{N} \forall^0 \varepsilon \forall n \in \mathbb{N} : \ |a_{n\varepsilon}| \leq \rho_{\varepsilon}^{q}$.

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Also, in next slides we write $\sigma \geq \rho^*$ whenever $\exists Q_{\sigma,\rho} \in \mathbb{N} \forall^0 \varepsilon$: $\sigma_{\varepsilon} \geq \rho_{\varepsilon}^{Q_{\sigma,\rho}}$.

Definition: Hyper-power series

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A hyperseries of the form ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n$ with $\sigma \geq \rho^*$ is hyper-power series such that $(a_n)_{n\in\mathbb{N}} \in {}^{\rho}\widetilde{\mathbb{R}}[\![x-c]\!]$. Here ${}^{\rho}\widetilde{\mathbb{R}}[\![x-c]\!] = \{(a_n)_{n\in\mathbb{N}} \mid (a_n(x-c)^n)_{n\in\mathbb{N}} \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}\}$.

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If x, c are in ${}^{\rho}\widetilde{\mathbb{R}}$ also $a_n \in {}^{\rho}\widetilde{\mathbb{R}}$ for every n. Then $(a_n)_{n\in\mathbb{N}} \in {}^{\rho}\widetilde{\mathbb{R}}[\![x-c]\!]$, if $\exists N \in \mathbb{N} : |x-c| \leq -N \log d\rho$.

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Lemma

If $x, c \in {}^{\rho}\widetilde{\mathbb{R}}$ and $(a_n)_n \in {}^{\rho}\widetilde{\mathbb{R}}_{\mathrm{u}}$ so if $\forall^0\varepsilon, \forall n \in \mathbb{N}$, we have $a_{n\varepsilon} \sim_\rho \bar{a}_{n\varepsilon}$, $x_\varepsilon \sim_\rho \bar{x}_\varepsilon$ and $c_\varepsilon \sim_\rho \bar{c}_\varepsilon$. Then if $(a_n)_{n\in\mathbb{N}} \in {}^{\rho}\widetilde{\mathbb{R}}[\![x-c]\!]$ so $(\bar{a}_n)_{n\in\mathbb{N}} \in {}^{\rho}\widetilde{\mathbb{R}}[\![x-c]\!]$.

Radius of Convergence

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If ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n x^n$ *converges at* x_i *then it converges for all* $|x| \leq |x_i|$.

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Let ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n$ *be a hyper-power series, if* $\exists r = sup\{|x-c|, {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n converges\}$, *then the hyper-power series converges absolutely for* $0 \leq |x-c| < r$.

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Definition

Let ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n$ be a hyper-power series then the radius of convergence R of hyper-power series ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n$ is defined as $R = sup\{|x-c| : {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n converges\}$, if exist. If the supremum does not exist, we can say that a hyper-power series converges at $\bar{x}$, hence also at all x such that $|x-c| \leq |\bar{x}-c|$.

Examples

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Theorem

For the hyper-power series ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-c)^n$, *let us define* A *and* ρ *by* $A = \limsup |a_n|^{1/n}$ *for* $n \in {}^{\sigma}\widetilde{\mathbb{N}}$, $\rho = 1/A$, *if this* $\limsup$ *exist in sharp topology. Then* ρ *is the radius of convergence.*

Analytic GSF

Theorem

If $f(x) = {}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} a_n (x-c)^n$ *converges under* r *then* $f \in {}^{\rho}\mathcal{GC}^{\infty}(B_r(c), {}^{\rho}\widetilde{\mathbb{R}})$.

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Definition

Let $\sigma \geq \rho^*$, a GSF $f \in {}^{\rho}\mathcal{GC}^{\infty}(U, {}^{\rho}\widetilde{\mathbb{R}})$ is real analytic with respect to σ and ρ on a sharply open set U, if $\forall\, x_0 \in U\, \exists (a_n)_n \in {}^{\rho}\widetilde{\mathbb{R}}\, \exists r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} :\ B_r(x_0) \subseteq U$, $f(x) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-x_0)^n$. It is denoted by $f \in {}^{\rho}\mathcal{GC}^{\omega}(U, {}^{\rho}\widetilde{\mathbb{R}})$

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Theorem

If $f \in \mathcal{C}^{\omega}(\Omega)$ *then* $\exists\, \sigma \geq \rho^{\star}$ *such that it's embedding is an analytic GSF.*

Basic properties of hyper-power series

Lemma

If two hyper-power series, ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n x^n$ *and* ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n x^n$ *converges at* x_1 *and* x_2 *respectively, then their summation converges at* $\min(x_1, x_2)$.

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Multiplication and division of hyper-power series

Theorem

If $f(x) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-x_0)^n$ *and* $g(x) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} b_n(x-x_0)^n$ *are two hyper-power series, then* $f(x)\cdot g(x) = {}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{i\geq 0}^{n} a_i b_{n-i}(x-x_0)^n$.

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$$d_{n\varepsilon} = \frac{1}{b_{0\varepsilon}^{n+1}} det \begin{bmatrix} a_{n\varepsilon} & b_{1\varepsilon} & b_{2\varepsilon} & \dots & b_{n\varepsilon} \\ a_{n-1\varepsilon} & b_{0\varepsilon} & b_{1\varepsilon} & \dots & b_{n-1\varepsilon} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_2 & 0 & b_{0\varepsilon} & \dots & b_{n-2} \\ a_1 & 0 & 0 & \dots & b_{0\varepsilon} \end{bmatrix}.$$

Integration and differentiation of hyper-power series

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Bounds of hyper-power series

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A hyper-power series ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n(x-\alpha)^n$ *has radius of convergence* r*, iff for each* $0 < R < r$*, there exist a constant* $0 < C_R < 1$*, such that* $|a_j| \leq \frac{C_R}{R^j}$, $\forall j \in \mathbb{N}$.

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Theorem

Let $f \in {}^{\rho}\mathcal{GC}^{\infty}(U, {}^{\rho}\widetilde{\mathbb{R}})$ *for some open interval* U*. The function* $f \in {}^{\rho}\mathcal{GC}^{\omega}(U, {}^{\rho}\widetilde{\mathbb{R}})$*, iff for each* $\alpha \in U$*, there is an open set* $J \in {}^{\rho}\widetilde{\mathbb{R}}$ *with* $\alpha \in J \subseteq U$ *and constants* $C \in (0,1) \subseteq {}^{\rho}\widetilde{\mathbb{R}}$, $R \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ *such that the derivatives of* f *satisfy* $|f_j(x)| \leq C\frac{j!}{R^j}, \forall x \in J, \forall j \in \mathbb{N}$.

Principle of Permanence: Lemma

Lemma

If f and g are analytic GSF on an open interval U and if there is a point x_0 in U such that $f^j(x_0) = g^j(x_0), \forall j \in \mathbb{N}$. Then $f(x) = g(x)$, $\forall x \in U$, if the following conditions are satisfied

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If f and g are analytic GSF on an open interval U and if there is an open set $W \subseteq U$ such that, $f(x) = g(x), \forall x \in W$, then $f(x) = g(x), \forall x \in U$, if

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If f is an analytic CGF, then it is also an analytic GSF.

Future Work

1. Complex analytic GSF.

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2. Uniform convergence of complex analytic GSF.

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2. Uniform convergence of complex analytic GSF.
3. Cauchy-Kowalevsky theorem using (2) and Picard-Lindlöf theorem for PDE.

Contact

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Thank you for your attention!