Analytic pseudo-differential calculus via the Bargmann transform

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Plan of the talk

1. Pseudo-differential operators and something about modulation spaces
2. Test functions, distributions and expansions
3. Images under the Bargmann transform
4. Analytic pseudo-differential and integral operators

Important contributors to the topic, e.g.:

W. Bauer, F. A. Berezin, L. A. Coburn, N. Lerner

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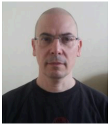
The talk is based on the following:

- J. Toft *Images of function and distribution spaces under the Bargmann transform*, J. Pseudo-Differ. Oper. Appl. **8** (2017), 83–139.
- N. Teofanov, J. Toft *Pseudo-differential calculus in a Bargmann setting*, Ann. Acad. Sci. Fenn. Math. 45 (2020), 227–257.
- N. Teofanov, J. Toft, P. Wahlberg *Some features on analytic pseudo-differential calculus* (Ongoing project)



N. Teofanov

P. Wahlberg

Pseudo-Differential Operators (ΨDO)

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$$\operatorname{Op}_A(a)f(x) = (2\pi)^{-d} \iint a(x-A(x-y),\xi)e^{i\langle x-y,\xi\rangle} f(y)\, dyd\xi, \quad f \in \mathscr{S}(\mathbf{R}^d).$$

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Normal representation: $A = 0$, i.e. $a(x,D) = \operatorname{Op}_0(a)$

Weyl quantization: $A = \frac{1}{2}\cdot I$, i.e. $\operatorname{Op}^w(a) = \operatorname{Op}_{\frac{1}{2}\cdot I}(a)$.

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Partial differential equations:

$$a(x,\xi) = \sum_{|\alpha|\leqslant N} a_\alpha(x)\xi^\alpha \quad \Leftrightarrow \quad a(x,D) = \sum_{|\alpha|\leqslant N} a_\alpha(x)D^\alpha, \qquad D_j = \frac{1}{i}\frac{\partial}{\partial x_j}.$$

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Natural assumption: a is smooth and e.g.

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \lesssim \omega(x,\xi)(1+|\xi|)^{-|\beta|}.$$

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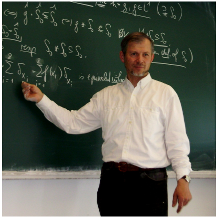
This leads to Modulation Spaces.

Modulation spaces (Feichtinger)

The Fourier transform and Short-time Fourier transform:

$$\widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int f(y) e^{-i\langle y,\xi\rangle}\, dy, \qquad V_\phi f(x,\xi) = (2\pi)^{-\frac{d}{2}} \int f(y)\overline{\phi(y-x)} e^{-i\langle y,\xi\rangle}\, dy.$$



H. Feichtinger

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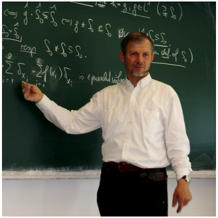

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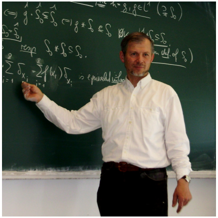
- f in the modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, iff

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \left(\int \left(\int |V_\phi f(x,\xi)\omega(x,\xi)|^p\, dx \right)^{q/p} d\xi \right)^{1/q} < \infty.$$



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Some properties (Feichtinger, Gröchenig, . . .)

Let $M^{p,q}_{s,t} = M^{p,q}_{(\omega)}$, $M^{p,q} = M^{p,q}_{0,0}$ when $\omega(x,\xi) = \langle x\rangle^t \langle \xi\rangle^s$, $\langle x\rangle = (1+|x|^2)^{1/2}$.

- $M^{2,2}_{s,0} = L^2_s$ and $M^{2,2}_{0,s} = H^2_s$
- $\|\mathscr{F} f\|_{M^{p,p}_{(\omega)}} \asymp \|f\|_{M^{p,p}_{(\omega)}}$ when $\omega(-x,\xi) = \omega(\xi,x)$.
- $M^{p,q}_{(\omega)}$ independent of ϕ (Usually)

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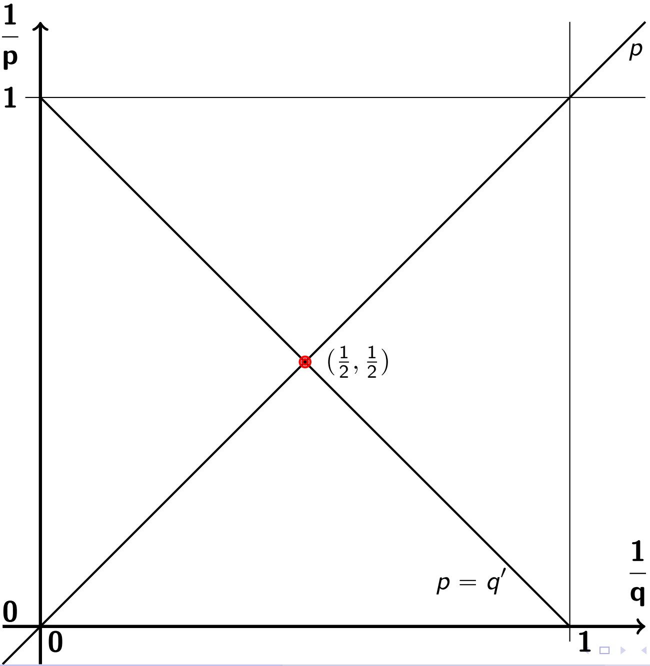
- $M^{p,q}_{(\omega)}$ independent of ϕ (Usually)
- if $1 \leqslant p,q < \infty$, then $(M^{p,q}_{(\omega)})' = M^{p',q'}_{(1/\omega)}$ $(1/p + 1/p' = 1)$
- if $p_1 \leqslant p_2$, $q_1 \leqslant q_2$ then $M^{p_1,q_1}_{(\omega)} \subseteq M^{p_2,q_2}_{(\omega)}$
- Convenient discretization properties with Gabor frames
- Convenient embeddings between modulation spaces, Lebesgue spaces and Besov spaces



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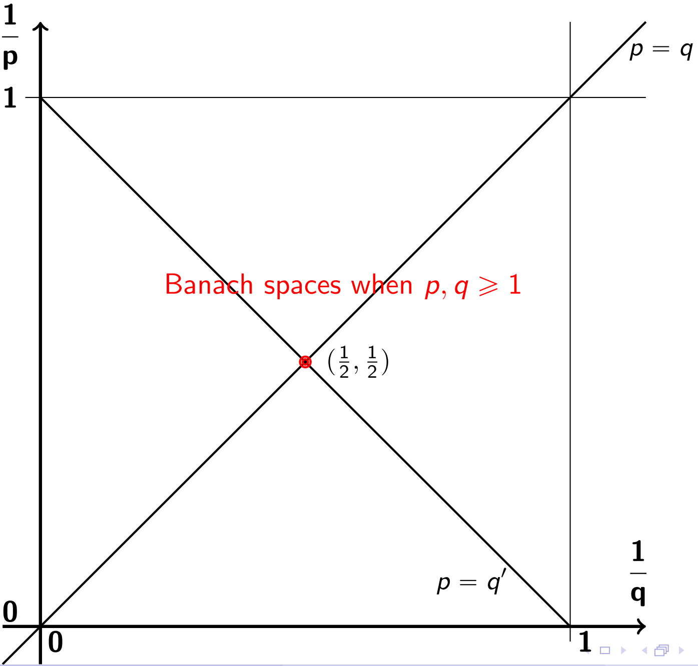

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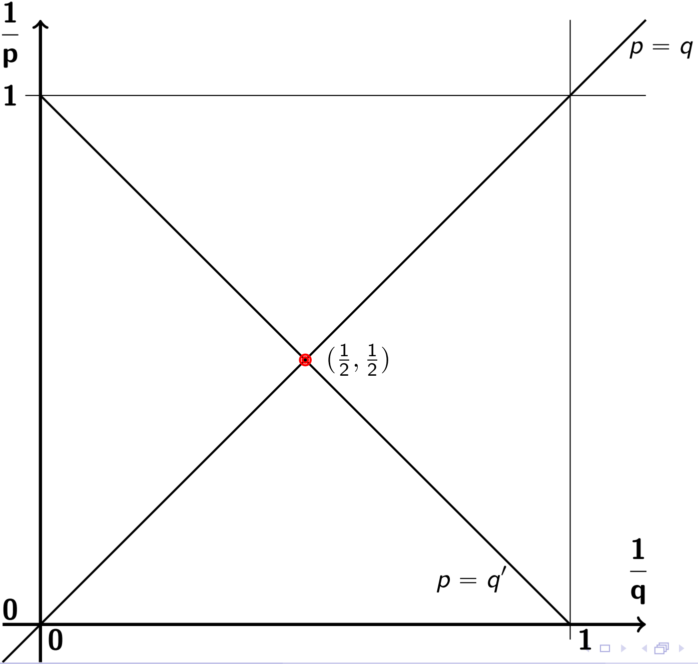

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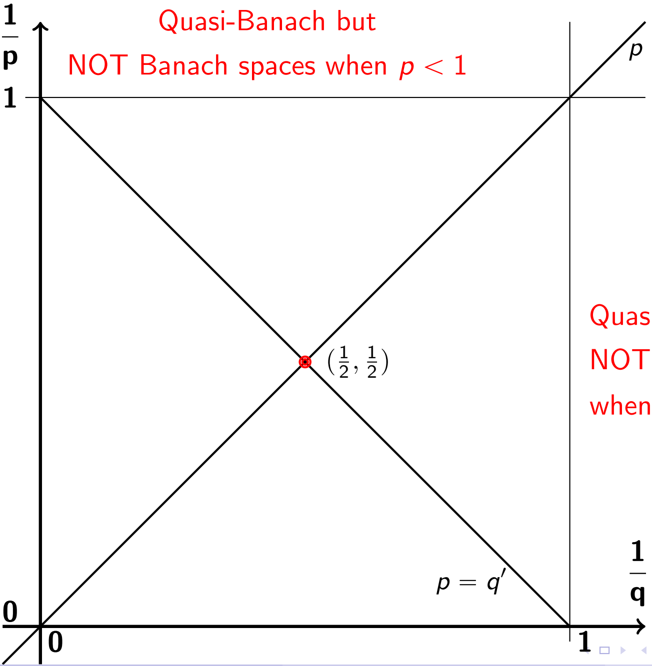

Quasi-Banach but

NOT Banach spaces

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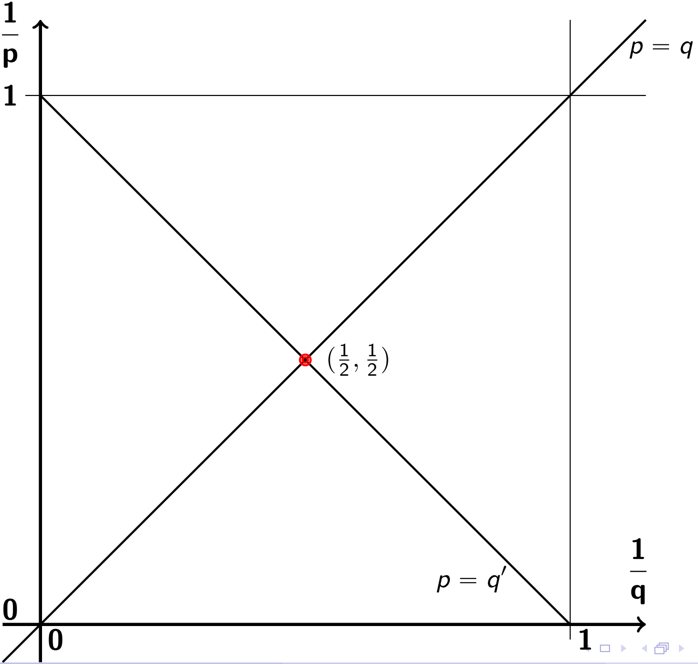

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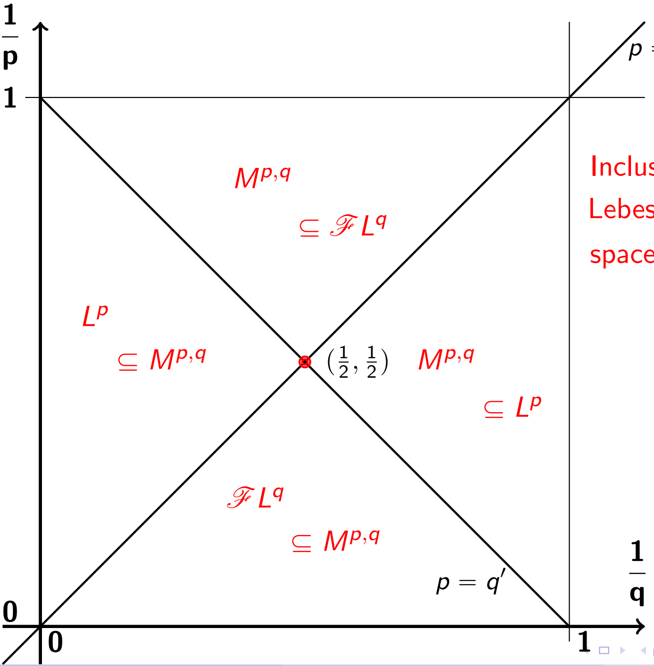

Inclusions betweeen Lebesgue and modulation spaces $p, q \geqslant 1$

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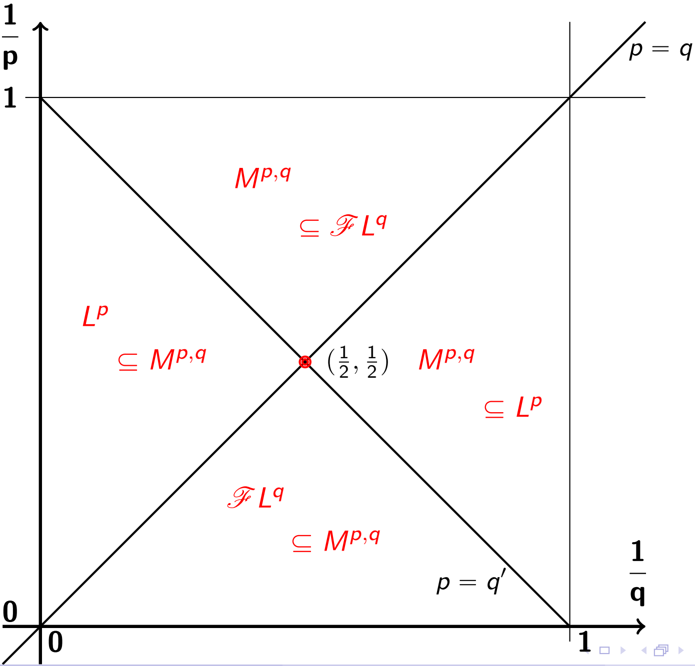

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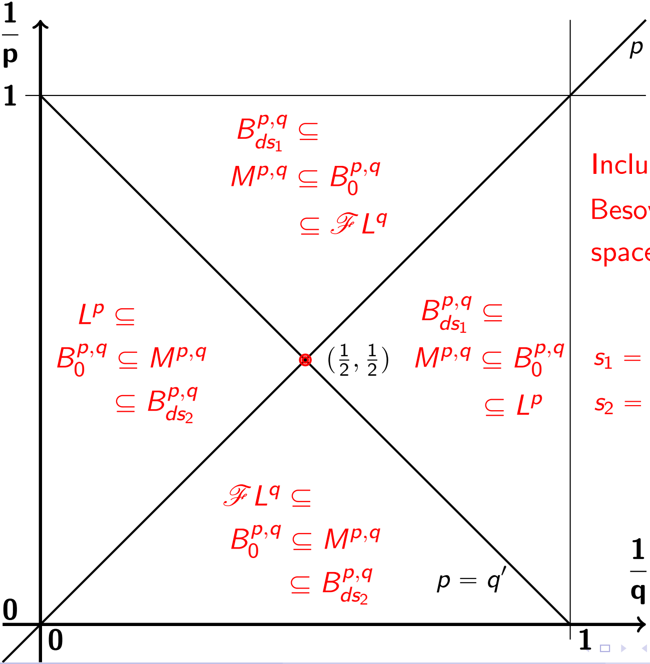

Inclusions betweeen Besov and modulation spaces $p, q > 0$

$$s_1 = \frac{1}{q} - \min(\frac{1}{p}, 1 - \frac{1}{p}) \geqslant 0$$
$$s_2 = \frac{1}{q} - \max(\frac{1}{p}, 1 - \frac{1}{p}) \leqslant 0$$

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Pilipović spaces

Hermite function h_α with respect to $\alpha \in \mathbf{N}^d$ is given by

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Formal Hermite function expansions:

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Let $s, \sigma > 0$.

- The Pilipović space of Roumieu / Beurling type, $\mathcal{H}_s(\mathbf{R}^d)$ / $\mathcal{H}_{0,s}(\mathbf{R}^d)$, consists of all f in (*) such that $|c(\alpha)| \lesssim e^{-r|\alpha|^{\frac{1}{2s}}}$ holds for some / every $r > 0$.

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We also let

$$\mathcal{H}_0(\mathbf{R}^d) = \text{All finite Hermite series expansions in (*)},$$

$$\mathcal{H}_0'(\mathbf{R}^d) = \text{All formal Hermite series expansions in (*)}.$$

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By letting $\mathbf{R}_\flat = \mathbf{R}_+ \cup \{\flat_\sigma\}$ with convention

$$s < \flat_{\sigma_1} < \flat_{\sigma_2} < \frac{1}{2}, \quad \text{when} \quad s < \frac{1}{2},\ \sigma_1 < \sigma_2$$

it follows

$$\mathcal{H}_0 \overset{Dense}{\hookrightarrow} \mathcal{H}_{0,s_1} \overset{Dense}{\hookrightarrow} \mathcal{H}_{s_1} \overset{Dense}{\hookrightarrow} \mathcal{H}_{0,s_2}, \qquad s_1, s_2 \in \mathbf{R}_\flat,\ s_1 < s_2.$$

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For Gelfand-Shilov spaces $\mathcal{S}_s$ and Σ_s, Pilipović proved 1986:

$$\mathcal{H}_s = \mathcal{S}_s = \{\, f\, ;\, \|(|x|^2 - \Delta)^N f\|_{L^\infty} \lesssim h^N N!^{2s} \text{ for some } h > 0 \,\}, \qquad s \geqslant \frac{1}{2},$$

$$\mathcal{H}_{0,s} = \Sigma_s = \{\, f\, ;\, \|(|x|^2 - \Delta)^N f\|_{L^\infty} \lesssim h^N N!^{2s} \text{ for every } h > 0 \,\}, \qquad s > \frac{1}{2}.$$

But... $\Sigma_{1/2} = \{0\} \neq \{\, f\, ;\, \|(|x|^2 - \Delta)^N f\|_{L^\infty} \lesssim h^N N! \text{ for every } h > 0 \,\} = \mathcal{H}_{0,1/2}.$

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Recent extension:

Thm. (T. 2017)

Let $0\leqslant s\in\mathbf{R}$. Then:

$$\mathcal{H}_s=\{\,f\,;\ \|(|x|^2-\Delta)^N f\|_{L^\infty}\lesssim h^N N!^{2s} \text{ for some } h>0\,\},$$

and

$$\mathcal{H}_{0,s}=\{\,f\,;\ \|(|x|^2-\Delta)^N f\|_{L^\infty}\lesssim h^N N!^{2s} \text{ for every } h>0\,\}.$$

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We let $\mathcal{H}'_s(\mathbf{R}^d)$ / $\mathcal{H}'_{0,s}(\mathbf{R}^d)$ be the set of all f in (*) such that $|c(\alpha)| \lesssim e^{+r|\alpha|^{\frac{1}{2s}}}$, $s \in \mathbf{R}_+$, and $|c(\alpha)| \lesssim r^{|\alpha|}\alpha!^{+\frac{1}{2\sigma}}$, $s = \flat_\sigma$, for every / some $r > 0$.

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We let $\mathcal{H}_s'(\mathbf{R}^d)$ / $\mathcal{H}_{0,s}'(\mathbf{R}^d)$ be the set of all f in (*) such that $|c(\alpha)| \lesssim e^{+r|\alpha|^{\frac{1}{2s}}}$, $s \in \mathbf{R}_+$, and $|c(\alpha)| \lesssim r^{|\alpha|}\alpha!^{+\frac{1}{2\sigma}}$, $s = \flat_\sigma$, for every / some $r > 0$.

Then $\mathcal{H}_s'$ for $s \geqslant 0$ and $\mathcal{H}_{0,s}'$ for $s > 0$ are the duals of $\mathcal{H}_s$ and $\mathcal{H}_{0,s}$ under $(\,\cdot\,,\,\cdot\,)_{L^2}$.

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$$(\mathfrak{V}_d f)(z) = \pi^{-d/4} \int_{\mathbf{R}^d} \exp\Big(-\frac{1}{2}(\langle z,z\rangle + |y|^2) + 2^{1/2}\langle z,y\rangle\Big) f(y)\, dy.$$

Here $z = (z_1, \dots, z_d) \in \mathbf{C}^d$, $\quad \langle z,w\rangle = \sum_{j=1}^{d} z_j w_j, \quad (z,w) = \langle z, \overline{w}\rangle.$

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- $A^2(\mathbf{C}^d)$ is the Hilbert space of entire analytic functions such that

$$\|F\|_{A^2} \equiv \Big(\int_{\mathbf{C}^d} |F(z)|^2 \, d\mu(z)\Big)^{1/2} < \infty.$$

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- A^2-scalar product: $(F, G)_{A^2} = \int_{\mathbf{C}^d} F(z)\overline{G(z)}\, d\mu(z).$

V. Bargmann 1961 - Mapping properties

He proved:

- $\mathfrak{V}_d$ is a bijective isometry from $L^2(\mathbf{R}^d)$ to $A^2(\mathbf{C}^d)$.
- $\mathfrak{V}_d h_\alpha = e_\alpha(z) \equiv \dfrac{z^\alpha}{(\alpha!)^{1/2}}$. Hence $\mathfrak{V}_d$ maps ON-basis $\{h_\alpha(x)\}$ in L^2 into the ON-basis $\{e_\alpha(z)\}$ in A^2.

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- Reproducing kernel:

$$(\Pi_A F)(z) = \int_{\mathbf{C}^d} e^{(z,w)} F(w)\, d\mu(w), \qquad F \text{ admissible.}$$

Then

$$(\Pi_A F)(z) = F(z), \quad F \in A^2, \quad d\mu(z) = \pi^{-d} e^{-|z|^2}\, d\lambda(z).$$

Spaces of power series expansions

In the most general situation we consider the power series expansions

$$F(z) = \sum_{\alpha \in \mathbf{N}^d} c(\alpha) e_\alpha(z), \quad z \in \mathbf{C}^d, \; c(\alpha) \in \mathbf{C}, \; e_\alpha(z) = \frac{z^\alpha}{(\alpha!)^{1/2}}. \qquad (*)$$

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Smaller spaces:

$\mathcal{A}_0(\mathbf{R}^d)$, the set of all analytic polynomials $F(z)$ in (*)

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$$|c(\alpha)| \lesssim \begin{cases} e^{-r|\alpha|^{\frac{1}{2s}}}, & s \in \mathbf{R}_+ \\ r^{|\alpha|} \alpha!^{-\frac{1}{2\sigma}}, & s = \flat_\sigma \end{cases}$$

for some (every) $r > 0$.

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By the definitions it follows that

$$
\begin{aligned}
\mathfrak{V}_d &: \mathcal{H}_{0,s}(\mathbf{R}^d) \to \mathcal{A}_{0,s}(\mathbf{C}^d),\\
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are bijective.

Characterizations of certain spaces of power series

Any entire function F is equal to a power series expansion $\sum_\alpha c(\alpha)e_\alpha$ such that

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for every $r > 0$. This implies $\mathcal{A}'_{\flat_1}(\mathbf{C}^d) = A(\mathbf{C}^d)$.

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From the definitions it now follows for $s \geqslant \frac{1}{2}$ and $s_0 < \frac{1}{2}$:

$$\mathcal{A}_{0,s_0}(\mathbf{C}^d) \subseteq \mathcal{A}_{s_0}(\mathbf{C}^d) \subseteq \mathcal{A}_{0,s}(\mathbf{C}^d) \subseteq \mathcal{A}_s(\mathbf{C}^d)$$

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What about those spaces which are contained in $A(\mathbf{C}^d)$??

Identifications with spaces of analytic functions

For $\quad s_0 < \frac{1}{2}, \quad s \geqslant \frac{1}{2}, \quad \langle z \rangle = 1 + |z| \quad$ (Recall: $s_0 < \flat_\sigma < \frac{1}{2}$):

The tiny planets (smaller than Gelfand-Shilov):

$$\mathcal{A}_{0,s_0} \ (\mathcal{A}_{s_0}) = \{\, F \in A\, ;\ |F(z)| \lesssim e^{r(\log \langle z \rangle)^{\frac{1}{1-2s_0}}},\ \text{for every (some) } r > 0 \,\},$$

$$\mathcal{A}_{0,\flat_\sigma} \ (\mathcal{A}_{\flat_\sigma}) = \{\, F \in A\, ;\ |F(z)| \lesssim e^{r|z|^{\frac{2\sigma}{\sigma+1}}},\ \text{for every (some) } r > 0 \,\},$$

$$\mathcal{A}_{0,\frac{1}{2}} = \{\, F \in A\, ;\ |F(z)| \lesssim e^{r|z|^2},\ \text{for every } r > 0 \,\},$$

The Gelfand-Shilov world:

$$\mathcal{A}_{0,s} \, / \, (\mathcal{A}_s) = \{\, F \in A\, ;\ |F(z)| \lesssim e^{\frac{|z|^2}{2} - r|z|^{\frac{1}{s}}},\ \text{for every (some) } r > 0 \,\},\ s \neq \frac{1}{2},$$

$$\mathcal{A}_s' \ (\mathcal{A}_{0,s}') = \{\, F \in A\, ;\ |F(z)| \lesssim e^{\frac{|z|^2}{2} + r|z|^{\frac{1}{s}}},\ \text{for every (some) } r > 0 \,\},$$

Beyond Gelfand-Shilov life:

$$\mathcal{A}_{0,\frac{1}{2}}' = \{\, F \in A\, ;\ |F(z)| \lesssim e^{r|z|^2},\ \text{for some } r > 0 \,\},$$

$$\mathcal{A}_{\flat_\sigma}' \ (\mathcal{A}_{0,\flat_\sigma}') = \{\, F \in A\, ;\ |F(z)| \lesssim e^{r|z|^{\frac{2\sigma}{\sigma-1}}},\ \text{for every (some) } r > 0 \,\},\ \sigma > 1,$$

$$\mathcal{A}_{\flat_1}' = A \quad (= A(\mathbf{C}^d)), \qquad \mathcal{A}_{0,\flat_1}' = \bigcup_{R>0} A(B_R(0)).$$

Analytic pseudo-differential and integral operators

Let $a(z,w)$ and $K(z,w)$, $z,w \in \mathbf{C}^d$, be suitable, analytic in z.

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1. The analytic pseudo-differential operator $\operatorname{Op}_{\mathfrak{V}}(a)$ is:

$$(\operatorname{Op}_{\mathfrak{V}}(a)F)(z) = \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z,w)}\, d\mu(w), \quad F \in \mathcal{A}_0(\mathbf{C}^d).$$

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2. The (analytic) kernel operator T_K is:

$$(T_K F)(z) = \int_{\mathbf{C}^d} K(z,w)F(w)\,d\mu(w), \quad F \in \mathcal{A}_0(\mathbf{C}^d).$$

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Examples and remarks:

We have $T_K = \operatorname{Op}_{\mathfrak{V}}(a)$ when $K(z,w) = a(z,w)e^{(z,w)}$.

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Examples and remarks: If $a(z,w)e^{-(\frac{1}{2}+r)(|z|^2+|w|^2)} \in L^1(\mathbf{C}^{2d})$ for every $r > 0$, then there is a unique $a_0(z,w)$ such that

- $(z,w) \mapsto a_0(z,\overline{w})$ is entire (belongs to $A(\mathbf{C}^{2d})$);
- $a_0(z,w)e^{-(\frac{1}{2}+r)(|z|^2+|w|^2)} \in L^1(\mathbf{C}^{2d})$ for every $r > 0$;
- $\operatorname{Op}_{\mathfrak{V}}(a) = \operatorname{Op}_{\mathfrak{V}}(a_0)$.

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Examples and remarks:

$$\operatorname{Op}_{\mathfrak{V}}(a)F(z) = \sum_{|\alpha|\leqslant N} a_\alpha(z)(\partial_z^\alpha F)(z), \quad a(z,w) = \sum_{|\alpha|\leqslant N} a_\alpha(z)\overline{w}^\alpha.$$

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It follows that if $\qquad b(x,\xi) = \sum_{|\alpha+\beta| \leqslant N} c_1(\alpha,\beta)x^\alpha \xi^\beta,$

there is a unique $\qquad a(z,w) = \sum_{|\alpha+\beta| \leqslant N} c_2(\alpha,\beta)z^\alpha \overline{w}^\beta$

such that $\qquad \operatorname{Op}_{\mathfrak{V}}(a) = \mathfrak{V}_d \circ \operatorname{Op}(b) \circ \mathfrak{V}_d^{-1}.$

Analytic pseudo-differential and integral operators

Let $a(z,w)$ and $K(z,w)$, $z,w \in \mathbf{C}^d$, be suitable, analytic in z.

1. The analytic pseudo-differential operator $\operatorname{Op}_{\mathfrak{V}}(a)$ is:

$$(\operatorname{Op}_{\mathfrak{V}}(a)F)(z) = \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z,w)}\, d\mu(w), \qquad F \in \mathcal{A}_0(\mathbf{C}^d).$$

Examples and remarks:

Analytic pseudo-differential and integral operators

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Examples and remarks:

If $a(z,w)$ is analytic, then

$$(\operatorname{Op}_{\mathfrak{V}}(a)F)(z) = a(z,z)F(z).$$

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Examples and remarks:

If χ is the characteristic function of a polydisc and $a(z,w) = \chi(w)$, then $\operatorname{Op}_{\mathfrak{V}}(a)$ is bijective between suitable $\mathcal{A}_s(\mathbf{C}^d)$ spaces. Some sorts of analytic Paley-Wiener properties Nabizadeh-Pfeuffer-T. (2018).

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Examples and remarks:

Analytic pseudo-differential operator are often called Wick operators or Berezin operators.

Kernel theorems and related mapping properties

$$(\mathrm{Op}_{\mathfrak{V}}(a)F)(z) = \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z,w)}\,d\mu(w), \qquad (T_K F)(z) = \int_{\mathbf{C}^d} K(z,w)F(w)\,d\mu(w).$$

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In what follows we let

$$\widehat{\mathcal{A}}_s'(\mathbf{C}^{2d}) = \{\, K(z,w)\,;\,(z,w) \mapsto K(z,\overline{w}) \in \mathcal{A}_s'(\mathbf{C}^{2d})\,\},$$

and similarly for other spaces.

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and similarly for other spaces.

We also let $\mathcal{L}(V_1,V_2)$ be the set of all linear continuous mappings from the topological vector space V_1 to the topological vector space V_2.

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and similarly for other spaces.

Thm. (by Kernel theorems for nuclear spaces)

Let $s_1 \in \mathbf{R}_\flat$ and $s_2 \in \overline{\mathbf{R}}_\flat$. The map $K \mapsto T_K$ is bijective

- from $\widehat{\mathcal{A}}_{0,s_1}(\mathbf{C}^{2d})$ to $\mathcal{L}(\mathcal{A}_{0,s_1}'(\mathbf{C}^d),\mathcal{A}_{0,s_1}(\mathbf{C}^d))$, and
  from $\widehat{\mathcal{A}}_{0,s_1}'(\mathbf{C}^{2d})$ to $\mathcal{L}(\mathcal{A}_{0,s_1}(\mathbf{C}^d),\mathcal{A}_{0,s_1}'(\mathbf{C}^d))$.
- from $\widehat{\mathcal{A}}_{s_2}(\mathbf{C}^{2d})$ to $\mathcal{L}(\mathcal{A}_{s_2}'(\mathbf{C}^d),\mathcal{A}_{s_2}(\mathbf{C}^d))$, and
  from $\widehat{\mathcal{A}}_{s_2}'(\mathbf{C}^{2d})$ to $\mathcal{L}(\mathcal{A}_{s_2}(\mathbf{C}^d),\mathcal{A}_{s_2}'(\mathbf{C}^d))$.

Kernel theorems and related mapping properties

$$(\operatorname{Op}_{\mathfrak{V}}(a)F)(z) = \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z,w)}\,d\mu(w), \quad (T_K F)(z) = \int_{\mathbf{C}^d} K(z,w)F(w)\,d\mu(w).$$

$$\mathcal{L}(\mathcal{A}_{s_2}', \mathcal{A}_{s_2}) = \{\, T_K \,;\, K \in \widehat{\mathcal{A}}_{s_2} \,\}, \qquad \mathcal{L}(\mathcal{A}_{s_2}, \mathcal{A}_{s_2}') = \{\, T_K \,;\, K \in \widehat{\mathcal{A}}_{s_2}' \,\} \quad \text{etc...}$$

Kernel theorems and related mapping properties

$$(\mathrm{Op}_{\mathfrak{V}}(a)F)(z) = \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z,w)}\,d\mu(w), \quad (T_K F)(z) = \int_{\mathbf{C}^d} K(z,w)F(w)\,d\mu(w).$$

$\mathcal{L}(\mathcal{A}'_{s_2}, \mathcal{A}_{s_2}) = \{\, T_K \,;\, K \in \widehat{\mathcal{A}}_{s_2} \,\}, \qquad \mathcal{L}(\mathcal{A}_{s_2}, \mathcal{A}'_{s_2}) = \{\, T_K \,;\, K \in \widehat{\mathcal{A}}'_{s_2} \,\}$ etc...

Thm. Teofanov-T. (2019)

Let $t \in \mathbf{C}$, $s_1 \in \mathbf{R}_\flat$, $s_1 \leqslant \frac{1}{2}$, and $s_2 \in \overline{\mathbf{R}_\flat}$, $s_2 < \frac{1}{2}$. Then $K(z,w) \mapsto K(z,w)e^{t(z,w)}$ is a continuous bijection on

$\widehat{\mathcal{A}}'_{0,s_1}(\mathbf{C}^{2d})$ and on $\widehat{\mathcal{A}}'_{s_2}(\mathbf{C}^{2d})$.

Kernel theorems and related mapping properties

$$(\mathrm{Op}_{\mathfrak{V}}(a)F)(z) = \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z,w)}\,d\mu(w), \quad (T_K F)(z) = \int_{\mathbf{C}^d} K(z,w)F(w)\,d\mu(w).$$

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By combining this with the earlier kernel theorems:

Kernel theorems and related mapping properties

$$(\mathrm{Op}_{\mathfrak{V}}(a)F)(z) = \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z,w)}\,d\mu(w), \quad (T_K F)(z) = \int_{\mathbf{C}^d} K(z,w)F(w)\,d\mu(w).$$

$\mathcal{L}(\mathcal{A}_{s_2}', \mathcal{A}_{s_2}) = \{\, T_K\,;\, K \in \widehat{\mathcal{A}}_{s_2} \,\}$, $\quad \mathcal{L}(\mathcal{A}_{s_2}, \mathcal{A}_{s_2}') = \{\, T_K\,;\, K \in \widehat{\mathcal{A}}_{s_2}' \,\}$ etc. . .

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$$(\mathrm{Op}_{\mathfrak{V}}(a)F)(z) = \int_{\mathbf{C}^d} a(z,w)F(w)e^{(z,w)}\,d\mu(w), \quad (T_KF)(z) = \int_{\mathbf{C}^d} K(z,w)F(w)\,d\mu(w).$$

$\mathcal{L}(\mathcal{A}'_{s_2},\mathcal{A}_{s_2}) = \{\, T_K \,;\, K \in \widehat{\mathcal{A}}_{s_2} \,\}$, $\qquad \mathcal{L}(\mathcal{A}_{s_2},\mathcal{A}'_{s_2}) = \{\, T_K \,;\, K \in \widehat{\mathcal{A}}'_{s_2} \,\}$ etc...

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Let $t \in \mathbf{C}$, $s_1 \in \mathbf{R}_\flat$, $s_1 \leqslant \frac{1}{2}$, and $s_2 \in \overline{\mathbf{R}_\flat}$, $s_2 < \frac{1}{2}$. Then $K(z,w) \mapsto K(z,w)e^{t(z,w)}$ is a continuous bijection on $\widehat{\mathcal{A}}'_{0,s_1}(\mathbf{C}^{2d})$ and on $\widehat{\mathcal{A}}'_{s_2}(\mathbf{C}^{2d})$.

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Let $s_1 \in \mathbf{R}_\flat$, $s_1 \leqslant \frac{1}{2}$, and $s_2 \in \overline{\mathbf{R}_\flat}$, $s_2 < \frac{1}{2}$. Then

- $\mathcal{L}(\mathcal{A}_{0,s_1}(\mathbf{C}^d),\mathcal{A}'_{0,s_1}(\mathbf{C}^d)) = \{\, \mathrm{Op}_{\mathfrak{V}}(a) \,;\, a \in \widehat{\mathcal{A}}'_{0,s_1}(\mathbf{C}^{2d}) \,\}$.
- $\mathcal{L}(\mathcal{A}_{s_2}(\mathbf{C}^d),\mathcal{A}'_{s_2}(\mathbf{C}^d)) = \{\, \mathrm{Op}_{\mathfrak{V}}(a) \,;\, a \in \widehat{\mathcal{A}}'_{s_2}(\mathbf{C}^{2d}) \,\}$.

Analytic Ψdo with Lebesgue conditions on their symbols

- Let $L^{\mathbf{p}}(\mathbf{C}^d) \asymp L^{\mathbf{p}}(\mathbf{R}^{2d})$ be the mixed Lebesgue space with respect to $\mathbf{p} \in [1,\infty]^{2d}$,

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  that is, $\omega_0 > 0$ and $\omega_0, 1/\omega_0 \in L^\infty_{loc}(\mathbf{C}^d)$.

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- Let $B^{\mathbf{p}}_{(\omega_0)}(\mathbf{C}^d) = \{\, F\, ;\, F(z)e^{-\frac{1}{2}\cdot|z|^2}\omega_0(\sqrt{2}\cdot \overline{z}) \in L^{\mathbf{p}}(\mathbf{C}^d)\,\}$.

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- Let $B^{\mathbf p}_{(\omega_0)}(\mathbf C^d) = \{\, F\, ;\ F(z)e^{-\frac12\cdot|z|^2}\omega_0(\sqrt2\cdot\overline z) \in L^{\mathbf p}(\mathbf C^d)\,\}$.
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- Let $B^{\mathbf p}_{(\omega_0)}(\mathbf C^d) = \{\, F\, ;\; F(z)e^{-\frac 12 \cdot |z|^2}\omega_0(\sqrt 2\cdot \overline z) \in L^{\mathbf p}(\mathbf C^d)\,\}$.
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Recently, results of the following type appeared

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- Let $B^{\mathbf{p}}_{(\omega_0)}(\mathbf{C}^d) = \{\, F\, ;\, F(z)e^{-\frac{1}{2}\cdot |z|^2}\omega_0(\sqrt{2}\cdot \overline{z}) \in L^{\mathbf{p}}(\mathbf{C}^d)\,\}$.
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Thm. Teofanov-T. (2019)

Suppose $K \in \widehat{A}(\mathbf{C}^{2d})$,

$$G_{K,\omega}(z+w,z) \in L^{p,q}(\mathbf{C}^{2d}), \quad G_{K,\omega}(z,w) = K(z,w)\cdot e^{-\frac{1}{2}(|z|^2+|w|^2)}\omega(\sqrt{2}\,\overline{z},\sqrt{2}\,w),$$

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Then T_K is continuous from $A^{\mathbf{p}_1}_{(\omega_1)}(\mathbf{C}^d)$ to $A^{\mathbf{p}_2}_{(\omega_2)}(\mathbf{C}^d)$.

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Then T_K is continuous from $A^{\mathbf{p}_1}_{(\omega_1)}(\mathbf{C}^d)$ to $A^{\mathbf{p}_2}_{(\omega_2)}(\mathbf{C}^d)$.

By putting some restrictions on ω, ω_j and taking the counter image of the previous result with respect to the Bargmann transform

$$\left(\mathfrak{V}_d \ : \ M^{\mathbf{p}}_{(\omega)}(\mathbf{R}^d) \to A^{\mathbf{p}}_{(\omega)}(\mathbf{C}^d) \quad \text{bijective} \right)$$

one gets well-known results of continuity results of real Ψdo on modulation spaces, like

Analytic Ψdo with Lebesgue conditions on their symbols

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Then T_K is continuous from $A^{\mathbf{p}_1}_{(\omega_1)}(\mathbf{C}^d)$ to $A^{\mathbf{p}_2}_{(\omega_2)}(\mathbf{C}^d)$.

Thm. Gröchenig-Heil, T.

Suppose

$$\frac{1}{\mathbf{p}_1}-\frac{1}{\mathbf{p}_2}=1-\frac{1}{p}-\frac{1}{q}, \quad q\leqslant \mathbf{p}_2\leqslant p, \quad \frac{\omega_2(x,\xi+\eta)}{\omega_1(x+y,\xi)}\lesssim \omega(x,\xi,\eta,y), \quad a\in M^{p,q}_{(\omega)}(\mathbf{R}^{2d}).$$

Then $\mathrm{Op}(a)$ is continuous from $M^{\mathbf{p}_1}_{(\omega_1)}(\mathbf{R}^d)$ to $M^{\mathbf{p}_2}_{(\omega_2)}(\mathbf{R}^d)$.

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1. Characterizations of global elliptic operators (N. Teofanov, T., P. Wahlberg):

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and let

$$a(z,w) = \sum_{|\alpha+\beta|\leqslant N} c_2(\alpha,\beta)z^\alpha\overline{w}^\beta, \qquad a_0(z,w) = \sum_{|\alpha+\beta|=N} c_2(\alpha,\beta)z^\alpha\overline{w}^\beta$$

be the uniquely defined polynomials given by $\operatorname{Op}_{\mathfrak{V}}(a) = \mathfrak{V}_d \circ \operatorname{Op}(b) \circ \mathfrak{V}_d^{-1}$.

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be the uniquely defined polynomials given by $\operatorname{Op}_{\mathfrak{V}}(a) = \mathfrak{V}_d \circ \operatorname{Op}(b) \circ \mathfrak{V}_d^{-1}$.

Then the following conditions are equivalent:

- $\operatorname{Op}(b)$ is elliptic ;
- $b_0(x,\xi) \neq 0$ when $(x,\xi) \neq (0,0)$;
- $a_0(z,z) \neq 0$ when $0 \neq z \in \mathbf{C}^d$.

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2. Continuity of analytic ΨDO on Orlicz spaces of analytic functions (T., R. Üster)

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3. Transition of symbol classes from real ΨDO to analytic ΨDO (N. Teofanov, T., P. Wahlberg)

Thank you for your attention.