Extension operators for spaces of smooth functions and Whitney jets

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Joint work with Leonhard Frerick and Enrique Jordá

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Whitney norms $|f|_{m,K} = \|f\|_{m,K} +$ sup of Taylor remainders for $x, y \in K, |\alpha| \le m$.

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Theorem (FJW '16)

$K \subseteq \mathbb{R}^d$ admits a Whitney extension operator which extends to $\mathscr{E}^n(K) \to C^n(\mathbb{R}^d)$ for all $n \in \mathbb{N}_0$ if and only if

- $d = 1$: $\exists\, \varrho \in (0,1)$ $\forall\, x_0 \in K$, $\varepsilon \in (0,1)$ $\exists\, x_1 \in K$ s.t. $\varrho\varepsilon < |x_0 - x_1| < \varepsilon$.
- Cantor set

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- Cantor set , Lipschitz boundary (Stein '70), Sierpiński triangle,...

Whitney type operators

- Whitney's operator

$$E_n(f)(x) = \begin{cases} f^{(0)}(x) & x \in K \\ \sum_{i \in \mathbb{N}} \varphi_i(x) T^n_{y(i)} f(x) & x \notin K \end{cases}$$

$(\varphi_i(x))_{i\in\mathbb{N}}$ Whitney partition of unity for $\mathbb{R}^d \setminus K$, $y(i) \in K$ minimizes $r_i =$dist(supp$(\varphi_i), K) \cong$diam(supp$(\varphi_i))$, $\|\partial^\alpha \varphi_i\| \leq c r_i^{-|\alpha|}$

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- Principle idea: Replace coefficients $f^{(\beta)}(y)$ of $T^n_y f$ by $\int_K f^{(0)} d\mu_{\beta,y,r_i}$ with suitable measures and estimate $\|E_n(f) - \tilde{E}(f)\|_n$
- $\tilde{E}$ is called a Whitney type operator

Theorem (FJJW '20)

WEP $\Leftrightarrow$ There is a Whitney type extension operator (& precise continuity estimates $\|\tilde{E}(f)\|_n \leq c_n |f|_{\sigma(n),K}$ in terms of the measures)

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- Bruna '80: $C_b^{(M)}(\mathbb{R}^d) \to \mathscr{E}^{(M)}(K)$ surjective. Hence there exists an extension operator. (Bonet-Braun-Meise-Taylor, Franken,...) $\to$ talk of Armin Rainer

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