Reconstruction of the one-dimensional thick distribution theory

Yunyun Yang

Hefei University of Technology

yangyunyun@hfut.edu.cn

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One-dim thick distribution theory (Estrada, Fulling)

Definition

The "thick test function space" is defined as the topological vector space consists of all compactly supported functions which are smooth on $\mathbb{R} \setminus \{a\}$, and whose one-sided derivatives at $x = a$ exist.

Denoted as $\mathscr{D}_{*,a}(\mathbb{R})$.

Given a proper topology, it is a TVS, with $\mathscr{D}(\mathbb{R})$ its closed subspace.

Definition

The dual of $\mathscr{D}_{*,a}(\mathbb{R})$ is the "thick distribution space", $\mathscr{D}'_{*,a}(\mathbb{R})$

By the Hahn-Banach Theorem, there is a projection from $\mathscr{D}_{*,a}(\mathbb{R})$ to $\mathscr{D}(\mathbb{R})$.

Example

$\delta_+(x)$:

$$< \delta_+(x), \phi(x) >= \phi_+(0) := \lim_{x \to 0^+} \phi(x)$$

Clearly, the projection of $\delta_+(x)$ onto the usual distribution space is $\delta(x)$.

Note: The lifting is NOT unique.

Higher-dim thick distribution theory (Yang, Estrada)

Definition

Let $\mathscr{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ denote the vector space of all smooth functions ϕ defined in $\mathbb{R}^n \setminus \{\mathbf{a}\}$, with support of the form $K \setminus \{\mathbf{a}\}$, where K is compact in $\mathbb{R}^n$, that admits a strong asymptotic expansion of the form

$$\phi(\mathbf{a}+\mathbf{x}) = \phi(\mathbf{a}+r\mathbf{w}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w})\, r^j, \quad \text{as } \mathbf{x} \to \mathbf{0}, \tag{1}$$

where $m \in \mathbb{Z}$, and where a_j are smooth functions of $\mathbf{w}$, that is, $a_j \in \mathscr{D}\left(\mathbb{S}^{n-1}\right)$. We call $\mathscr{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ the space of test functions on $\mathbb{R}^n$ with a thick point located at $\mathbf{x} = \mathbf{a}$. It is sometimes convenient to take $\mathbf{a} = \mathbf{0}$; we denote $\mathscr{D}_{*,\mathbf{0}}(\mathbb{R}^n)$ by $\mathscr{D}_*(\mathbb{R}^n)$.

Recall that

Definition

Let ϕ be defined in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. We say that ϕ has the asymptotic expansion $\sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j$ as $\mathbf{x} \to \mathbf{0}$ if for all $M \geq m, M \in \mathbb{Z}$,

$$\lim_{r \to 0^+} \left| \phi(\mathbf{x}) - \sum_{j=m}^{M} a_j(\mathbf{w}) r^j \right| r^{-M} = 0, \quad \text{uniformly on } \mathbf{w} \in \mathbb{S}. \tag{2}$$

In this case we write $\phi(\mathbf{x}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j$ as $\mathbf{x} \to \mathbf{0}$.

With a proper topology, $\mathscr{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ is a TVS:

Definition

Let m be a fixed integer and K a compact of subset of Whose interior contains $\mathbf{a}$. $\mathscr{D}_{*,\mathbf{a}}^{[m;K]}(\mathbb{R}^n)$ consists of those test functions whose expansion begins at m, and whose support is in K.

Definition

Let m be a fixed integer and K a compact subset of $\mathbb{R}^n$ whose interior contains $\mathbf{a}$. The topology of $\mathscr{D}_{*,\mathbf{a}}^{[m;K]}(\mathbb{R}^n)$ is given by the seminorms $\left\{\|\|_{q,s}\right\}_{q>m,s\geq 0}$ defined as

$$\|\phi\|_{q,s} = \sup_{\mathbf{x}+\mathbf{a}\in K} \sup_{|\mathbf{p}|\leq s} r^{-q} \left| (\partial/\partial\mathbf{x})^{\mathbf{p}} \phi(\mathbf{a}+\mathbf{x}) - \sum_{j=m-|\mathbf{p}|}^{q-1} a_{j.\mathbf{p}}(\mathbf{w})\, r^j \right|,$$

where $\mathbf{x} = r\mathbf{w}, \mathbf{p} \in \mathbb{N}^n$, and $(\partial/\partial\mathbf{x})^{\mathbf{p}} \phi(\mathbf{a}+\mathbf{x}) \sim \sum_{j=m-|\mathbf{p}|}^{\infty} a_{j.\mathbf{p}}(\mathbf{w})\, r^j$. The topology of $\mathscr{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ is the inductive limit topology of the $\mathscr{D}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$ as $K \nearrow \infty$. and $m \searrow -\infty$.

Definition

The space of distributions on $\mathbb{R}^n$ with a thick point at $\mathbf{x} = \mathbf{a}$ is the dual space of $\mathscr{D}_{*,a}\left(\mathbb{R}^n\right)$. We denoted it by $\mathscr{D}'_{*,a}\left(\mathbb{R}^n\right)$, or just as $\mathscr{D}'_{*}\left(\mathbb{R}^n\right)$ when $\mathbf{a} = \mathbf{0}$.

inclusion map $i : \mathscr{D}\left(\mathbb{R}^n\right) \hookrightarrow \mathscr{D}_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$.
projection map $\pi : \mathscr{D}'_{*,a}\left(\mathbb{R}^n\right) \to \mathscr{D}'\left(\mathbb{R}^n\right)$ (Since $\mathscr{D}\left(\mathbb{R}^n\right)$ is closed in $\mathscr{D}_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$.)

Example (Thick delta functions of degree q)

Let $g(\mathbf{w})$ is a distribution in $\mathbb{S}^{n-1}$. The thick delta function of degree q, denoted as $g\delta_*^{[q]}$, acts on a thick test function $\phi(\mathbf{x})$ as

$$\left\langle g\delta_*^{[q]}, \phi \right\rangle = \frac{1}{C_{n-1}} \langle g(\mathbf{w}), a_q(\mathbf{w}) \rangle,$$

where $\phi(r\mathbf{w}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j$ as $\mathbf{x} \to 0$, and $C_{n-1} = \int_{\mathbb{S}^{n-1}} d\sigma(\mathbf{w})$ is the surface area of the (n-1)-dimensional unit sphere.
We consider the special case when $q = 0, g(\mathbf{w}) \equiv 1$, we denote it $\delta_*^{[0]} := \delta_*$. One could easily check that

$$\pi(\delta_*) = \delta,$$

is the famous Dirac delta function.

Reconstruction of the one-dimensional case

Asymptotic expansions with respect to r

Now let us view the line segment $[-1,1]$ as 1-dimensional unit ball, and the boundary, i.e., the two points at -1 and 1 as the "0 dimensional unit sphere". We denote the two boundary points $-\mathbf{1}$ and $\mathbf{1}$, respectively. We denote the set $\{-\mathbf{1},\mathbf{1}\}$ as $\mathbb{S}^0$, in accordance to the name "0 dimensional unit sphere".

Thus we can generalize the concept of "functions on the unit sphere" to "functions on the 0 dimensional unit sphere": that is, a function from two points to $\mathbb{R}$. Notice that the two points are disconnected, and any functions from one point to $\mathbb{R}$ is just a constant.

Now we can express the $\mathbb{R}\backslash\{0\}$ as $\mathbb{R}\backslash\{0\} \subset \mathbb{S}^0 \times \mathbb{R}_{\geqslant 0} : x = (\mathbf{w}, r)$, where $r = |x|$; $\mathbf{w} = \mathbf{1}$ when $x > 0$ and $\mathbf{w} = -\mathbf{1}$ when $x < 0$. That is, if $r > 0, (\mathbf{1}, r)$ denotes all positive numbers while $(-\mathbf{1}, r)$ denotes all negative numbers. Notice there are two points in $\mathbb{S}^0 \times \mathbb{R}_{\geqslant 0}$ with $r = 0$, that is, $(\mathbf{1}, 0)$ and $(-\mathbf{1}, 0)$.

Notice that $\mathbb{S}^0$ has the natural discrete topology. We endow the space $\mathbb{S}^0 \times \mathbb{R}_{\geq 0}$ with the product topology. We endow $\mathbb{S}^0 \times \mathbb{R}_+$ with the product topology, and it is not hard to see that $\mathbb{R} \setminus \{0\}$ is homeomorphic to $\mathbb{S}^0 \times \mathbb{R}_+$.

Definition

Let $r = |x|$, we say a function $f(x) = f(\mathbf{w}, r)$ defined on $\mathbb{R}\backslash\{0\}$ has an asymptotic expansion $\sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j$, as $x \to 0$, where $\mathbf{w} \in \mathbb{S}^0$, $a_i(\mathbf{w})$ is a function on $\mathbb{S}^0$, if

$$\lim_{r \to 0^+} \left| f(x) - \sum_{j=m}^{M} a_j(\mathbf{w}) r^j \right| r^{-M} = 0, \quad \text{uniformly on } \mathbf{w} \in \mathbb{S}^0. \tag{3}$$

In this case we write $f(x) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j$ as $x \to 0$. In fact, we can interchange $x \to 0$ with $r \to 0^+$ here.

Example

The Heaviside function is given as

$$H(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x < 0 \end{cases} .$$

Let us write it in the above notation, then

$$H(x) = a_0(\mathbf{w}) = \begin{cases} 1 & \text{when } \mathbf{w} = \mathbf{1} \\ 0 & \text{when } \mathbf{w} = -\mathbf{1} \end{cases} , \tag{4}$$

and it admits an asymptotic expansion $H(x) \sim a_0(\mathbf{w})$ as $x \to 0$.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. It has a Taylor expansion at the origin: $f(x) \sim \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$. It may not converge. But it is an asymptotic expansion as $x \to 0$. By the Lemma above, we have an asymptotic expansion

$$f \sim \sum_{j=0}^{\infty} a_{2j} r^{2j} + a_{2j+1}(\mathbf{w}) r^{2j+1} \quad \text{as } r \to 0^+,$$

where $a_{2j} = \frac{f^{(2j)}(0)}{(2j)!}$ and

$$a_{2j+1}(\mathbf{w}) = \begin{cases} \frac{f^{(2j+1)}(0)}{(2j+1)!} & \text{when } \mathbf{w} = \mathbf{1} \\ -\frac{f^{(2j+1)}(0)}{(2j+1)!} & \text{when } \mathbf{w} = -\mathbf{1} \end{cases}.$$

Theorem

If $f(x) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j$ *as* $x \to 0$*, then the term-by-term derivative with respect to x of the expasion takes the following form in* $\mathbb{S}^0 \times \mathbb{R}_+$ *:*

$$\sum_{j=m}^{\infty} \frac{d\left(a_j(\mathbf{w}) r^j\right)}{dx} = \sum_{j=m-1}^{\infty} a_{j,1}(\mathbf{w}) r^j, \tag{5}$$

$$\text{where } a_{j,1}(\mathbf{w}) = \begin{cases} a_{j+1}(\mathbf{w})(j+1) & \text{when } \mathbf{w} = \mathbf{1} \\ -a_{j+1}(\mathbf{w})(j+1) & \text{when } \mathbf{w} = -\mathbf{1} \end{cases}. \tag{6}$$

Theorem

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$$\text{where } a_{j,1}(\mathbf{w}) = \begin{cases} a_{j+1}(\mathbf{w})(j+1) & \text{when } \mathbf{w} = \mathbf{1} \\ -a_{j+1}(\mathbf{w})(j+1) & \text{when } \mathbf{w} = -\mathbf{1} \end{cases}. \tag{6}$$

Example

The above theorem shows that the usual derivative (NOT the distributional derivative) of the Heaviside function is 0 on $\mathbb{S}^0 \times \mathbb{R}_+ = \mathbb{R}\setminus\{0\}$. Here we can clearly distinguish between the "usual derivative" and the "distributional derivative". Since we know the famous fact that the "distributinal derivative" of the Heaviside function is the Dirac delta function.

Proof.

Let us first discuss $\frac{d\left(a(\mathbf{w})r^j\right)}{dx}$, where $\mathbf{w} \in \mathbb{S}^0$, $a(\mathbf{w})$ is a function on $\mathbb{S}^0$. Clearly, $a(\mathbf{w})\, r^j$ can be viewed as a function on x when $r = |x| \neq 0$. Thus, it is legal to talk about "derivative with respect to x" in the usual sense.

Now we discuss the derivative $\frac{d\left(a(\mathbf{w})r^j\right)}{dx}$, $j \neq 0$, at $x_0 > 0$. Denote the coordinate of x_0 at $\mathbb{S}^0 \times \mathbb{R}_{\geq 0}$ as $(\mathbf{1}, x_0)$. Since $\mathbb{S}^0$ is endowed with the discrete topology, there is a small neighborhood of $(\mathbf{1}, x_0)$, denoted as $\{\mathbf{1}\} \times [x_0 - \delta, x_0 + \delta]$, on which the function $a(\mathbf{w})\, r^j$ equals $a(\mathbf{1})\, x^j$, where $a(\mathbf{1})$ is the value of the function $a(\mathbf{w})$ at $\mathbf{w} = \mathbf{1}$, namely, a constant. Thus we have

$$\left.\frac{d\left(a(\mathbf{w})\, r^j\right)}{dx}\right|_{x=x_0} = \left.\frac{d\left(a(\mathbf{1})\, x^j\right)}{dx}\right|_{x=x_0} = a(\mathbf{1})\, j x^{j-1}\big|_{x=x_0}.$$

The case when $x_0 < 0$ is similar.

When $j = 0$, a similar analysis shows that $\left.\frac{d(a(\mathbf{w}))}{dx}\right|_{x=x_0} = 0$ when $x_0 \neq 0$.

Comment:

When $x = 0$, $a(\mathbf{w})$ can be viewed as a multi-valued function on x. On the other hand, if $j > 0$, then $a(\mathbf{w})\, r^j = 0$ at $x = 0$. We can talk about the so called "left-derivative" and "right-derivative" of $a(\mathbf{w})\, r^j$, $j > 0$ at $x = 0$. It is not hard to see, the "left-derivative" and "right-derivative" are both 0 when $j > 1$; and the "right-derivative" equals $a(\mathbf{1})$ while the "left-derivative" equals $-a(-\mathbf{1})$ when $j = 1$, at $x = 0$.

Reconstruction of the space of test functions on $\mathbb{R}$ with a thick point

Definition

Let $\mathcal{D}_{*,a}(\mathbb{R})$ denote the vector space of all compactly supported smooth functions ϕ defined in $\mathbb{R}\backslash\{a\}$, that admit a strong asymptotic expansion of the form

$$\phi(a+x) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w})\, r^j, \qquad \text{as } x \to 0. \tag{7}$$

where $a_j(\mathbf{w})$ is a function on $\mathbb{S}^0$ as defined above in definition 10. We

call $\mathcal{D}_{*,a}(\mathbb{R})$ "the space of test functions on $\mathbb{R}$ with a thick point located at $x=a$". We denote $\mathcal{D}_{*,0}(\mathbb{R})$ as $\mathcal{D}_{*}(\mathbb{R})$.

Recall that a thick test function defined in [Estrada, Fulling; Int. J. Appl. Math. Stat. (2007)] is a compactly supported function ϕ with domain $\mathbb{R}$, smooth in $\mathbb{R}\setminus\{a\}$, and at $x=a$ all its one-sided derivatives,

$$\phi^{(n)}(a\pm 0)=\lim_{x\to a^{\pm}}\phi^{(n)}(x), \quad \forall n\in\mathbb{N},$$

exist. Here let us introduce a different notation $\mathscr{D}_{*,a}^{old}(\mathbb{R})$ to denote the space of such functions. One can see that any function in $\mathscr{D}_{*,a}^{old}(\mathbb{R})$ admits a strong asymptotic expansion

$$\phi(a+x)\sim\sum_{j=0}^{\infty}a_j(\mathbf{w})\,r^j, \quad \text{as } x\to 0, \tag{8}$$

$$\textit{where } a_j(\mathbf{w})=\begin{cases}\frac{\phi^{(j)}(a+0)}{j!} & \text{when } \mathbf{w}=\mathbf{1}\\ (-1)^j\frac{\phi^{(j)}(a-0)}{j!} & \text{when } \mathbf{w}=-\mathbf{1}\end{cases}.$$

In particular, if $\phi(x)$ has a jump discontinuity at $x=a$, then in the expansion (8), $a_0(\mathbf{1})\neq a_0(-\mathbf{1})$.

Lemma

Thus there are natural inclusion maps :

$$\mathscr{D}\left(\mathbb{R}\right) \hookrightarrow \mathscr{D}_{*,a}^{old}\left(\mathbb{R}\right) \hookrightarrow \mathscr{D}_{*,a}\left(\mathbb{R}\right). \tag{9}$$

With the following topology, $\mathscr{D}_{*,a}^{old}\left(\mathbb{R}\right) \subseteq \mathscr{D}_{*,a}\left(\mathbb{R}\right)$, as a closed subspace. Moreover, $\mathscr{D}_{*,a}^{old}\left(\mathbb{R}\right)$ is closed in $\mathscr{D}_{*,a}\left(\mathbb{R}\right)$ with respect to derivatives.

Definition

Let m be a fixed integer and K a compact subset of $\mathbb{R}$ whose interior contains a. The topology of $\mathscr{D}_{*,a}^{[m,K]}(\mathbb{R})$ is given by the seminorms $\left\{\|\|_{q,s}\right\}_{s\geq 0}$ defined as

$$\|\phi\|_{q,s} = \sup_{x+a\in K} \sup_{0\leq p\leq s} r^{-q} \left| (d/dx)^p \phi(a+x) - \sum_{j=m-p}^{q-1} a_{j,p}(\mathbf{w}) r^j \right|,$$

where $(d/dx)^p \phi(a+x) \sim \sum_{j=m-p}^{q-1} a_{j,p}(\mathbf{w}) r^j$. The topology of $\mathscr{D}_{*,a}^{[m]}(\mathbb{R})$ is the inductive limit topology of the $\mathscr{D}_{*,a}^{[m,K]}(\mathbb{R})$ as $K \nearrow \infty$. The topology of $\mathscr{D}_{*,a}(\mathbb{R})$ is the inductive limit topology of $\mathscr{D}_{*,a}^{[m]}(\mathbb{R})$ as $m \searrow -\infty$.

Space of distributions on $\mathbb{R}$ with a thick point

Definition

The space of distributions on $\mathbb{R}$ with a thick point at $x = a$ is the dual space of $\mathscr{D}_{*,a}(\mathbb{R})$. We denote it $\mathscr{D}'_{*,a}(\mathbb{R})$, or just as $\mathscr{D}'_{*}(\mathbb{R})$ when $a = 0$. We call the elements of $\mathscr{D}'_{*,a}(\mathbb{R})$ "thick distributions".

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Let $\pi : \mathscr{D}'_{*,a}(\mathbb{R}) \to \mathscr{D}'(\mathbb{R})$, be the projection operator, dual to the inclusion $i : \mathscr{D}(\mathbb{R}) \hookrightarrow \mathscr{D}_{*,a}(\mathbb{R})$. Since $\mathscr{D}(\mathbb{R})$ is closed in $\mathscr{D}_{*,a}(\mathbb{R})$, by the Hanh-Banach theorem we have the following result.

Theorem

Let f be any distribution in $\mathscr{D}'(\mathbb{R})$, then there exist thick distributions $g \in \mathscr{D}'_{*,a}(\mathbb{R})$ such that $\pi(g) = f$.

Naturally, if $f \in \mathscr{D}'(\mathbb{R})$ then there are infinitely many thick distributions g with $\pi(g) = f$.

Before giving examples of thick distributions, recall that by the convention of the discrete measure, the "integral" of a function on a discrete set is just the summation of the function over these discrete points, we have

$$\int_{S^0} \phi(\mathbf{w})\, d\sigma(\mathbf{w}) = \phi(\mathbf{1}) + \phi(-\mathbf{1}).$$

Before giving examples of thick distributions, recall that by the convention of the discrete measure, the "integral" of a function on a discrete set is just the summation of the function over these discrete points, we have

$$\int_{S^0} \phi(\mathbf{w})\, d\sigma(\mathbf{w}) = \phi(\mathbf{1}) + \phi(-\mathbf{1}).$$

Example

For example, for the Heaviside function as in equation (4): $H(x) = a_0(\mathbf{w})$,

$$\int_{S^0} a_0(\mathbf{w})\, d\sigma(\mathbf{w}) = 1. \tag{10}$$

For a constant function $\phi(\mathbf{w}) \equiv 1$,

$$\int_{S^0} 1 d\sigma(\mathbf{w}) = \int_{S^0} d\sigma(\mathbf{w}) = 2. \tag{11}$$

Thus one can discuss the ”double integral” on $\mathbb{S}^0 \times \mathbb{R}_+$ if it exists:

$$\begin{aligned}\int_{S^0}\int_0^{+\infty} \phi(\mathbf{w}, r)\, dr d\sigma(\mathbf{w}) &= \int_0^{+\infty} [\phi(\mathbf{1}, r) + \phi(-\mathbf{1}, r)]\, dr \\ &= \int_0^{+\infty} \phi(x)\, dx + \int_0^{-\infty} \phi(x)\, d(-x) \\ &= \int_0^{+\infty} \phi(x)\, dx + \int_{-\infty}^{0} \phi(x)\, dx.\end{aligned} \tag{12}$$

Clearly, if $\phi \in \mathscr{D}(\mathbb{R})$ is a usual test function, then

$$\int_{S^0}\int_0^{+\infty} \phi(\mathbf{w}, r)\, dr d\sigma(\mathbf{w}) = \int_{-\infty}^{+\infty} \phi(x)\, dx,$$

is just a normal integral over $\mathbb{R}$.

Definition

Let f be a locally integrable function definied in $\mathbb{R}\backslash\{a\}$. The thick distribution $Pf\,(f\,(x))$ is defined as

$$\begin{aligned}\langle Pf\,(f\,(x))\,,\phi\,(x)\rangle &= F.p.\int_{-\infty}^{+\infty} f\,(x)\,\phi\,(x)\,dx \qquad (13)\\ &= F.p.\lim_{\varepsilon\to 0^{+}}\int_{|x-a|\geq\varepsilon} f\,(x)\,\phi\,(x)\,dx, \quad \phi\in\mathscr{D}_{*,a}\,(\mathbb{R})\,,\end{aligned}$$

provided that the finite part integrals exist for all $\phi\in\mathscr{D}_{*,a}\,(\mathbb{R})$.

Similar to the higher dimensional case, although the finite part limit is not defined for all locally integrable functions f, $Pf\left(f\left(x\right)\right)$ is defined in many important and interesting cases.

Example

Since $F.p.\int_0^A r^{\alpha}dr = A^{\alpha+1}/\left(\alpha+1\right), \alpha \neq -1, F.p.\int_0^A r^{-1}dr = \log A$, we obtain that if $\lambda \notin \mathbb{Z}$ then

$$\begin{aligned}
&\left\langle Pf\left(\left|x-a\right|^{\lambda}\right), \phi\left(x\right)\right\rangle \\
&= \int_{\left|x-a\right|\geq A} \left|x-a\right|^{\lambda} \phi\left(x\right)dx \\
&+ \int_{\left|x-a\right|<A} \left|x-a\right|^{\lambda}\left(\phi\left(x\right) - \sum_{j\leq -\operatorname{Re}\lambda-1} a_j\left(w\right)\left|x-a\right|^{j}\right)dx \\
&+ \sum_{j\leq -\operatorname{Re}\lambda-1}\left(a_j\left(\mathbf{1}\right)+a_j\left(-\mathbf{1}\right)\right)\frac{A^{\lambda+j+1}}{\lambda+j+1},
\end{aligned} \tag{14}$$

Example (conti..)

while if $\lambda = k \in \mathbb{Z}$ then

$$\left\langle Pf\left(|x-a|^{\lambda}\right), \phi(x)\right\rangle$$

$$= \int_{|x-a|\geq A} |x-a|^{k}\phi(x)\,dx \tag{15}$$

$$+ \int_{|x-a|<A} |x-a|^{k}\left(\phi(x) - \sum_{j\leq -k-1} a_j(w)\,|x-a|^{j}\right)dx$$

$$+ \sum_{j<-k-1}\left(a_j(\mathbf{1}) + a_j(-\mathbf{1})\right)\frac{A^{\lambda+j+1}}{\lambda+j+1} + \left(a_{-k-1}(\mathbf{1}) + a_{-k-1}(-\mathbf{1})\right)\log A.$$

Formulas (14) and (15) hold for any $A > 0$. The finite part is needed for all λ in the space of thick distributions $\mathscr{D}'_{*,a}(\mathbb{R})$.

Example (Heaviside Function)

This example will be the "finite part regularization" of the Heaviside function. The regulariztion is needed because of the singularity of the thick test functions: let $\phi(x) \in \mathscr{D}_*(\mathbb{R})$, the $Pf(H(x)) \in \mathscr{D}'_*(\mathbb{R})$ is defined as:

$$\langle Pf(H(x)), \phi(x)\rangle$$

$$= \int_{|x-a|\geq A} H(x)\,\phi(x)\,dx + \int_{|x-a|<A} H(x)\left(\phi(x) - \sum_{j\leq -1} a_j(w)\,r^j\right)dx \tag{16}$$

$$+ \sum_{j<-1} a_j(\mathbf{1})\,\frac{A^{j+1}}{j+1} + a_{-1}(\mathbf{1})\log A.$$

Definition (thick delta function of degree q)

Let $g(\mathbf{w})$ be a distribution in $\mathbb{S}^0$, the thick delta function of degree q, denoted as $g\delta_*^{[q]}$, acts on a thick test function $\phi(x)$ as

$$\left\langle g\delta_*^{[q]}, \phi \right\rangle_{\mathscr{D}'_*(\mathbb{R})\times\mathscr{D}_*(\mathbb{R})} = \frac{1}{C_0}\left\langle g(\mathbf{w}), a_q(\mathbf{w})\right\rangle_{\mathscr{D}'_*(\mathbb{S})\times\mathscr{D}_*(\mathbb{S})},$$

where $\phi(x) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j$,as $x \to 0$, and $C_0 = 2$.

In particular, if $g(x) \equiv 1, q = 0$ then we obtain the one-dimensional "plain thick delta function" δ_*, given as

$$\langle \delta_*, \phi \rangle_{\mathscr{D}'_*(\mathbb{R})\times\mathscr{D}_*(\mathbb{R})} = \frac{1}{C_0}\int_{\mathbb{S}^0} a_0(\mathbf{w})\, d\sigma(\mathbf{w}) = \frac{a_0(\mathbf{1})}{2} + \frac{a_0(-\mathbf{1})}{2}.$$

Remark:

1. If $\phi \in \mathscr{D}(\mathbb{R})$ is a usual test function, then

$$\langle \pi(\delta_*), \phi \rangle = \langle \delta_*, i(\phi) \rangle = \frac{\phi(0)}{2} + \frac{\phi(0)}{2} = \phi(0),$$

hence $\pi(\delta_*) = \delta$.

2. If $\phi \in \mathscr{D}_*^{old}(\mathbb{R})$, $\pi' : \mathscr{D}'_*(\mathbb{R}) \to \mathscr{D}_*'^{old}(\mathbb{R})$, let $\phi_+(0)$ denote $\lim_{x\to 0^+} \phi(x)$ and $\phi_-(0)$ denote $\lim_{x\to 0^-} \phi(x)$. Then

$$\left\langle \pi'(\delta_*), \phi \right\rangle = \frac{1}{2}\phi_+(0) + \frac{1}{2}\phi_-(0)$$

Example

Let $g_\lambda(\mathbf{w})$ be a distribution in $\mathbb{S}^0$:
$\langle g_\lambda(\mathbf{w}), a(\mathbf{w})\rangle = 2\lambda a(\mathbf{1}) + 2(1-\lambda) a(-\mathbf{1})$,
where $0 \leq \lambda \leq 1$ is a constant. Then

$$\left\langle g_\lambda \delta_*^{[q]}, \phi \right\rangle_{\mathscr{D}'_*(\mathbb{R}) \times \mathscr{D}_*(\mathbb{R})} = \lambda a_q(\mathbf{1}) + (1-\lambda) a_q(-\mathbf{1}).$$

In particular, if $\lambda = 1$,

$$\left\langle g_1 \delta_*^{[q]}, \phi \right\rangle_{\mathscr{D}'_*(\mathbb{R}) \times \mathscr{D}_*(\mathbb{R})} = a_q(\mathbf{1}).$$

If $\phi \in \mathscr{D}_*^{old}(\mathbb{R})$, then

$$\left\langle \pi'(g_1 \delta_*), \phi \right\rangle = \phi_+(0).$$

Algebraic and analytic operations in $\mathscr{D}'_{*,a}(\mathbb{R})$

1. $\langle f+\lambda g,\phi\rangle = \langle f,\phi\rangle + \lambda\langle g,\phi\rangle$.
2. $\langle f(x+c),\phi(x)\rangle = \langle f(x),\phi(x-c)\rangle$.
3. $\langle f(cx),\phi(x)\rangle = \frac{1}{|c|}\langle f(x),\phi(x/c)\rangle$.
4. $\langle \psi\rho,\phi\rangle := \langle \rho,\psi\phi\rangle$. ψ is called a “multiplier” of $\mathscr{D}_{*,a}(\mathbb{R})$ and $\mathscr{D}'_{*,a}(\mathbb{R})$.

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Example

The Heaviside function $H(x) = a_0(\mathbf{w}) = \begin{cases} 1 & \text{when } \mathbf{w} = \mathbf{1} \\ 0 & \text{when } \mathbf{w} = -\mathbf{1} \end{cases}$ is NOT a multiplier of $\mathscr{D}'(\mathbb{R})$, but it is a multiplier of $\mathscr{D}'_{*}(\mathbb{R})$.

Definition (derivatives)

If $f \in \mathscr{D}'_{*,a}(\mathbb{R})$ then its thick distributional derivative d^*f/dx is defined as

$$\left\langle \frac{d^*f}{dx}, \phi \right\rangle = -\left\langle f, \frac{d\phi}{dx} \right\rangle, \quad \phi \in \mathscr{D}_{*,a}(R).$$

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Remark:

$$\frac{d^*(\psi f)}{dx} = \frac{d\psi}{dx} f + \psi \frac{d^*f}{dx}, \quad f \in \mathscr{D}'_{*,a}(\mathbb{R}), \psi \in \mathscr{E}_{*,a}(\mathbb{R}).$$

Example

Now let us compute the thick distributional derivative of the Heaviside function. Now suppose $d\phi/dx$ has asymptotic expansion $d\phi/dx \sim \sum_{j=m}^{+\infty} b_j(\mathbf{w})\, r^j$ and ϕ has the asymptotic expansion $\phi \sim \sum_{j=m+1}^{+\infty} a_j(\mathbf{w})\, r^j$. Then

$$\left\langle \frac{d^*(Pf(H(x)))}{dx}, \phi \right\rangle = -\left\langle Pf(H(x)), \frac{d\phi}{dx} \right\rangle$$

$$= -\int_A^{+\infty} \frac{d\phi}{dx} dx - \int_0^A \left(\frac{d(\phi(x))}{dx} - \sum_{j\leq -1} b_j(w)\, r^j \right) dx$$

$$- \sum_{j<-1} b_j(\mathbf{1}) \frac{A^{j+1}}{j+1} - b_{-1}(\mathbf{1}) \log A$$

$$= \phi(A) - \phi(A) + a_0(\mathbf{1}) + 0 = a_0(\mathbf{1}) = \langle g_1 \delta_*, \phi \rangle_{\mathscr{D}'_*(\mathbb{R}) \times \mathscr{D}_*(\mathbb{R})}$$

Thus the derivative of the Heaviside function is $g_1 \delta_*$.

Example (conti..)

Now consider the projection of the derivative of the Heaviside function $d^*(Pf(H(x)))/dx = g_1\delta_*$ onto the usual distribution space $\mathscr{D}'(\mathbb{R})$:

$$\langle \pi(g_1\delta_*), \phi\rangle = \langle g_1\delta_*, i(\phi)\rangle = \phi(0) = \langle \delta, \phi\rangle . \tag{17}$$

Keep in mind that $\pi(Pf(H(x))) = H(x)$, the usual Heaviside function. Hence $\pi(d^*(Pf(H(x)))/dx) = \delta(x) = \overline{d}(\pi(Pf(H(x))))/dx$ as expected.

An application

Problem

Paskusz [IEEE Trans. Ed. 43 (2000)] pointed out that the following proof is problematic, where $H(x)$ is the usual Heaviside function: Since $H(x) = H^2(x)$, taking the distributional derivative on both sides, we have $\delta(x) = 2H(x)\delta(x)$. Hence $H(x)\delta(x) = \frac{1}{2}\delta(x)$. However, if we multiply $H(x)$ on both sides again we will get $\frac{1}{2}\delta(x) = H(x)\delta(x) = H^2(x)\delta(x) = \frac{1}{2}H(x)\delta(x) = \frac{1}{4}\delta(x)$, hence we have $\frac{1}{2} = \frac{1}{4}$, which is clearly wrong.

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The key observation of this mistake is that $H(x) \cdot H(x)$ is not a well-defined distribution, that is, $\left\langle H^2(x), \phi(x)\right\rangle_{\mathscr{D}'(\mathbb{R})\times\mathscr{D}(\mathbb{R})} = \langle H(x), H(x)\phi(x)\rangle$ is not well-defined since $H(x)\phi(x)$ is not a usual test function in $\mathscr{D}(\mathbb{R})$: it has a jump discontinuity. Thus we cannot simply apply the distributional derivative on both sides of the equaion $H(x) = H^2(x)$.

Solution

In the sense of thick distributions we can restate this whole story in a rigorous way thus to avoid the mistake $\frac{1}{2} = \frac{1}{4}$.

In fact, it's easy to see that $H(x)$ *is a multiplier of the thick distributions, i.e.* $H(x)f(x) \in \mathscr{D}'_*(\mathbb{R})$ *for any* $f(x) \in \mathscr{D}'_*(\mathbb{R})$. *Thus* $H(x) \cdot H(x)$ *should be viewed as a multiplier times a thick distribution:* $H(x) \cdot Pf(H(x))$. *Then*

$$\begin{aligned}\frac{d^*(H(x) \cdot Pf(H(x)))}{dx} &= \frac{d(H(x))}{dx} Pf(H(x)) + H(x) \frac{d^*(Pf(H(x)))}{dx} \\ &= 0 + g_1 \delta_*\end{aligned}$$

And observe that $\pi\left(\frac{d^*(H(x) \cdot Pf(H(x)))}{dx}\right) = \delta$.
On the other hand, it is clear that $H(x) \cdot Pf(H(x)) = Pf(H(x))$, *taking derivatives on both sides yields*

$$g_1 \delta_* = g_1 \delta_*.$$

Thank you!