Invertibility of matrix type operators of infinite order with exponential off-diagonal decay

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We study the decay rate of the entries in the inverse of an invertible matrix type operators $A = (A_{s,t})_{\Lambda\times\Lambda} \in \mathcal{L}(\ell^2(\Lambda))$ such that

$$(\exists C_p > 0) \ \ |A_{s,t}| \leq C_p e^{-pd(s,t)} \quad \text{for all } s, t \in \Lambda,$$

where

- the index set $\Lambda \subseteq \mathbb{R}^d$ is a lattice, i.e. $\Lambda = G\mathbb{Z}^d$ for some non-singular matrix G;
- the function $d : \Lambda^2 \to \mathbb{R}$ is a distance on Λ.

We call this decay *exponential off-diagonal decay of order p*.

Outline

Recall, for non-zero $g \in L^2(\mathbb{R}^d)$ the set of time-frequency shifts $\mathcal{G}(g, \Lambda) = \{g_s : s \in \Lambda\}$, with

$$g_s = g_{s_1, s_2} = \pi(s_1, s_2)g = e^{2\pi i s_2 x} g(x - s_1), \quad s = (s_1, s_2) \in \Lambda = G\mathbb{R}^{2d}$$

is a Gabor frame, if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{s \in \Lambda} |\langle f, g_s \rangle|^2 \leq B\|f\|_2^2, \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

Then there is also a dual window $h \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(h, \Lambda)$ is a frame, and every $f \in L^2(\mathbb{R}^d)$ possesses the frame expansions

$$f = \sum_{s \in \Lambda} \langle f, g_s \rangle h_s = \sum_{s \in \Lambda} \langle f, h_s \rangle g_s.$$

The Gabor decomposition of an operator T is then as follows

$$Tf(x) = \sum_{s \in \Lambda} \sum_{t \in \Lambda} \langle Tg_t, g_s \rangle c_t h_s, \quad \text{with } c_t = \langle f, h_t \rangle.$$

The infinite matrix $(A_{s,t})_{s,t \in \Lambda} = \{\langle Tg_t, g_s \rangle\}_{s,t \in \Lambda}$ is the Gabor matrix of the operator T.

E. Cordero, F. Nicola, L. Rodino: Exponentially sparse representations of Fourier integral operators, Rev. Mat. Iberoam. 31(2) (2015) 461–476.

Let T be FIO:

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,\eta)} \sigma(x,\eta) \hat{f}(\eta) d\eta$$

with non-degenerate phase function Φ satisfying

$$|\partial^\alpha \Phi(z)| \leq cC^{|\alpha|}(\alpha!)^r, \quad \alpha \in \mathbb{N}^{2d},\ |\alpha| \geq 2,\ z = (x,\eta) \in \mathbb{R}^{2d},$$

and amplitude σ satisfying

$$|\partial^\alpha \sigma(z)| \leq cC^{|\alpha|}(\alpha!)^r, \quad \alpha \in \mathbb{N}^{2d},\ z = (x,\eta) \in \mathbb{R}^{2d}.$$

Let $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be the canonical transformation associated to Φ. In the generic case $r \geq 1$, fix a window $g \in S^{r/2}_{r/2}(\mathbb{R}^d)$. Then for some $\varepsilon > 0$

$$|A_{s,t}| \leq ce^{-\varepsilon |s-\chi(t)|^{1/r}}.$$

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Let $\mathcal{G}(k, \Lambda)$ be a Gabor frame and let T be the frame operator given by $Tf = \sum_{r\in\Lambda}\langle f, k_r\rangle k_r,\ f \in L^2(\mathbb{R}^d)$. Then

$$Tf = \sum_{s\in\Lambda}\sum_{t\in\Lambda}\sum_{r\in\Lambda}\langle g_t, k_r\rangle\langle k_r, h_s\rangle c_t g_s, \quad \text{with } c_t = \langle f, h_t\rangle.$$

The infinite matrix

$$(A_{s,t})_{s,t\in\Lambda} = \left\{\sum_{r\in\Lambda}\langle g_t, k_r\rangle\langle k_r, h_s\rangle\right\}_{s,t\in\Lambda} = \{\langle Tg_t, h_s\rangle\}_{s,t\in\Lambda}$$

is the Gabor matrix of the operator T.

Then Gabor frame $\mathcal{G}(k, \Lambda)$ is polynomially or (sub-)exponentially localized with respect to $\mathcal{G}(g, \Lambda)$ if corresponding Gabor matrix A is of polynomial or (sub-)exponential off-diagonal decay.

$m-$banded matrices and Jaffard's theorem

A matrix $A = (A_{s,t})_{\Lambda\times\Lambda}$ is $m-$banded if $A_{s,t} = 0$ for $s, t \in \Lambda$, $|s - t| > m$. It is proved that for an $m-$banded invertible matrix operator $A \in \mathcal{L}(\ell^2(\Lambda))$ with $(A^{-1}_{s,t})_{\Lambda\times\Lambda} = A^{-1} \in \mathcal{L}(\ell^2(\Lambda))$, the following estimate holds true

$$|A^{-1}_{s,t}| \leq Ce^{\frac{1}{m}(\ln(1-\frac{2}{\sqrt{\kappa}+1})|s-t|)} \quad \text{for all } s, t \in \Lambda,$$

where C, m and κ are a certain positive constants.

S. Demko, W. F. Moss, P. W. Smith, Decay rates for inverses of band matrices, Math. of Comput. 43(168) (1984) 491–499.

Denote by $\mathcal{E}_\gamma$, $\gamma > 0$, the space of matrices $A = (A_{s,t})_{\Lambda\times\Lambda}$ whose entries satisfy:

$$(\exists C_{A,\gamma} \geq 1) \;\; |A_{s,t}| \leq C_{A,\gamma}e^{-\gamma d(s,t)} \quad \text{for all } s, t \in \Lambda.$$

Let $A : \ell^2(\Lambda) \to \ell^2(\Lambda)$ be an invertible matrix on $\ell^2(\Lambda)$. If $A \in \mathcal{E}_\gamma$, then $A^{-1} \in \mathcal{E}_{\gamma_1}$ for some $\gamma_1 \in (0, \gamma)$.

S. Jaffard, Properiétés des matrices 'bien loclisées' prés de leur diagonale et queleques applications, Ann. Inst. H. Poincaré Anal. Non Linéaire 7(5) (1990) 461–476.

polynomial and sub-exponential off-diagonal decay

Assume that $\rho : [0,\infty) \to [0,\infty)$ is a strictly increasing concave and normalised ($\rho(0) = 0$) function that satisfies $\lim_{\xi\to\infty} \rho(\xi)/\xi = 0$, e.g. $\rho(\xi) = c\xi^{\beta}$, $\beta \in (0,1)$, $\xi \geq 0$.
Let $k > d$ and let the weight v be given by $v(x) = e^{\rho(|x|)}(1+|x|)^k$, $x \in \mathbb{Z}^d$. Then if there exists $C > 0$ such that

$$|A_{s,t}| \leq Cv(s-t)^{-1}, \quad s,t \in \mathbb{Z}^d,$$

then there is $C_1 > 0$ such that

$$|A^{-1}_{s,t}| \leq C_1 v(s-t)^{-1}, \quad s,t \in \mathbb{Z}^d.$$

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Example

Consider the matrix $A = I - \Gamma$, where I is the identity matrix and the elements of $\Gamma = (\Gamma_{s,t})_{\mathbb{Z}\times\mathbb{Z}}$ are given by $\Gamma_{s,t} = e^{-1/k}$ if $s+1=t$, $s,t \in \mathbb{Z}$ and $\Gamma_{s,t} = 0$ otherwise, i.e.

$$A = I - \Gamma = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 1 & -e^{-1/k} & 0 & 0 & \dots \\ \dots & 0 & 1 & -e^{-1/k} & 0 & \dots \\ \dots & 0 & 0 & 1 & -e^{-1/k} & \dots \\ \dots & 0 & 0 & 0 & 1 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Clearly, for every $\gamma > 0$ there exists $C_\gamma > 0$ such that

$$|A_{s,t}| \leq C_\gamma e^{-\gamma|s-t|}, \qquad s,t \in \mathbb{Z};$$

i.e. A belongs to $\bigcap_{\gamma>0} \mathcal{E}_\gamma$.
As $\|\Gamma\| = e^{-1/k} < 1$, A^{-1} exists and $A^{-1} = \sum_{n=0}^{\infty} \Gamma^n$.

Example

The entries in the inverse $A^{-1} = (B_{s,t})_{\mathbb{Z}\times\mathbb{Z}}$ are as follows:
$B_{s,t} = e^{-(1/k)|s-t|}$ for $s \leq t,\ s,t \in \mathbb{Z}$ and $B_{s,t} = 0$ for $s > t,\ s,t \in \mathbb{Z}$;

$$A^{-1} = B = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 1 & e^{-1/k} & e^{-2/k} & e^{-3/k} & \dots \\ \dots & 0 & 1 & e^{-1/k} & e^{-2/k} & \dots \\ \dots & 0 & 0 & 1 & e^{-1/k} & \dots \\ \dots & 0 & 0 & 0 & 1 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Hence, one obtains

$$|B_{s,t}| = e^{-(1/k)|s-t|}, \quad \text{for} \quad s \leq t,\ s,t \in \mathbb{Z},$$

so $A^{-1} \in \mathcal{E}_\gamma$ only for $\gamma \leq 1/k$.

- The super-exponential decay can not give the spectral invariance. Let $\beta > 1$ and $|A_{s,t}| \leq C_p e^{-p|s-t|^\beta}$. The infinite matrix in this example satisfies this condition but its inverse has only exponential decay.

Classification with respect to decay rate

Consider the class of matrix type operator A satisfying the estimate

$$|A_{s,t}| \leq C\omega, \;\; s, t \in \Lambda.$$

Then for

- $\omega = \omega_p = (1 + |s - t|)^{-k/2}$, for some $k > 0$, the class of matrices with polynomial off-diagonal decay is spectrally invariant;
- $\omega = \omega_{se} = e^{-k|s-t|^{\beta}}$ for some $\beta \in (0, 1)$, the class of matrices with sub-exponential off-diagonal decay is spectrally invariant;
- $\omega = \omega_e = e^{-k|s-t|}$, (i.e. the class of matrices with exponential off-diagonal decay), A^{-1} is with exponential off-diagonal decay but of different order;
- $\omega = \omega_{Se} = e^{-k|s-t|^{\beta}}$ for some $\beta > 1$, (i.e. the class of matrices with super-exponential off-diagonal decay), A^{-1} is not even with super-exponential off-diagonal decay, in general.

Recall that spectral invariance (or Wiener type property) means that if A is invertible in ℓ^2, then its inverse has the same off-diagonal decay order.

- Let the index set $\Lambda \subseteq \mathbb{R}^d$ be a lattice, i.e. $\Lambda = G\mathbb{Z}^d$ for some non-singular matrix G;
- endowed with a distance $d : \Lambda^2 \to \mathbb{R}$ which satisfies the following assumption:

$$m_\varepsilon := \sup_{s \in \Lambda} \sum_{t \in \Lambda} e^{-\varepsilon d(s,t)} < \infty, \quad \forall \varepsilon > 0.$$

Note that $m_\varepsilon \geq 1$, $\forall \varepsilon > 0$, the function $\varepsilon \mapsto m_\varepsilon$, $(0, \infty) \to [1, \infty)$, is decreasing and $m_\varepsilon \to \infty$, as $\varepsilon \to 0^+$.

- Let $A : \ell^2(\Lambda) \to \ell^2(\Lambda)$ be matrix type operator denoted by $A = (A_{s,t})_{\Lambda \times \Lambda}$ and so $A \in \mathcal{L}(\ell^2(\Lambda))$.

Theorem 1

Let $A = (A_{s,t})_{\Lambda\times\Lambda} : \ell^2(\Lambda) \to \ell^2(\Lambda)$ be an invertible matrix on $\ell^2(\Lambda)$ with inverse $(A^{-1}_{s,t})_{\Lambda\times\Lambda} = A^{-1} \in \mathcal{L}(\ell^2(\Lambda))$.
Assume that $A \in \mathcal{E}_\gamma$, that is, there exists $C_\gamma \geq 1$ so that

$$|A_{s,t}| \leq C_\gamma e^{-\gamma d(s,t)} \quad \text{for all} \quad s,t \in \Lambda.$$

Then there exist constants $\gamma_1 \in (0,\gamma)$ and $C_{A,\gamma_1} > 0$ such that

$$|A^{-1}_{s,t}| \leq C_{A,\gamma_1} e^{-\gamma_1 d(s,t)} \quad \text{for all} \quad s,t \in \Lambda.$$

Furthermore,

$$\gamma_1 = \min\left\{\delta, \frac{(\gamma'-\delta)\ln(1/r)}{\ln\left(\tilde{C}C_\gamma^2 r^{-1} m^2_{(\gamma-\gamma')/2}\right)}\right\}, \quad C_{A,\gamma_1} = \frac{2C_\gamma m_{\gamma-\gamma_1}}{(1-r)\|A\|^2}$$

where $\gamma', \delta \in (0,\gamma)$, $0 < \delta < \gamma' < \gamma$ are arbitrary, $r = \|\operatorname{Id} - \|A\|^{-2}AA^*\|$ and $\tilde{C} = 1 + \|A\|^{-2}$.

- Recall, for arbitrary $k \in \mathbb{Z}_+$, there is a matrix $A \in \bigcap_{\gamma>0} \mathcal{E}_\gamma$ such that $A^{-1} \in \mathcal{E}_{\gamma_1}$ only for $\gamma_1 \leq 1/k$. So, at best, one can claim that if $A \in \bigcap_{\gamma>0} \mathcal{E}_\gamma$, the inverse $A^{-1} \in \bigcup_{\gamma>0} \mathcal{E}_\gamma$.

Assume that the matrix type operator $A = (A_{s,t})_{\Lambda\times\Lambda} : \ell^2(\Lambda) \to \ell^2(\Lambda)$ is such that $A \in \bigcap_{p\in\mathbb{N}} \mathcal{E}_p$, i.e. A satisfies

$$(\forall p \geq 1)(\exists C_p \geq 1)\ |A_{s,t}| \leq C_p e^{-pd(s,t)}, \text{ for all } s,t \in \Lambda. \tag{1}$$

Let φ be a strictly increasing function $\varphi : [1,\infty) \to [1,\infty)$ which satisfies the following condition:

$$\limsup_{p\to\infty} \frac{C_p}{e^{p\varphi(p)}} = K \in [1,\infty). \tag{2}$$

- If the function φ is bounded, then there exists $k_0 \geq 1$ such that $A_{s,t} = 0$ for $d(s,t) > k_0$ (i.e., the band limited case).
- If the function φ is unbounded, i.e. $\lim_{p\to\infty} \varphi(p) = \infty$, then

$$|A_{s,t}| \leq K_1 e^{p(\varphi(p)-d(s,t))}, \quad \text{for all } s,t \in \Lambda,\ p \geq 1,$$

with $K_1 = \sup_{p\in[1,\infty)} C_p e^{-p\varphi(p)} \geq 1$.

For the next theorem we assume $\varphi : [1,\infty) \to [1,\infty)$ has the following properties:

(*a*) $\lim_{p\to\infty} \varphi(p) = \infty$ and $\varphi(1) = 1$,

(*b*) there exists $a > 1$ such that $\varphi(\xi p) \leq \xi^{a-1}\varphi(p),\ p,\xi \in [1,\infty)$.

The fact that φ is strictly increasing together with (a) and (b) implies that φ is continuous and consequently bijective.

Example.
$\varphi(p) = p^\alpha$, $\alpha > 0$ and $\varphi(p) = \ln(p + e - 1)$
Products of such functions also satisfy the above conditions.

Theorem 2

Assume the matrix type operator $A = (A_{s,t})_{\Lambda\times\Lambda} : \ell^2(\Lambda) \to \ell^2(\Lambda)$ is invertible with $(A^{-1}_{s,t})_{\Lambda\times\Lambda} = A^{-1}$ being its inverse.
If A satisfies (1) and (2), with $\varphi : [1,\infty) \to [1,\infty)$ a strictly increasing function which satisfies (2), (a) and (b), then there exist $C_A, b > 0$ such that

$$|A^{-1}_{s,t}| \leq C_A e^{-bd(s,t)}, \quad \text{for all } s,t \in \Lambda.$$

Furthermore,

$$b = \frac{\ln(1/r)}{\ln(\tilde{C}K_1^2 m_1^2 r^{-1}) + 2\cdot 4^a} \text{ and } C_A = \frac{2C_2 m_1}{(1-r)\|A\|^2},$$

where $r = \|\,\mathrm{Id} - \|A\|^{-2}AA^*\|$, $\tilde{C} = 1 + \|A\|^{-2}$, $K_1 = \sup\limits_{p\in[1,\infty)} C_p e^{-p\varphi(p)}$.

Thank you for your attention!

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