Inverse problems in anisotropic elasticity and waves

techniques from microlocal analysis, algebraic geometry, Finsler geometry

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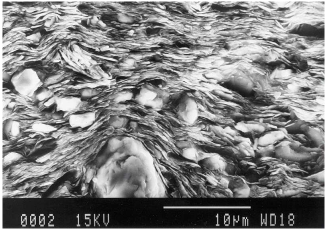

ISAAC 2021

background

anisotropy, exploration and global seismology

Anderson & Dziewonski (1982), Crampin (1984), Nataf, Nakanish & Anderson (1986), Helbig (1994)

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Finsler geometry

Antonelli, Ingarden & Matsumoto (1993), Červený (2001), Yajima & Nagahama (2009), Clayton (2015) smooth stiffness tensor $c_{ijk\ell} = c_{ijk\ell}(x)$; symmetries

$$c_{ijk\ell}(x) = c_{jik\ell}(x) = c_{k\ell ij}(x)$$

smooth density $\rho = \rho(x)$, normalized elastic moduli

$$a_{ijk\ell} = \frac{c_{ijk\ell}}{\rho}$$

elastic wave operator

$$P_{i\ell} = \delta_{i\ell}\frac{\partial^2}{\partial t^2} - \sum_{j,k} \frac{\partial}{\partial x^j} a_{ijk\ell}(x) \frac{\partial}{\partial x^k} + \text{lower order terms}$$

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principal symbol, *Christoffel matrix*

$$\Gamma_{i\ell}(x,p) := \sum_{j,k} a_{ijk\ell}(x)p_j p_k \qquad \text{we also write } \xi \text{ for } p$$

symmetric, positive definite for every $(x,p) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$

matrix representation reflecting major symmetries

Voigt notation

tensor	$a_{ijk\ell}$	11	22	33	$23, 32$	$31, 13$	$12, 21$
matrix	a_{ij}	1	2	3	4	5	6

$a_{ji} = a_{ij}$: 21 parameters $\qquad \Gamma(x, p) = \Gamma(x, p, a) = \Gamma(p, a)$

three positive eigenvalues $G^m(x,p),\ m \in \{1,2,3\}$ eigenvectors $\Gamma q^m = G^m q^m$

$$G^1(x,p) > G^m(x,p), \qquad m \in \{2,3\}$$

level set of G^1 for x fixed is convex

level sets of G^2 and G^3 must have points in common: D_c (degenerate) slowness surface

$$G = \sum_{i,\ell} \Gamma_{i\ell} q_i q_\ell = \sum_{i,j,k,\ell} a_{ijk\ell} q_i q_\ell p_j p_k$$

avoid points at which $D_p G$ vanishes

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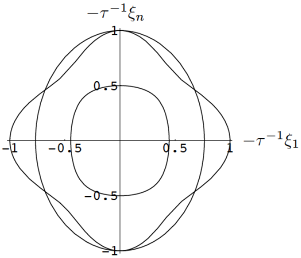
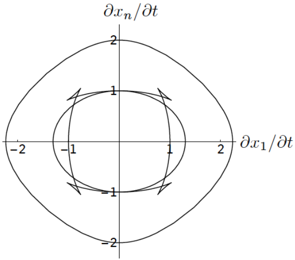
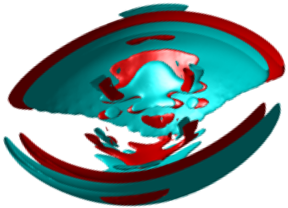
continuous function $f(x,p) := \sqrt{G^1(x,p)}$ τ dual to t

- $f : \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}) \to (0,\infty)$ is smooth, real analytic on the fibers
- for every $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $s \in \mathbb{R}$ it holds that $f(x,sp) = |s| f(x,p)$
- for every $(x,p) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$ the Hessian of $\frac{1}{2} f^2$ is symmetric and positive definite with respect to p



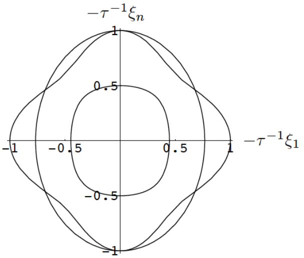


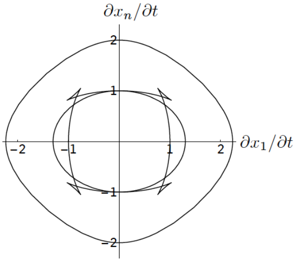


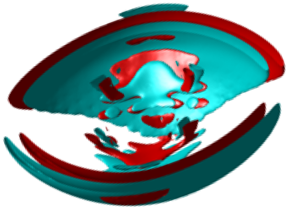
parameters vs surfaces (sheets), $p = \tau^{-1}\xi$

(tilted) transverse isotropy

qP, *qSV*, *qSH*







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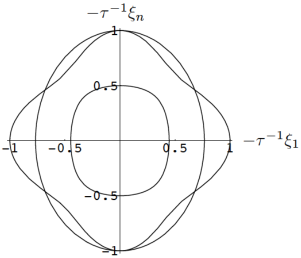
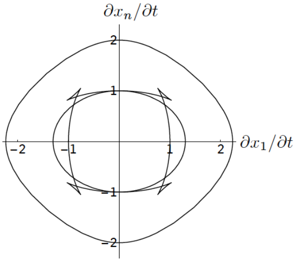
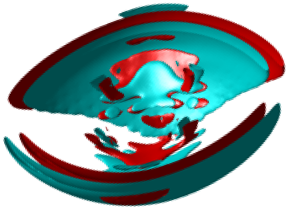
no symmetry

qP, qSV, qSH

elastic waves $\mathbb{R}^3$







parameters vs surfaces (sheets), $p = \tau^{-1}\xi$

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no symmetry

propagation of singularities governed by Hamiltonians $H(x, \xi)$, bicharacteristics

geodesic ray transform, (tensor) tomography

Herglotz (1905), Wiechert & Zoeppritz (1907), Mukhometov (1975, 1982, 1987), Pestov & Sharafutdinov (1988), Anikonov & Romanov (1997), Paternain, Salo & Uhlmann (2013) surfaces, *Uhlmann & Vasy (2016)*

boundary rigidity

Michel (1981), Muhometov & Romanov (1978), Besson, Courtois & Gallot (1995), Stefanov & Uhlmann (1998), Lassas, Sharafutdinov & Uhlmann (2003), Pestov & Uhlmann (2005), Stefanov, Uhlmann & Vasy (2017)

boundary distance data

Kurylev (1997), *Katsuda, Kurylev & Lassas (2007)*, Ivanov (2010)

broken scattering relation data

Kurylev, Lassas & Uhlmann (2010)

(compact) Riemannian manifolds

elastic anisotropy through the lens of algebraic geometry

recovery of stiffness tensor from one slowness sheet

inverse problem

slowness surface

$$S_x = \{p \in T_x^* \mathbb{R}^3 \colon \det(\Gamma(x, p) - I) = 0\}$$

is algebraic

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is algebraic

Question

given a number of points on a slowness surface, can we

1. *reconstruct the entire slowness surface?*
2. *reconstruct the stiffness tensor that gave rise to it?*

Theorem (dH, Ilmavirta, Lassas & Várilly-Alvarado)

For most choices of triclinic (that is, totally anisotropic materials) the slowness surface at any point is irreducible.

anisotropy

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consequence: for most anisotropic materials, slownes surfaces can be reconstructed from a small number of data points

bonus: for most anisotropic materials, the stiffness tensor can be recovered uniquely (algorithmically!) from a slowness surface

under the hood: algebraic geometry - irreducibility

$$
\begin{array}{ccl}
\mathbf{S} := \{(p,a)\colon\ \det(\Gamma(p,a) - I) = 0\} & \leftarrow & \text{all orthorhombic slowness surfaces at once (“moduli space”)} \\
\cap & & \\
\mathbb{C}^3_{(p_1,p_2,p_3)} \times \mathbb{C}^9_{(a_{11},a_{12},a_{13},a_{22},a_{23},a_{33},a_{44},a_{55},a_{66})} & & \\
\Big\downarrow pr_2 & & \\
\mathbb{C}^9 & \leftarrow & \text{given a point } a \in \mathbb{C}^9\text{, the preimage } pr_2^{-1}(a) \text{ is the slowness surface associated to the stiffness tensor that } a \text{ represents}
\end{array}
$$

after compactifying the set-up, we show that the subset of $pr_2(\mathbf{S})$ with irreducible preimages is (Zariski) open in $\mathbb{C}^9$

under the hood: algebraic geometry - irreducibility

- the subset of $pr_2(\mathbf{S})$ with irreducible preimages is (Zariski) open in $\mathbb{C}^9$
- a single irreducible slowness surface would witness non-emptiness
- we use reduction modulo a prime number + Galois theory, and properties of finite fields

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- the subset of $pr_2(\mathbf{S})$ with irreducible preimages is (Zariski) open in $\mathbb{C}^9$
- a single irreducible slowness surface would witness non-emptiness
- we use reduction modulo a prime number + Galois theory, and properties of finite fields

$$\begin{aligned} 5000424p_1^6 &+ 14708322p_1^4p_2^2 + 18410832p_1^4p_3^2 - 105178p_1^4 + 14702184p_1^2p_2^4 \\ &+ 38400324p_1^2p_2^2p_3^2 - 209855p_1^2p_2^2 + 15221384p_1^2p_3^4 - 227260p_1^2p_3^2 \\ &+ 647p_1^2 + 4975872p_2^6 + 20903748p_2^4p_3^2 - 107584p_2^4 + 17362658p_2^2p_3^4 \\ &- 249931p_2^2p_3^2 + 668p_2^2 + 3317072p_3^6 - 81254p_3^4 + 583p_3^2 - 1 = 0 \end{aligned}$$

is irreducible over $\mathbb{F}_{56} \implies$ is irredicible over $\mathbb{C}$ particular mineral

under the hood: algebraic geometry - irreducibility

Theorem (Grothendieck, EGA IV.3, 1966)

Let $f : X \to Y$ be a proper morphism, flat and of finite presentation. Then the set of $y \in Y$ such that the fiber X_y is geometrically integral is open.

all orthorhombic stiffness tensors at once ("moduli space")

$$\downarrow$$

$$\mathbf{A} \subset \mathbb{C}^9_{(a_{11},a_{12},a_{13},a_{22},a_{23},a_{33},a_{44},a_{55},a_{66})}$$

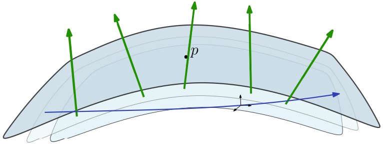
$$\Big\downarrow \chi$$

$\mathbb{C}^{20}_{c_0,\ldots,c_{20}}$ $\qquad$ $\chi(a) =$ vector of coefficients of corresponding slowness surfaces

- show: image of ϕ is 9-dimensional and generically $1-1$
- reconstruction of pre-image: Gröbner bases

boundary rigidity, bicharacteristic curves point of view

recovery of TTI parameters



invariant setting, parametrization

- invariant setting based on Riemannian geometry
- a given background metric g_0, which is typically the Euclidean metric; dual metric G_0

invariant setting, parametrization

- invariant setting based on Riemannian geometry
- a given background metric g_0, which is typically the Euclidean metric; dual metric G_0
- parameterization of transversely isotropic materials:
  - material constants $a_{11}, a_{13}, a_{33}, a_{55}$ and a_{66}
  - axis of isotropy, which can be encoded by a vector field, or a one form ω
- use orthogonal coordinates relative to the metric g_0, aligning the axis of isotropy with the third coordinate axis

qSH

(squared) Riemannian dual metric (defining the slowness surface)

$$G = G_{qSH} = a_{66}(x)|\xi'|^2 + a_{55}(x)\xi_3^2 = a_{66}(x)G_0 + (a_{55}(x) - a_{66}(x))\xi_3^2$$

corresponds to a Riemannian metric

$$g = g_{qSH} = a_{66}(x)^{-1}\,|dx'|^2 + a_{55}(x)^{-1}\,dx_3^2 = a_{66}(x)^{-1}g_0 + (a_{55}(x)^{-1} - a_{66}(x)^{-1})\,dx_3^2$$

invariantly

$$g = \alpha g_0 + (\beta - \alpha)\omega \otimes \omega$$

metric is a rank one perturbation of a conformal multiple of the background (say, Euclidean) metric, with $\alpha = a_{66}^{-1}$, $\beta = a_{55}^{-1}$ functions on the base manifold

- g determines the span of ω if $\beta \neq \alpha$
- kernel of ω is well-defined (at any point in the manifold) as the 2-dimensional subspace of the tangent space restricted to which g is a constant multiple of g_0

- integrability – its kernel is an integrable hyperplane distribution, which means that $\mathrm{Ker}\,\omega$ is the tangent space of a smooth family of submanifolds
- locally level sets of a function f, ω is a smooth multiple of df

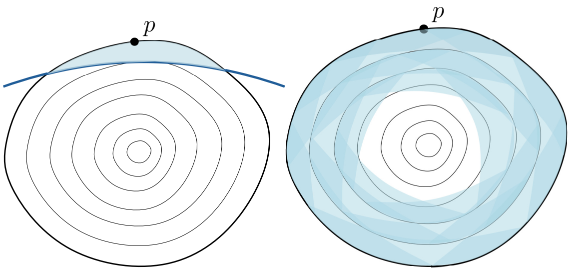
$$g = \alpha g_0 + \gamma\, df \otimes\, df$$

- this corresponds to some "stratification" given by the level sets of f

convex foliation

(Uhlmann & Vasy, '16)

- (M, g) is an n-dimensional Riemannian manifold with boundary, $n \geq 3$, and assume that ∂M is strictly convex (in the second fundamental form sense) with respect to the g at some $p \in \partial M$
- x is a smooth function with non-vanishing differential whose level sets are strictly concave from the superlevel sets, and $\{\mathrm{x} \geq 0\} \cap M \subset \partial M$



(Hamiltonian flow $(X(t), \Xi(t))$: $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{x} \circ X)(t) = 0$ implies $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathrm{x} \circ X)(t) > 0$)

determining the symmetry axis

(dH, Uhlmann & Vasy, '19)

Theorem

consider the class of elastic problems in which $\operatorname{Ker}\omega = \operatorname{Ker} df$ *is an integrable hyperplane distribution on a manifold with boundary* M*, with* ω *not conormal to* ∂M *and not orthogonal to* $N^*\partial M$ *relative to* G_0

then, under the local, resp. global, convexity conditions for Riemannian determination (up to diffeomorphisms), f, α, β *are locally, resp. globally, determined by the qSH travel times, resp. qSH lens relations, and the labelling of the level sets of* f *at the boundary*

there is *no* diffeomorphism freedom in this problem, unlike for the boundary rigidity problem in Riemannian geometry!

remaining material parameters, a_{11}, a_{13} and a_{33}

at a point with coordinates g_0-orthogonal and such that the isotropy axis is aligned with the $\widetilde{x}_3$ axis the Hamiltonians for the other waves take the form

$$G_{qP/qSV} = (a_{11} + a_{55})|\widetilde{\xi}'|^2 + (a_{33} + a_{55})\widetilde{\xi}_3^2 \pm \sqrt{\left((a_{11} - a_{55})|\widetilde{\xi}'|^2 + (a_{33} - a_{55})\widetilde{\xi}_3^2\right)^2 - 4E^2|\widetilde{\xi}'|^2\widetilde{\xi}_3^2}$$

where

$$E^2 = (a_{11} - a_{55})(a_{33} - a_{55}) - (a_{13} + a_{55})^2$$

assumption

$$\max\{a_{55}, a_{66}\} < \min\{a_{11}, a_{33}\}$$

microlocally weighted ray transform along “curves” (projected Hamiltonian flow)

determination of a_{11}, a_{33} and E^2?

triplications, ..

multiple points in the cotangent space potentially corresponding to the same tangent vector via the Hamilton map

Lemma

suppose that either $H = H_+ = G_{qP}$*, or instead* $H = H_- = G_{qSV}$ *and*

$$a_{33}(a_{11} - a_{55}) > (a_{13} + a_{55})^2$$

then the map $\widetilde{\xi} \to \mathfrak{H}_{\widetilde{x}}(\widetilde{\xi}) = \sum_j \frac{\partial H}{\partial \widetilde{\xi}_j} \partial_{\widetilde{x}_j}$ *has an invertible differential at* $\widetilde{\xi}_3 = 0$*, and indeed the level sets of* H *are strictly convex (from the sublevel sets) nearby*

Remark

if $E^2 \geq 0$*, the right-hand side is* $\leq (a_{11} - a_{55})(a_{33} - a_{55})$*, so the inequality in the statement of the lemma is automatically true* if the transverse isotropy orthogonal planes are close to the tangent spaces to a convex foliation, then the material is non-degenerate relative to the convex foliation

Definition

a transversely isotropic material is non-degenerate relative to a convex foliation (concave from the superlevel sets for G_{qSV}) if for each point x and each vector v tangent to the convex foliation at the point x there is a covector ξ in the cotangent space over x such that $\mathfrak{H}_x(\xi) = v$ and the map $\mathfrak{H}_x$ has invertible differential at ξ, with $\mathfrak{H}_x$ arising from G_{qSV}

a transversely isotropic material is non-degenerate provided the statement above holds for all v (and not just v tangent to a particular convex foliation)

tangent plane to the artifical boundary

Lemma

if the gradient ∇f *of the transverse isotropy foliation function is not parallel to the artificial boundary, points with* $\widetilde{\xi}' = 0$ *cannot give rise to vectors tangent to the artificial boundary under Hamiltonian map* $\mathfrak{H}_x$

Theorem

assume that ∇f is neither parallel, nor orthogonal to the artificial boundary with respect to g_0; assume moreover that the transversely isotropic material is non-degenerate relative to a convex foliation if qSV data are used below, with convexity of the foliation always understood with respect to G_{qP}, resp. G_{qSV}, if qP, resp. qSV data are used below

then the modified and localized "normal operators" arising from the Stefanov-Uhlmann formula (pseudolinearization) are in Melrose's scattering pseudodifferential operator algebra

furthermore, the boundary principal symbol is elliptic at finite *points of the scattering cotangent bundle for any one of E^2, a_{11}, a_{33} from the qP travel data, and for E^2 (as well as a_{11} and a_{33} if $E^2 > 0$) from the qSV travel data; furthermore, for a_{11} from the qP-travel time data standard principal symbol ellipticity also holds*

Theorem

for a_{33}, E^2 from the qP or qSV travel data, as well as for E^2 and one of a_{11} and a_{33} jointly from the qP and qSV data, the standard principal symbol is not *elliptic, rather vanishes in a non-degenerate quadratic manner along the span of df at fiber infinity in the scattering cotangent bundle*

- in general, for the normal operator's standard principal symbol computation at a point $\zeta \in T_x^* M$, one takes a weighted average of certain quantities evaluated at covectors for which the Hamilton vector field for the relevant polarization is annihilated by ζ
- if $\zeta = df$ is in the axis direction, the tangent vectors involved in the integration correspond to covectors in the g_0-orthogonal plane, that is, with vanishing $\widetilde{\xi}_3$ coordinate, and there the qP and qSV wave speeds are insensitive to a_{33}, E^2 as these appear with a prefactor $\widetilde{\xi}_3^2$

Finsler geometry point of view

recovery of “slowness surfaces”

elastic waves, bicharacteristics, Finsler geodesics

$f = \sqrt{G^1}$ is a convex norm on the cotangent space

a Finsler manifold is a differentiable manifold M equipped with $F_x \colon T_x M \to [0, \infty)$ for each x with the properties of a norm; combining the functions on separate fibers gives rise to the Finsler function $F \colon TM \to [0, \infty)$, which is continuous on TM and smooth enough on $TM \setminus 0$

- F^* is related to F as (Legendre transform)

$$F_x(v(x,p)) = F_x^*(p) \quad \text{with} \quad v(x,p) = \frac{\partial F_x^*}{\partial p}(p)$$

elastic waves, bicharacteristics, Finsler geodesics

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elastic waves, bicharacteristics, geodesics

- for any $x \in M$, $v \in T_x M \setminus 0$, in local coordinates

$$\left(\frac{1}{2}\frac{\partial}{\partial v^i}\frac{\partial}{\partial v^j}F^2(x,v)\right)_{i,j=1}^{n} =: \left(g_{ij}(x,v)\right)_{i,j=1}^{n}$$

  the *local Riemannian metric*
- the length of any curve is defined using Riemannian metric associated with its tangent direction; the infinitesimal travel time $\mathrm{d}t$ is determined by the local Riemannian metric g_{ij}

$$\mathrm{d}t^2 = F^2(x,\dot{x}) = g_{ij}(x,\dot{x})\dot{x}^j\dot{x}^i$$

let I be a closed interval; $\gamma \colon I \to M$, with a constant speed $F(\dot{\gamma}(t)) \equiv c \geq 0$, is a geodesic of Finsler manifold (M,F) if $\gamma(t)$ solves the system of geodesic equations

$$\ddot{\gamma}^i(t) + 2\mathcal{G}^i(\dot{\gamma}(t)) = 0, \quad \mathcal{G}^i(x,v) = \frac{1}{4}\sum_{j,k,l} g^{il}(x,v)\left\{2\frac{\partial g_{jl}(x,v)}{\partial x^k} - \frac{\partial g_{jk}(x,v)}{\partial x^l}\right\}v^j v^k$$

$\mathcal{G}^i \colon TM \to \mathbb{R}$

distance and geodesics on Finsler manifolds

let $x, y \in M$ and let $C_{x,y}$ denote the collection of all piecewise C^1 paths from x to y

$$d_F(x, y) := \inf \left\{ \mathcal{L}(c) := \int_0^1 F(c(t), \dot{c}(t))\, dt \ : \ c \in C_{x,y} \right\}$$

Finsler isomorphism

we call a map $\Psi : (M_1, F_1) \to (M_2, F_2)$ a Finslerian isomorphism, if it is a diffeomorphism that satisfies

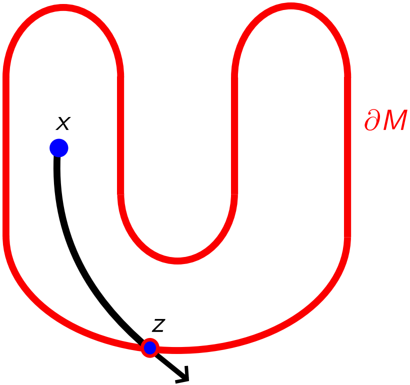
$$F_1(v) = F_2(\Psi_* v), \quad v \in TM_1$$

where Ψ_* is the push forward of Ψ

Finslerian boundary distance function

(M, F) is smooth compact n-dimensional, $n \geq 2$, Finsler manifold with boundary and $x \in M^{int}$

boundary distance function $r_x \colon \partial M \to \mathbb{R}, \quad r_x(z) := d_F(x, z)$



boundary distance data:
$(\partial M, \ \{r_x : x \in M^{int}\})$

direction is from x to z

inverse problem of Finslerian boundary distance functions

let (M_i, F_i), $i = 1, 2$ be compact smooth n-dimensional $(n \geq 2)$ Finsler manifolds with boundary

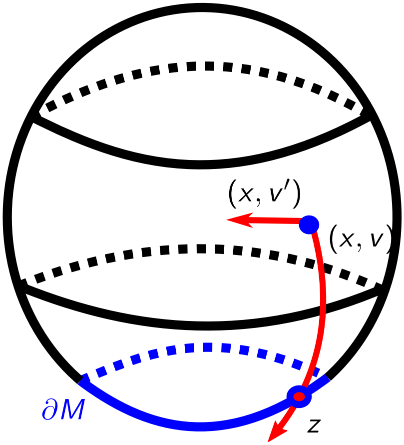
boundary distance data of (M_1, F_1) and (M_2, F_2) agree if $\exists\, \phi \colon \partial M_1 \to \partial M_2$, diffeomorphism such that

$$\{d_{F_1}(x, \cdot) \colon \partial M_1 \to [0, \infty) \mid x \in M_1^{int}\} = \{d_{F_2}(y, \phi(\cdot)) \colon \partial M_1 \to [0, \infty) \mid y \in M_2^{int}\}$$

inverse problem: are (M_1, F_1) and (M_2, F_2) Finsler isometric if their boundary distance data agree? *not quite*

obstruction for the uniqueness

define set $\mathfrak{G}(M, F)$ so that for $(x, v) \in \mathfrak{G}(M, F) \subset TM$ the geodesic $\gamma_{x,v}$ is a distance minimizer until it exits M at $z \in \partial M$



- we can reconstruct $F(x, v)$ since $(x, v) \in \mathfrak{G}(M, F)$ is related to $d_F(x, z)$
- we cannot reconstruct $F(x, v')$ since geodesic $\gamma_{x,v'}$ is trapped
- we can "modify" F in neighborhood of (x, v') without changing $d_F(x, \cdot)|_{\partial M}$

Theorem (dH, Ilmavirta, Lassas, Saksala)

let (M_i, F_i), $i = 1, 2$ *be smooth, connected, compact Finsler manifolds with smooth boundary; if the boundary distance data of* (M_1, F_1) *and* (M_2, F_2) *agree, then there is a diffeomorphism* $\Psi\colon M_1 \to M_2$ *such that* Ψ *on* ∂M_1 *coincides with* ϕ

the sets $\overline{\mathfrak{G}(M_1, F_1)}$ *and* $\overline{\mathfrak{G}(M_1, \Psi^* F_2)}$ *coincide and in this set* $F_1 = \Psi^* F_2$

for any $(x, v) \in TM_1^{int} \setminus \overline{\mathfrak{G}(M_1, F_1)}$ *there exists a smooth Finsler function* $F_3 : TM_1 \to [0, \infty)$ *so that* $d_{F_1}(p, z) = d_{F_3}(p, z)$ *for all* $p \in M_1$ *and* $z \in \partial M_1$ *but* $F_1(x, v) \neq F_3(x, v)$

Theorem (dH, Ilmavirta, Lassas, Saksala)

let (M_i, F_i), $i = 1, 2$ be smooth, connected, compact Finsler manifolds with smooth boundary; if the boundary distance data of (M_1, F_1) and (M_2, F_2) agree, then there is a diffeomorphism $\Psi\colon M_1 \to M_2$ such that Ψ on ∂M_1 coincides with ϕ

the sets $\overline{\mathfrak{G}(M_1, F_1)}$ and $\overline{\mathfrak{G}(M_1, \Psi^* F_2)}$ coincide and in this set $F_1 = \Psi^* F_2$

for any $(x, v) \in TM_1^{int} \setminus \overline{\mathfrak{G}(M_1, F_1)}$ there exists a smooth Finsler function $F_3 : TM_1 \to [0, \infty)$ so that $d_{F_1}(p, z) = d_{F_3}(p, z)$ for all $p \in M_1$ and $z \in \partial M_1$ but $F_1(x, v) \neq F_3(x, v)$

anisotropic elasticity

Theorem (dH, Ilmavirta, Lassas, Saksala)

let (M_i, F_i), $i = 1, 2$ be smooth, connected, compact Finsler manifolds with smooth boundary; if the boundary distance data of (M_1, F_1) and (M_2, F_2) agree, and if Finsler function F_i is fiberwise real analytic, then there exists a Finslerian isometry $\Psi\colon (M_1, F_1) \to (M_2, F_2)$ such that Ψ on ∂M_1 coincides with ϕ

strategy of the proof

proof of optimality consists of three steps

- reconstruction of topology
- reconstruction of smooth structure
- reconstruction of Finsler structure

broken scattering relation

recovery of “slowness surfaces”

Definition

A Finsler manifold with boundary is said to have a strictly convex foliation if there is a smooth function $f: M \to \mathbb{R}$ so that

- $f^{-1}\{0\} = \partial M$, $f^{-1}(0, S] = \mathrm{int}(M)$, $f^{-1}(S)$ has empty interior
- for each $s \in [0, S)$ the set $\Sigma_s := f^{-1}(s)$ is a strictly convex smooth surface in the sense that $\mathrm{d}f \neq 0$ and any geodesic γ, having initial conditions in $T\Sigma_s$, satisfies $\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\gamma(t))|_{t=0} < 0$.

broken scattering relation

- let $\varphi_t \colon SM \to SM$ be the geodesic flow
- the natural projection of the tangent bundle will be denoted by $\pi \colon TM \to M$
- the boundary of the sphere bundle is

$$\partial SM = \{v \in SM; \pi(v) \in \partial M\}$$

- identify the inward-pointing part of this boundary

$$\partial_{\mathrm{in}} SM \; = \; \{v \in \partial SM; \langle v, \nu \rangle_{\nu} > 0\}$$

where ν is the inward pointing normal vector field and $\langle v, \nu \rangle_{\nu} = g_{ij}(\nu)\nu^i v^j$

- identify the outward-pointing part

$$\partial_{\mathrm{out}} SM = \{v \in \partial SM; \langle v, \nu \rangle_{\nu} < 0\}$$

broken scattering relation

Definition

Let (M, F) be a Finsler manifold with boundary. For each $t > 0$ we define a relation R_t on $\partial_{\text{in}} SM$ so that $v R_t w$ if and only if there exist two numbers $t_1, t_2 > 0$ for which $t_1 + t_2 = t$ and $\pi(\varphi_{t_1}(v)) = \pi(\varphi_{t_2}(w))$. We call this relation the broken scattering relation.

the broken scattering relation, that is, the lengths of the broken geodesics, determine uniquely the isometry type of a Finsler manifold

compatible diffeomorphism

Definition

Let M_1 and M_2 be two smooth manifolds with boundary. We say that a diffeomorphism $\Xi\colon \partial TM_1 \to \partial TM_2$ is compatible with a diffeomorphism $\phi\colon \partial M_1 \to \partial M_2$ if Ξ is a linear isomorphism on every fiber and satisfies $\Xi(T\partial M_1) = T\partial M_2$ and $\Xi|_{T\partial M_1} = \mathrm{d}\phi$.

Theorem (dH, Ilmavirta, Lassas & Saksala)

Let (M_i, F_i), $i \in \{1,2\}$ be two compact Finsler manifolds of dimension larger or equal to 3, with boundary. We assume the following:

(i) *Both Finsler functions F_1 and F_2 are reversible.*

(ii) *The manifolds (M_i, F_i), $i \in \{1,2\}$ have strictly convex foliations in the sense of definition 11.*

(iii) *There are diffeomorphisms $\phi\colon \partial M_1 \to \partial M_2$ and $\Xi\colon \partial TM_1 \to \partial TM_2$ that are compatible.*

(iv) $F_1 = F_2 \circ \Xi$ *on* ∂TM_1.

(v) *For any two vectors $v, w \in \partial_{\mathrm{in}} SM_1$ and $t > 0$ we have $v R_t^{(1)} w$ if and only if $\Xi(v) R_t^{(2)} \Xi(w)$, where $R_t^{(i)}$ is the broken scattering relation on (M_i, F_i).*

Then there is a diffeomorphism $\Psi\colon M_1 \to M_2$ that is an isometry in the sense of $F_1 = F_2 \circ \mathrm{d}\Psi$, which satisfies $\Psi|_{\partial M_1} = \phi$, and $\mathrm{d}\Psi|_{\partial TM_1} = \Xi$.

uniqueness

in the elastic setting no foliation condition is needed; it can be replaced with fiberwise analyticity

perspective

- elasticity as interplay of microlocal analysis, algebraic geometry and Finsler geometry
- anisotropic elasticity (polarizations) – *more rigidity* than in the Riemannian case
- boundary rigidity beyond TTI (in dimension 3) is still open
- finite set of *almost simultaneous* point sources: approximate recovery of a manifold for the simple Riemannian case – generalization?