The canonical isomorphism between $T^k T^* M$ and $T^* T^k M$

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Abstract — We discuss a natural symplectic structure on $T^k T^* M$ and a natural $k$th-order almost tangent structure on $T^* T^k M$. The main result concerns the construction of a vector bundle isomorphism $\psi_k : T^k T^* M \rightarrow T^* T^k M$ (over $T^k M$), which behaves naturally with respect to all structures of interest. We further use this result to prove that one can identify spaces such as $T^r T^s M$ and $T^s T^r M$ by a map which in coordinates simply consists in switching suffices.

1. Introduction. — The existence of a canonical isomorphism between $TT^* M$ and $T^* T M$ is well known and of fundamental importance in many applications. It is, for example, a key matter in Tulczyjew’s description of Lagrangian and Hamiltonian theory, in terms of a Lagrangian submanifold which is shared by two special symplectic structures on $TT^* M$ [1]. It is our belief that the canonical isomorphism between $T^k T^* M$ and $T^* T^k M$, which we will discuss in the present note, has a number of interesting features in its own right and will be valuable for clarifying certain aspects of the approach to higher-order mechanics described in [2] and [3].

2. Notations and Preliminaries. — We seek conformity in notations with some of our earlier work (see e.g. [4] and [5]), but slight adaptations are always necessary. Let $\tau_k : T^k N \rightarrow N$ denote the tangent bundle of order $k$ of some manifold $N$. For $\ell < k$, we have projection operators $\tau_{\ell}^k : T^k N \rightarrow T^\ell N$. It is known that a smooth map $\phi : N \rightarrow N'$ induces a map $\phi^{(k)} : T^k N \rightarrow T^k N'$. Thus, for example, the manifold $T^k T^* M$ is fibred over $T^k M$ (as a vector bundle) through the map $\pi^{(k)}_M$, induced by the cotangent bundle projection $\pi_M : T^* M \rightarrow M$. As in [4] and [6], let $T$ denote the canonical inclusion $T : T^k N \rightarrow TT^{k-1} N$ and $d_T$ the total derivative operator which turns functions and forms on $T^\ell N$ into corresponding objects on $T^{\ell+1} N$. Repeated application of this operator on a 1-form $\alpha$ on $N$ results in a 1-form $d_T^k \alpha$ on $T^k N$. A tangent bundle of order $k$ such as $T^k N$ comes naturally equipped with a type (1,1) tensor field, the so-called vertical endomorphism, which we will denote by $S^{(k)}_N$. We recall from [4] the following commutation relations: $\phi^{(k+1)^*} \circ d_T = d_T \circ \phi^{(k)^*}$, $\tau^{k+1}_k \circ d_T = d_T \circ \tau_k^*$, and also, as a map on 1-forms,
\( S^{(k+1)}_N \circ d_T = d_T \circ S^{(k)}_N + \tau^{k+1}_N \). From these properties it is easy to deduce that 
\( \phi^{(k)} \circ d_T^k = d_T^k \circ \phi^* \) and 
\( S^{(k)}_N \circ d_T^k = k \tau^{k-1}_N \circ d_T^{k-1} \) (where for \( k = 1 \), \( d_T^0 \) is to be regarded as the identity operator). Other canonical objects which are soon to be discussed are the canonical 1-form of a cotangent bundle (notation: \( \theta M \) for \( T^*M \)), and dilation vector fields. For notational distinction we will write e.g. \( \Delta^M \) for the dilation field on \( T^*M \) and \( \Delta^k \) for the dilation field on \( T^kN \). Finally, we will indicate the complete lift of a geometrical object to its cotangent bundle with a tilde (as in [5]), while for the complete lift of a vector field \( X \) on \( N \) to a vector field on \( T^kN \) we will write \( \tilde{X} \) (as in [4]).

3. Main Results. — Representing coordinates on \( T^*M \) as \( (q^a, p_a) \) \((a = 1, \ldots, n = \dim M)\), we will denote the corresponding natural coordinates on \( T^kT^*M \) by \( (q^a_i, p_a|_i) \) \((i = 0, \ldots, k)\) and the coordinates on \( T^*T^kM \) by \( (q^a_i, p^a_i) \).

The space \( T^kT^*M \) obviously carries a kth-order tangent structure. On the other hand, starting from \( \theta M \), its kth-order total derivative produces a globally defined 1-form on \( T^kT^*M \), which in coordinates reads
\[
d_T^k \theta_M = \sum_{i=0}^k \binom{k}{i} p_{a|_i} dq^a_i
\]
(with summation over \( a \) from 1 to \( n \) understood). It is clear from this expression that the exterior derivative yields a non-degenerate (exact) 2-form, which proves the following statement.

**Theorem.** \( (T^kT^*M, dd_T^k \theta_M) \) is a symplectic manifold.

Next we turn to the space \( T^*T^kM \) which has a natural symplectic structure and can further be endowed with a type (1,1) tensor field via the complete lift of \( S^{(k)}_M \).

In coordinates:
\[
\tilde{S}^{(k)}_M = \sum_{i=1}^k \frac{\partial}{\partial q^a_i} \otimes dq^a_{i-1} + \sum_{i=1}^k \frac{\partial}{\partial p^a_{i-1}} \otimes dp^a_i.
\]

**Theorem.** \( \tilde{S}^{(k)}_M \) is an integrable kth-order almost tangent structure on \( T^*T^kM \).

**Proof.** Inspection of the above coordinate expression shows that for \( S = \tilde{S}^{(k)}_M \), ker \( S_j \) and im \( S^{k-j} \) are both spanned by the coordinate vector fields
\[
\left\{ \frac{\partial}{\partial q^a_{k-j}}, \ldots, \frac{\partial}{\partial q^a_k}, \frac{\partial}{\partial p^a_0}, \ldots, \frac{\partial}{\partial p^a_k} \right\} \quad \text{for} \quad j = 0, \ldots, k.
\]
The vanishing of the Nijenhuis tensor \( N_S \) is obvious because the coefficients in that same expression are all constants. Thus, all requirements for an integrable kth-order almost tangent structure are verified.

We now come to the generalization of the diffeomorphism \( TT^*M \leftrightarrow T^*TM \).
Theorem. — There is a vector bundle isomorphism \( \psi_k : T^k T^*M \rightarrow T^* T^k M \), which is both a symplectomorphism and an isomorphism of kth-order almost tangent structures.

The vector bundle structures we are referring to here are, respectively, the fibrations \( \pi_M^{(k)} : T^k T^*M \rightarrow T^*M \) and \( \pi_{T^k M} : T^* T^k M \rightarrow T^k M \). Observe first that the coordinate expression of \( d_{T^k \theta_M} \) clearly shows that this 1-form vanishes on vectors which are vertical with respect to \( \pi_M^{(k)} \). This justifies the following intrinsic construction.

Definition. — For each \( Q \in T^k T^*M \) with projection \( q = \pi_M^{(k)}(Q) \), we define \( \psi_k(Q) \in T^* T^k M \) to be the covector at \( q \), determined by the relation:

\[
\forall \zeta_q \in T_q T^k M, \quad \langle \zeta_q, \psi_k(Q) \rangle = \langle \xi_q, (d_{T^k \theta_M})_q \rangle,
\]

where \( \xi_q \in T_q T^k T^*M \) is any vector with the property \( T_{\pi_M^{(k)}}(\xi_q) = \zeta_q \).

It is clear from this definition that \( \pi_{T^k M} \circ \psi_k = \pi_M^{(k)} \). Moreover, we have: \( \forall \xi_Q \in T_q T^k T^*M \),

\[
\langle \xi_Q, (\psi_k \ast \psi_{T^k M})_Q \rangle = \langle T \psi_k(\xi_Q), (\theta_{T^k M})_{\psi_k(Q)} \rangle = \langle T \pi_{T^k M} \circ T \psi_k(\xi_Q), \psi_k(Q) \rangle = \langle T \pi_M^{(k)}(\xi_Q), \psi_k(Q) \rangle = \langle \xi_Q, (d_{T^k \theta_M})_Q \rangle,
\]

where we have used the definition of \( \theta_{T^k M} \). It follows that \( \psi_k \ast \theta_{T^k M} = d_{T^k \theta_M} \). Since the coordinate expression for \( \theta_{T^k M} \) reads \( \theta_{T^k M} = \sum_{i=0}^k p_a^i d q_a^i \), we see rightaway that the map \( \psi_k \) in coordinates is given by

\[
(q_i^a, p_{a|i}) \overset{\psi_k}{\mapsto} (q_i^a, p_a^i = \binom{k}{i} p_{a|i-k})
\]

and is truly a vector bundle isomorphism. Finally, if \( \psi_{k*} \) stands for the push forward operation on tensor fields, we have

\[
\psi_{k*} \frac{\partial}{\partial p_{a|i}} = \binom{k}{i} \frac{\partial}{\partial p_{a|i-k}}, \quad \psi_{k*} dp_{a|i-1} = \binom{k}{i-1}^{-1} dp_{a|i+k+1}.
\]

The kth-order tangent structure on \( T^k T^*M \) is determined by

\[
S_{T^* M}^{(k)} = \sum_{i=1}^k i \frac{\partial}{\partial q_i^a} \otimes dq_i^a + \sum_{i=1}^k i \frac{\partial}{\partial p_{a|i}} \otimes dp_{a|i-1}
\]

and it is now easy to verify that \( \psi_{k*} S_{T^* M}^{(k)} = S_M^{(k)} \), which completes the proof of our main theorem.
The geometrical structure of a tangent bundle (of any order) is, in a way, fully determined by the almost tangent structure and its associated dilation field. The dilation field associated to $S^{(k)}_M$ on $T^kT^*M$ is the vector field

$$\Delta^{(k)}_{T^*M} = \sum_{i=1}^k q^a_i \frac{\partial}{\partial q^a_i} + \sum_{i=1}^k p^a_i \frac{\partial}{\partial p^a_i}.$$ 

Obviously, in view of the diffeomorphism $\psi_k$, the dilation field associated to the almost tangent structure $\tilde{S}^{(k)}_M$ on $T^*T^kM$ is the vector field

$$\psi_k^* \Delta^{(k)}_{T^*M} = \sum_{i=1}^k q^a_i \frac{\partial}{\partial q^a_i} + \sum_{i=0}^{k-1} (k - i) p^a_i \frac{\partial}{\partial p^a_i}.$$ 

It is of some interest to find the relation between this vector field and other dilation fields which naturally live on $T^*T^kM$. There are in fact two such dilation fields, namely the complete lift of the one on $T^kM$ and the dilation field of $T^*T^kM$ as a cotangent bundle. Their coordinate expressions read

$$\tilde{\Delta}^{(k)}_M = \sum_{i=1}^k q^a_i \frac{\partial}{\partial q^a_i} - \sum_{i=1}^k p^a_i \frac{\partial}{\partial p^a_i}, \quad \Delta^{*}_{T^kM} = \sum_{i=0}^k p^a_i \frac{\partial}{\partial p^a_i}.$$ 

The following correspondence now can easily be verified.

**Theorem.** — The dilation field associated to the almost tangent structure $\tilde{S}^{(k)}_M$ on $T^*T^kM$ is given by $\psi_k^* \Delta^{(k)}_{T^*M} = \tilde{\Delta}^{(k)}_M + k \Delta^{*}_{T^kM}$.

4. **Remarks.** — a) For $k = 1$ we recover the known results. Our present intrinsic definition of $\psi_1$ then coincides with the one in [7].

b) Note that the $\psi_k$-maps induce a projection of cotangent bundles of higher-order tangent bundles: we can define

$$\rho^{k-1}_k : T^*T^kM \longrightarrow T^*T^{k-1}M, \quad \rho^{k-1}_k = \psi_{k-1} \circ \tau^{k-1}_k \circ \psi^{-1}_k.$$ 

It is easy to show, using the main theorem and results of section 2, that

$$\tilde{S}^{(k)}_M (\theta_{T^*M}) = k \rho^{k-1}_k \theta_{T^{k-1}M}.$$ 

c) We know that $\theta_M$ is uniquely determined by the property $\alpha^* \theta_M = \alpha$, for any 1-form $\alpha$ on $M$, regarded as a section of $T^*M$. The induced section $\alpha^{(k)}$ of $\pi^{(k)}_M$ is easily seen to have the property $\alpha^{(k)*} d_T T^k \theta_M = d_T T^k (\alpha^* \theta_M) = d_T T^k \alpha$. It follows that $\psi_k \circ \alpha^{(k)} = d_T T^k \alpha$, regarded as a section of $\pi_{T^kM}$.

5. **Generalization of the canonical involution of $TTM$.** — Tulczyjew’s construction of the map $\psi_1$ [1] was based on the canonical involution of $TTM$. Roughly speaking, we now want to turn the arguments around and use the fact
that we already have $\psi_k : T^kT^*M \rightarrow T^*T^kM$ at our disposal, to define a canonical diffeomorphism between $T^kT^*M$ and $TT^kM$. This will then be used to initialize an induction process.

A point $z \in T^kT^*M$ is the $k$-velocity of a curve $\gamma(t)$ in $TM$. Let $q \in T^kM$ denote the point $\tau_M^{(k)}(z)$, where $\tau_M : TM \rightarrow M$ is the tangent bundle projection. The point of $TT^kM$ which we want to associate to $z$ is going to be the vector $\zeta_q \in T_qT^kM$, determined by the condition:

$$\forall \alpha_q \in T_q^*T^kM : \langle \zeta_q, \alpha_q \rangle = \frac{d^k}{dt^k} \langle \gamma(t), \chi(t) \rangle \bigg|_{t=0},$$

where $\chi(t)$ is a curve in $T^*M$, representing the $k$-velocity $\psi_k^{-1}(\alpha_q) \in T^kT^*M$ and satisfying $\tau_M(\gamma(t)) = \pi_M(\chi(t)) \ \forall t$.

To see what this means in coordinates, let us denote the coordinates of $z$ as $(q_{0,i}^a, q_{0,i}^b)$. A representative curve $\gamma(t)$ then is given by

$$\gamma(t) = \left( \sum_{i=0}^k \frac{1}{i!} q_{0,i}^a t^i, \sum_{i=0}^k \frac{1}{i!} q_{0,i}^b t^i \right).$$

The element $\zeta_q$ we look for will have coordinates $(q_{0,0}^a = q_{0,i}^a, q_{0,1}^a)$. Its pairing with an arbitrary $\alpha_q = (q_i^a, p_i^b)$ is given by $\sum_{i=0}^k q_{i,1}^a p_i^b$ and the defining relation will of course have to determine the $q_{0,1}^a$. We have $\psi_k^{-1}(\alpha_q) = \left(q_i^a, (k)^{-1}p_i^{k-i}\right)$ so that a representation of an appropriate $\chi(t)$ reads:

$$\chi(t) = \left( \sum_{i=0}^k \frac{1}{i!} q_i^a t^i, \sum_{i=0}^k \frac{(k-i)!}{k!} p_i^{k-i} t^i \right).$$

The right-hand side of the defining relation is precisely the coefficient of $(1/k!)t^k$ in the product of the second components of $\gamma(t)$ and $\chi(t)$ and is easily found to be $\sum_i q_i^a p_i^b$. It follows that the map $T^kT^*M \rightarrow TT^kM$ simply consists in switching suffices: $q_{r,i}^a \mapsto q_{0,r}^a \quad (i = 0, \ldots, k; \ r = 0, 1)$.

For the induction, assume that we know about the identification $T^sT^*M = T^sT^*M$ for some $s$ and all $r$ and that it consists of switching suffices (the case $s = 1$ having just been proved). Using the canonical injection of $T^{s+1}M$ into $TT^sM$ we then obtain the chain $T^sT^{s+1}M \subset T^sTT^sM = TT^sT^*M = T^sT^*M$, which shows that there is an injective map $T^sT^{s+1}M \rightarrow TT^sT^*M$. A schematic coordinate representation of this map is obtained as follows (the ranges of the different indices are $i = 0, \ldots, s + 1; \ j = 0, \ldots, r; \ m = 0, \ldots, s; \ \ell = 0, 1$):

$$(q_{i,j}^a) \mapsto (q_{m,\ell,j}^a) \quad \text{where} \quad q_{m,\ell,j}^a = q_{i,j}^a \quad \text{with} \quad i = m + \ell$$

$$(q_{m,\ell,j}^a) \mapsto (q_{m,\ell,j}^a) \mapsto (q_{j,m,\ell}^a).$$
Since the image point in $TT^*T^*M$ satisfies $q^{a}_{j,m,\ell} = q^{a}_{j,m',\ell'}$ when $m + \ell = m' + \ell'$, it is actually a point of the submanifold $T^{s+1}T^*M$, which means that there is a final identification: $(q^{a}_{j,m,\ell}) \mapsto (q^{a}_{j,i})$.

We conclude that there is a natural identification of $T^rT^*M$ and $T^sT^*M$ for all $r$ and $s$. For a derivation of this result in a more abstract setting, see e.g. [8].

ACKNOWLEDGEMENTS. — This research is supported by NATO, under the Collaborative Research Grants Programme. One of us (M.C.) wishes to thank the Belgian National Fund for Scientific Research for support which made a longer stay possible at the Instituut voor Theoretische Mechanica (Gent).

REFERENCES


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