A geometric characterisation of subvarieties of $\mathcal{E}_6(\mathbb{K})$ related to the ternions and sextonions

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Abstract

The main achievement of this paper is a geometric characterisation of certain subvarieties of the Cartan variety $\mathcal{E}_6(\mathbb{K})$ over an arbitrary field $\mathbb{K}$. The characterised varieties arise as Veronese representations of certain ring projective planes over quadratic subalgebras of the split octonions $\mathbb{O}'$ over $\mathbb{K}$ (among which the sextonions, a 6-dimensional non-associative algebra). We describe how these varieties are linked to the Freudenthal-Tits magic square, and discuss how they would even fit in, when also allowing the sextonions and other "degenerate composition algebras" as the algebras used to construct the square.

Keywords: Veronese varieties, ring geometries, composition algebras, Freudenthal-Tits magic square, E6
MSC 2010 classification: 14N05, 51A45, 51B25,51C05, 51E24

1 Introduction

The characterisation which forms the core of this paper could be carried out without knowing the existence of the Freudenthal-Tits magic square (FTMS). However, the latter carries both the idea and motivation for it, in the sense that the characteristic behaviour of the varieties of the FTMS (in particular, its second row) hints at the existence of similarly behaving varieties across the borders of the square (leaving the non-degenerate world). This gives rise to an extended version of square; in particular of the split version of its second row, the varieties of which are exactly the ones we wish to study and characterise. Below, we explain this in more detail.

1.1 Context: Characterisations related to the FTMS

The FTMS is a 4×4 array of, depending on the viewpoint, Lie algebras, Dynkin diagrams, buildings, projective varieties. Our viewpoint will be geometric in the sense of Tits [14], and over an arbitrary field $\mathbb{K}$. The square can be constructed using a pair of composition algebras $(\mathbb{A}_1, \mathbb{A}_2)$ over $\mathbb{K}$. The algebra $\mathbb{A}_1$ indexes the rows and indicates the rank of the varieties in that row; the algebra $\mathbb{A}_2$ indexes the columns and encodes the algebraic structure over which the varieties in that column are defined. In the geometric version of the square that we consider, the algebra $\mathbb{A}_1$ is always split, whereas $\mathbb{A}_2$ can be either division or split, giving rise to a 'non-split' and 'split' version of the square, respectively. Let us illustrate this by zooming in on the second row, which will be most relevant for this paper.

*Supported by the Fund for Scientific Research - Flanders (FWO - Vlaanderen)

– In the non-split version, this row contains projective planes over division composition algebras over $\mathbb{K}$;
– In the split version, the three last entries of this row are the Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$ (Dynkin type $\mathsf{A}_2 \times \mathsf{A}_2$), the line Grassmannian variety $\mathcal{G}_{6,2}(\mathbb{K})$ (Dynkin type A_5) and the Cartan variety $\mathcal{E}_6(\mathbb{K})$ (Dynkin type E_6) (i.e., the projective version of the 27-dimensional module of the split exceptional group of Lie type E_6), respectively. Abstractly, these are ring projective planes over the split composition algebras over $\mathbb{K}$, and the mentioned varieties can be obtained by taking the Veronese representation of these planes (see Section 10 of [3]). The first entry, which coincides with the non-split case, could thus be seen as the quadric Veronese variety $\mathcal{V}_2(\mathbb{K})$ (Dynkin type A_2), i.e., the Veronese representation of the projective plane over $\mathbb{K}$.

The work of J. Schillewaert H. Van Maldeghem (e.g., [12, 6]) recently culminated in a common characterisation of the Veronese representations of the varieties of the second row of the FTMS [2]. These Veronese varieties are point sets in a projective space equipped with a family of quadrics of a certain kind, depending on the composition algebra. Their characterisation was achieved by means of three elementary axioms, and was accomplished among an infinite family of objects consisting of points and arbitrary (non-degenerate) quadrics in a projective space $\mathbb{P}$ over $\mathbb{K}$. The fact that such a general characterisation singles out exactly the varieties of the FTMS demonstrates the latter's special behaviour once more. It is especially remarkable that this can be done for the split and non-split version simultaneously. The dichotomy of the composition algebras (division/split) translates geometrically in the fact that in the above-mentioned Veronese varieties, the quadrics are either all line-free (i.e., of minimal Witt index) or all hyperbolic (i.e., of maximal Witt index), respectively.

1.2 Motivation: Characterisations across the borders of the FTMS

Inspired by a low-dimensional test case elaborated in [11], the author and H. Van Maldeghem extended the above setting to certain 'degenerate' composition algebras $\mathbb{B}$ [4]. These algebras $\mathbb{B}$ are setwise given by $\mathbb{A} \oplus t\mathbb{A}$, where $\mathbb{A}$ is an associative division composition algebra over $\mathbb{K}$ and t an indeterminate with $t^2 = 0$, and satisfy the Cayley-Dickson multiplication formulas (for example, when $\mathbb{A} = \mathbb{K}$, this yields the dual numbers over $\mathbb{K}$). Equivalently, $\mathbb{B}$ is the result of applying the Cayley-Dickson process to $\mathbb{A}$ with 0 as a primitive element; we will hence refer to $\mathbb{B}$ by $\mathsf{CD}(\mathbb{A}, 0)$. Just like the composition algebras, $\mathsf{CD}(\mathbb{A}, 0)$ is quadratic and alternative (since $\mathbb{A}$ is associative), and its norm form $a + tb \mapsto \mathsf{N}(a)$ (where N is the norm form of $\mathbb{A}$) is multiplicative, though degenerate. When taken to the above setting, where the quadrics are determined by the norm form, this translates geometrically to projective varieties equipped with *degenerate* quadrics whose base is a line-free quadric. Using similar axioms as in [2], it was shown in [4] that point-sets equipped with such quadrics (a priori inside arbitrary dimensions) arise from the Veronese representation of a projective Hjelmslev plane defined over an algebra $\mathsf{CD}(\mathbb{A}, 0)$ with $\mathbb{A}$ an associative division composition algebra. The current paper investigates the following question:

Question. *What happens for the algebras setwise given by $\mathbb{A} \oplus t\mathbb{A}$, where $\mathbb{A}$ is an associative composition algebra over $\mathbb{K}$ which is* not *division?*

A first essential difference with the former case is that we should not only consider the algebras $\mathsf{CD}(\mathbb{A}, 0)$. Indeed, also the *ternions* $\mathbb{T}$, a non-commutative 3-dimensional subalgebra of the split quaternions $\mathbb{H}'$, and *sextonions* $\mathbb{S}$, a strictly alternative 6-dimensional subalgebra of the split octonions $\mathbb{O}'$, can be written as $\mathbb{L}' \oplus t\mathbb{L}'$ (where $\mathbb{L}' = \mathbb{K} \times \mathbb{K}$) and $\mathbb{H}' \oplus t\mathbb{H}'$, respectively (cf. Proposition 2.2). The way we convey it, this gives rise to an additional layer for the FTMS. Denoting division and split composition algebras of dimensions 2,4 over $\mathbb{K}$ by $\mathbb{L}$ and $\mathbb{L}'$, $\mathbb{H}$ and $\mathbb{H}'$, respectively, we see this second layer of the second row as ring projective planes over the following algebras.

$\dim_{\mathbb{K}} \mathbb{A}$	1	2	3	4	6	8
non-split	/	$\mathsf{CD}(\mathbb{K},0)$	/	$\mathsf{CD}(\mathbb{L},0)$	/	$\mathsf{CD}(\mathbb{H},0)$
split	/	$\mathsf{CD}(\mathbb{K},0)$	ternions $\mathbb{T}$	$\mathsf{CD}(\mathbb{L}',0)$	sextonions $\mathbb{S}$	$\mathsf{CD}(\mathbb{H}',0)$

A second difference is that it turns out that the Veronese variety associated to the octonion algebra $\mathsf{CD}(\mathbb{H}',0)$, where $\mathbb{H}'$ are the split quaternions, behaves differently compared to the ones associated to the other algebras in that series. It does not fit in, not in any natural way. This manifests itself in some sense in the fact that, opposed to the Veronese varieties associated to the other algebras in that row, it cannot be seen as a subvariety of the Cartan variety $\mathcal{E}_6(\mathbb{K})$.

The link between the FTMS and the sextonions $\mathbb{S}$ was already explored in [17] by Westbury, who suggested to extend the FTMS (which he considers as a square of complex semisimple Lie algebras) by adding a row/column between the third and the fourth one. Around the same time, also Landsberg and Manivel considered this intermediate Lie algebra between $\mathfrak{e}_7$ and $\mathfrak{e}_8$ in [8]. In Section 8 of [8], they in particular study some (algebraic) geometric properties of the sextonionic plane, i.e., a Veronese variety associated to $\mathbb{S}$. Our approach on the other hand starts from the (incidence) geometric properties of this Veronese variety and its smaller siblings related to $\mathbb{T}$ and $\mathsf{CD}(\mathbb{L}',0)$ (the one related to $\mathsf{CD}(\mathbb{K},0)$ is viewed as a part of the non-split case). By means of three elementary axioms, a natural extension of the ones used in [2] etc., we characterise these varieties. We also provide two additional ways of viewing these varieties: on the one hand, constructed from two (dual) representations of the non-degenerate varieties they are composed of (involving $\mathcal{S}_{2,2}(\mathbb{K})$ and $\mathcal{G}_{6,2}(\mathbb{K})$) (cf. Section 3.2.2), and on the other hand, as subvarieties of the 26-dimensional projective $\mathcal{E}_6(\mathbb{K})$-variety, obtained by slicing it with certain subspaces of dimension $11, 14$ or 20 (cf. Section 2.4). This hence also gives us additional insight in the geometric structure of $\mathcal{E}_6(\mathbb{K})$.

1.3 Main result: Characterisation of the Veronese varieties related to the "new" split second row of the FTMS

Stating the main result (Theorem 3.6) requires more notation and a slightly technical set-up, so we refer to Section 3.3 for that. For the purpose of this introduction, we prefer a simplified set-up, by which means we can explain a related characterisation, proved in [12]. With this, we cannot only informally situate the current main result, but also point out similarities and differences.

Consider a set of points X in a projective space $\mathbb{P}^N(\mathbb{K})$, with $N \in \mathbb{N} \cup \{\infty\}$, equipped with a family Ξ of subspaces of $\mathbb{P}^N(\mathbb{K})$ (of arbitrary yet fixed projective dimension $d+1 < \infty$), $|\Xi| \geq 2$, such that, for each $\xi \in \Xi$, the intersection $X(\xi) := X \cap \xi$ is a parabolic or hyperbolic quadric generating ξ (i.e., the maximal isotropic subspaces on $X(\xi) := X \cap \xi$ have projective dimension $\lfloor \frac{d}{2} \rfloor$). Then the pair (X, Ξ) is called a *split Veronese set (of type d)* if the following axioms are satisfied:

(SV1) Each pair of distinct points $p_1, p_2 \in X$ is contained in a member of Ξ;
(SV2) If ξ_1, ξ_2 are distinct members of Ξ, then $\xi_1 \cap \xi_2 \subseteq X$;
(SV3) for each point $x \in X$, the subspace $T_x := \langle T_x(X(\xi)) \mid x \in \xi \in \Xi \rangle$ has dimension at most $2d$.

Theorem 1.1 (Main Result/Corollary of [12]) *Let (X, Ξ) be a split Veronese set of type $d \geq 1$ in $\mathbb{P}^N(\mathbb{K})$. Then if $\dim T_x = 2d$ for at least one point $x \in X$, we have $d \in \{1, 2, 4, 8\}$ and (X, Ξ) is projectively unique and the resulting varieties are exactly the (Veronese) varieties of the split version of the second row of the FTMS, mentioned above. If $\dim T_x < 2d$ but $T_x = \langle T_x(\xi_1), T_x(\xi_2) \rangle$ for at least one point $x \in X$ and ξ_1, ξ_2 in Ξ with $x \in \xi_1 \cap \xi_2$, then (X, Ξ) is either $\mathcal{S}_{1,2}(\mathbb{K})$ (a subvariety of $\mathcal{S}_{2,2}(\mathbb{K})$) or $\mathcal{G}_{5,2}(\mathbb{K})$ (a subvariety of $\mathcal{G}_{6,2}(\mathbb{K})$).*

A natural way to define Veronese varieties related to $\mathbb{T}$, $\mathsf{CD}(\mathbb{L}',0)$, $\mathbb{S}$ and $\mathsf{CD}(\mathbb{H}',0)$ is by giving an affine description (see Section 2.4). A study of these reveals that, except for the variety associated to $\mathsf{CD}(\mathbb{H}',0)$, they also come with a set of points and quadrics "more or less" satisfying axioms (SV1), (SV2) and (SV3) above. It was not obvious to see why this was more or less, but it turns out that, this time, the structure is not homogenous: there are two types of points and two types of quadrics. Taking this into account when rephrasing axioms (SV1) up to (SV3), the resulting axioms are nothing but a natural extension of them (cf. Section 3.1). It remains slightly mysterious why the variety related to $\mathsf{CD}(\mathbb{H}',0)$ does not satisfy these axioms, not even more or less. As hinted at above, it stands out from the other varieties in that it is not a subvariety of $\mathcal{E}_6(\mathbb{K})$, which is explained by the fact that all other algebras under consideration are subalgebras of the split octonions $\mathbb{O}'$, whereas $\mathsf{CD}(\mathbb{H}',0)$ is not. Axioms (SV1) and (SV2) (and also their natural extensions) imply that the convex closure of two points of X should form a quadric (corresponding to a member of Ξ). However, for the variety related to $\mathsf{CD}(\mathbb{H}',0)$, the convex closure of two points is not a quadric anymore, it rather is a bunch of quadrics. One could argue that it is not a surprise that octonions come with different behaviour, though we did not anticipate this. Indeed, in the non-split case, treated in [4], the Veronese variety related to the octonion algebra $\mathsf{CD}(\mathbb{H},0)$ behaves as do the Veronese varieties related to $\mathsf{CD}(\mathbb{L},0)$ and $\mathsf{CD}(\mathbb{K},0)$ (with notation as above).

If we denote by X and Z the two types of point sets, and by Ξ and Θ the two types of subspaces intersecting $X \cup Z$ in certain quadrics, and call (X, Z, Ξ, Θ) a *dual split Veronese set* if it satisfies the axioms extending (SV1), (SV2) and (SV3) as explained above, then informally the main result reads as follows (where we only exclude the field with two elements, see Remark 3.7).

Main Result–informal statement *If (X, Z, Ξ, Θ) is a dual split Veronese set in $\mathbb{P}^N(\mathbb{K})$, where $\mathbb{K}$ is an arbitrary field with $|\mathbb{K}| > 2$, then, up to projectivity and up to projection from a subspace contained in each member of $\Xi \cup \Theta$, either Θ is empty and then (X, Ξ) is a split Veronese set, or Θ is non-empty and there are four possibilities, all of which are subvarieties of $\mathcal{E}_6(\mathbb{K})$; three of them can be obtained as a Veronese variety associated to one of $\mathbb{T}, \mathsf{CD}(\mathbb{L}',0), \mathbb{S}$, the fourth and smallest case is a subvariety of the Veronese variety associated to $\mathbb{T}$.*

1.4 Structure of the paper

In **Section 2** the "degenerate composition algebras" are formally introduced and discussed to the extent that we will need them. Afterwards, the Veronese varieties associated to $\mathbb{T}$, $\mathsf{CD}(\mathbb{L}',0)$, $\mathbb{S}$ and $\mathsf{CD}(\mathbb{H}',0)$ are defined and studied briefly. In Proposition 2.10, we show that, apart from the last one, they all satisfy properties that naturally generalise (SV1), (SV2) and (SV3) above.

In **Section 3**, the axiomatic set-up for dual split Veronese sets is given, as is a purely geometric description of certain families of varieties that we will encounter later on, each of them containing examples of dual split Veronese sets. This geometric description does not rely on an underlying algebraic structure, but is of course in accordance with the coordinate description given in Section 2. In **Section 3.3** we state the formal version of our main result and describe the examples.

In **Section 4** we prove some basic properties, with the help of which we arrive in **Section 5** to an inductive approach in terms of point-residues. See Proposition 5.16 and Table 1 for a schematic overview. Of the 6 cases that we obtain in Proposition 5.16, 3 lead to actual examples of dual split Veronese sets, the other 3 do not.

Since there is some similarity among the existing cases, and also among the non-existing cases, we do not treat all of them in full detail. We choose to focus on the case that leads to the Veronese variety related to the sextonions $\mathbb{S}$. This turns out to fit in the larger class of so-called *dual line Grassmannians*, see subsection 3.2.2. To deal with these geometries (which we do in **Section 7**), we first need to have a full understanding of their point-residue, which is a so-called *half dual Segre variety*, see subsection 3.2.1. The latter class of geometries is treated in **Section 6** and contains the

Veronese variety related to $\mathbb{T}$. In the final **Section 8** we treat the reamining cases, in particular, we show that $\mathcal{V}_2(\mathbb{K}, \mathbb{S})$ does not occur as a point-residue of some other dual split Veronese set.

2 "Degenerate composition algebras" and associated Veronese varieties

Henceforth, let $\mathbb{K}$ be an arbitrary field. Seeing that composition algebras and (non-degenerate) quadratic alternative algebras are equivalent notions, we will use the latter setting to incorporate the degenerate case. In the literature, one does not find too much on quadratic alternative algebras with a possibly non-trivial radical ánd without restriction on the characteristic of $\mathbb{K}$. What follows is a combination of elements from [9, 10] (allowing characteristic 2, but restricting to the non-degenerate case) and [7] (where the possibilities for the radical are examined, excluding characteristic 2). For a more extended version, we refer to Chapter 4 of [1].

2.1 Quadratic alternative algebras and their radical

Let $\mathbb{A}$ be a unital quadratic alternative $\mathbb{K}$-algebra, i.e., the associator $[a,b,c] := (ab)c - a(bc)$ yields a trilinear alternating map and each $a \in \mathbb{A}$ satisfies a quadratic equation $x^2 - \mathsf{T}(a)x + \mathsf{N}(a) = 0$, where the *trace* $\mathsf{T} : \mathbb{A} \to \mathbb{K} : a \mapsto \mathsf{T}(a)$ is a linear map with $\mathsf{T}(1) = 2$ and the *norm* $\mathsf{N} : \mathbb{A} \to \mathbb{K} : a \mapsto \mathsf{N}(a)$ is a quadratic map with $\mathsf{N}(1) = 1$. The canonical involution associated to $\mathbb{A}$ is given by the map $\mathbb{A} \to \mathbb{A} : x \mapsto \overline{x} := \mathsf{T}(x) - x$, which is indeed an involutive anti-automorphism ($\overline{xy} = \overline{y}\,\overline{x}$ for all $x, y \in \mathbb{A}$), fixing $\mathbb{K}$. Note that $\mathsf{N}(a) = a\overline{a}$ for each $a \in \mathbb{A}$. The bilinear form f associated to the quadratic form N is given by $f(x,y) = \mathsf{N}(x+y) - \mathsf{N}(x) - \mathsf{N}(y) = x\overline{y} + y\overline{x}$. Its *radical* is the set $\mathsf{rad}(f) = \{x \in \mathbb{A} \mid f(x,y) = 0\ \forall y \in \mathbb{A}\}$. We call $\mathbb{A}$ *non-degenerate* if its norm form N is non-degenerate, i.e., if N is anisotropic on $\mathsf{rad}(f)$, so if $\{r \in \mathsf{rad}(f) \mid \mathsf{N}(r) = 0\}$ is trivial. We call the latter set the *radical* R of $\mathbb{A}$. One could also describe R as the *nil radical of* $\mathbb{A}$, which is the maximal ideal of $\mathbb{A}$ with the property that each of its elements is nilpotent.

We note in passing that the non-degenerate quadratic alternative $\mathbb{K}$-algebras $\mathbb{A}$ with $\dim_{\mathbb{K}}(\mathbb{A}) < \infty$ can all be produced using the *Cayley-Dickson doubling process*. An extended version of this process (in which the primitive element is allowed to be 0) also produces *degenerate* algebras, see also Section 4.2 of [1]. We use the notation $\mathsf{CD}(\mathbb{A}, 0)$ for the result of one application of the extended Cayley-Dickson process on $\mathbb{A}$, where $\mathsf{CD}(\mathbb{A}, 0)$ is setwise equal to $\mathbb{A} + t\mathbb{A}$, where t is an indeterminate with $t^2 = 0$. Addition is natural, multiplication goes as follows: $(a + tb) \cdot (c + td) = ac + t(\overline{a}d + cb)$ for $a, b, c, d \in \mathbb{A}$. The radical is given by $t\mathbb{A}$, so it is generated by t.

Our interest goes out to the quadratic alternative algebras $\mathbb{A}$ for which R is, as a $\mathbb{K}$-algebra, generated by a single element. Since R is a 2-sided ideal of $\mathbb{A}$ and $\mathbb{A}(\mathbb{A}r) = \mathbb{A}r = r\mathbb{A} = (r\mathbb{A})\mathbb{A}$ for each $r \in R$, this is equivalent to requiring that R is a principal ideal of $\mathbb{A}$.

2.2 Non-degenerate split quadratic alternative algebras

Let $\mathbb{A}$ be a non-degenerate quadratic alternative algebra. It is well known that its norm form N is either anisotropic on $\mathbb{A}$ or hyperbolic (i.e., has maximal Witt index). In the former case, $\mathbb{A}$ is a division algebra, since $x \in \mathbb{A}$ is invertible if and only if $\mathsf{N}(x) \neq 0$, in the latter case $\mathbb{A}$ is called *split*.

Two non-degenerate quadratic algebras are isomorphic if and only if their respective norm forms are equivalent quadratic forms. So, since any two hyperbolic quadratic forms in the same (even) dimension are equivalent, all non-degenerate split quadratic alternative algebras over $\mathbb{K}$ with the same dimension over $\mathbb{K}$ are isomorphic. This allows us to speak of *the* non-degenerate split quadratic alternative algebras over $\mathbb{K}$, which we will refer to as $\mathbb{K}$, $\mathbb{L}'$, $\mathbb{H}'$ and $\mathbb{O}'$. They can be described as follows (independently of the characteristic), see for instance [5].

Fact 2.1 *Let $\mathbb{A}$ be a non-degenerate split quadratic alternative algebra over a field $\mathbb{K}$. Then $\mathbb{A}$ is isomorphic to either $\mathbb{K}$, $\mathbb{K}\times\mathbb{K}$, the 2×2-matrices $\mathcal{M}_2(\mathbb{K})$ over $\mathbb{K}$ or the split octonions $\mathsf{CD}(\mathcal{M}_2(\mathbb{K}),1)$.*

The split octonions, being non-associative, cannot be given by ordinary matrices and their ordinary multiplication. Zorn's vector-matrices however are a special way of writing the split octonions as 2×2-matrices $M(a,b,c,d,x,y,z,u)$, the off-diagonal elements of which are vectors:

$$\mathbb{O}' \cong \left\{ \begin{pmatrix} a & \begin{bmatrix} b & x & y \end{bmatrix} \\ \begin{bmatrix} c \\ z \\ u \end{bmatrix} & d \end{pmatrix} : a,b,c,d,x,y,u,z \in \mathbb{K} \right\},$$

and the multiplication is given as follows (with the usual dot product and vector product):

$$\begin{pmatrix} a & \boldsymbol{v} \\ \boldsymbol{w} & d \end{pmatrix}\begin{pmatrix} a' & \boldsymbol{v'} \\ \boldsymbol{w'} & d' \end{pmatrix} = \begin{pmatrix} aa' + \boldsymbol{v}\cdot\boldsymbol{w'} & a\boldsymbol{v'} + d'\boldsymbol{v} + \boldsymbol{w}\times\boldsymbol{w'} \\ a'\boldsymbol{w} + d\boldsymbol{w'} - \boldsymbol{v}\times\boldsymbol{v'} & dd' + \boldsymbol{v'}\cdot\boldsymbol{w} \end{pmatrix},$$

and norm (which can be seen as an element of $\mathbb{K}$ indeed)

$$\begin{pmatrix} a & \boldsymbol{v} \\ \boldsymbol{w} & d \end{pmatrix}\overline{\begin{pmatrix} a & \boldsymbol{v} \\ \boldsymbol{w} & d \end{pmatrix}} = \begin{pmatrix} a & \boldsymbol{v} \\ \boldsymbol{w} & d \end{pmatrix}\begin{pmatrix} d & -\boldsymbol{v} \\ -\boldsymbol{w} & a \end{pmatrix} = \begin{pmatrix} ad-bc-xz-yu & \boldsymbol{0} \\ \boldsymbol{0} & ad-bc-xz-yu \end{pmatrix}.$$

2.3 Split quadratic alternative algebras with a level 1 degeneracy

Let $\mathbb{A}$ be a degenerate quadratic alternative unital $\mathbb{K}$-algebra with a non-trivial radical R. In general, one can show that $\mathbb{A}$ contains a non-degenerate quadratic associative unital algebra $\mathbb{B}$ such that $\mathbb{A} = \mathbb{B}\oplus R$. The next proposition, the proof of which is based on methods occurring in [10] to classify the composition algebras, deals with the special case where R is a principal ideal.

Proposition 2.2 (Theorem 4.4.1 of [1]) *Let $\mathbb{A}$ be a degenerate quadratic alternative $\mathbb{K}$-algebra whose radical R is a principal ideal (t) for some $t\in\mathbb{A}\setminus\{0\}$. Then there exists a non-degenerate quadratic subalgebra $\mathbb{B}$ of $\mathbb{A}$ containing 1 such that $\mathbb{A} = \mathbb{B}\oplus t\mathbb{B}$. Moreover, if $\mathbb{B}$ is split, then either $\mathbb{A}$ is isomorphic to $\mathsf{CD}(\mathbb{B},0)$ where $\mathbb{B}\in\{\mathbb{K},\mathbb{L}',\mathbb{H}'\}$, or $\dim_{\mathbb{K}}(\mathbb{A})\in\{3,6\}$. In the latter case, $\mathbb{A}$ is isomorphic to the following respective quotients of $\mathsf{CD}(\mathbb{B},0)$:*

(a) *the upper triangular 2×2-matrices over $\mathbb{K}$ (the* ternions $\mathbb{T}$*);*
(b) $\{M(a,b,c,d,0,y,z,0) \mid a,b,c,d,y,z\in\mathbb{K}\}$ *(the* sextonions $\mathbb{S}$*);*

If $\mathbb{B}$ is split and $\dim_{\mathbb{K}}(\mathbb{A}) < 8$, then $\mathbb{A}$ is isomorphic to a subalgebra of the split octonions $\mathbb{O}'$.

Notation. We will refer to the algebras $\mathbb{A} = \mathbb{B}\oplus t\mathbb{B}$ of the above proposition as *"split quadratic alternative algebras with a level 1 degeneracy"*.

2.4 Veronese varieties associated to $\mathbb{T}$, $\mathbb{H}'$, $\mathbb{S}$ and $\mathbb{O}'$

Let $\mathbb{A}$ be a split quadratic alternative $\mathbb{K}$-algebra with a level 1 degeneracy, i.e., an algebra as in Proposition 2.2: $\mathbb{T}$, $\mathsf{CD}(\mathbb{L}',0)$, $\mathbb{S}$ or $\mathsf{CD}(\mathbb{H}',0)$. To each of those, we associate a plane Veronese variety $\mathcal{V}_2(\mathbb{K},\mathbb{A})$, by which we mean the image the Veronese map of the following point-line geometry $\mathsf{G}_2(\mathbb{K},\mathbb{A})$: the points (resp. lines) are given by the triples in $\mathbb{A}$ such that there is a left (resp. right) $\mathbb{A}$-linear combination that gives 1, incidence is containment. The geometry $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is an instance of a *ring projective plane*, a "projective plane" with the ring $\mathbb{A}$ as coordinatising structure (see for instance [15]). Since $\mathbb{A}$ contains many non-invertible elements, it is hard to give a list of all points and lines of $\mathsf{G}_2(\mathbb{K},\mathbb{A})$. So instead we start with an affine part of $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ (the affine part consisting of points of the form $(1,B,C)$) and use the following *partial Veronese map* ρ, with $d := \dim_{\mathbb{K}}(\mathbb{A})$:

$$\rho : \mathbb{A}\times\mathbb{A} \to \mathbb{P}^{3d+2}(\mathbb{K}) : (1,B,C) \mapsto (1, B\overline{B}, C\overline{C}, BC, C, B)$$

Remark 2.3 Usually, the Veronese map takes (y,z) to $(1, y\overline{y}, z\overline{z}, y\overline{z}, z, \overline{y})$, but we can change z to $\overline{z}$, and then obtain $(1, y\overline{y}, z\overline{z}, yz, \overline{z}, \overline{y})$, which linearly transforms into the above definition.

If $|\mathbb{K}| > 2$, a calculation shows that a line L of $\mathbb{P}^{3d+2}(\mathbb{K})$ containing three points of $\rho(\mathbb{A}\times\mathbb{A})$ has all its points in $\rho(\mathbb{A}\times\mathbb{A})$, except for the unique point on L in the hyperplane H_0 given by the equation $X_0 = 0$. As a first step, we add the points $L \cap H_0$ for such lines L. Repeated steps of this process (one could show that two steps suffice) yield a point-set which is *projectively closed*: each line of $\mathbb{P}^{3d+2}(\mathbb{K})$ is either contained in it, or meets it in at most two points. If $|\mathbb{K}| = 2$, one can also define this closure, but we do not do this effort as we will not consider $\mathbb{F}_2$ (see Remark 3.7).

Definition 2.4 For $|\mathbb{K}| > 2$, we the Veronese variety $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ as the *projective closure* of $\rho(\mathbb{A}\times\mathbb{A})$.

Proposition 2.5 (Corollary 10.42 of [3]) *Let $|\mathbb{K}| > 2$. Then the Veronese varieties $\mathcal{V}_2(\mathbb{K},\mathbb{L}')$, $\mathcal{V}_2(\mathbb{K},\mathbb{H}')$, $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$ are isomorphic to the Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$, the line Grassmannian $\mathcal{G}_{6,2}(\mathbb{K})$, the Cartan variety $\mathcal{E}_6(\mathbb{K})$, respectively.*

We introduce some terminology.

Terminology. Let $\mathbb{P}^N(\mathbb{K})$ denote the N-dimensional projective space over $\mathbb{K}$ generated by the points of $\mathcal{V}_2(\mathbb{K},\mathbb{A})$. In general, we call two points of $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ *collinear* if the line of $\mathbb{P}^N(\mathbb{K})$ determined by them is fully contained in $\mathcal{V}_2(\mathbb{K},\mathbb{A})$; in this case, the line is called a *singular line*. Also, the *convex closure* of two non-collinear points x,y of $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ is given by the points of $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ which are contained in the smallest projective subspace of $\mathbb{P}^N(\mathbb{K})$ that contains all shortest paths between x and y (where shortest paths are viewed in the incidence graph). In $\mathcal{V}_2(\mathbb{K},\mathbb{O}') \cong \mathcal{E}_6(\mathbb{K})$, the convex closure of any pair of non-collinear points is isomorphic to a hyperbolic quadric (of rank 5) in $\mathbb{P}^9(\mathbb{K})$, referred to as a a *symp* (following the parapolar spaces terminology, [13]).

2.4.1 The case $\mathbb{A} = \mathbb{S}$

We use $\mathbb{O}' \cong \{M(a,b,c,d,x,y,z,u) \mid a,b,c,d,x,y,z,u \in \mathbb{K}\}$ (the Zorn matrices, see Subsection 2.2) and $\mathbb{S} \cong \{M(a,b,c,d,0,y,z,0) \mid a,b,c,d,y,z \in \mathbb{K}\}$. Note that $\mathbb{S}' := M(a,b,c,d,x,0,0,u) \mid a,b,c,d,x,u \in \mathbb{K}\} \cong \mathbb{S}$ and $\mathbb{H}' \cong \{M(a,b,c,d,0,0,0,0) \mid a,b,c,d \in \mathbb{K}\}$, so we may asume $\mathbb{H}' = \mathbb{S}\cap\mathbb{S}' \subseteq \mathbb{O}'$. If $\mathbb{A}' \subseteq \mathbb{A}$ then $\rho(\mathbb{A}'\times\mathbb{A}') \subseteq \rho(\mathbb{A}\times\mathbb{A})$ and hence also $\mathcal{V}_2(\mathbb{K},\mathbb{A}') \subseteq \mathcal{V}_2(\mathbb{K},\mathbb{A})$. So $\mathcal{V}_2(\mathbb{K},\mathbb{H}')$ is a subgeometry of both $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S})$, and the latter two are subgeometries of $\mathcal{V}_2(\mathbb{K},\mathbb{O}') \cong \mathcal{E}_6(\mathbb{K})$ (cf. Proposition 2.5). Take $B = M(a,b,c,d,x,y,z,u)$ and $C = M(a',b',c',d',x',y',z',u')$ in $\mathbb{O}'$. Then $\rho(B,C) = (X_{0-26})$, where $X_{i-j} = (x_i,...,x_j)$, and we have

$$\begin{aligned}
X_{0-2} &= (1, ad-bc-zx-uy, a'd'-b'c'-z'x'-u'y'),\\
X_{3-6} &= (aa'+bc'+xz'+yu', d'b+ab'+zu'-z'u, a'c+dc'+x'y-xy', dd'+b'c+x'z+y'u),\\
X_{7-10} &= (ax'+d'x+uc'-cu', d'y+ay'+cz'-c'z, a'z+dz'+by'-b'y, a'u+du'+b'x-bx'),\\
X_{11-26} &= (a',b',c',d',x',y',z',u',a,b,c,d,x,y,z,u).
\end{aligned}$$

Note that, if $B,C \in \mathbb{S}$ (i.e., $x = y = x' = y = 0$), then $x_i = 0$ for $i \in \{7,10,15,18,23,26\} =: J$; likewise, if $B,C \in \mathbb{S}'$, then $x_i = 0$ for $i \in \{8,9,16,17,24,25\} =: J'$. Let $(e_0,...,e_{26})$ be the standard basis of $\mathbb{P}^{26}(\mathbb{K})$. Put $I = \{0,...,26\}\setminus J$ and $I' = \{0,...,26\}\setminus J'$ and define $Y := \langle e_i \mid i \in J\rangle$ and $Y' = \langle e_i \mid i \in J'\rangle$, and finally $F := \langle e_i : i \in I\cap I'\rangle$. Clearly, $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ is contained in the 20-dimensional subspace $\langle F,Y\rangle = \langle e_i \mid i \in I\rangle$ and $\mathcal{V}_2(\mathbb{K},\mathbb{H}')$ is contained in F. See also Figure 1. In the following fact, we gather important properties of the variety $\mathcal{V}_2(\mathbb{K},\mathbb{S})$.

Fact 2.6 (Subsection 5.2.1 of [1]) (1) *The variety $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ is the intersection of the 20-space $\langle F, Y\rangle$ and $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$.*

(2) *Likewise, $F\cap\mathcal{V}_2(\mathbb{K},\mathbb{O}') = F\cap\mathcal{V}_2(\mathbb{K},\mathbb{S})$ coincides with $\mathcal{V}_2(\mathbb{K},\mathbb{H}')$ and is hence isomorphic to the line Grassmannian $\mathcal{G} := \mathcal{G}_{6,2}(\mathbb{K})$ by Proposition 2.5.*

(3) *The subspace Y is a singular subspace of $\mathcal{V}_2(\mathbb{K},\mathbb{S})$, which is not contained in $\rho(\mathbb{S}\times\mathbb{S})$, so it is "at infinity".*

(4) *Each point p of $\mathcal{V}_2(\mathbb{K},\mathbb{S})\setminus Y$ corresponds to a unique point p_G of $\mathcal{G}$ and the set of points of $\mathcal{V}_2(\mathbb{K},\mathbb{S})\setminus Y$ corresponding with this point p_G forms an affine 4-space, whose 3-space at infinity U_p belongs to Y.*

(5) *In view of the foregoing and the transitivity properties in $\mathcal{G}$, we have transitivity on the set of pairs of collinear points of $\mathcal{V}_2(\mathbb{K},\mathbb{S})\setminus Y$ and on the set of non-collinear points of $\mathcal{V}_2(\mathbb{K},\mathbb{S})\setminus Y$.*

(6) *The correspondence between $\mathcal{G}$ and Y, taking a point p of $\mathcal{G}$ to the 3-space U_p, is a linear duality.*

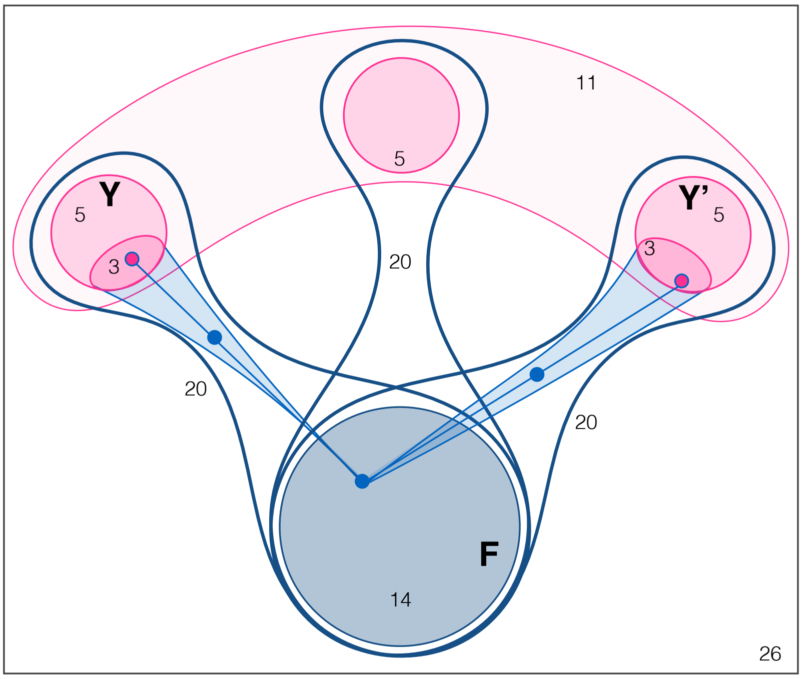

Figure 1: A schematic representation of varieties isomorphic to $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ (living in $\langle F, Y\rangle \cong \langle F, Y'\rangle \cong \mathbb{P}^{20}(\mathbb{K})$), sharing the variety $\mathcal{V}_2(\mathbb{K},\mathbb{H}')\cong\mathcal{G}_{6,2}(\mathbb{K})$ (living in $F\cong\mathbb{P}^{14}(\mathbb{K})$), viewed inside $\mathcal{V}_2(\mathbb{K},\mathbb{O}')\cong\mathcal{E}_6(\mathbb{K})$ (living in $\langle F, Y, Y'\rangle\cong\mathbb{P}^{26}(\mathbb{K})$). Y and Y' are singular 5-spaces of $\mathcal{E}_6(\mathbb{K})$.

The following proposition, whose proof is largely based on the correspondence given in (6) of Fact 2.6, lists the possibilities for the intersection of $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ with a symp of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')\cong\mathcal{E}_6(\mathbb{K})$.

Proposition 2.7 (Proposition 5.2.11 of [1]) *Let Σ be a symp of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$ such that $\zeta := \Sigma\cap\mathcal{V}_2(\mathbb{K},\mathbb{S})$ contains a pair of non-collinear points of $\mathcal{V}_2(\mathbb{K},\mathbb{S})$. Then either*

(i) $Y\cap\Sigma$ is a line L, in which case ζ is a cone with 1-dimensional vertex L and base isomorphic to a hyperbolic quadric in $\mathbb{P}^5(\mathbb{K})$, and hence $\zeta = L^\perp\cap\Sigma$;

(ii) $Y\cap\Sigma$ is a 4-space, in which case $\zeta = \Sigma$.

In both cases, the convex closure (viewed in $\mathcal{V}_2(\mathbb{K},\mathbb{S})$) of two non-collinear points of ζ is ζ.

The symps of $\mathcal{V}_2(\mathbb{K},\mathbb{S})$. We define $\Xi := \{\Sigma\cap\mathcal{V}_2(\mathbb{K},\mathbb{S}) \mid \dim(Y\cap\Sigma) = 1\}$, secondly we put $\Theta := \{\Sigma\cap\mathcal{V}_2(\mathbb{K},\mathbb{S}) \mid \dim(Y\cap\Sigma) = 4\}$.

2.4.2 The case $\mathbb{A} = \mathsf{CD}(\mathbb{L}', 0)$

Here we use that $\mathbb{H}'' := \mathsf{CD}(\mathbb{L}', 0)$ is isomorphic to $\{M(a, 0, 0, d, 0, y, z, 0) \mid a, d, y, z \in \mathbb{K}\}$. Clearly, $\mathbb{H}'' \subseteq \mathbb{S}$ and hence $\mathcal{V}_2(\mathbb{K}, \mathbb{H}'')$ belongs to $\mathcal{V}_2(\mathbb{K}, \mathbb{S})$ and is, as one can easily verify, contained in the 14-space $\langle e_i \mid i \in I \setminus \{4, 5, 12, 13, 20, 21\}\rangle$, which can also be given as $\langle Y, F' \rangle$ where F' is an 8-space in F. The subspace Y is a singular 5-space of $\mathbb{V}_2(\mathbb{K}, \mathbb{H}'')$ as well. As in the first case, $\mathbb{V}_2(\mathbb{K}, \mathbb{S}) \cap \langle Y, F' \rangle = \mathbb{V}_2(\mathbb{K}, \mathbb{H}'')$ and $\mathbb{V}_2(\mathbb{K}, \mathbb{H}'') \cap F' = \mathbb{V}_2(\mathbb{K}, \mathbb{L}') =: \mathcal{S}$, and the latter is isomorphic to the Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$ by Proposition 2.5.

Now, let $U \cong \mathbb{P}^5(\mathbb{K})$ be (an abstract) projective space whose line Grassmannian gives $\mathcal{G}$. Then the Segre variety $\mathcal{S}$, as subvariety of $\mathcal{G}$, arises as the set of lines of U intersecting two given disjoint planes π_1 and π_2 non-trivially. The correspondence between $\mathcal{G}$ and Y (as given in (6) of Fact 2.6) implies that Y is isomorphic to the dual of U, and hence the lines of U intersecting both π_1 and π_2 non-trivially correspond to 3-spaces having a line in common with two planes Z_1 and Z_2 in Y, and these 3-spaces all arise as $U_p = p^\perp \cap Y$, for some point $p \in \mathcal{S}$. We then have the analogue of Proposition 2.7.

Proposition 2.8 (Proposition 5.2.15 of[1]) *Let Σ be a symp of $\mathcal{V}_2(\mathbb{K}, \mathbb{O}')$ such that $\zeta := \Sigma \cap \mathcal{V}_2(\mathbb{K}, \mathbb{H}'')$ contains at least two non-collinear points of $(\mathcal{V}_2(\mathbb{K}, \mathbb{H}'') \setminus Y) \cup Z_1 \cup Z_2$. Then either*

- *(i) $Y \cap \Sigma$ is a line V, in which case ζ is a cone with vertex V and base isomorphic to a hyperbolic quadric in $\mathbb{P}^3(\mathbb{K})$ (so $\zeta \subseteq L^\perp \cap \Sigma$); or,*
- *(ii) $Y \cap \Sigma$ is a 4-space W generated by a line V_i in Z_i and the plane Z_j, with $\{i, j\} = \{1, 2\}$, in which case ζ is a cone with vertex V_i and base isomorphic to a hyperbolic quadric in $\mathbb{P}^5(\mathbb{K})$.*

In both cases, the convex closure (viewed in $\mathcal{V}_2(\mathbb{K}, \mathbb{H}'')$) of two non-collinear points of ζ is ζ.

The symps of $\mathcal{V}_2(\mathbb{K}, \mathbb{H}'')$. We put $\Xi := \{\Sigma \cap \mathcal{V}_2(\mathbb{K}, \mathbb{H}'') \mid \dim(Y \cap \Sigma) = 1\}$ and $\Theta := \{\Sigma \cap \mathcal{V}_2(\mathbb{K}, \mathbb{H}'') \mid Y \cap \Sigma$ is a 4-space containing Z_1 or $Z_2\}$.

2.4.3 The case $\mathbb{A} = \mathbb{T}$

Here we use $\mathbb{T} \cong \{M(a, 0, 0, d, 0, y, 0, 0) \mid a, d, y \in \mathbb{K}\}$, and we obtain that $\mathcal{V}_2(\mathbb{K}, \mathbb{T})$ arises as the intersection of $\mathcal{V}_2(\mathbb{K}, \mathbb{H}'')$ with the 11-space generated by the Segre variety $\mathcal{S}$ in F and by the plane Z_1 in Y. Again, we have:

Proposition 2.9 (Proposition 5.2.18 of[1]) *Let Σ be a symp of $\mathcal{V}_2(\mathbb{K}, \mathbb{O}')$ such that $\zeta := \Sigma \cap \mathcal{V}_2(\mathbb{K}, \mathbb{T})$ contains at least two non-collinear points of $\mathcal{V}_2(\mathbb{K}, \mathbb{T})$. Then either*

- *(i) $\Sigma \cap Z_1$ is a point V, in which case ζ is a cone with vertex V and base isomorphic to a hyperbolic quadrangle over $\mathbb{K}$;*
- *(ii) $\Sigma \cap Z_1 = Z_1$, in which case ζ is isomorphic to a hyperbolic quadric in $\mathbb{P}^5(\mathbb{K})$.*

In both cases, the convex closure (viewed in $\mathcal{V}_2(\mathbb{K}, \mathbb{T})$) of two non-collinear points of ζ is ζ.

The symps of $\mathcal{V}_2(\mathbb{K}, \mathbb{T})$. We put $\Xi := \{\Sigma \cap \mathcal{V}_2(\mathbb{K}, \mathbb{T}) \mid \dim(Z_1 \cap \Sigma) = 0\}$ and $\Theta := \{\Sigma \cap \mathcal{V}_2(\mathbb{K}, \mathbb{T}) \mid Z_1 \subseteq \Sigma\}$.

2.4.4 The case $\mathbb{A} = \mathsf{CD}(\mathbb{H}', 0)$

As alluded to before, $\mathcal{V}_2(\mathbb{K}, \mathsf{CD}(\mathbb{H}', 0))$ does not exhibit the same behaviour as its three siblings $\mathcal{V}_2(\mathbb{K}, \mathbb{T})$, $\mathcal{V}_2(\mathbb{K}, \mathsf{CD}(\mathbb{L}', 0))$ and $\mathcal{V}_2(\mathbb{K}, \mathbb{S})$. One thing that goes wrong for instance, is the following. Firstly, the schematic representation (cf. Figure 1) of the embedding of varieties isomorphic to

$\mathcal{V}_2(\mathbb{K},\mathbb{S})$ and containing the same line Grassmannian variety $\mathcal{V}_2(\mathbb{K},\mathbb{H}')$, is still applicable in this case. Consider the 5-spaces Y, which are pairwise disjoint singular 5-spaces in an 11-dimensional subspaces. A calculation shows that these are on a regulus of $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{H}',0))$ (meaning that for each point of Y there is a unique line of $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{H}',0))$ which meets the other such 5-spaces), whereas in $\mathcal{V}_2(\mathbb{K},\mathbb{S})$, these 5-spaces were pairwise opposite (meaning that no point of one of them is on a line of $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ with a point of one of the others). This is the reason why the convex closure of two non-collinear points of $\mathcal{V}_2(\mathbb{K},\mathbb{H}')$ is not a quadric, as opposed to the situation in $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ (cf. Propositions 2.7, 2.8 and 2.9).

2.5 Properties of the Veronese varieties $\mathcal{V}_2(\mathbb{K},\mathbb{T})$, $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S})$

We show some properties satisfied by each of the varieties $\mathcal{V}_2(\mathbb{K},\mathbb{T})$, $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ (which we will later on use as their characterising properties). In case of $\mathcal{V}_2(\mathbb{K},\mathbb{T})$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S})$, we define Z as the points of the subspace Y; in $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ we define Z as the union of the two subspaces Z_1 and Z_2 and Y as $\langle Z_1, Z_2\rangle$. In the three varieties $\mathcal{V}_2(\mathbb{K},\cdot)$, we set X equal to the points in $\mathcal{V}_2(\mathbb{K},\cdot)\setminus Y$. Recall the definitions of Ξ and Θ.

Proposition 2.10 *The Veronese varieties $\mathcal{V}_2(\mathbb{K},\mathbb{T})$, $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ satisfy the following three properties:*

(S1) *Each pair of distinct points $p_1, p_2 \in X\cup Z$ is contained in a member of $\Xi\cup\Theta$;*
(S2) *for each pair of distinct members $\zeta_1,\zeta_2\in\Xi\cup\Theta$, the intersection $\zeta_1\cap\zeta_2$ is a singular subspace;*
(S3) *for each point $x\in X$, there exists ξ_1,ξ_2 in Ξ such that $T_x = \langle T_x(\xi_1), T_x(\xi_2)\rangle$.*

Proof Consider $\mathcal{V}_2(\mathbb{K},\mathbb{A})$, where $\mathbb{A}\in\{\mathbb{T},\mathsf{CD}(\mathbb{L}',0),\mathbb{S}\}$.

(S1) If p_1 and p_2 are non-collinear, then they determine a unique symp Σ of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$, which by assumption intersects $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ in two non-collinear points, so $\Sigma\cap\mathcal{V}_2(\mathbb{K},\mathbb{O}')\in\Xi\cup\Theta$ by Propositions 2.9, 2.8 and 2.7. If p_1 and p_2 are on a line, then we can always find a point p_3 in $X\cup Z$ which is collinear to p_1 and not to p_2. Then the symp of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$ containing p_2 and p_3 also contains p_1 and the same argument as above applies.

(S2) This is immediate as each member of $\Xi\cup\Theta$ is contained in a symp of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$ and two symps of the latter intersect in a singular subspace, which at its turn will intersect $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ in a singular subspace.

(S3) This can be shown by using the correspondence between the "base" variety (the part contained in F) and the subspace Y (see Observation (6)), together with the properties of the base variety. □

3 Dual split Veronese sets

We start with recalling some properties of quadrics and setting notation, as we will need this all the time. Firstly, a *(non-degenerate) quadric* Q in $\mathbb{P}^n(\mathbb{K})$, $n\in\mathbb{N}$, is the null set of an (irreducible) quadratic homogeneous polynomial in the (homogeneous) coordinates of points of $\mathbb{P}^n(\mathbb{K})$. A line $L\not\subseteq Q$ of $\mathbb{P}^n(\mathbb{K})$ either meets Q in 0, 1 or 2 points. Two points x, x' of Q are *collinear* if $xx'\subseteq Q$. A subspace S of $\mathbb{P}^N(\mathbb{K})$ is called *singular* (w.r.t. Q) if $S\subseteq Q$. The *rank* of Q is one more than the maximum dimension of a singular subspace. The rank is sometimes also called the Witt index.

Let x be any point of Q. Then a line of $\mathbb{P}^n(\mathbb{K})$ containing x that is either singular or meets Q in x only is called a *tangent line (to Q at x)*. The union $T_x(Q)$ of all tangent lines to Q at x is a subspace of $\mathbb{P}^n(\mathbb{K})$ with $\dim T_x(Q)\geq n-1$. Put $V := \{x\in Q\mid T_x(Q)=\mathbb{P}^n(\mathbb{K})\}$. Then V is a subspace of $\mathbb{P}^n(\mathbb{K})$, called the *vertex* of Q. If V is empty, then Q is non-degenerate; if not, then Q is a cone with vertex V and base a non-degenerate quadric Q'.

A property that we will often use is the *one-or-all* property: for each point $x \in Q$ and line $L \subseteq Q$ with $x \notin L$, the point x is collinear either to a unique point of L or to all points of L. Finally, a quadric is called *hyperbolic* if a submaximal singular subspace (i.e., of dimension $r-2$ where r is the rank) is contained in exactly to maximal singular subspaces (i.e., of dimension $r-1$). This means that the set of maximal singular subspaces falls naturally in two families, where two maximal singular subspaces S_1 and S_2 are in the same family if and only if $\dim S_i$ has the same parity as $\dim(S_1 \cap S_2)$. We can now introduce the quadrics that we will be working with.

Definition 3.1 Let R, V be integers with $V \geq -1$ and $R \geq 1$. An (R, V)-*cone* C is a cone with a V-dimensional vertex and as base a non-degenerate hyperbolic quadric of rank $R+1$; C without its vertex is called an (R, V)-*tube*.

Given a tube, there is a unique cone containing the tube, i.e., the vertex can be determined.

3.1 Definition

As mentioned after Propositions 2.7, 2.8 and 2.9, there are two types of symps. Hence we need to work with two families of (R, V)-tubes with a different behaviour, 'ordinary tubes' (Ξ) and 'special tubes' (Θ); dually, we work with 'ordinary points' (X) and 'special points' (Z). Informally speaking, an ordinary tube consists of ordinary points only; special points occur in the vertex of ordinary tubes and in special tubes and their vertices. The definition for the dual split Veronese sets uses the following set-up. When reading this, one may think of X and Z as disjoint sets, this is not a requirement but we will prove in Lemma 4.3 the even stronger statement that X and $\langle Z \rangle$ are disjoint if (X, Z, Ξ, Θ) is a (pre-)DSV.

Definition 3.2 Let r, v, r', v', N be integers which are at least -1 with $r' > r \geq 1$ and put $d := 2r + v + 1$ and $d' := 2r' + v' + 1$. An $(r, v, r', v'; N)$-*system* is a quadruple (X, Z, Ξ, Θ), where X and Z are point sets of $\mathbb{P}^N(\mathbb{K})$ with $Y := \langle Z \rangle$ and $\langle X, Z \rangle = \mathbb{P}^N(\mathbb{K})$, and Ξ is a collection of $(d+1)$-spaces of $\mathbb{P}^N(\mathbb{K})$ with $|\Xi| > 1$ and Θ is a possibly empty collection of $(d'+1)$-spaces of $\mathbb{P}^N(\mathbb{K})$ such that:

- For each $\xi \in \Xi$, the intersection $XY(\xi) := (X \cup Y) \cap \xi$ is an (r, v)-cone C_ξ, $Y(\xi) := Y \cap \xi$ is the vertex of C_ξ and $X(\xi) := X \cap \xi$ is the (r, v)-tube $C_\xi \setminus Y(\xi)$;
- for each $\theta \in \Theta$, the intersection $XY(\theta) := (X \cup Y) \cap \theta$ is an (r', v')-cone C_θ, $Z(\theta) := Z \cap \theta$ the union of the vertex V_θ of C_θ and some r'-space of $XY(\theta) \setminus V_\theta$; the intersection $Y(\theta) := Y \cap \theta$ is the maximal singular subspace $\langle Z(\theta) \rangle$ of C_θ and $X(\theta) := X \cap \theta$ is $C_\theta \setminus Y(\theta)$.

A subspace S of $\mathbb{P}^N(\mathbb{K})$ is called *singular* if all its points are contained in $X \cup Y$. For each point $x \in X$, we denote by T_x the subspace spanned by all singular lines through x.

Definition 3.3 An $(r, v, r', v'; N)$-system (X, Z, Ξ, Θ) is called a **dual split Veronese set** (DSV for short) with parameters (r, v, r', v') if the following axioms are satisfied:

(S1) Each pair of distinct points $p_1, p_2 \in X \cup Z$ is contained in a member of $\Xi \cup \Theta$;

(S2) the intersection $\zeta_1 \cap \zeta_2$ of two distinct members $\zeta_1, \zeta_2 \in \Xi \cup \Theta$ is singular;

(S3) for each $x \in X$, therare are ξ_1, ξ_2 in Ξ with $x \in \xi_1 \cap \xi_2$ and $T_x = \langle T_x(\xi_1), T_x(\xi_2) \rangle$.

If (X, Z, Ξ, Θ) satisfies (S1) and (S2), then we call it a **dual split *pre*-Veronese set** (pre-DSV for short).

Remark 3.4 If Θ is empty then r' and v' can be anything, so we omit them and speak of an $(r, v'; N)$-system and of a (pre-)DSV with parameters (r, v) instead.

3.2 Examples

As was proven in Proposition 2.10, the Veronese varieties $\mathcal{V}_2(\mathbb{K}, \mathbb{T})$, $\mathcal{V}_2(\mathbb{K}, \mathsf{CD}(\mathbb{L}', 0))$ and $\mathcal{V}_2(\mathbb{K}, \mathbb{S})$ are DSVs. Their parameters are discussed below. In this section, we describe more general classes of $(r, v, r', v'; N)$-systems which contain the Veronese varieties, as we will encounter those throughout the proof.

3.2.1 (Half) dual Segre varieties

The Veronese variety $\mathcal{V}_2(\mathbb{K}, \mathbb{T})$ defined using the ternions $\mathbb{T}$ is an example of what we will call a *half dual Segre variety*; the Veronese variety $\mathcal{V}_2(\mathbb{K}, \mathsf{CD}(\mathbb{L}', 0))$ using the degenerate quaternions $\mathsf{CD}(\mathbb{L}', 0)$ is an example of so-called *dual Segre variety.* The adjective 'half' will become clear when the dual Segre varieties are introduced. Both classes of examples hence use a Segre variety, which we recall below. Let ℓ and k be natural numbers with $\ell, k \geq 1$ and put $m := (\ell + 1)(k + 1) - 1$.

Segre varieties with two families of maximal singular subspaces. The Segre variety $\mathcal{S}_{\ell,k}(\mathbb{K})$ is the set of points in the image of the *Segre map* σ:

$$\sigma : \mathbb{P}^\ell(\mathbb{K}) \times \mathbb{P}^k(\mathbb{K}) \to \mathbb{P}^m(\mathbb{K}) : ((x_0, .., x_\ell), (y_0, ..., y_k)) \mapsto (x_i y_j)_{0 \leq i \leq \ell, 0 \leq j \leq k}.$$

This product can be visualised in $\mathbb{P}^m(\mathbb{K})$ by taking an ℓ-space Π_ℓ and a k-space Π_k intersecting each other in precisely a point, and considering $\Pi_\ell \times \Pi_k$. There are two natural families of maximal singular subspaces, where two maximal singular subspaces belong to the same family if and only if they are disjoint. This is a pre-DSV with Θ and Z empty and parameters $(1, -1)$.

Half dual Segre varieties. Inside $\mathbb{P}^{m+\ell+1}(\mathbb{K})$, we consider a Segre variety $\mathsf{S} := \mathcal{S}_{\ell,k}(\mathbb{K})$ and an ℓ-space Y complementary to $\langle \mathsf{S} \rangle$. Let S be any ℓ-space of S and χ_S a linear duality between S and Y, which hence takes a point of S to a hyperplane of Y. We extend χ_S to a map χ from all points of S to Y as follows. Given a point $x \in \mathsf{S} \setminus S$, let x_S be the unique point in S collinear to x; if $x \in S$ then $x_S := x$. Then we put $\chi(x) := \chi_S(x_S)$. We use this to define a $(1, \ell-2, \ell, -1; m+\ell+1)$-system (X, Z, Ξ, Θ). The set X is defined as $\{\langle x, \chi(x)\rangle \setminus \chi(x) \mid x \in \mathsf{S}\}$ and $Z := Y$[1]. A member of Ξ has as base a hyperbolic quadrangle (for a generic member, this can be thought of as inside S) and as vertex a $(\ell - 2)$-space of Y; a member of Θ has empty vertex and is a hyperbolic quadric of rank $\ell + 1$, and one of its maximal singular subspaces is Y. We call this a *half dual Segre variety* and denote it by $\mathcal{HDS}_{\ell,k}(\mathbb{K})$. One can verify that $\mathcal{V}_2(\mathbb{K}, \mathbb{T})$ is isomorphic to $\mathcal{HDS}_{2,2}(\mathbb{K})$ (compare with the description given in Subsection 2.4).

Dual Segre varieties. Inside $\mathbb{P}^{m+2\ell+2}(\mathbb{K})$, we consider a Segre variety $\mathsf{S} := \mathcal{S}_{\ell,\ell}(\mathbb{K})$, and in an $(2\ell + 1)$-space Y complementary to it, we take two disjoint subspaces Z_1 and Z_2 of dimension ℓ. As above, let S_1 be any ℓ-space of S, and also take any ℓ-space S_2 of S which intersects S_1 in a point. For $i = 1, 2$, let χ_{S_i} be a linear duality between S_i and Z_i, thus taking a point of S_i to a hyperplane of Z_i. We extend the maps χ_{S_1} and χ_{S_2} to a map χ from all points of S to $\langle Z_1, Z_2\rangle$ by defining, for a point x of S, its image $\chi(x)$ as $\langle \chi_{S_1}(x_{S_1}), \chi_{S_2}(x_{S_2})\rangle$, where x_{S_i} is equal to x if $x \in S_i$or it is the unique point in S_i collinear to x if $x \notin S_i$. We again use this to define a $(1, 2\ell-1, \ell, \ell-1; m+2\ell+2)$-system (X, Z, Ξ, Θ). The set X is defined as $\{\langle x, \chi(x)\rangle \setminus \chi(x) \mid x \in \mathsf{S}\}$, and $Z := S_1 \cup S_2$. A member of Ξ has as base a hyperbolic quadrangle (for a generic member, this can be thought of as inside S) and its vertex is a $(2\ell-1)$-space of Y generated by a hyperplane of S_1 and a hyperplane of S_2; a member of Θ has as vertex a hyperplane of S_1 or S_2 and is a hyperbolic

[1]The difference between Z and Y is only apparent in the dual Segre varieties.

quadric of rank $\ell+1$, and S_2 or S_1, respectively, is one of its maximal singular subspaces. This is called a *dual Segre variety*, which we will denote by $\mathcal{DS}_{\ell,\ell}(\mathbb{K})$. One can verify that $\mathcal{DS}_{2,2}(\mathbb{K})$ is isomorphic to $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$.

We add that the half dual Segre variety $\mathcal{HDS}_{\ell,\ell}(\mathbb{K})$ can be obtained from the dual Segre variety $\mathcal{DS}_{\ell,\ell}(\mathbb{K})$ by projecting the latter from the subspace Z_2.

3.2.2 Dual line Grassmannians

The Veronese variety $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ defined using the sextonions $\mathbb{S}$ is an example of what we will call a *dual line Grassmannian variety*. This hence builds upon ordinare line Grassmannians, see below. Let n be a natural number with $n\geq 2$ and put $m=\frac{1}{2}(n^2+n)-1$.

Line Grassmannians of projective spaces. The line Grassmannian $\mathcal{G}_{n+1,2}(\mathbb{K})$ of $\mathbb{P}^n(\mathbb{K})$ is the set of points in $\mathbb{P}^m(\mathbb{K})$ obtained by taking the images of all lines of $\mathbb{P}^n(\mathbb{K})$ under the Plücker map

$$\mathsf{pl}:(\langle x_0,x_1,...,x_n),(y_0,y_1,...,y_n)\rangle\mapsto\left(\begin{vmatrix}x_i & x_j\\ y_i & y_j\end{vmatrix}\right)_{0\leq i<j\leq n}.$$

This is a pre-DSV with Θ and Z empty and parameters $(2,-1)$.

Dual line Grassmannians. Consider, inside $\mathbb{P}^{m+n+1}(\mathbb{K})$, an n-space Y and a complementary subspace F of dimension m. In F, take a line Grassmannian $\mathcal{G}:=\mathcal{G}_{n+1,2}(\mathbb{K})$, which is the image under the Plücker map pl of a certain n-dimensional projective space $\mathbb{P}$. Let $\chi':\mathbb{P}\to Y$ be a linear duality, and note that each line of $\mathbb{P}$ corresponds to a $(n-2)$-space of Y. As such, we can define a map χ between $\mathcal{G}$ and Y which is defined by, for each point $x\in\mathcal{G}$, taking x to $\chi'(\mathsf{pl}^{-1}(x))$. The set X is defined as $\{\langle x,\chi(x)\rangle\setminus\chi(x)\mid x\in\mathcal{G}\}$, and $Z:=Y$. Also here, we use this to define a $(2,n-4,n-1,-1;m+n+1)$-system (X,Z,Ξ,Θ). A member of Ξ has as base a hyperbolic quadric of rank 3 (for a generic member, this can be thought of as inside $\mathcal{G}$) and its vertex is a $(n-4)$-space of Y; a member of Θ is a non-degenerate hyperbolic quadric of rank n, i.e., with empty vertex, and it has a hyperplane of Y as one of its maximal singular subspaces. This is called a *dual line Grassmannian*, which we will denote by $\mathcal{DG}_{n+1,2}(\mathbb{K})$. One can verify that $\mathcal{DG}_{6,2}(\mathbb{K})$ is isomorphic to $\mathcal{V}_2(\mathbb{K},\mathbb{S})$.

We do not explicitly claim that the (half) dual Segre varieties and the dual line Grassmannians with general parameters are pre-DSVs, as this is not our main concern, we merely describe these geometries as they will arise naturally when studying the pre-DSVs. Yet it should be conceivable that they satisfy (S1) and (S2). They will in general not satisfy (S3), because for large enough parameters, the tangent space in a point is bigger than the subspace generated by the tangent spaces in that point of two symps through that point. For the specific parameters which correspond to Veronese varieties (or subvarieties of those), we have:

Proposition 3.5 *The varieties $\mathcal{HDS}_{2,2}(\mathbb{K})$, $\mathcal{DS}_{2,2}(\mathbb{K})$ and $\mathcal{DG}_{6,2}(\mathbb{K})$ are isomorphic to $\mathcal{V}_2(\mathbb{K},\mathbb{T})$, $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S})$, respectively, and hence they are dual split Veronese sets, with respective parameters $(1,0,2,-1)$, $(1,1,2,1)$, $(2,1,4,-1)$. The variety $\mathcal{HDS}_{2,1}(\mathbb{K})$ is a subvariety of $\mathcal{HDS}_{2,2}(\mathbb{K})$ and is also a dual split Veronese sets.*

Proof The isomorphisms follow from the description of $\mathcal{V}_2(\mathbb{K},\mathbb{T})$, $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ in Section 2.4: the decomposition into a subspace Y and a "base variety" (which is the Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$ in the smallest two cases and the line Grassmannian $\mathcal{G}_{6,2}(\mathbb{K})$ in the largest case) is given, together with the structure of $p^\perp\cap Y$ for each point p in the base variety. The fact that $\mathcal{V}_2(\mathbb{K},\mathbb{T})$, $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S})$ are DSVs follows from Proposition 2.10. By definition, $\mathcal{HDS}_{2,1}(\mathbb{K})$ is a $(1,0,2,-1;8)$-system. Since it is clearly are contained in $\mathcal{HDS}_{2,2}(\mathbb{K})$ it is straightforward to check that it inherites Axioms (S1) and (S2). A verification shows that also axiom (S3) holds. $\square$

3.3 Main result

Main Result 3.6 *Let (X, Z, Ξ, Θ) be a dual split Veronese set with parameters (r, v, r', v') where $\langle X, Z\rangle = \mathbb{P}^N(\mathbb{K})$ for some arbitrary field $\mathbb{K}$ with $|\mathbb{K}| > 2$. If $|\Theta| \geq 1$, then X is projectively equivalent to a cone with a vertex V' of dimension v' over one of the following varieties:*

(i) A half dual Segre variety $\mathcal{HDS}_{2,k}(\mathbb{K})$, where $k \in \{1, 2\}$, which is a dual split Veronese set with parameters $(1, 0, 2, -1)$;

(ii) A dual line Grassmannian variety $\mathcal{DG}_{6,2}(\mathbb{K})$, which is a dual split Veronese set with parameters $(2, 1, 4, -1)$,

or X is projectively equivalent to:

(iii) A dual Segre variety $\mathcal{DS}_{2,2}(\mathbb{K})$, with parameters $(1, 1, 2, 1)$.

In particular, these varieties are subvarieties of the Veronese variety $\mathcal{V}_2(\mathbb{K}, \mathbb{O}')$ over the split octonions $\mathbb{O}'$. Apart from $\mathcal{HDS}_{2,1}(\mathbb{K})$, all of them are a Veronese variety $\mathcal{V}_2(\mathbb{K}, \mathbb{A})$ for some split quadratic alternative algebra $\mathbb{A}$ whose radical is a principal ideal and every such Veronese variety occurs.

Lemma 4.14 explains why in the above we speak of a cone with vertex V'. A brief structure the proof is given in Subsection 1.4, for more details on its inductive nature, see Section 5.

Remark 3.7 We exclude the field of two elements in the above because already one of the very preliminary lemmas (Lemma 4.1) might fail if $|\mathbb{K}| = 2$. An alternative approach is required. Seeing the high cost and low benefits, we did not pursue this. We know of no counterexamples.

The next theorem contains the special case in which Θ is empty. This case reduces almost immediately to the Main Result of [12], see also Theorem 1.1. On the one hand, we mention this because it shows that the current result extends this characterisation in a natural way, on the other hand, we will need this in the course of the proof of Main Result 3.6.

Theorem 3.8 *Let (X, Z, Ξ, Θ) be a dual split Veronese set with parameters (r, v, r', v') where $\langle X, Z\rangle = \mathbb{P}^N(\mathbb{K})$ for some arbitrary field $\mathbb{K}$. If $|\Theta| = 0$, then all members of Ξ have $\langle Z\rangle$ as their vertex and X is projectively equivalent to a cone with a vertex $\langle Z\rangle$ over one of the following varieties:*

$(r = 1)$ *a Segre variety $\mathcal{S}_{1,2}(\mathbb{K})$ in $\mathbb{P}^5(\mathbb{K})$ or $\mathcal{S}_{2,2}(\mathbb{K})$ in $\mathbb{P}^8(\mathbb{K})$;*

$(r = 2)$ *a line Grassmannian $\mathcal{G}_{5,2}(\mathbb{K})$ in $\mathbb{P}^9(\mathbb{K})$ or $\mathcal{G}_{6,2}(\mathbb{K})$ in $\mathbb{P}^{14}(\mathbb{K})$;*

$(r = 4)$ *the Cartan variety $\mathcal{E}_6(\mathbb{K})$ in $\mathbb{P}^{26}(\mathbb{K})$.*

Proof This is proven in Lemma 4.15 and Proposition 4.16. □

4 Preliminaries

For now, we do not invoke axiom (S3), since many properties can be deduced without it.

Standing hypothesis. Throughout, let (X, Z, Ξ, Θ) be a pre-DSV with parameters (r, v, r', v') with $\langle X, Z\rangle = \mathbb{P}^N(\mathbb{K})$ for an arbitrary field $|\mathbb{K}| > 2$.

Notation. Recall that a subspace S of $\mathbb{P}^N(\mathbb{K})$ is called *singular* if $S \subseteq X \cup Y$. If moreover $S \subseteq X$, then we call S an *X-space*. Note that each $\xi \in \Xi$ has r-dimensional X-spaces and each $\theta \in \Theta$ has r'-dimensional X-spaces. Two subspaces S_1, S_2 are called *collinear* if there is a singular subspace containing them; we denote this by $S_1 \perp S_2$.

4.1 Basic properties

We start with some direct consequences of Axioms (S1) and (S2). The following refines (S1).

Lemma 4.1 *Suppose p_1 and p_2 are non-collinear points of $X \cup Z$. Then there is a unique member of $\Xi \cup \Theta$ containing them, denoted by $[p_1, p_2]$. If p_1 or p_2 belongs to Z, then $[p_1, p_2] \in \Theta$.*

Proof By (S1), there exists a $\zeta \in \Xi \cup \Theta$ containing p_1 and p_2. If $\zeta' \in \Xi \cup \Theta$ also contains p_1 and p_2, and $\zeta \neq \zeta'$, then (S2) implies that $\zeta \cap \zeta'$ is a singular subspace containing p_1, p_2. This contradicts the fact that p_1 and p_2 are not collinear. If $\zeta \in \Xi$, then by Definition 3.2, $Y(\zeta) = Y \cap \zeta$ is the vertex of ζ and its points are hence collinear to all points of $XY(\zeta)$. So if $p_1 \in Z \subseteq Y$, then p_1 would be collinear to p_2, a contradiction; likewise if $p_2 \in Z$. $\square$

Lemma 4.2 *The set X is non-empty and each $x \in X$ is contained in a member of $\Xi \cup \Theta$.*

Proof Take any $\zeta \in \Xi \cup \Theta$ (recall that Ξ is non-empty by Definition 3.2). Then $X(\zeta) = X \cap \zeta \subseteq X$ is non-empty. Now take any $x \in X$. If $x \in X(\zeta)$, the statement follows, so suppose $x \notin X(\zeta)$. Take any point $x' \in X(\zeta)$. By (S1), x and x' are cotnained in a member of $\Xi \cup \Theta$. $\square$

Lemma 4.3 *The set X is disjoint from Y.*

Proof Suppose for a contradiction that p is contained in both X and Y. Lemma 4.2 yields a $\zeta \in \Xi \cup \Theta$ containing p. By Definition 3.2, $X(\zeta)$ and $Y(\theta)$ are disjoint, a contradiction. $\square$

Lemma 4.4 *A line of $\mathbb{P}^N(\mathbb{K})$ containing at least three points of $X \cup Y$ is singular. A singular line of $\mathbb{P}^N(\mathbb{K})$ containing a point of X, contains at most one point of Y.*

Proof Let L be line of $\mathbb{P}^N(\mathbb{K})$ with $|L \cap (X \cup Y)| \geq 3$. Observe that, if L contains two points of Y, then $L \subseteq Y$ since the latter is a subspace (generated by the points of Z) by definition. So we may assume that L contains at least two points x_1, x_2 of X. By (S1), these are contained in a member ζ of $\Xi \cup \Theta$. Then $L \subseteq \zeta$. Since a non-singular line in ζ would meet the quadric $XY(\zeta)$ in at most two points, it follows that L is singular. If L contains a point of X, then by Lemma 4.3, $L \nsubseteq Y$. Since Y is a subspace, $L \cap Y$ is at most one point. $\square$

Lemma 4.5 *Let L_1 and L_2 be two singular lines both not contained in Y, intersecting each other in a unique point s. Then either $\langle L_1, L_2 \rangle$ is a singular plane or it is contained in a unique member of $\Xi \cup \Theta$, denoted $[L_1, L_2]$.*

Proof For $i = 1, 2$, $L_i \nsubseteq Y$ means that L_i contains at most one point of Y, according to Lemma 4.4. Therefore, and since $|\mathbb{K}| > 2$, we can take distinct X-points $x_i, x_i' \in L_i \setminus \{s\}$, $i = 1, 2$. Suppose first that x_1 is not collinear to x_2. Then they determine a unique member $[x_1, x_2] \in \Xi \cup \Theta$ by Lemma 4.1. By Lemma 4.4, x_1 and x_2 are the only points of $X \cup Y$ on x_1x_2. Since $\langle L_1, L_2 \rangle$ is a plane of $\mathbb{P}^N(\mathbb{K})$, the line $x_1'x_2'$ intersects x_1x_2 in a point p, distinct from x_1, x_2. By the foregoing, $p \notin X \cup Y$ and as such the line $x_1'x_2'$ is not singular. So also x_1', x_2' determine a unique member $[x_1', x_2'] \in \Xi \cup \Theta$. If $[x_1, x_2]$ and $[x_1', x_2']$ are distinct, then by (S2), $p \in [x_1, x_2] \cap [x_1', x_2'] \subseteq X \cup Y$, a contradiction. Hence $[x_1, x_2]$ and $[x_1', x_2']$ coincide and therefore they contain $L_1 = x_1x_1'$ and $L_2 = x_2x_2'$. Since each member of $\Xi \cup \Theta$ containing $L_1 \cup L_2$ in particular contains x_1 and x_2, it follows from Lemma 4.1 that $[x_1, x_2]$ is the unique member of $\Xi \cup \Theta$ containing $L_1 \cup L_2$, and hence also $\langle L_1, L_2 \rangle$. Next, suppose that $x_1 \perp x_2$ for each pair of X-points $x_i \in L_i \setminus \{s\}$, $i = 1, 2$. Let $p \in \langle L_1, L_2 \rangle \setminus (L_1 \cup L_2)$ be arbitrary. As $|\mathbb{K}| > 2$, there is a line through p meeting L_1 and L_2 in distinct X-points. By assumption, these X-points are collinear and hence $p \in X \cup Y$. We conclude that the plane $\langle L_1, L_2 \rangle$ is singular indeed. $\square$

Definition 4.6 For each point $p \in X \cup Y$, we denote by $p^\perp$ the union of all singular lines through p with at most one point in Y (i.e., not entirely contained in Y).

The following lemma expresses that, for $\zeta \in \Xi \cup \Theta$, the quadric $XY(\zeta)$ can be obtained as the convex closure of any two of its X-points, i.e., any shortest path between two non-collinear points of $XY(\zeta)$ using lines that are not fully contained in Y is contained in $XY(\zeta)$.

Lemma 4.7 *Let $\zeta \in \Xi \cup \Theta$ and $p \in (X \cup Y) \setminus \zeta$ arbitrary. Then $p^\perp \cap \zeta$ is a singular subspace. In other words, a point of $X \cup Y$ (resp. X) collinear to two non-collinear points of $X(\zeta)$ (resp. $XY(\zeta)$), belongs to ζ.*

Proof If $|p^\perp \cap \zeta| \leq 1$, the first assertion is trivial. So suppose $|p^\perp \cap \zeta| \geq 2$ and let p_1, p_2 be distinct points in $p^\perp \cap \zeta$. Put $L_i := pp_i$, $i = 1, 2$. Then L_1 and L_2 are singular lines, both not contained in Y (by definition of $p^\perp$). If p_1 and p_2 are not collinear, then Lemma 4.5 implies that $p \in [L_1, L_2] = [p_1, p_2] = \zeta$, a contradiction. So p_1 and p_2 are collinear. Put $L = p_1p_2$ and let p_3 be a third point on L. We show that $L_3 := pp_3$ is singular too. Indeed, if p and p_3 are not collinear, then it follows from Lemma 4.5 that $[p, p_3]$ contains the plane $\langle p, L\rangle$. Since the quadric $XY([p, p_3])$ satisfies the 1-or-all axiom, it follows that p is collinear to p_3 after all, a contradiction. We conclude that $L_3 \in p^\perp$ and hence $p^\perp \cap \zeta$ is a singular subspace. The other statement is a direct consequence of the first one. $\square$

We can extend Lemma 4.5 to higher-dimensional subspaces.

Lemma 4.8 *Let S_1 and S_2 be two singular subspaces of dimension k, with $k \geq 1$, both not contained in Y, intersecting each other in a $(k-1)$-space S. Then either $\langle S_1, S_2\rangle$ is a singular $(k+1)$-space or it is contained in a unique member of $\Xi \cup \Theta$ (denoted by $[S_1, S_2]$).*

Proof If $k = 1$, this follows from Lemma 4.5, so let $k > 1$. Observe that $S_1 \nsubseteq Y$ implies that $S_1 \setminus S \nsubseteq Y$, likewise for S_2. So there are X-points $x_1 \in S_1 \setminus S$ and $x_2 \in S_2 \setminus S$. Suppose first that x_1 and x_2 are not collinear. Then they are contained in a unique member $[x_1, x_2] \in \Xi \cup \Theta$ by Lemma 4.1. By Lemma 4.7, $S \subseteq [x_1, x_2]$ since $S \subseteq X \cup Y$ is collinear to x_1 and x_2. Therefore, $\langle S_1, S_2\rangle = \langle S, x_1, x_2\rangle$ is contained in $[x_1, x_2]$. Since each member of $\Xi \cup \Theta$ containing $\langle S_1, S_2\rangle$ contains x_1, x_2, uniqueness follows.

So we may suppose that x_1 and x_2 are collinear for any pair of X-points $x_1 \in S_1 \setminus S$ and $x_2 \in S_2 \setminus S$. Let p be any point in $\langle S_1, S_2\rangle \setminus (S_1 \cup S_2)$. Take an X-point $x_1 \in S_1 \setminus S$. Then the line x_1p intersects $S_2 \setminus S$ in a point p_2. If $p_2 \in X$, then $x_1 \perp p_2$ and hence $p \in x_1p_2 \subseteq X \cup Y$. So suppose $p_2 \in Y$. Observe that this implies that $S \nsubseteq Y$, for otherwise $\langle S, p_2\rangle \subseteq Y$, a contradiction. So we can take an X-point x_1' in $S_1 \setminus (S \cup \{x_1\})$ such that the line x_1x_1' intersects S in an X-point x. Let p_2' be the point in $S_2 \setminus S$ on $x_1'p$. The lines x_1x_1' and x_2x_2' intersect each other in the point $x \in S$, since they are contained in the plane $\langle x_1, x_1', p\rangle$. It follows that $p_2' \in X$, for otherwise $x \in p_2p_2' \subseteq Y$, contradicting Lemma 4.3. So $p \in x_2'p_2' \subseteq X \cup Y$. Therefore, $\langle S_1, S_2\rangle$ is singular. $\square$

Definition 4.9 For each X-space S, we denote by Y_S the set of points of Y collinear to (all points p of) S, i.e., $Y_S := \bigcap_{p \in S}(p^\perp \cap Y)$.

Corollary 4.10 *For each X-space S of dimension $k \geq 0$, Y_S is a subspace of Y and $\langle S, Y_S\rangle$ is a singular subspace.*

Proof If $|Y_S| \leq 1$, there is nothing to prove. So suppose y_1, y_2 are distinct points in Y_S. Note that $y_1y_2 \subseteq Y$ because Y is a subspace by definition. We show that $\langle S, y_1, y_2\rangle$ is a singular subspace, which in particular implies that $y_1y_2 \subseteq Y_S$. Put $S_1 = \langle y_1, S\rangle$ and $S_2 = \langle y_2, S\rangle$. Then S_1 and S_2 are not contained in Y since they contain $S \subseteq X$ (and X and Y are disjoint). Hence, by Lemma 4.8, $\langle S_1, S_2\rangle = \langle S, y_1, y_2\rangle$ is either a singular subspace (in which case we are done), or $S_1 \cup S_2$ is contained in a member ζ of $\Xi \cup \Theta$. In the second case, the 1-or-all axiom in the quadric $XY(\zeta)$ implies that $\langle S, y_1, y_2\rangle$ is a singular subspace. Since $y_1, y_2 \in Y_S$ were arbitrary, this already shows that Y_S is a subspace of Y. Now take any point p in $\langle S, Y_S\rangle \setminus S \cup Y_S$. Then p is contained in a line xy with $x \in S$ and $y \in Y_S$ and hence, since $x \perp y$, we have $p \in xy \subseteq X \cup Y$. So $\langle S, Y_S\rangle$ is singular. □

Lemma 4.11 *Let S be a singular k-space with $k \geq 1$ and suppose $\zeta \in \Xi \cup \Theta$ is such that $S \cap \zeta$ is a subspace of dimension $k-1$ not contained in Y and not a maximal singular subspace of $XY(\zeta)$. Then there is a $\zeta' \in \Xi \cup \Theta$ containing S such that $\zeta' \cap \zeta$ is not collinear to S.*

Proof Since $S \cap \zeta$ is not a maximal singular subspace of ζ, there are non-collinear singular k-spaces S_1 and S_2 in ζ containing $S \cap \zeta$. Moreover, since $S \cap \zeta \nsubseteq Y$, also $S \setminus \zeta$, $S_1 \setminus S \cap \zeta$ and $S_2 \setminus S \cap \zeta$ are not contained in Y. If both $\langle S, S_1\rangle$ and $\langle S, S_2\rangle$ were singular, then an X-point of $S \setminus \zeta$ (recall $S \nsubseteq Y$) would be collinear to a pair of non-collinear X-points of $S_1 \setminus S$, $S \setminus S_2$, violating Lemma 4.7. So we may assume that $\langle S, S_1\rangle$ is not singular. By Lemma 4.8, $\langle S, S_1\rangle$ is contained in a unique $\zeta' \in \Xi \cup \Theta$. By construction, $\zeta' \cap \zeta$ contains S_1, which is not collinear to S. □

We record a special case of the previous lemma. Observe that a crucial difference between Ξ and Θ is that for each $\xi \in \Xi$, each point of $Y(\xi)$ is collinear to each point of $X(\xi)$, whereas for each X-point of $\theta \in \Theta$ there is a point $y \in Y(\theta)$ not collinear to it.

Corollary 4.12 *Let L be a singular line with a unique point $y \in Y$, intersecting some $\zeta \in \Xi \cup \Theta$ in an X-point x. Then there is a $\theta \in \Theta$ containing L and an X-line of ζ.*

Proof We verify that L and ζ satisfy the assumptions of Lemma 4.11 (with L in the role of S). Indeed, $\dim(L \cap \zeta) = 0$ and $L \cap \zeta \nsubseteq Y$ since $L \cap \zeta = \{x\} \subseteq X$. Moreover, by assumption, $r' > r \geq 1$ so x is not a maximal singular subspace in ζ. Lemma 4.11 yields a $\zeta' \in \Xi \cup \Theta$ containing L such that $\zeta \cap \zeta'$ is not collinear to L. In particular, $\zeta \cap \zeta'$ strictly contains x. So let L' be a line of $\zeta \cap \zeta'$ containing x, with L' not collinear to L. If L' contains a point $y' \in Y$, then $y, y' \in Y_x$ and hence $\langle x, y, y'\rangle = \langle L, L'\rangle$ is a singular subspace by Corollary 4.10, contradicting the fact that L and L' are not collinear. So $L' \subseteq X$. Inside ζ', the point y is not collinear to L', so by Lemma 4.1, $\zeta' \in \Theta$. □

4.2 Projections of (X, Z, Ξ, Θ) and the case where Θ is empty

Let S be a subspace of $\mathbb{P}^N(\mathbb{K})$ and F a subspace complementary to it.

Definition 4.13 The projection of (X, Z, Ξ, Θ) from S onto F is induced by the following map.

$$\rho_S : X \to F : x \mapsto \langle Y, x\rangle \cap F.$$

If S is clear from the context, we write ρ instead of ρ_S. We denote $\rho(\Xi) := \{\rho(\xi) \mid \xi \in \Xi\}$, likewise for Θ, and $\rho(X, Z, \Xi, \Theta)$ abbreviates $(\rho(X), \rho(Z), \rho(\Xi), \rho(\Theta))$.

Writing $V^* = \bigcap_{x \in X} Y_x \leq Y$, we show in the next lemma that it suffices to study the projection $\rho_{V^*}(X, Z, \Xi, \Theta)$, and then afterwards conclude that (X, Z, Ξ, Θ) is a cone with vertex V^* over $\rho_{V^*}(X, Z, \Xi, \Theta)$.

Lemma 4.14 *Suppose (X, Z, Ξ, Θ) is a (pre-)DSV with parameters (r, v, r', v'). Let $V^* = \bigcap_{x \in X} Y_x$ be the subspace of Y collinear to all points of X, and put $v^* = \dim V^*$. Then $\rho_{V^*}(X, Z, \Xi, \Theta)$ is a (pre-)DSV with parameters $(r, v - v^* - 1, r', v' - v^* - 1)$ inside $\mathbb{P}^{N - v^* - 1}(\mathbb{K})$.*

Proof If all points of X are collinear to V^*, then all members of Ξ and Θ have V^* in their vertex (cf. Lemma 4.7). A straightforward verification shows that the projection of (X, Z, Ξ, Θ) from V^* satisfies (Si) if (X, Z, Ξ, Θ) does, for each $i \in \{1, 2, 3\}$. $\square$

In case Θ is empty, we can actually show that $V^* = Y$.

Lemma 4.15 *Suppose Θ is empty. Then Y is collinear to all points of X and each member of Ξ has Y as its vertex.*

Proof Suppose for a contradiction that $x \in X$ is a point with Y_x a strict subspace of Y (cf. Corollary 4.10). Since Z generates Y, there is a point z of Z in $Y \setminus Y_x$. Since x and z are not collinear, Lemma 4.1 implies that $[x, z] \in \Theta$. Since Θ is empty, this is a contradiction. So $Y_x = Y$ for each point $x \in X$. It then follows from Lemma 4.7 that each $\xi \in \Xi$ has Y as its vertex: considering two non-collinear points $x_1, x_2 \in X(\xi)$, we have $Y(\xi) = Y_{x_1} \cap Y_{x_2} = Y$. $\square$

Since we now gathered everything we need to deal with the case where Θ is empty, we finish the proof of Theorem 3.8 here.

Proposition 4.16 *Let (X, Z, Ξ, Θ) be a dual split Veronese set with Θ empty. Then the projection $(\rho_Y(X), \rho_Y(\Xi))$ is isomorphic to one of the following varieties: a Segre variety $\mathcal{S}_{1,2}(\mathbb{K})$ or $\mathcal{S}_{2,2}(\mathbb{K})$ $(r = 1)$, a line Grassmannian $\mathcal{G}_{5,2}(\mathbb{K})$ or $\mathcal{G}_{6,2}(\mathbb{K})$ $(r = 2)$ or the variety $\mathcal{E}_6(\mathbb{K})$ $(r = 4)$.*

Proof Recall that Lemma 4.15 says $Y_x = Y$ for each $x \in X$. By Lemma 4.14, the projection $(\rho_Y(X), \rho_Y(\Xi))$ (where we omit the empty sets Z and Θ) satisfies axioms (S1), (S2) and (S3). We claim that $(\rho_Y(X), \rho_Y(\Xi))$ is a split Veronese set of type $2r$. Indeed, it is clear from Definition 3.2 that each $\xi \in \rho_Y(\Xi)$ is a subspace of dimension $2r + 1$ which meets $\rho_Y(X)$ in a quadric whose maximal singular subspaces have dimension r. Moreover, Axioms (S1) and (S2) are identical to Axioms (SV1) and (SV2), respectively. Now take any $x \in \rho_Y(X)$. By Axiom (S3), the tangent space T_x is generated by $T_x(\xi_1)$ and $T_x(\xi_2)$ for two members ξ_1, ξ_2 of $\rho_Y(\Xi)$ containing x. Therefore, since $\dim T_x(\xi_i) = 2r$ for $i = 1, 2$, it follows that $\dim T_x \leq 4r$, and hence (SV3) is satisfied. Whence the claim. By Theorem 1.1, the proposition follows. $\square$

Another projection that we will frequently use, is from the subspace Y. In this context, we also consider the connection between $\rho_Y(X)$ and Y:

Definition 4.17 The *connection* map between $\rho_Y(X)$ and Y (recalling $Y_x = x^\perp \cap Y$) is defined as follows:

$$\chi : \rho_Y(X) \mapsto Y : \rho(x) \mapsto Y_x.$$

We show some general properties on ρ_Y and χ (in particular that χ is well defined).

Lemma 4.18 *Put $\rho = \rho_Y$.*

(i) For each $x \in X$, $\rho^{-1}(\rho(x)) = \langle x, Y_x \rangle \cap X$ and hence χ is well defined;
(ii) for each $\xi \in \Xi$, $\rho(X(\xi))$ is a non-degenerate hyperbolic quadric of rank $r + 1$;
(iii) for each $\theta \in \Theta$, $\rho(X(\theta))$ is a singular subspace of dimension r'.

Proof *(i)* If $\rho(x) = \rho(x')$ for points $x, x' \in X$ with $x \neq x'$, then $\langle x, x', Y \rangle$ contains Y as a hyperplane. Therefore, the line xx' meets Y in a point $y \in Y$, which by Lemma 4.4 means that xx' is singular. In particular, $y \in Y_x$, and so $x' \in \langle x, Y_x \rangle \cap X$ indeed. Conversely it is clear that all points of the latter set are mapped onto the same point by ρ. Consequently, $\langle x', Y_{x'} \rangle = \langle x, Y_x \rangle$, so in particular $Y_{x'} = Y_x$, from which we conclude that χ is well defined. Assertions *(ii)* and *(iii)* are obvious noting that, for each $\xi \in \Xi$, $Y \cap \xi$ is the vertex of ξ and for each $\theta \in \Theta$, $Y \cap \theta$ is a maximal singular subspace of θ. $\square$

Remark 4.19 We will work towards the situation where $(\rho_Y(X), \emptyset, \rho_Y(\Xi), \emptyset)$ is a (pre-)DSV. For this, several things need to be established. Indeed, if p_1, p_2 are two non-collinear points of $\rho(X)$, and $x_i, x_i' \in \rho^{-1}(p_i)$ for $i = 1, 2$, then x_1 and x_2 and also x_1' and x_2' are non-collinear points of X, but it remains to be proven that $\rho([x_1, x_2]) = \rho([x_1', x_2'])$. It also requires some work to determine the inverse image of a singular line of $\rho(X)$.

5 The inductive approach: point-residues

In view of Lemma 4.14, we may assume that no point of Y is collinear to all points of X; and by Proposition 4.16, we may assume that Θ is non-empty. Summarized:

Standing hypothesis. We recall that (X, Z, Ξ, Θ) is a pre-DSV with parameters (r, v, r', v') with $\langle X, Z \rangle = \mathbb{P}^N(\mathbb{K})$ for an arbitrary field $|\mathbb{K}| > 2$, in which no point of Y is collinear to all points of X and such that Θ is non-empty.

In this section we introduce the point-residue of (X, Z, Ξ, Θ) for a point $x \in X$. This amounts to considering all singular lines through x and all members of Ξ and Θ through x. As a preparation, we start by studying the members of Θ containing a given X-space.

5.1 Members of Θ containing a point $x \in X$

For $x \in X$, let Θ_x denote the set of members of Θ containing x.

Lemma 5.1 *Each X-point x, $|\Theta_x| \geq 1$.*

Proof Suppose for a contradiction that $x \in X$ is not contained in any member of Θ. Observe that Z (and hence also Y) is non- empty, since Θ is non-empty by our standing hypothesis and $Z(\theta)$ is non-empty for $\theta \in \Theta$ (cf. Definition 3.2). So take any point $z \in Z$. If x and z are not collinear, then Lemma 4.1 implies that $[x, z] \in \Theta$, so $x \perp z$. Since $Y = \langle Z \rangle$, Corollary 4.10 implies that $Y_x = Y$. By Lemma 4.14, there is a point $x' \in X$ with $Y_{x'}$ a strict subspace of Y (if not, then each point of the non-emtpy subspace Y is collinear to all points of X, contradicting our standing hypothesis). By (S1) and our assumption, there is a $\xi \in \Xi$ containing x and x', and its vertex $Y(\xi)$ is contained in $Y_{x'}$. Take a point $y \in Y \setminus Y(\xi)$. Then the singular line xy has y as unique point in Y and meets ξ in the point x. Corollary 4.12 then implies that xy contained in some $\theta \in \Theta$ together with an X-line of ξ through x. This contradiction shows the lemma. $\square$

Next, we show that, if $|\Theta_{x^*}| = 1$ for some point $x^* \in X$, then $|\Theta_x| = 1$ for every $x \in X$.

Lemma 5.2 *If $|\Theta_{x^*}| = 1$ for some point $x^* \in X$, then $Y(\theta) = Y$ for each $\theta \in \Theta$. In particular, $\dim Y = r' + v' + 1$ and $\dim Y_x = r' + v'$. Also, $|\Theta_x| = 1$ for every $x \in X$.*

Proof Let θ^* be the unique member of Θ_{x^*}. Suppose for a contradiction that $Y(\theta^*) \subsetneq Y$. Since $Y = \langle Z \rangle$, this means that there is a point $z \in Z \setminus Y(\theta^*)$. If z and x^* are not collinear, then by Lemma 4.1, $[z, x^*]$ is a second member of Θ containing x^*, a contradiction. So z and x^* are collinear. Now let L_1 and L_2 be two X-lines in θ^* containing x^* with L_1 and L_2 non-collinear (cf. Definition 3.2 and $r' > r \geq 1$). Take points $x_i \in L_i \setminus \{x^*\}$ with $i = 1, 2$. Then x_1 and x_2 are not collinear (otherwise $\langle x^*, x_1, x_2 \rangle = \langle L_1, L_2 \rangle$ is singular after all). If z would be collinear to both x_1 and x_2, then Lemma 4.7 yields $z \in [x_1, x_2] = \theta^*$, a contradiction. Renumbering if necessary, z is not collinear to x_1. By Lemma 4.5, $[x^* z, L_1]$ is a member of $\Xi \cup \Theta$ and by Lemma 4.1, $[x^* z, L_1]$ belongs to Θ. Again we found a second member of Θ containing x^*. We conclude that $Y(\theta^*) = Y$.

In particular, $\dim Y = \dim Y(\theta^*) = r' + v' + 1$. If $\theta \in \Theta$ is arbitrary, then $Y(\theta) = Y$ follows from $\dim Y(\theta) = r' + v' + 1$. Now let $x \in X$ be arbitrary. Lemma 5.1 guarantees the existence of at least one $\theta \in \Theta$ containing x. Suppose for a contradiction that $\theta' \in \Theta \setminus \theta$ contains x too. Then $Y(\theta) = Y(\theta') = Y$ by the above. Observe that Y_x is a hyperplane of the maximal singular subspace $Y = Y(\theta)$ of θ. As above, this means that there is a point $z \in Z$ which is not contained in Y_x but which is contained in both θ and θ'. By Lemma 4.1, $\theta = [x, z] = \theta'$. □

We go one step further by showing that $|\Theta| > 1$, so in particular, if $|\Theta_x| = 1$ for each $x \in X$, it cannot be that it is the same member of Θ for each point $x \in X$.

Lemma 5.3 *The set Θ contains at least 2 elements.*

Proof Suppose for a contradiction that $\Theta = \{\theta\}$ (recall that Θ is non-empty by our standing hypothesis). Then by Lemma 5.1, each point of X is contained in some member of Θ, so $X(\theta) = X$. However, there is a $\xi \in \Xi$ by assumption. Let x_1, x_2 be two non-collinear points of $X(\xi)$. By the foregoing, x_1, x_2 are also contained in θ. So $\xi = [x_1, x_2] = \theta$ by Lemma 4.1, a contradiction because Θ and Ξ are disjoint (cf. their description in Definition 3.2 and the fact that X and Y are disjoint, see Lemma 4.3). □

5.2 Members of Θ containing an X-line

Definition 5.4 An X-line contained in 0, 1 or at least 2 members of Θ is called a 0-line, a 1-line or a 2-line, respectively.

It turns out that the nature of an X-line L can be expressed in terms of Y_L and Y_x with $x \in L$ (cf. Definition 4.9 and Corollary 4.10).

Lemma 5.5 *Let L be an X-line and x any of point of L. For each subspace $H \subseteq Y_x$ such that Y_L is a hyperplane of H, there is a unique $\theta_{H,L} \in \Theta$ containing L and with $\theta_{H,L} \cap Y_x = H$. Conversely, each $\theta \in \Theta$ containing L coincides with $\theta_{H,L}$ for some subspace H as described and in particular, $Y_L \subseteq \theta$. Consequently:*

(i) L is a 0-line if and only if $Y_x = Y_L$, in which case $Y_x = Y_{x'}$ for each $x' \in L$;
(ii) L is a 1-line if and only if Y_L is a hyperplane of Y_x;
(iii) L is a 2-line if and only if Y_L is strictly contained in a hyperplane of Y_x.

Proof Suppose x, L and H are as described and take a point $y \in H \setminus Y_L$. Since $y \in Y_x \setminus Y_L$, the line xy is singular and is not contained in a singular plane with L. So, by Lemma 4.5, $[xy, L]$ is the unique member of $\Xi \cup \Theta$ containing L and y. Lemma 4.1 implies that $[xy, L] \in \Theta$ (since $y \notin Y_L$). According to Corollary 4.10, $\langle x, Y_x \rangle$ and $\langle L, Y_L \rangle$ are singular subspaces, and hence so is their intersection $\langle x, Y_L \rangle$. Let x' be any point of $\langle x, Y_L \rangle \cap X$, i.e., a point of $\langle x, Y_L \rangle \setminus Y_L$. Then x' is collinear to all points of L (since $x' \in \langle L, Y_L \rangle$) and all points of xy (since $x' \in \langle x, Y_x \rangle$), so Lemma 4.7 implies that x' belongs to $[xy, L]$. As $x' \in \langle x, Y_L \rangle \setminus Y_L$ was arbitrary, and as ζ is a subspace, also Y_L and $\langle Y_L, y \rangle$ are contained in $[xy, L]$. It follows that $[xy, L]$ is the unique member of Θ containing L and H, and is hence also denoted by $\theta_{H,L}$. Inside the quadric $XY(\theta_{H,L})$, the subspace $Y_x \cap \theta_{H,L}$ (which contains H) and is a hyperplane of the maximal singular subspace $Y(\theta_{H,L})$ of $\theta_{H,L}$; the subspace $Y_L \cap \theta_{H,L}$ (which coincides with Y_L) is a hyperplane of $Y_x \cap \theta_{H,L}$. Since Y_L is a hyperplane of H, it follows that $Y_x \cap \theta_{H,L} = H$. This shows the first statement.

Next, suppose $\theta \in \Theta$ contains L and let y be a point in $Y(\theta)$ collinear to x but not collinear to L. Then $\theta = [xy, L]$ by Lemma 4.1. As in the previous paragraph, $[xy, L]$ contains $\langle y, Y_L \rangle$ and hence, putting $H := \langle y, Y_L \rangle$, it follows that $\theta = \theta_{H,L}$. This shows the second statement.

We conclude that the number of members of Θ through L depends on the number of subspaces of Y_x containing Y_L as a hyperplane. So $Y_L = Y_x$ is equivalenbt with the fact that there are no members of Θ through L; and as $x \in L$ was arbitrary, $Y_{x'} = Y_L = Y_x$ for each $x' \in L$. $\square$

5.3 X-planes and higher dimensions

Surprisingly, no X-plane is contained in more than one member of Θ.

Lemma 5.6 *Two members of Θ sharing an X-plane coincide.*

Proof Suppose θ_1, θ_2 are distinct members of Θ which share an X-plane π. Then they also share an X-line L. By Lemma 5.5, $\theta_1 \cap \theta_2$ contains $\langle L, Y_L \rangle$. Since this is a maximal singular subspace in both θ_1 and θ_2, it follows that $\theta_1 \cap \theta_2 = \langle L, Y_L \rangle$ and hence $\pi \subseteq \langle L, Y_L \rangle$, a contradiction. $\square$

5.4 Point-residues and the inductive approach

Recall that, for each point $x \in X$, we denote by T_x the subspace generated by all singular lines through x. We now introduce the point-residue, as hinted at in the beginning of this section.

Definition 5.7 For each $x \in X$, we define the *point-residue* $\mathrm{Res}_X(x) := (X_x, Z_x, \Xi_x, \Theta_x)$ as follows. Take any hyperplane H_x of T_x containing Y_x and not containing x. We define X_x as the set of points in H_x that are on an X-line with x and $Z_x := Y_x \cap Z$; furthermore, Ξ_x is the set $\{T_x(\xi) \cap H_x \mid x \in \xi \in \Xi\}$ and Θ_x as $\{T_x(\theta) \cap H_x \mid x \in \theta \in \Theta\}$. Choosing a different hyperplane H'_x in T_x leads to an isomorphic point-residue (with canonical isomorphism).

Remark 5.8 Consider a point $x \in X$ and $\zeta \in \Xi \cup \Theta$ with $x \in X(\zeta)$. Let ζ_x be the corresponding member $\Xi_x \cup \Theta_x$, i..e, $\zeta_x = T_x(\zeta) \cap H_x$. Then $X_x(\zeta_x) := X_x \cap \zeta_x$ is isomorphic to the point-residue $\mathrm{Res}_{X(\zeta)}(x)$ of $X(\zeta)$ as a (degenerate) hyperbolic quadric. Conversely, given ζ_x, we can recover ζ by taking the convex closure (cf. Lemma 4.7) of two non-collinear points of $X_x(\zeta_x)$ (note that these points exist since there are non-collinear X-lines through x in ζ, as $r' > r \geq 1$). Note that, if $\zeta \in \Xi$, then ζ_x is an $(r-1, v)$-tube and if $\zeta \in \Theta$, then ζ_x is an $(r'-1, v')$-tube. Therefore, we will only consider the point-residue if $r > 1$ (recall that $r' > r \geq 1$ so automatically, $r' > 1$).

We give an example to clarify the definition and to point out why we need the residues.

Example 5.9 Consider the dual line Grassmannian $\mathcal{G}_{6,2}(\mathbb{K}) \cong \mathcal{V}_2(\mathbb{K}, \mathbb{S})$, as introduced in Section 3.2.2, which is a DSV (X, Z, Ξ, Θ) with parameters $(2, 1, 4, -1)$. Later we show (see Lemma 7.1 and further) that, for any $x \in X$, the point-residue $(X_x, Z_x, \Xi_x, \Theta_x)$ is the half dual Segre variety $\mathcal{DS}_{3,1}(\mathbb{K})$, which is a $(1, 1, 3, -1; 11)$-system (it is also a pre-DSV). This can be made conceivable as follows. The variety $\mathcal{G}_{6,2}(\mathbb{K})$ is composed of a line Grassmannian $\mathcal{G} = \mathcal{G}_{6,2}(\mathbb{K})$ and a 5-space Y. By transitivity (see Fact 2.6(5)), we may assume that $x \in \mathsf{G}$. By definition, $\dim Y_x = 3$, and on the other hand $\mathrm{Res}_{\mathcal{G}}(x)$ is isomorphic to $\mathcal{S}_{3,1}(\mathbb{K}) =: \mathsf{S}$. Furthermore, for each X_x-point x' it turns out that $Y_x \cap Y_{x'}$ is a plane of Y_x, yielding the desired correspondence between the points of S and the (hyper)planes of Y_x. We leave the details to the interested reader.

As explained in Remark 5.8, we will not consider point-residues if $r = 1$. In case $r \geq 2$, we want to show that $\mathrm{Res}_X(x)$ is a pre-DSV as well. For that, we first need that $\langle Z_x \rangle = \langle Y_x \rangle$.

Lemma 5.10 *For each $x \in X$, $\langle Z_x \rangle = Y_x$.*

Proof Recall that Y_x is a subspace by Corollary 4.10. Suppose for a contradiction that $S := \langle Z_x \rangle$ is properly contained in Y_x. Let $\theta \in \Theta$ through x be arbitrary (cf. Lemma 5.1). Take an X-line L through x in θ (which exists since $r' > r \geq 1$). By Lemma 5.5, Y_L is stricty contained in Y_x. Since two strict subspaces of Y_x do not cover Y_x and hence there is a point $y \in Y_x \setminus (Y_L \cup S)$. According to Lemmas 4.5 and 4.1, $\theta' := [xy, L] \in \Theta$. Recall from Definition 3.2 that $Y(\theta')$ is generated by an r'-space $M \subseteq Z$ and the vertex $V \subseteq Z$. Now $x \in \theta'$ is collinear to a hyperplane of $Y(\theta)$, which obviously contains V. Therefore x is collinear to only a hyperplane M' of M (otherwise $x \perp Y(\theta)$), and hence $x^\perp \cap Y(\theta)$ is the hyperplane of $Y(\theta)$ generated by V and M'. We conclude that $x^\perp \cap Y(\theta') = \langle x^\perp \cap Z(\theta') \rangle$. As the former subspace contains y and the latter is contained in S, this contradicts $y \notin S$. The conclusion follows. $\square$

We can show that $\mathrm{Res}_X(x)$ is a pre-DSV, and that it also satisfies the properties mentioned in the standing hypothesis.

Proposition 5.11 *Let (X, Z, Ξ, Θ) be a pre-DSV with parameters (r, v, r', v') with $r \geq 2$, $|\Theta| \geq 1$ and such that no point of Y is collinear to X. For each $x \in X$, $(X_x, Z_x, \Xi_x, \Theta_x)$ is a pre-DSV in $\mathbb{P}^{N_x}(\mathbb{K})$ with $N_x = \dim(T_x) - 1$ with parameters $(r-1, v, r'-1, v')$ and $|\Theta_x| \geq 1$, and is such that no point of Y_x is collinear to X_x. If (S3) holds in (X, Z, Ξ, Θ), then $N_x \leq 2d - 1$.*

Proof We first look at the ambient projective space. A singular line containing x is either an X-line, which has a unique point in X_x, or it has a unique point in Y_x. Since T_x is generated by the singular lines containing x, it follows that $T_x = \langle x, X_x, Y_x \rangle$. By Lemma 5.10 we also have $\langle Z_x \rangle = Y_x$. So we obtain that X_x and Z_x generate the projective space H_x and hence $H_x \cong \mathbb{P}^{N_x}(\mathbb{K})$ with $N_x = \dim(T_x) - 1$ (so if (S3) holds, then indeed $N_x \leq 2d - 1$).

A straightforward verification (see also Remark 5.8) tells us that each $\zeta \in \Xi \cup \Theta$ with $x \in \zeta$, intersects the sets X_x, Y_x and Z_x as described in Definition 3.2, and that its parameters are $(r-1, v, r'-1, v')$.

Since (S2) holds in (X, Z, Ξ, Θ), it also holds in $\mathrm{Res}_X(x)$, as the projection of a singular subspace containing x is a singular subspace itself. Next, we show that also (S1) holds in $\mathrm{Res}_X(x)$. Take two points p_1, p_2 in $X_x \cup Z_x$. These points correspond to two singular lines L_1 and L_2 containing x, so both lines are not contained in Y. By Lemma 4.5, either $[L_1, L_2] \in \Xi \cup \Theta$, in which case (S1) follows; or $\langle L_1, L_2 \rangle$ is a singular plane π. In the latter case, (S1) implies the existence of a member ζ of $\Xi \cup \Theta$ containing L_1. If ζ also contains L_2, we are good, so suppose it does not. Since $r, r' \geq 2$, there is a singular plane π' in ζ through L_1 not collinear to L_2 (cf. Lemma 4.7). By Lemma 4.8, π and π' determine a unique member of $\Xi \cup \Theta$ containing L_1 and L_2, so also in this case (S1) follows.

Finally, suppose for a contradiction that $y \in Y_x$ collinear to all points of X_x. Take any $x' \in X$ not collinear to x. Then by Lemma 4.1, $[x, x'] \in \Xi \cup \Theta$. By assumption, y is collinear to all points of X_x and to x, and hence y is collinear to $x^\perp \cap X([x, x'])$. Since each point of Y_x is trivially collinear to y, in fact $x^\perp \cap XY([x, x']) \subseteq y^\perp \cap XY([x, x'])$. This is only possible if xy contains a point of the vertex of $XY([x, x'])$, and hence y belongs to the vertex of $XY[x, x']$. Therefore, $x' \perp y$. As $x' \in X \setminus x^\perp$ was arbitrary, we conclude that all X-points are collinear to y. $\square$

The fact that there are no X-planes contains in multiple members of Θ, translates to the following—where we not only look at a point-residue but also at a point-residue of the point-residue (a line residue in fact).

Lemma 5.12 *Let $x \in X$ be arbitrary and $x' \in X_x$ too. Put $X_{xx'} := (X_x)_{x'}$, et cetera. Then:*

- *If $r \geq 2$, then in $(X_x, Z_x, \Xi_x, \Theta_x)$, each X_x-line is contained in at most one member of Θ_x;*
- *If $r \geq 3$, then in $(X_{xx'}, Z_{xx'}, \Xi_{xx'}, \Theta_{xx'})$, each $X_{xx'}$-point is contained in a unique member of $\Theta_{xx'}$.*

Proof Recall that if $r \geq 2$, then $(X_x, Z_x, \Xi_x, \Theta_x)$ is a pre-DSV with parameters $(r-1, v, r'-1, v')$ by Proposition 5.11, and hence if $r \geq 3$, also $(X_{xx'}, Z_{xx'}, \Xi_{xx'}, \Theta_{xx'})$ is a pre-DSV, by the same token.

• Suppose $r \geq 2$. Let L be an X_x-line in $\mathrm{Res}_X(x)$. Then, viewed in (X, Z, Ξ, Θ), each line xu with $u \in L$ is an X-line by Definition 5.7. Therefore, the plane $\langle u, L\rangle$ is an X-plane in (X, Z, Ξ, Θ). By Lemma 5.6, $\langle u, L\rangle$ is contained in at most one member of Θ. Consequently, L is contained in at most one member of Θ_x.

• Next, suppose $r \geq 3$. Since a point of $X_{xx'}$ corresponds to an X-plane of (X, Z, Ξ, Θ), there is again at most one member of $\Theta_{xx'}$ containing it. On the other hand, there is at least one member of $\Theta_{xx'}$ contianing it, by Lemma 5.1. $\square$

This has a two very strong consequences.

Corollary 5.13 *If there is a point of X contained in a unique member of Θ, then $r = 1$.*

Proof Suppose for a contradiction that $r \geq 2$ and $|\Theta_x| = 1$ for some $x \in X$. Consider the point-residue $(X_x, Z_x, \Xi_x, \Theta_x)$, which is a pre-DSV with parameters $(r-1, v, r'-1, v')$ by Proposition 5.11. The fact that $|\Theta_x| = 1$ contradicts Lemma 5.3. We obtain that $r = 1$. $\square$

Corollary 5.14 *A pre-DSV (X, Z, Ξ, Θ) with parameters (r, v, r', v') has $r \leq 3$.*

Proof Suppose that $r \geq 3$. Take a point $x \in X$ and a point $x' \in X_x$. Then $(X_{xx'}, Z_{xx'}, \Xi_{xx'}, \Theta_{xx'})$ is a pre-DSV with parameters $(r-2, v, r'-2, v')$ by Proposition 5.11. By Lemma 5.12, each point of $X_{xx'}$ is contained in a unique member of $\Theta_{xx'}$. So by Corollary 5.13, we obtain that $r - 2 = 1$, i.e., $r = 3$. $\square$

By the second item of Lemma 5.12 it will hence be crucial to study the pre-DSVs in which each point of X is contained in a unique member of Θ, in which case we already know that $r = 1$ by Corollary 5.13. Before pursuing this, we note that we can say something more about the X-lines too, if $r \geq 2$.

Lemma 5.15 *If $r \geq 2$, then either all X-lines are 1-lines or all X-lines are 2-lines.*

Proof Let L be an X-line and let x be any point of L. Consider $(X_x, Z_x, \Xi_x, \Theta_x)$, which is a pre-DSV with parameters $(r-1, v, r'-1, v')$ by Lemma 5.11. In here, L corresponds to a point $x' \in X_x$. By Lemma 5.1, there is a member of Θ_x containing x'. This means that there is a member of Θ containing L. So L is either a 1-line or a 2-line. Assume that it is a 1-line, i.e., is is contained in a unique member of Θ_x. Then all points of X_x are contained in a unique member of Θ_x by Lemma 5.2. So each X-line through x is a 1-line. For any point x'' on such a 1-line through X, the same argument yields that each X-line through x'' is a 1-line. By connectivity, we conclude that each X-line is a 1-line. The lemma follows. $\square$

5.5 Case distinction

Based on the previous subsection, we summarize the possibilities to investigate in Table 1, as is shown in Proposition 5.16. For each X-line L and each X-plane π, let Θ_L and Θ_π denote the set of members of Θ containing L and π, respectively, in analogy with the notation Θ_x for $x \in X$ (although we do admit that we use this notation twice, but the meaning should be clear from the context).

Proposition 5.16 *The possibilities for the pre-DSV (X, Z, Ξ, Θ) with parameters (r, v, r', v') are as listed in Table 1.*

$r=3$	$\forall X$-plane π, $\lvert\Theta_\pi\rvert = 1$		
$\downarrow$	$\downarrow$		
$r=2$	$\forall X$-line L, $\lvert\Theta_L\rvert = 1$	$\forall X$-line L, $\lvert\Theta_L\rvert \geq 2$	
$\downarrow$	$\downarrow$	$\downarrow$	
$r=1$	$\forall X$-point x, $\lvert\Theta_x\rvert = 1$ $\forall X$-line L, $\lvert\Theta_L\rvert \in \{0,1\}$	$\forall X$-point x, $\lvert\Theta_x\rvert \geq 2$ $\forall X$-line L, $\lvert\Theta_L\rvert \in \{0,1\}$	$\forall X$-point X, $\lvert\Theta_x\rvert \geq 2$ $\exists X$-line L, $\lvert\Theta_L\rvert \geq 2$

Table 1: The possible situations to consider, where the residue of a pre-DSV gives a pre-DSV with properties as listed one cell below. The ones in red will turn out not to lead to examples of DSVs (where also (S3) holds).

Proof Firstly, it follows from Corollary 5.14 that $r \leq 3$. Moreover, if $r = 3$, then the second item of Lemma 5.12 tells us that $|\Theta_\pi| = 1$ for each X-plane π. Next, suppose $r = 2$. Then it follows from Lemma 5.15 that all X-line are either 1-lines or 2-lines; and by the foregoing, only the former situation can occur as the residue of a pre-DSV. Finally, suppose $r = 1$. Then we know that $|\Theta_x| \geq 1$ for each point $x \in X$ by Lemma 5.1. In case the pre-DSV occurs as a residue, then we know by the first item of Lemma 5.12 that there are no 2-lines. Note that, if L is a 2-line, then for each $x \in L$ we have $|\Theta_x| \geq 2$, so by Lemma 5.2, it follows that $|\Theta_x| \geq 2$ for all $x \in X$. $\square$

We only treat the first column in full detail, since that contains the most interesting cases. For the other columns we will be brief and give precise references to [1], also because the techniques are very similar to the ones applied in the first column (it is not entirely straightforward, otherwise the cases might just as well be treated simultanously, but at many points there is an overlap).

Overview. In Section 6, we start by the first column in case $r = 1$, which will lead us to the half dual Segre varieties $\mathcal{HDS}_{r',k}(\mathbb{K})$ (which will be a DSV if $r' = 2$ and $k \in \{1,2\}$). In Section 7, we treat the first column entry in case $r = 2$, which will lead us to the dual Line Grassmannian varieties $\mathcal{DG}_{r'+2,2}(\mathbb{K})$ (which will be a DSV if $r' = 4$). In Section 8, we treat all remaining cases. We mention already that the case $r = 1$ of the second column leads to the dual Segre variety $\mathcal{DS}_{r',r'}(\mathbb{K})$ (which will be a DSV if $r' = 2$).

6 The half dual Segre varieties

As explained in Subsection 5.5 (see also Proposition 5.11), one of the cases to be treated is the one where $r = 1$ and $|\Theta_x| = 1$ for each $x \in X$. So we summarize:

Standing hypothesis. Throughout this section, (X, Z, Ξ, Θ) is a pre-DSV with parameters $(1, v, r', v')$ with $\langle X, Z\rangle = \mathbb{P}^N(\mathbb{K})$ for an arbitrary field $|\mathbb{K}| > 2$, in which no point of Y is collinear to all points of X and such that for each $x \in X$, there is a unique member of Θ containing x, which we denote by θ^x.

We start with a basic but useful observation.

Lemma 6.1 *Each singular line containing x and not contained in θ^x is an X-line which is contained in no member of Θ (a 0-line). Each singular line in θ^x containing x either has a unique point in Y or it is an X-line contained in a unique member of Θ (a 1-line).*

Proof Let L be a singular line containing x. Suppose first that $L \nsubseteq \theta^x$. If L would not be an X-line, then it contains a unique point of Y and hence it is contained in θ^x since $Y = Y(\theta^x)$ by Lemma 5.2. So L is an X-line. If it would not be a 0-line, then we obtain at least two members of Θ containing x, contradicting our standing hypothesis $|\Theta_x| = 1$. Next, suppose $L \subseteq \theta^x$. If L contains a point of Y, there is nothing to show, so suppose it is an X-line. Since it is contained in θ^x and since θ^x is the unique member of Θ containing x, it follows that L is a 1-line. $\square$

6.1 The parameters v, r', v'

We already showed in Lemma 5.2 that each $\theta \in \Theta$ has $Y(\theta) = \theta$. We continue by showing that $v' = -1$, and then it follows that Y has dimension r', and coincides with Z. We can also deduce v in terms of r'.

Lemma 6.2 *We have $v' = -1$, $Y = Z$ and $\dim Y = r'$.*

Proof Take any $\theta \in \Theta$. We show that all points of X are collinear to the vertex $V(\theta)$ of θ, which then by our standing assumption leads to $V(\theta)$ being empty. Let $x \in X$ be arbitrary. If $x \in X(\theta)$, we trivially have that x is collinear to $V(\theta)$, so suppose $x \notin X(\theta)$. Suppose first that x is collinear to some point $x' \in X(\theta)$. By Lemma 6.1, xx' is a 0-line. It follows from Lemma 5.5(i) that $Y_{x'} = Y_x$. Since x' is collinear to the vertex $V(\theta)$ of θ, we obtain that $V(\theta) \subseteq Y_x$ and hence x is collinear to $V(\theta)$ indeed.

Finally, suppose that x is not collinear to any point of $X(\theta)$. Taking $x' \in X(\theta)$ arbitrary, Lemma 4.1 implies that $[x, x'] \in \Xi \cup \Theta$, and since $|\Theta_{x'}| = 1$ by assumption, we obtain $[x, x'] \in \Xi$. Let L_1 and L_2 be two non-collinear X-lines of $[x, x']$ through x'. For $i = 1, 2$, we claim that L_i is collinear to $V(\theta)$. Indeed, if $L_i \subseteq \theta$ then $L_i \perp V(\theta)$ by definition, and if $L_i \nsubseteq \theta$ then $L_i \perp V(\theta)$ by the first paragraph. By Lemma 4.7, $V(\theta)$ is contained in the vertex of $[x, x']$ and, in particular, $x \perp V(\theta)$. This shows that $V(\theta)$ is collinear to all points of X and therefore is empty, so $v' = -1$.

It now follows from Definition 3.2 that $Y(\theta)$ is a singular r'-space of θ which is contained in Z. Since $Y = Y(\theta)$ by Lemma 5.2, the lemma follows. $\square$

Lemma 6.3 *Suppose θ_1, θ_2 are distinct member of Θ and suppose $x_1 \in X(\theta_1)$ and $x_2 \in X(\theta_2)$. Then $Y_{x_1} = Y_{x_2}$ if and only if x_1 and x_2 are collinear; if $Y_{x_1} \neq Y_{x_2}$, then $[x_1, x_2] \in \Xi$. In the latter case, the vertex of $[x_1, x_2]$ is $Y_{x_1} \cap Y_{x_2}$. In particular, $v = r' - 2$.*

Proof By Lemma 5.2 $Y(\theta_1) = Y(\theta_2) = Y$ and $Y(\theta_i)$ is a maximal singular subspace in θ_i, $i = 1, 2$ (cf. Definition 3.2). So by (S2), $\theta_1 \cap \theta_2 = Y$. Note that $\dim Y = r'$ by Lemma 6.2.

Suppose first that $Y_{x_1} \neq Y_{x_2}$. Then x_1 and x_2 are not collinear, for otherwise x_1x_2 would be a 0-line by Lemma 6.1, and then $Y_{x_1} = Y_{x_2}$ by Lemma 5.5(i), a contradiction. Lemma 4.1 implies $[x_1, x_2] \in \Xi \cup \Theta$, and since $x_1x_2 \nsubseteq \theta_1$, we obtain $[x_1, x_2] \in \Xi$ (recall $|\Theta_{x_1}| = 1$). By Lemma 4.7, the vertex of $[x_1, x_2]$ is $Y_{x_1} \cap Y_{x_2}$, which has dimension $r' - 2$ since it is the intersection of two hyperplanes of Y.

Note that Lemma 5.3 garantees that $|\Theta| \geq 2$, and that, as can be verified easily, points $x_1 \in X(\theta_1)$ and $x_2 \in X(\theta_2)$ with $Y_{x_1} \neq Y_{x_2}$ exist. So we may conclude from the previous paragraph that $v = r' - 2$.

Finally, suppose that $Y_{x_1} = Y_{x_2}$. If x_1 and x_2 are not collinear, then the above argument shows that $[x_1, x_2] \in \Xi$ and has vertex $Y_{x_1} \cap Y_{c_2} = Y_{x_1}$, contradicting $v = r' - 2$. So x_1x_2 is a singular line. $\square$

6.2 The X-lines containing a common a point of X

Let $x \in X$ be arbitrary. In this subsection, we show that the 0-lines containing x consistute a maximal singular subspace. Note that Lemma 5.5(i) tells us that, for each x' which is on a 0-line with x, we have $Y_x = Y_{x'}$. Moreover, since $Y = Y(\theta^x)$ (cf. Lemma 5.2), Y_x is a hyperplane of Y. In the following definition, we work more generally with any hyperplane of Y.

Definition 6.4 For any hyperplane H of Y, we define

$$\pi(H) := \{x \in X \mid Y_x = H\} \cup H.$$

We show that $\pi(H)$ is a non-empty maximal singular subspace, which in particular shows that each hyperplane occurs as Y_x for some $x \in X$.

Lemma 6.5 *Let H be a hyperplane of Y. Then $\pi(H)$ is a (maximal) singular subspace, and if $x \in \pi(H) \cap X$ then also $\langle x, Y_x \rangle \subseteq \pi(H)$. Moreover, $\pi(H)$ intersects each $\theta \in \Theta$ in a maximal singular subspace of θ of the form $\langle x, H \rangle$ with $x \in X(\theta)$.*

Proof Suppose x is a point of $\pi(H) \cap X$. Then $Y_x = H$ since both are hyperplanes of Y. Let $x' \in \langle x, Y_x \rangle \cap X$ be arbitrary. Since $\langle x, Y_x \rangle$ is a singular subspace by Lemma 4.10, x' is collinear to Y_x, and hence $Y_x = Y_{x'} = H$. So $x' \in \pi(H)$ too. Next, take any $\theta \in \Theta$. Since $Y(\theta) = Y$ by Lemma 5.2, $X(\theta)$ contains a point x with $Y_x = H$. For such a point x, $\langle x, H \rangle \subseteq X(\theta) \cap \pi(H)$ by the foregoing. By (S2) and the fact that $\langle x, H \rangle$ is a maximal singular subspace of $X(\theta)$, we obtain $\langle x, H \rangle = X(\theta) \cap \pi(H)$.

We claim that $\pi(H)$ is a subspace. Let p, q be any pair of distinct points of $\pi(H)$. We show that $pq \subseteq \pi(H)$. First, suppose $p, q \in X$, so $Y_p = Y_q = H$. If $\theta^p \neq \theta^q$, then pq is a singular line by Lemma 6.3, even a 0-line by Lemma 6.1. According to Lemma 5.5, $Y_r = Y_p = H$ for each point $\in pq$, so $r \in \pi(H)$. So suppose $\theta^p = \theta^q$. Then since Y and $\langle p, H \rangle$ are the only two maximal singular subspace of θ^p containing H, we obtain $pq \subseteq \langle p, H \rangle$. Now $\langle p, H \rangle \subseteq \pi(H)$ by the first paragraph. Second, suppose $p \in X$ and $q \in Y$. Then $Y_p = H$ and $q \in H$, so again $pq \subseteq \langle p, H \rangle \subseteq \pi(H)$. Finally, suppose $p, q \in Y$, i.e., $p, q \in H$. Then $y_1 y_1 \subseteq H \subseteq \pi(H)$ since H is a subspace of Y. The claim follows. Note that $\pi(H)$ is singular since $\pi(H) \subseteq X \cup Y$ by definition. Maximality follows also from the definition: no other point of X is collinear to H, and no point of Y is collinear to any point of $\pi(H) \cap X$. $\square$

Notation. For $x \in X$, we denote the subspace $\pi(Y_x)$ by π^x. Note that by definition, $\pi^x = \pi^{x'}$ for any $x' \in \pi^x \cap X$ since $Y_x = Y_{x'}$ for such x'. We define Π as the set $\{\pi(H) \mid H \text{ a hyperplane of } Y\}$, or equivalently, the set $\{\pi^x \mid x \in X\}$.

Corollary 6.6 *Let $x \in X$ be arbitrary. Then each X-line through x is contained in exactly on of θ^x, π^x.*

Proof Let L be any X-line through x and suppose L is not contained in θ^x. If L is contained in a member of Θ, then this would be a second member of Θ through x, violating our standing hypothesis. So, L is a 0-line and by Lemma 5.5, $Y_{x'} = Y_x$ for all points $x' \in L$. So, by definition of $\pi^x = \pi(Y_x)$, we obtain $L \subseteq \pi^x$. By Lemma 6.5, $\theta^x \cap \pi^x = \langle x, Y_x \rangle$ and hence $\theta^\cap \pi^x$ contains no X-lines. $\square$

Recall that $r = 1$ and hence, for each point $x \in X(\xi)$ for $\xi \in \Xi$, there are exactly two maximal singular subspaces of $XY(\xi)$ containing x (of the form $\langle L_1, V \rangle$ and $\langle L_2, V \rangle$, where V is the vertex of ξ and L_1 and L_2 are X-lines).

Lemma 6.7 *Let $\xi \in \Xi$ be arbitrary. Then for each $x \in X(\xi)$, the two maximal singular subspaces of $XY(\xi)$ containing x are given by $\theta^x \cap \xi$ and $\pi^x \cap \xi$. If $x_1, x_2 \in X(\xi)$ are non-collinear points, then $\theta^{x_1} \cap \pi^{x_2} = \langle x^*, Y(\xi)\rangle$ for some X-point $x^* \in x_1^\perp \cap x_2^\perp \subseteq \xi$.*

Proof Let V denote the vertex $Y(\xi)$ of ξ and suppose L_1 and L_2 are X-lines of ξ containing x such that $\langle L_1, V\rangle$ and $\langle L_2, V\rangle$ are the two maximal singular subspaces of ξ containing x. Note that this means that L_1 and L_2 are not collinear, so $[L_1, L_2] = \xi$ by Lemma 4.5. According to Corollary 6.6, each of L_1, L_2 is contained in θ^x or π^x. If both are contained in θ^x, then $\xi = [L_1, L_2] = \theta^x$, a contradiction; if both are contained in π^x, then $L_1 \perp L_2$ by Lemma 6.5, a contradiction. So renumbering if necessary, $L_1 \subseteq \theta^x$ and $L_2 \subseteq \pi^x$. Since θ^x contains Y, it in particular contains V; and π^x by definition contains Y_x and hence V. By maximality, we obtain $\theta^x \cap \xi = \langle L_1, V\rangle$ and $\pi^x \cap \xi = \langle L_2, V\rangle$. The first statement follows.

Now let x_1, x_2 be non-collinear points of $X(\xi)$ and note that $Y_{x_1} \cap Y_{x_2} = V$ by Lemma 4.7. By the previous paragraph, $\theta^{x_1} \cap \xi$ is of the form $\langle L_1, V\rangle$, where L_1 is an X-line of ξ through x_1; likewise $\pi^{x_2} \cap \xi$ is of the form $\langle L_2, V\rangle$, where L_2 is an X-line of ξ through x_2. Let x^* be the unique point of L_1 collinear to x_2. Note that $L_2^* := xx_2$ is an X-line, for otherwise $x^* \in \langle x_2, V\rangle \cap x_1^\perp = V$. By the previous paragraph and the fact that $\theta^{x_1} = \theta^{x^*}$, we obtain that L_2^* belongs to $\pi^{x^*} = \pi^{x_2}$. Therefore, $L_2^* \subseteq \langle L_2, V\rangle$ and hence $\langle L_1, V\rangle \cap \langle L_2, V\rangle = \langle x^*, V\rangle$ indeed. □

6.3 The projection $\rho_Y(X)$ and its connection to Y

We consider the projection $\rho := \rho_Y$ of (X, Z, Ξ, Θ) from Y onto the subspace of F complementary to Y in $\mathbb{P}^N(\mathbb{K})$, and also the connection map χ between $\rho(X)$ and Y (cf. Definitions 4.13 and 4.17).

Recall that the inverse image under ρ of a point $\rho(x)$ with $x \in X$ is given by $\langle x, Y_x\rangle \cap X$ by Lemma 4.18. Our first aim is to determine the inverse image of a singular line of $\rho(X)$, i.e., a line of F which is contained in $\rho(X)$.

Lemma 6.8 *Suppose $\rho(x_1)$ and $\rho(x_2)$ determine a singular line of $\rho(X)$, for $x_1, x_2 \in X$. Let $x_i' \in \rho^{-1}(\rho(x_i))$ be arbitrary, for $i = 1, 2$. Then:*

- (*i*) *there is a unique $\zeta \in \Theta \cup \Pi$ containing $x_1' \cup x_2'$, and $\rho^{-1}(\rho(x_1)) \cup \rho^{-1}(\rho(x_2)) \subseteq \zeta$, with $\zeta \in \Theta$ if and only if $Y_{x_1} \neq Y_{x_2}$;*
- (*ii*) *there is an $x_2'' \in \rho^{-1}(\rho(x_2))$ such that $x_1'x_2''$ is an X-line;*
- (*iii*) *If $\zeta \in \Theta$, then $\{Y_x \mid \rho(x) \in \langle \rho(x_1), \rho(x_2)\rangle\}$ is the set of all $(r'-1)$-spaces through the $(r'-2)$-space $Y_{x_1} \cap Y_{x_2}$ inside Y.*

Proof As mentioned just before the statement of the proof, $\rho^{-1}(\rho(x_i)) = \langle x_i, Y_{x_i}\rangle \cap X$ for $i = 1, 2$. In particular, $Y_{x_i} = Y_{x_i'}$. We distinguish two cases.

Suppose first that $Y_{x_1} = Y_{x_2} =: H$. Then $\pi(H)$ is the unique member of $\Pi \cup \Theta$ containing x_1' and x_2'. Moreover, $\pi(H)$ contains $\langle x_1', H\rangle$ and $\langle x_2', H\rangle$ by Lemma 6.5. Assertion (*i*) follows in this case. Since $\pi(H)$ is a singular subspace by Lemma 6.5, so is its subspace $\langle x_1', x_2', H\rangle$. As such, the line $x_1'x_2''$ is an X-line for each point $x_2'' \in \langle x_2, H\rangle \cap X = \rho^{-1}(\rho(x_2))$. Assertion (*ii*) follows in this case.

Next, suppose $Y_{x_1} \neq Y_{x_2}$. Recall that $\dim Y = r'$ by Lemma 6.2 and that $r' \geq 2$ since $r' > r = 1$ by Definition 3.2. Consider the subspace $\langle x_1, x_2, Y\rangle$ and note that $\langle x_1, x_2, Y\rangle = \langle x_1, x_2, Y_{x_1}, Y_{x_2}\rangle$. Let x_3 be an X-point with $\rho(x_3)$ on $\langle \rho(x_1), \rho(x_2)\rangle \setminus \{\rho(x_1), \rho(x_2)\}$. Then $x_3 \in \langle x_1, x_2, Y\rangle \cap X \setminus (\langle x_1, Y_{x_1}\rangle \cup \langle x_2, Y_{x_2}\rangle)$. Since $\rho(x_1) \neq \rho(x_2)$, we have $\dim(\langle x_1, x_2, Y\rangle) = \dim Y + 2 = r' + 2$. Inside $\langle x_1, x_2, Y\rangle$, we hence see that the $(r'+1)$-space $\langle x_3, x_1, Y_{x_1}\rangle$ intersects the r'-space $\langle x_2, Y_{x_2}\rangle$ in an $(r'-1)$-space M_2 with $M_2 \cap Y$ equal to the $(r'-2)$-space $Y_{x_1} \cap Y_{x_2}$. The r'-space $\langle M_2, x_3\rangle$ intersects $\langle x_1, Y_{x_1}\rangle$ in an $(r'-1)$-space M_1 with $M_1 \cap Y = Y_{x_1} \cap Y_{x_2}$. Since $r' - 1 \geq 1$, it follows that there is a point $x_1^* \in M_1 \cap X$. Let x_2^* denote the unique point $x_1^* x_3 \cap M_2$. Then $x_1^* x_2^*$ is an X-line: it is singular by Lemma 4.4 since it contains at least 3 points of $X \cup Y$, and if it would contain a point of Y, then it is contained in $\langle x_1, Y_{x_1} \rangle$, contradicting $\rho(x_1) \neq \rho(x_3)$. By Corollary 6.6, the X-line $x_1^* x_2^*$ belongs to a member ζ of $\Theta \cup \Pi$. Since $Y_{x_1} \neq Y_{x_2}$, $\zeta \in \Theta$. Uniqueness follows from $\zeta = \theta^{x_1^*}$. Since ζ contains Y, we have that ζ contains $\langle x_1^*, Y_{x_1} \rangle \cup \langle x_2^*, Y_{x_2} \rangle$. Assertion (i) follows. Viewed inside the quadric $XY(\zeta)$, the point x_1' is collinear to a hyperplane M_2' of $\langle x_2, Y_{x_2} \rangle$, distinct from Y_{x_2}. For any point $x_2'' \in M_2' \cap X$, the line $x_1' x_2''$ is hence an X-line. Assertion (ii) follows also in this case.

(iii) Suppose $\zeta \in \Theta$. Let L be an X-line in ζ with $\rho(L) = \langle \rho(x_1), \rho(x_2) \rangle$ (possible by (ii)). Inside the quadric $XY(\zeta)$, we see that $Y_L = Y_{x_1} \cap Y_{x_2}$ and that the collinearity relation $x \mapsto Y_x$ gives a bijection between the points of L and the $(r'-1)$-spaces of Y containing Y_L. Finally, ρ induces a bijection between the points of L and the points of $\rho(L)$. Composing thse bijections, assertion (iii) follows. □

Next, we treat the case where x_1 and x_2 are non-collinear points of some $\xi \in \Xi$.

Lemma 6.9 *Suppose $x_1, x_2 \in X$ are non-collinear points with $\xi = [x_1, x_2] \in \Xi$ and put $V = Y(\xi)$. Let $x_i' \in \rho^{-1}(\rho(x_i))$ for $i = 1, 2$. Then x_1' and x_2' are non-collinear and $\xi' := [x_1', x_2']$ has vertex V and $\rho(X(\xi')) = \rho(X(\xi))$.*

Proof Since $[x_1, x_2] \in \Xi$, we have $\theta^{x_1} \neq \theta^{x_2}$. It follows from Lemma 6.3 that $Y_{x_1} \neq Y_{x_2}$ and that V is the $(r'-2)$-space $Y_{x_1} \cap Y_{x_2}$. Let x_1', x_2' be as described. By Lemma 4.18, $x_i' \in \langle x_i, Y_{x_i} \rangle \cap X$ and $Y_{x_i'} = Y_{x_i}$ for $i = 1, 2$. Suppose for a contradiction that $x_1' x_2'$ is singular. Then $x_1' x_2'$ is an X-line, for otherwise $x_2' \in \langle x_1', Y_{x_1} \rangle$ and hence $\rho(x_2') = \rho(x_1')$, a contradiction. By Lemma 6.6, there is a $\theta \in \Theta$ containing the X-line $x_1' x_2'$. Now, if θ contains x_1', x_2' then θ also contains $\langle x_1, Y_{x_1} \rangle$ and $\langle x_2, Y_{x_2} \rangle$ (even if x_1' and x_2' would not be collinear). However, this means by Lemma 4.1 that $\theta = [x_1, x_2] = \xi$, a contradiction. So x_1' and x_2' are not collinear, and it also follows that $[x_1', x_2'] \in \Xi$. Put $\xi' = [x_1', x_2']$. Note that the vertex $Y(\xi')$ of ξ' is given by $Y_{x_1'} \cap Y_{x_2'} = Y_{x_1} \cap Y_{x_2} = V$ too.

We show that $\rho(X(\xi)) = \rho(X(\xi'))$. First observe that $\theta^{x_1'} = \theta^{x_1}$ and $\pi^{x_2} = \pi^{x_2'}$, because $x_1' \in \langle x_1, Y_{x_1} \rangle \subseteq \theta^{x_1}$ and $Y_{x_2'} = Y_{x_2}$. It then follows from Lemma 6.7 that there are points $x_{12} \in x_1^\perp \cap x_2^\perp \subseteq \xi$ and $x_{12}' \in x_1'^\perp \cap x_2'^\perp \subseteq \xi'$ such that

$$\langle x_{12}, V \rangle = \theta^{x_1} \cap \pi^{x_2} = \theta^{x_1'} \cap \pi^{x_2'} = \langle x_{12}', V \rangle.$$

We claim that $\rho(x_{12}) = \rho(x_{12}')$: indeed, $Y_{x_2} = Y_{x_{12}}$ because $x_2, x_{12} \in \pi^{x_2}$, likewise $Y_{x_2'} = Y_{x_{12}'}$, and since $Y_{x_2} = Y_{x_2'}$ because $\rho(x_2) = \rho(x_2')$, we hence obtain that $\langle x_{12}, Y_{x_{12}} \rangle = \langle x_{12}', Y_{x_{12}'} \rangle$.

Next, we claim that the images under ρ of the lines $x_1 x_{12}$ and $x_1' x_{12}'$ coincide, likewise for $x_2 x_{12}$ and $x_2' x_{12}'$. We look in θ^{x_1}, which contains the X-lines $x_1 x_{12}$ and $x_1' x_{12}'$. By Lemma 6.8(iii), each point $u_1 \in x_1 x_{12}$ corresponds to a unique point $u_1' \in x_1' x_{12}'$ such that $Y_{u_1} = Y_{u_1'}$. Since $\pi^{u_1} = \pi^{u_1'}$ shares exactly $\langle u_1, Y_{u_1} \rangle$ with $\theta^{u_1} = \theta^{x_1}$ by Lemma 6.5, we obtain that $\langle u_1, Y_{u_1} \rangle = \langle u_1', Y_{u_1'} \rangle$. As such, $\rho(u_1) = \rho(u_1')$. Likewise, both X-lines $x_{12} x_2$ and $x_{12}' x_2'$ are contained in π^{x_2}. Take any point $u_2 \in x_{12} x_2$ and consider the r'-space $\langle u_2, Y_{u_2} \rangle = \langle u_2, Y_{x_2} \rangle$. The latter intersects the line $x_{12}' x_2'$ in a unique point u_2' and hence $\langle u_2', Y_{u_2'} \rangle = \langle u_2', Y_{x_2} \rangle = \langle u_2, Y_{u_2} \rangle$. We obtain $\rho(u_2) = \rho(u_2')$. The claim follows.

Finally, let u be an arbitrary point on $X(\xi) \setminus (\langle x_{12} x_2, V \rangle \cup \langle x_{12} x_1, V \rangle)$. Then u is collinear to unique points u_1 on $x_{12} x_1$ and u_2 on $x_{12} x_2$. Let u_1' be the unique point on $x_{12}' x_1'$ corresponding to u_1 via ρ, likewise for u_2'. Similarly as above, one can deduce that $\theta^{u_2} = \theta^{u_2'}$ and $\pi^{u_1} = \pi^{u_2'}$, and that there is a point $u' \in \xi'$ such that

$$\langle u, V \rangle = \theta^{u_2} \cap \pi^{u_1} = \theta^{u_2'} \cap \pi^{u_1'} = \langle u', V \rangle,$$

and finally that $\rho(u) = \rho(u')$. We conclude that $\rho(X(\xi)) \subseteq \rho(X(\xi'))$, and switching the roles of ξ and ξ' we obtain $\rho(X(\xi)) = \rho(X(\xi'))$. $\square$

Notation. We denote by L the set of X-lines.

Recall the notion of a Segre variety from Sections 3.2.1.

Proposition 6.10 *The point-line geometry $\mathcal{S} := (\rho(X), \rho(\mathsf{L}))$ is an injective projection of the Segre variety $\mathcal{S}_{r',k}(\mathbb{K})$ where $k = \dim(\pi) - r'$ for any $\pi \in \Pi$. Moreover,*

- *(i) for each maximal singular subspace S in $\mathcal{S}$, there is a unique $\zeta_S \in \Theta \cup \Pi$ with $\rho(X(\theta_S)) = S$.*
- *(ii) the sets $\rho(\Theta) := \{\rho(X(\theta)) \mid \theta \in \Theta\}$ and $\rho(\Pi) := \{\rho(X(\pi)) \mid \pi \in \Pi\}$ are the two natural families of maximal singular subspaces of $\mathcal{S}$.*
- *(iii) for each hyperbolic quadrangle G of $\mathcal{S}$, there is a unique v-space V in Y such that there is a $\xi \in \Xi$ with vertex V and $\rho(X(\xi)) = G$.*

Proof We determine the maximal singular subspaces of the point-line geometry $\mathcal{S} = (\rho(X), \rho(\mathsf{L}))$. Let $\zeta \in \Theta \cup \Pi$ be arbitrary. By Lemma 4.18 and the fact that members of Π are singular subspaces (cf. Lemma 6.5), $\rho(X(\zeta))$ is a singular subspace of $\mathcal{S}$, which we denote by S_ζ. We aim to prove that S_ζ is a maximal singular subspace and that each maximal singular subspace of $\mathcal{S}$ arises like this; moreover, we show that $\mathcal{S}$ is the direct product of S_θ and S_π for any pair $\theta \in \Theta$ and $\pi \in \Pi$. We proceed in a few steps.

Claim 1: each point $p \in \rho(X) \setminus S_\zeta$ is collinear to at most one point of S_ζ.
Suppose for a contradiction that there is a $p \in \rho(X) \setminus S_\zeta$ collinear to distinct points s_1 and s_2 of S_ζ. Take $x \in X$ with $\rho(x) = p$ and put $i = 1, 2$. Since p and s_i determine a singular line of $\rho(X)$, Lemma 6.8(ii) yields a point $x_i \in \rho^{-1}(s_i)$ such that xx_i is an X-line. As also s_1 and s_2 determine a singular line of $\rho(X)$, (i) of the same lemma implies that $x_1, x_2 \in \zeta$. Note that $x \notin \zeta$ since $p \notin S_\zeta$. Suppose first that $\zeta \in \Theta$. According to Corollary 6.6, the X-line xx_i, for $i = 1, 2$, is contained in $\pi^{x_i} = \pi^x$. This means that $\pi^{x_1} = \pi^x$ and $\theta^{x_1} = \zeta$ share the line x_1x_2. Lemma 6.5 then implies that x_1x_2 contains a point of Y, and hence $s_1 = \rho(x_1) = \rho(x_2) = s_2$, a contradiction. If $\zeta \in \Pi$, the argument is analogous. This shows the claim.

Note that the previous claim implies that S_ζ is a maximal singular subspace of $\mathcal{S}$ indeed. The following claim in particular shows that each maximal singular subspace is of this form.

Claim 2: each point p of $\rho(X)$ is contained in two maximal singular subspaces, namely S_{θ^x} and S_{π^x}, where x is any X-point with $\rho(x) = p$, and $S_{\theta^x} \cap S_{\pi^x} = \{p\}$.
Let S be any maximal singular subspace of $\mathcal{S}$ containing a line L with $p \in L \in \rho(\mathsf{L})$. By Lemma 6.8$(i)$, L is contained in S_ζ for a unique $\zeta \in \Theta \cup \Pi$, and ζ contains $\rho^{-1}(p)$, which means that $\zeta = \theta^x$ or $\zeta = \pi^x\}$, where x is any point in $\rho^{-1}(p)$. Claim 1 implies that each point of $S \setminus L$ (if any) is contained in S_ζ. It follows that p is not a maximal singular subspace itself, that S_{π^x} and S_{θ^x} are the unique maximal singular subspaces of $\mathcal{S}$ containing p and that $S_{\pi^x} \cap S_{\theta^x}$ does not contain a line of L. The claim follows, and so does assertion (i).

Recall that $\rho(\Theta) := \{S_\theta \mid \theta \in \Theta\}$ and $\rho(\Pi) := \{S_\pi \mid \pi \in \Pi\}$. We will write S^p_Θ for S_{θ^x} and S^p_Π for S_{π^x}, to denote the two maximal singular subspaces of $\mathcal{S}$ containing p, to avoid the use of the point $x \in \rho^{-1}(p)$.

Claim 3: Two maximal singular subspaces S_1 and S_2 of $\mathcal{S}$ are disjoint if S_1, S_2 both belong to $\rho(\Theta)$ or to $\rho(\Pi)$, and intersect in a unique point otherwise. It follows from Claim 2 that no point is contained nor in two members of $\rho(\Theta)$, neither in two members of $\rho(\Pi)$. It remains to show that $S_1 = S_\theta$, for any $\theta \in \Theta$, and $S_2 = S_\pi$, for any $\pi \in \Pi$, intersect non-trivially. This follows from Lemma 6.5, as it says that $\theta \cap \pi$ meet in a subspace $\langle x, Y_x \rangle$ for some $x \in X$ by. By Claim 2, $S_1 \cap S_2$ coincides with $\rho(x)$. The claim follows.

We can now determine the structure of $\mathcal{S}$.

Claim 4: $\rho(X)$ is the direct product of S_θ and S_π for any $\theta \in \Theta$ and $\pi \in \Pi$.
Let $q \in \rho(X)$ be arbitrary. Consider the two maximal singular subspaces S^q_Θ and S^q_Π containing q. By Claim 3, $S^q_\Pi \cap S_\theta$ is a unique point, say q_θ; likewise, $S^q_\Theta \cap S_\pi$ is a unique point, say q_π. We show that the map

$$\rho(X) \mapsto S_\theta \times S_\pi : q \mapsto (q_\theta, q_\pi)$$

is bijective by determining its inverse. So suppose we are given a pair of points (t, p) with $t \in S_\theta$ and $p \in S_\pi$. Then one can verify that the point $q := S^t_\Pi \cap S^p_\Theta$ is such that $t = q_\theta$ and $p = q_\pi$ (observe that $S^t_\Pi = S^q_\Pi$ and $S^p_\Theta = S^q_\Theta$). The claim follows.

Claim 5: $\rho(\mathsf{L})$ is the line set of the direct product geometry $S_\theta \times S_\pi$.
A line of the latter geometry either has the form $\{t\} \times L$ with t a point in S_θ and L a line in S_π, or the form $L \times \{p\}$ with L a line in S_θ and p a point in S_π. Let L be a line of $\rho(\mathsf{L})$. Considering any point $p \in L$, Claim 2 implies that L is contained in S^p_Θ or S^p_Π. Assume $L \subseteq S^p_\Theta$ (the case $L \subseteq S^p_\Pi$ is analogous). With the notation introduced in the previous paragraph, consider $L_\pi := \{q_\pi \mid q \in L\}$ and $L_\theta := \{q_\theta \mid q \in L\}$. Clearly, L_π is a unique point of S_π since $q_\pi = S^q_\Theta \cap S_\pi$ does not depend on the point $q \in L$ since $S^q_\Theta = S^p_\Theta$. We show that $L_\theta := \{q_\theta \mid q \in L\}$ is a line in S_θ. If $S^p_\Theta = S_\theta$, then $L_\theta = L$, so suppose $S^p_\Theta \neq S_\theta$. Let q be a point of $L \setminus \{p\}$. Note that p and q^θ are not collinear, since $q^\theta \notin S^p_\Theta \cup S^p_\Pi$. Take $x_1 \in \rho{-}1(p)$ and $x_2 \in \rho^{-1}(q_\theta)$. Then x_1 and x_2 are not collinear and do not belong to a member of Θ, for otherwise p and q^θ would be collinear. So $[x_1, x_2] \in \Xi$. Then $\rho([x_1, x_2])$ is a hyperbolic quadrangle in $\mathcal{S}$. By Claim 3, the unique hyperbolic quadrangle in $\mathcal{S}$ containing p and q^θ also contains L and L^θ and it also follows that the latter is a line. We have shown that L is a line of $S_\theta \times S_\pi$. A similar argument now shows that a line of $S_\theta \times S_\pi$ is a line in S_ζ for some $\zeta \in \Theta \cup \Pi$, and by Lemma 6.8$(ii)$, this line belongs to $\rho(\mathsf{L})$. The claim follows.

We have shown that $\mathcal{S} = (\rho(X), \rho(\mathsf{L}))$ is isomorphic to the direct product geometry $S_\theta \times S_\pi$ for $\theta \in \Theta$ and $\pi \in \Pi$. In particular, $k := \dim(S_\pi)$ does not depend on $\pi \in \Pi$; and hence $\mathcal{S} = (\rho(X), \rho(\mathsf{L}))$ is, as a point-line geometry, isomorphic to $\mathsf{A}_{r',1}(\mathbb{K}) \times \mathsf{A}_{k,1}(\mathbb{K})$. By the main result of [18], the Segre variety $\mathcal{S}_{r',k}(\mathbb{K})$ is the absolutely universal embedding of $\mathsf{A}_{r',1}(\mathbb{K}) \times \mathsf{A}_{k,1}(\mathbb{K})$. This means that $\mathcal{S} = (\rho(X), \rho(\mathsf{L}))$ is an injective projection of $\mathcal{S}_{r',k}(\mathbb{K})$.

For the final assertion, let Q be any hyperbolic quadrangle in $\mathcal{S}$. As in Claim 5, it can be deduced that Q is the direct product of a line in S_θ and a line in S_π. Also as in Claim 5, it follows that Q arises as the image under ρ of a member of Ξ. Assertion (iii) now follows from Lemma 6.9, since any pair of members of Ξ with the same image under ρ will have the same vertex indeed. This concludes the proof of the proposition. □

Remark 6.11 Note that $k \geq 1$: considering any $\theta \in \Theta$ and any point $x \in X \setminus \Theta$ (which exists, for otherwise $|\Xi| = 0$), Lemma 6.5 implies that π^x contains a point $x' \in X(\theta)$, and hence xx' is an X-line.

Corollary 6.12 $N \leq (r' + 1)(k + 2) - 1$.

Proof By Lemma 5.2, we know $\dim(Y) = r'$ and by Proposition 6.10, $\dim(F) \leq (r'+1)(k+1)-1$. Since F and Y generate $\mathbb{P}^N(\mathbb{K})$, we obtain $N \leq (r' + 1)(k + 2) - 1$. □

6.4 Mutants of the half dual Segre variety

In Proposition 6.10, we showed that $\mathcal{S} := (\rho(X), \rho(\mathsf{L}))$ is an injective projection of the Segre variety $\mathcal{S}_{r',k}(\mathbb{K})$. Next, we show that there is a subspace F^* of $\mathbb{P}^N(\mathbb{K})$ such that, if $F^* \cap X$ contains a subset X^* and $\Xi^* := \{\langle \xi \cap X^* \rangle \mid \xi \in \Xi\}$, the subgeometry (X^*, Ξ^*) is an injective projection of $\mathcal{S}_{r',k}(\mathbb{K})$ and is a pre-DSV itself (where Θ is empty). Possibly, F^* is not disjoint from Y. We will arrive naturally at a tweaked version of the half dual Segre variety, called a *mutant*.

To start, we consider the projections of the Segre variety mentioned above in a more general setting, by noting that the Segre variety is a pre-DSV with Θ empty.

Definition 6.13 Suppose that (X, Z, Ξ, Θ) is a pre-DSV in $\mathbb{P}^N(\mathbb{K})$ with parameters (r, v, r', v'). A subspace S of $\mathbb{P}^N(\mathbb{K})$ is called *legal with respect to* (X, Z, Ξ, Θ) if the projection of (X, Z, Ξ, Θ) from S is injective on $X \cup Z$ and yields a pre-DSV in $\mathbb{P}^{N'}(\mathbb{K})$, with $N' = N - \dim S - 1$ with parameters (r, v, r', v').

The following characterisation of legal projections comes in handy.

Lemma 6.14 *Suppose that (X, Z, Ξ, Θ) is a pre-DSV in $\mathbb{P}^N(\mathbb{K})$ with parameters (r, v, r', v'). Then a subspace S of $\mathbb{P}^N(\mathbb{K})$ is legal w.r.t. (X, Z, Ξ, Θ) if and only if S is disjoint from $\langle \zeta_1, \zeta_2 \rangle$ for any pair of $\zeta_1, \zeta_2 \in \Xi \cup \Theta$.*

Proof Suppose that S is disjoint from $\langle \zeta_1, \zeta_2 \rangle$ for any pair of $\zeta_1, \zeta_2 \in \Xi \cup \Theta$. Then in particular, S is disjoint from any member of $\Xi \cup \Theta$. We claim that S is injective on $X \cup Z$. Suppose for a contradiction that two points $p_1, p_2 \in X \cup Z$ project onto the same point. Then $\langle S, p_1 \rangle$ contains p_2 and hence the line p_1p_2 contains a point $s \in S$. By (S1), p_1, p_2 are contained in some $\zeta \in \Xi \cup \Theta$. However, by the foregoing, $\zeta \cap S =$. The claim follows. From this we already deduce that the projection of (X, Z, Ξ, Θ) is an $(r, v', r', v', N - \dim S - 1)$-system.

It follows from the previous paragraph that Axiom (S1) is preserved when projecting (X, Z, Ξ, Θ) from S. Consider two members $\zeta_1, \zeta_2 \in \Xi \cup \Theta$. Since S is disjoint from $\langle \zeta_1, \zeta_2 \rangle$, the intersection of the projections of ζ_1 and ζ_2 coincides with the projection of $\zeta_1 \cap \zeta_2$. Hence Axiom (S2) is preserved too.

Conversely, if there would be $\zeta_1, \zeta_2 \in \Xi \cup \Theta$ with $S \cap \langle \zeta_1, \zeta_2 \rangle \neq \emptyset$, then the projections of ζ_1 and ζ_2 would intersect in strictly more than the projection of $\zeta_1 \cap \zeta_2$, and Axiom (S2) would no longer hold. $\square$

In the smallest cases however, no non-trivial legal projections of the Segre variety occur:

Lemma 6.15 *The Segre varieties $\mathcal{S}_{\ell,k}(\mathbb{K})$ with $\ell \leq 3$ and $k \geq 1$ do not admit proper legal projections.*

Proof Since $\mathcal{S}_{\ell,k}(\mathbb{K})$ with $\ell < 3$ is contained in $\mathcal{S}_{3,k}(\mathbb{K})$ it suffices to show that the latter does not admit proper legal projections. The Segre variety $\mathcal{S}_{3,k}(\mathbb{K})$ is defined by the $4 \times (k+1)$ matrices over $\mathbb{K}$ of rank 1 in the projective space defined by the vector space of all $4 \times (k+1)$ matrices over $\mathbb{K}$. If A is such a matrix of rank 4, then A is the sum of four rank 1 matrices A_1, A_2, A_3 and A_4 which are pairwise not collinear. Let ξ_1 and ξ_2 be the respective members of Ξ determined by the pairs (A_1, A_2) and (A_3, A_4). Then $A \in \langle \xi_1, \xi_2 \rangle$ is not a legal point w.r.t. $\mathcal{S}_{3,k}(\mathbb{K})$. If A has rank 2 or 3, M is already the sum of two or three rank 1 matrices, respectively, and the same conclusion can be reached analogously. $\square$

We show that the subspace F^* and subset X^* of X alluded to above exist.

Proposition 6.16 *The set X contains a subset X^* such that, with induced line set L^*, the geometry $\Omega := (X^*, \mathsf{L}^*)$ is a legal projection of $\mathcal{S}_{r',k}(\mathbb{K})$, where containment gives a bijection between the natural families of r'-spaces and k-spaces of Ω and the sets Θ and Π, respectively. For each hyperbolic quadrangle Q in Ω, there is a unique $\xi \in \Xi$ with $X^* \cap \xi = Q$. Moreover, if $F^* = \langle X^* \rangle$, then*

(i) $\bigsqcup_{x \in X^*} \langle x, Y_x \rangle \setminus Y_x = X$ *and hence* $\langle F^*, Y \rangle = \mathbb{P}^N(\mathbb{K})$;

(*ii*) *Re-choose F such $F \subseteq F^*$. Then $F^* \cap Y$ and F are complementary in F^*, and the projection ρ^* of $F^* \cap X$ from $F^* \cap Y$ on F coincides with the projection of ρ restricted to $F^* \cap X$. In particular, if $F^* \cap Y = \emptyset$, then $X^* = X \cap F^*$ and ρ is the identity on X^*.*

(*iii*) *If $r' = 2$ and $k \in \{1, 2\}$, then $F^* \cap Y = \emptyset$.*

Proof By Proposition 6.10, $\mathcal{S} = (\rho(X), \rho(\mathsf{L}))$ is an injective projection of the Segre geometry $\mathcal{S}_{r',k}(\mathbb{K})$, and the latter's r'-dimensional subspaces are in $1-1$-correspondence with the members of Θ. We show that we can select an r'-dimensional space in $X(\theta)$ for each $\theta \in \Theta$, such that the union of these r'-spaces, equipped with induced line set, is isomorphic to $\mathcal{S} = (\rho(X), \rho(\mathsf{L}))$. We start however with the k-spaces, which are in $1-1$-correspondence with the members of Π.

Take $r'+1$ hyperplanes $Y_0, ..., Y_{r'}$ in Y such that they form a basis of the dual of Y (i.e., the dimension of the intersection of i members of them, with $0 \leq i \leq r'$, has dimension $r'-i$). For $t \in \{0, ..., r'\}$, let π_t be short for the subspace $\pi(Y_t) \in \Pi$ (cf. Lemma 6.5). Take any k-dimensional X-space S^0_Π in π_0 complementary to Y_0 and let $(x_{0,0}, ..., x_{0,k})$ be a basis of S^0_Π. For each $u \in \{0, ..., k\}$, let θ_u be short for $\theta^{x_{0,u}} \in \Theta$; and for each $t \in \{0, ..., r'\}$, put $\Pi_{t,u} := \pi_t \cap \theta_u$. Recall that Lemma 6.5 says that $\Pi_{t,u} := \langle x'_{t,u}, Y_t\rangle$ where $x'_{t,u}$ is any X-point in θ_u collinear to Y_t. In particular, $\dim \Pi_{t,u} = r'$.

Let $u \in \{0, ..., k\}$. We now choose points $x_{1,u}, ..., x_{r',u}$ in θ^u such that, together with $x_{0,u}$, they form an r'-dimensional subspace in $X(\theta^u)$. We choose these points consecutively. Suppose the points $x_{0,u}, ..., x_{i,u}$ have been chosen already, with $0 \leq i \leq r'-1$. We choose the point $x_{i+1,u}$. Consider θ_u, which by definition contains $\langle x'_{t,u}, Y_t\rangle$ for all $0 \leq t \leq r'$. Therefore, the subspace of the maximal singular subspace $\langle x'_{i+1,u}, Y_{i+1}\rangle$ of θ_u that is collinear to all $i+1$ points $x_{i',u}$ with $0 \leq i' \leq i$ has dimension at least $r'-i-1$, and it intersects Y_{i+1} in the $(r'-i-2)$-space $\bigcap_{0 \leq i' \leq i+1} Y_{i'}$. Noting that $r'-i-1 \geq 0$, there is a point $x_{i+1,u} \in \langle x'_{i+1,u}, Y_{i+1}\rangle$ collinear to $x_{i',u}$ for $0 \leq i' \leq i$. Define $S^u_\Theta := \langle x_{0,u}, ..., x_{r',u}\rangle$. Then S^u_Θ is an r'-space in $X(\theta^u)$ indeed, since $(S^u_\Theta)^\perp \cap Y = \bigcap_{i=0}^{r'} Y_i = \emptyset$ implies that the subspaces $S^u\Theta$ and Y are opposite subspaces in the quadric $XY(\theta^u)$, which is only possible if they are disjoint and have the same dimension.

Next, we show that, for $0 \leq t \leq r'$, the singular subspace $S^t_\Pi := \langle x_{t,0}, ..., x_{t,k}\rangle$ of π_t has dimension k. For $t = 0$ this is by construction. Let $t > 0$. We claim the subspace S^t_Π is isomorphic to S^0_Π, with isomorphism given by collinearity. We extend the correspondence $x_{0,u} \mapsto x_{t,u}$ between the $k+1$ points of S^0_Π and S^t_Π. As a first step, let x_0 be a point on a line between two such points of S^0_Π, to fix ideas, let $x \in x_{0,0}x_{0,1}$. Note that the line $x_{0,0}x_{0,1}$ is an X-line by construction; the line $x_{t,0}x_{t,1}$ is also an X-line, because $x_{t,0} \in \theta^0$ and $x_{t,1} \in \theta^1$ and if $x_{t,0}x_{t,1}$ would contain a point of Y, then $x_{t,0}x_{t,1} \subseteq \theta^0 \cap \theta^1 \subseteq Y$, a contradiction. The points $x_{0,0}$ and $x_{t,1}$ are not collinear, nor contained in a member of Θ, as otherwise $x_{t,1}$ would belong to π_0 or to θ_0 (and since $x_{t,1} \in \pi_t \cap \theta_1$, this is not the case). Hence $[x_{0,0}, x_{t,1}] \in \Xi$ by Lemma 4.1. Since $[x_{0,0}, x_{t,1}]$ is the convex closure of $x_{0,0}, x_{t,1}$ (cf. Lemma 4.7), it follows that $x_{0,1}$ and $x_{0,0}$ also belong to $[x_{0,0}, x_{t,1}]$. Therefore, x_0 is collinear to a unique point x_t on the line $x_{t,0}x_{t,1} \subseteq \pi_t$. Since θ^{x_0} meets π_t in $\langle x_1, Y_t\rangle$, x_t is the unique point of S^t_Π collinear to x_0. Continuing like this, it follows that each point $x_0 \in S^0_\Pi$ is collinear to a unique point of S^t_Π, moreover, each line of S^0_Π is mapped to an X-line of S^t_Π. We conclude that collinearity is an isomorphism between S^0_Π and S^t_Π and in particular, $S^t_\Pi \subseteq X$ and $\dim S^t_\Pi = k$. Observe that the bijective correspondence can equivalently be given by "being contained in the same member of Θ". Indeed, given the collinear points x_0 and x_t of the previous paragraph, we have $Y_{x_0} = Y_0$ and $Y_{x_t} = Y_t$ and hence x_0x_t is an X-line, which is not contained in $\pi_{x_0} = \pi_0$ and hence it is contained in θ^{x_0}. As above, θ^{x_0} meets S^t_Π exactly in the point x_t.

By the foregoing, "collinearity" and "being contained in the same member of Θ" are bijections between each pair of k-spaces S^t_Π and $S^{t'}_\Pi$. It then follows that for each point $x_0 \in S^0_\Pi$, the subspace $\langle x_0, \theta^{x_0} \cap S^1_\Pi, ..., \theta^{x_0} \cap S^{r'}_\Pi\rangle$ is a singular subspace. By the same argument as used above, it is an r'-dimensional X-space. So, if $\theta \in \Theta$ is arbitrary, then it meets S^t_Π in a unique point $x_t(\theta)$ and these points generate an r'-space in $X(\theta)$, which we denote by S_θ.

Finally, we can define X^* as the union of all r'-spaces S_θ where θ ranges over Θ. Taking any point $x^* \in X^*$, we by definition of X^* have that x^* belongs to a unique r'-space, namely $S^{\theta^{x^*}}$. Now consider π^{x^*} and let S^*_Π be the subspace generated by all points $\pi^{x^*} \cap S_\theta$ with $\theta \in \Theta$. Since it belongs to π^{x^*} it is a singular subspace, moreover it has dimension k and belongs to X because it is isomorphic to S^0_Π by the same argument as in the previous paragraph (using the correspondence "being contained in the same S_θ"). We conclude that x^* is contained in a unique singular r'-space of X^* (contained in $X(\theta^{x^*})$) and in a unique k-space of X^* (contained in $X(\pi^{x^*})$). Note that this implies that any two collinear points in X^* are on an X-line which is contained in X^*; the resulting line set we denote by L^*. Finally, since any $\pi \in \Pi$ shares a point with S^0_Θ, we see that π contains a k-space in X^* too. It is now straightforward to verify that (X^*, L^*), is isomorphic to $S^0_\Theta \times S^0_\Pi$. Then (X^*, L^*) is an injective projection of $\mathcal{S}_{r',k}(\mathbb{K})$. Consider a hyperbolic quadrangle Q in X^*. Then we already deduced above that a pair of non-collinear points of Q gives rise to a unique $\xi \in \Xi$, and $Q \subseteq \xi$. Moreover, since each point x of $X(\xi)$ corresponds to a unique point x_Q of Q (in the sense that $\langle x, Y(\xi)\rangle = \langle x_Q, Y(\xi)\rangle$), we have $X^* \cap \xi = Q$ and hence also $\langle Q\rangle \cap X^* = Q$. The resulting pair (X^*, Ξ^*) with $\Xi^* := \{\langle Q\rangle \mid Q \text{ hyperbolic quadrangle in } \Omega\}$ is a pre-DSV: it is an $(1,-1;N)$-system satisfying (S1) by the above, and (S2) holds because each $\langle Q\rangle \in \Xi^*$ corresponds to a unique $\xi \in \Xi$. So (X^*, L^*) is a legal projection of $\mathcal{S}_{r',k}(\mathbb{K})$. This shows the main assertion.

(i) Let $x \in X$ be arbitrary. Consider the quadric $XY(\theta^x)$, in which the X^*-space S_{θ^x} and Y are opposite r'-spaces. Hence there is a unique point $x^* \in S_{\theta^x}$ with $Y_x = Y_{x^*}$. So $x^* \in \pi_x \cap \theta_x = \langle x, Y_x\rangle$, and therefore $x \in \langle x^*, Y_{x^*}\rangle$. Note that x^* is the unique point of X^* in $\langle x^*, Y_{x^*}\rangle$ because, as mentioned above, collinear points in X^* determine an X-line. Since $x \in X$ was arbitrary and $F^* = \langle X^*\rangle$, the assertion follows.

(ii) Suppose $F \subseteq F^*$. Then F^* is generated by F and $F^* \cap Y$: if not, then F^* contains a subspace F' strictly containing F and disjoint from $F^* \cap Y$; and because F and Y are complementary, F' meets Y in a point outside $F^* \cap Y$, a contradiction. So F and $F^* \cap Y$ are, being disjoint, complementary subspaces of F^*. Take any $x \in F^* \cap X$. By definition, $\rho^*(x) = \langle x, F^* \cap Y\rangle \cap F$ and $\rho(x) = \langle x, Y\rangle \cap F$. Since $x \in \langle \rho^*(x), F^* \cap Y\rangle \subseteq \langle \rho^*(x), Y\rangle$, we get $\rho^*(x) \in \langle x, Y\rangle \cap F = \rho(x)$, so $\rho(x) = \rho^*(x)$. If $F^* \cap Y$ is empty, then ρ^* is the identity on $F^* \cap X$. Since $\rho(X)$ and X^* both are projections of $\mathcal{S}_{r',k}(\mathbb{K})$ in $F = F^*$, and $X^* = \rho(X^*) \subseteq \rho(X)$, we obtain that $X^* = \rho(X)$. In particular, if $x \in F^* \cap X$, then $x = \rho(x) \in X^*$. So $F^* \cap X = X^*$ in case $F^* \cap Y = \emptyset$.

(iii) Finally, suppose that $r' = 2$. First let $k = 1$. Then $(\rho(X), \rho(\mathsf{L})$ and $(X^*, \rho(\mathsf{L}^*)$ are injective projections of $\mathcal{S}_{2,1}(\mathbb{K})$. Since the latter contains disjoint planes, both $\langle \rho(X)\rangle$ and $F^* = \langle X^*\rangle$ have dimension 5 and hence $F = F^*$ so $F^* \cap Y = \emptyset$. So let $k = 2$. In this case, $(\rho(X), \rho(\mathsf{L})$ is an injective projection of $\mathcal{S}_{2,2}(\mathbb{K})$ and $(X^*, \rho(\mathsf{L}^*)$ is a legal projection of $\mathcal{S}_{2,2}(\mathbb{K})$. By Lemma 6.15, this means that $F^* = \langle X^*\rangle$ has dimension 8. Suppose for a contradiction that $F^* \cap Y$ contains a point y. Since $v = r' - 2 = 0$, it follows from Lemma 6.10(iii) that y occurs as the vertex of some member $\xi \in \Xi$. By the foregoing, $X^* \cap \xi$ is a hyperbolic quadrangle Q. The 4-space $\langle Q, y\rangle \subseteq \xi$ is hence contained in $F^* = \langle X^*\rangle$, but intersects X^* in precisely Q. Consider two planes $S^\pi = \pi \cap X^*$ and $S_\theta = \theta \cap X^*$, with $\pi \in \Pi$ and $\theta \in \Theta$, such that they are disjoint from Q (then Q arises as the direct product of a line in $S^\pi \setminus S_\theta$ and a line in $S^\theta \setminus S_\pi$). The 4-space $\langle S_\theta, S_\pi\rangle$ then has a point q in common with the 4-space $\langle Q, y\rangle$ (both are contained in F^*, which has dimension 8). By a dimension argument, there are unique lines $L_1 \subseteq S_\theta$ and $L_2 \subseteq S_\pi$ with $L_1 \cap L_2 = S_\theta \cap S_\pi$ such that $q \in \langle L_1, L_2\rangle$. Let Q' be the hyperbolic quadrangle determined by L_1 and L_2 and let $\xi' \in \Xi$ be the corresponding member. Then $\xi \cap \xi'$ contains q, and by (S2), $q \in X$. Now q belongs to the 3-space $[L_1, L_2]$ and to X, and we noted above that this means that $q \in X^*$. But then $q \in Q$, a contradiction. We conclude that $F^* \cap Y = \emptyset$ in this case too. $\square$

Finally, we will show that X is, up to projection from a subspace in Y, a *mutant* of the half dual Segre variety $\mathcal{HDS}_{\ell,k}(\mathbb{K})$, which we define below. Recall that in Section 3.2.1, we constructed the half dual Segre variety $\mathcal{HDS}_{\ell,k}(\mathbb{K})$ by means of an ℓ-space Y and a Segre variety $\mathsf{S} := \mathcal{S}_{\ell,k}(\mathbb{K})$ with Y and S complementary, but the construction for X and Ξ do not depend on the fact that Y and

S are disjoint.

Definition 6.17 Consider the half dual Segre variety $\mathcal{HDS}_{\ell,k}(\mathbb{K}) = (X, Z, \Xi, \Theta)$ associated to the Segre variety $\mathsf{S} := \mathcal{S}_{\ell,k}(\mathbb{K})$. Then:

- we may replace S by a legal projection of $\mathcal{S}_{\ell,k}(\mathbb{K})$
- we may choose $Y = Z$ in such a way that the projection of $\langle \mathsf{S} \rangle \cap X$ from $\langle \mathsf{S} \rangle \cap Y$ (onto a subspace of $\langle \mathsf{S} \rangle$ complementary to $\langle \mathsf{S} \rangle \cap Y$) yields an injective projection of S; the ambient projective space is afterwards restricted to $\langle \mathsf{S}, Y \rangle$.

The resulting structure is called a *mutant* of $\mathsf{HDS}_{\ell,k}(\mathbb{K})$.

6.5 Conclusion

We phrase the conclusion without the standing hypothesis, to deal with the situation in full generality. Only now we will invoke axiom (S3).

Theorem 6.18 *Let (X, Z, Ξ, Θ) be a pre-DSV with parameters (r, v, r', v') such that there is an $x \in X$ with $|\Theta_x| = 1$. Then $r = 1$, $v = r' - 2$ and:*

(i) Up to projection from a v'-space $V \subseteq Y$ collinear to all points of X, we obtain that X is the point set of a mutant of a half dual Segre variety $\mathcal{HDS}_{r',k}(\mathbb{K})$ for some $k \geq 1$, with Z as r'-space at infinity and $\Xi \cup \Theta$ as its symps;

(ii) if additionally, (X, Z, Ξ, Θ) satisfies (S3), then X is the point set of a half dual Segre variety $\mathcal{HDS}_{r',k}(\mathbb{K})$ with $r' = 2$ and $k \in \{1, 2\}$ and X is projectively unique.

Proof (i) Lemma 6.2 yields the v'-space V collinear to all points of X and Lemma 4.14 allows us to project from V, so that we only need to deal with the case where $v' = -1$. That $r = 1$ follows from Corollary 5.13 and $v = r' - 2$ follows from Lemma 6.3.

By Proposition 6.16, X contains a subset X^* such that X^*, with induced line set L^*, is a legal projection of $\mathcal{S}_{r',k}(\mathbb{K})$, and $X = \bigcup_{x \in X^*} \langle x, Y_x \rangle \setminus Y_x$. Therefore it suffices to show that the connection map $\chi : X^* \to Y : x \mapsto Y_x$ satisfies the properties mentioned in the definition of the half dual Segre varieties (cf. Subsection 3.2.1).

Take $\theta \in \Theta$. Then by Proposition 6.16, $S := \theta \cap X^*$ is an r'-space of X^*. Inside the quadric $XY(\theta)$, in which S and Y are opposite r'-spaces, it is clear that the restriction χ_S of χ to S coincides with the collinearity relation between S and Y, so χ_S is a linear duality between S and Y.

Take $x \in X$ arbitrary. If $x \notin S$, then there is a unique point $s^x \in S$ collinear to x. Since the line xs^x belongs to π^x, we have $Y_{s^x_\theta} = Y_x$, and hence $\chi(s^x) = \chi(x)$. We conclude that χ is indeed as described in Section 3.2.1. For each pair of non-collinear points p_1, p_2 of X, Lemmas 4.1 and 4.7 imply that the unique symp ζ through p_1, p_2, $XY(\zeta)$ coincides with the convex closure of p_1 and p_2 via singular lines not contained in Y. Assertion (i) follows.

(ii) By Proposition 6.10, $\rho(X) \subseteq F$ is an injective projection of a Segre variety $\mathcal{S}_{r',k}(\mathbb{K})$. Let $x \in X$ be arbitrary. We denote by $T^F_{\rho(x)}$ the set of $\rho(X)$-lines in F through $\rho(x)$ and by $T^F_{\rho(x)}(\xi)$ the tangent space to $\rho(X(\xi))$ at $\rho(x)$ for some $\xi \in \Xi$ with $\rho(x) \in \rho(X(\xi))$.

Axiom (S3) yields members $\xi_1, \xi_2 \in \Xi$ through x such that $T_x = \langle T_x(\xi_1), T_x(\xi_2) \rangle$. Since for $i = 1, 2$, $T_x(\xi_i)$ is generated by $Y(\xi_i)$ and a pair of non-collinear X-lines through x (which project on non-collinear $\rho(X)$-lines through $\rho(x)$) and since $Y_x \subseteq T_x$, we obtain that $T_x = \langle T_x(\xi_1), T_x(\xi_2) \rangle$ is equivalent with $Y_x = \langle Y(\xi_1), Y(\xi_2) \rangle$ and $T^F_{\rho(x)} = \langle T^F_{\rho(x)}(\xi_1), T^F_{\rho(x)}(\xi_2) \rangle$. On the other hand, $\dim T^F_{\rho(x)} = r' + k$ as the tangent space at $\rho(x)$ is generated by the two maximal singular subspaces

$\rho(\theta^x)$ and $\rho(\pi^x)$ of $\rho(X)$ through $\rho(x)$. Furthermore, since $r = 1$, $T^F_{\rho(x)}(\xi_1)$ and $T^F_{\rho(x)}(\xi_2)$ are just planes, which generate at most a 4-space in F, and so $r + k' \leq 2 + 2 = 4$. Recalling that $r' > r \geq 1$ by assumption and $k \geq 1$, as noted in Remark 6.11, we deduce that $(r', k) \in \{(3,1),(2,1),(2,2)\}$. However, if $k = 1$, then $\rho(\pi^x)$ is a line contained in both planes $T^F_{\rho(x)}(\xi_1)$ and $T^F_{\rho(x)}(\xi_2)$, and hence these planes generate at most a 3-space, so $r' + k = r' + 1 \leq 3$, excluding the possibility $(r', k) = (3, 1)$. So $r' = 2$. Since $v = r' - 2 = 0$ and $\dim Y_x = r' - 1 = 1$, the requirement $Y_x = \langle Y(\xi_1), Y(\xi_2)\rangle$ only implies that ξ_1 and ξ_2 have disjoint vertices.

Put $F^* = \langle X^* \rangle$. Since $\langle F^*, Y \rangle = \mathbb{P}^N(\mathbb{K})$ by Proposition 6.16(i), we may choose the subspace F complementary to Y such that $F \subseteq F^*$. Since $r' = 2$ and $k \in \{1, 2\}$, the variety $\mathcal{S}_{r',k}(\mathbb{K})$ does not admit legal projections (cf. Lemma 6.15), moreover $F^* \cap Y = \emptyset$ by Proposition 6.16(iv), i.e., $F = F^*$. So in this case, (X, Z, Ξ, Θ) is not a mutant of but a proper half dual Segre variety, with $r' = 2$ and $k \in \{1, 2\}$. For the uniqueness, up to projectivity: Y and F in $\mathbb{P}^N(\mathbb{K})$, Ω in F. Moreover, the projectivity χ_S between S and the dual of Y is unique up to a projectivity of Y. We conclude that X is projectively unique. □

The case where $|\Theta_x| \geq 1$ for some $x \in X$ hence leads us to the conclusion of Main Result 3.6(i).

7 The dual line Grassmannians

As explained in Subsection 5.5 (see also Proposition 5.11), one of the cases to be treated is the following (corresponding to the $r = 2$ case in the first column of Table 1).

Standing hypothesis. Throughout this section, (X, Z, Ξ, Θ) is a pre-DSV with parameters $(2, v, r', v')$ with $\langle X, Z \rangle = \mathbb{P}^N(\mathbb{K})$ for an arbitrary field $|\mathbb{K}| > 2$, in which no point of Y is collinear to all points of X, and such that $|\Theta_L| = 1$ for each X-line L.

Recall Definition 5.7, which introduced the point-residue of (X, Z, Ξ, Θ). As also explained in Subsection 5.5, our approach will be inductive. We make this formal in the next lemma. Recall that $r' > r \geq 2$ by assumption, so $r' \geq 3$.

Lemma 7.1 *For any $x \in X$, the point-residue $\mathrm{Res}_X(x) = (X_x, Z_x, \Xi_x, \Theta_x)$ is isomorphic to a mutant of a half dual Segre variety $\mathcal{HDS}_{r'-1,k_x}(\mathbb{K})$ for some $k_x \geq 1$. In particular, $v = r' - 3$ and $v' = -1$. Moreover, each $\theta \in \Theta$ containing x contains $Y_x = Z_x$.*

Proof By Proposition 5.11, the point residue $\mathrm{Res}_X(x) = (X_x, Z_x, \Xi_x, \Theta_x)$ is a pre-DSV with parameters $(1, v, r' - 1, v')$, and no point of Y_x is collinear to all points of X_x. Since $|\Theta_L| = 1$ for each X-line L, we have $|\Theta_{x'}| = 1$ for each $x' \in X_x$. The first part of the statement now follows from Theorem 6.18, and since no point of Y_x is collinear to all points of X_x, the v'-space V is empty, so $v' = -1$. Finally, by Lemmas 5.2 and 6.2, each member of Θ_x contains $Y_x = Z_x$ and therefore so does the corresponding member of Θ. □

For each $x \in X$, we define k_x as in the previous lemma. We start by relating $\dim Y$ to k_x, and show that the latter does not depend on $x \in X$.

Lemma 7.2 *The sets Y and Z coincide, and for any $x \in X$, $\dim Y = r' + k_x$.*

Proof Let $x \in X$ be arbitrary. By Lemma 7.1, the point-residue $\mathrm{Res}_X(x) = (X_x, Z_x, \Xi_x, \Theta_x)$ is isomorphic to a mutant of $\mathcal{HDS}_{r'-1,k_x}(\mathbb{K})$ and $v' = -1$. According to Proposition 6.16, X_x contains a legal projection Ω of the Segre variety $\mathcal{S}_{r'-1,k_x}(\mathbb{K})$. By the same proposition, containment gives a bijection between the $(r' - 1)$-spaces of Ω and the members of Θ_x, which at their turn correspond bijectively with the members of Θ containing x. Let I be an index set such that $\{\theta_i \in \Theta \mid i \in I\}$ ranges over all members of Θ containing x. Let $i \in I$ be arbitrary. Consider the $(r'-1)$-space S_i of Ω contained in θ_i. By Lemma 7.1, θ_i contains Y_x, and according to Lemma 6.2, $\dim Y_x = r'-1$. Since $v' = -1$ by Lemma 7.1, we have that $Y(\theta_i)$ is an r'-space contained in Z by Definition 3.2. The subspaces Y_x and S_i of θ_i are opposite $r'-1$-spaces, and hence the properties of the quadric $XY(\theta_i)$ imply that $Y(\theta_i)$ contains a unique point $z_i \in Z$ collinear to S_i. Clearly, $z_i \notin Y_x$, so z_i corresponds to the unique point $\langle Y_x, z_i\rangle$ of $\mathrm{Res}_Y(Y_x)$. Conversely, suppose we are given a point $z \in Z \setminus Y_x$. As x and z are not collinear, Lemma 4.1 yields $[x,z] \in \Theta$, and hence $[x,z]$ contains a subspace S_i of Ω for a unique $i \in I$. Therefore, $\langle Y_x, z_i\rangle = \langle Y_x, z\rangle$ for a unique $i \in I$. We conclude that the map $S_i \mapsto \langle Y_x, z_i\rangle$ as defined above is a bijection between the set of $(r'-1)$-spaces of Ω and the set $M := \{\langle Y_x, z\rangle \mid z \in Z\rangle$, which is a subset of the point set of $\mathrm{Res}_Y(Y_x)$. We show that this bijection is an isomorphism.

Let S be any k_x-space of Ω. Each point of S is contained in a unique member of Θ_x and hence corresponds to a unique point of M. We show that a line L of S corresponds to a (full) line of $M \subseteq \mathrm{Res}_Y(Y_x)$. Let J be the subset of I such that $\{S_j \mid j \in J\}$ corresponds to the members of Θ through x meeting L in a point. Consider the unique member θ of Θ containing the X-line L. Again, by Lemma 7.1, θ contains Y_{x_j} for any $j \in J$, so in particular $z_j \in \theta$. On the other hand, as L is contained in S, and S is contained in a unique member of Π of $\mathrm{Res}_X(x)$ (see Proposition 6.16 and Definition 6.4), there is an $(r'-2)$-space Y_L in Y_x collinear to all points of L. Viewed in the quadric $XY(\theta)$, the map $p_j \mapsto Y_{p_j}$ from the points of L to the $(r'-1)$-spaces containing Y_L is a bijection. For each $j \in J$, $z_j \in \theta$ is collinear to a unique point p_j from L and hence $z_j \in Y_{p_j}$. Projecting $Y(\theta)$ from Y_L gives a full line which is in bijective correspondence with $\{z_j \mid j \in J\}$. When projecting from $Y \supseteq Y_x$, we henco also obtain that $\{z_j \mid j \in J\}$ a full line of $\mathrm{Res}_Y(Y_x)$. We conclude that M is a subspace, isomorphic to S, and hence $\dim M = k_x$.

Since M contains all points of the form $\langle Y_x, z\rangle$ with $z \in Z$, as explained above, we obtain that M coincides with $\mathrm{Res}_Y(Y_x)$ since $Y = \langle Z\rangle$. So $\dim Y = \dim Y_x + k_x + 1 = r' + k_x$ indeed. Take any $y \in Y$. Then by the above, $y \in \langle Y_x, z_i\rangle$ for some $i \in I$, and since $Z(\theta_i) = Y(\theta_i) = \langle Y_x, z_i\rangle$ we obtain that $y \in Z$. So also $Y = Z$ follows. $\square$

Henceforth, we write k instead of k_x since the latter does not depend on $x \in X$. Next, we show that the mutual position of two points $x_1, x_2 \in X$ is reflected in the mutual position of Y_{x_1} and Y_{x_2}. This will in particular allow us to prove later on that $k = 1$.

Lemma 7.3 *Take two distinct points $x_1, x_2 \in X$. Then*

(i) x_1x_2 is a singular line with a unique point in Y $\Leftrightarrow$ $Y_{x_1} = Y_{x_2}$;
(ii) x_1, x_2 are non-collinear with $[x_1, x_2] \in \Theta$ or x_1x_2 is an X-line $\Leftrightarrow$ $\dim(Y_{x_1} \cap Y_{x_2}) = r'-2$;
(iii) x_1 and x_2 are non-collinear with $[x_1, x_2] \in \Xi$ $\Leftrightarrow$ $\dim(Y_{x_1} \cap Y_{x_2}) = r'-3$.

Proof Clearly, the possibilities for x_1, x_2 described in (i), (ii) and (iii) exhaust the mutual positions between x_1 and x_2, so it suffices to verify the "$\Rightarrow$"s.

(i) $\Rightarrow$: This is clear.

(ii), $\Rightarrow$: In both cases, there is a unique $\theta \in \Theta$ containing x_1, x_2 since each X-line is contained in a unique member of Θ by the standing hypothesis. Recall that the r'-space $Y(\theta)$ contains both Y_{x_1} and Y_{x_2} (cf. Lemma 7.1). Therefore either $\dim(Y_{x_1} \cap Y_{x_2}) = r'-2$ or $Y_{x_1} = Y_{x_2}$. Suppose for a contradiction that $Y_{x_1} = Y_{x_2}$. Since $XY(\theta)$ is a hyperbolic quadric, there are only two r'-spaces containing Y_{x_1}, namely $\langle x_1, Y_{x_1}\rangle$ and $Y(\theta)$. This means that $x_2 \in \langle x_1, Y_{x_1}\rangle$ and hence x_1x_2 is a singular line with a unique point in Y, a contradiction.

(iii), $\Rightarrow$: If $[x_1, x_2] = \xi \in \Xi$, then its vertex $Y(\xi)$ is exactly $Y_{x_1} \cap Y_{x_2}$. By Lemma 7.1 $v = r'-3$, and hence $\dim(Y_{x_1} \cap Y_{x_2}) = r'-3$. $\square$

We record an obvious but important consequence.

Corollary 7.4 *For all $x_1, x_2 \in X$, we have $Y_{x_1} = Y_{x_2} \Leftrightarrow \rho(x_1) = \rho(x_2)$.*

Proof By the previous lemma, $Y_{x_1} = Y_{x_2}$ is equivalent with x_1x_2 being a singular line with a unique point in Y, i.e., with $x_2 \in \langle x_1, Y_{x_1}\rangle$. By Lemma 4.18, this is at its turn equivalent with $\rho(x_1) = \rho(x_2)$. □

Lemma 7.3 becomes a powerful tool if we can show that each $(r'-1)$-space of Y occurs as Y_x for some $x \in X$:

Lemma 7.5 *For each $(r'-1)$-space H in Y, there is a point $x \in X$ such that $Y_x = H$, and all X-points collinear to H are precisely the points in $\langle x, H\rangle \setminus H$.*

Proof Take $x \in X$ arbitrary. Recall that $\dim Y = r' + k$ by Lemma 7.2, so in particular $\dim Y < \infty$. This implies that it suffices to show that, for each $(r'-1)$-space H' of Y with $\dim(Y_x \cap H') = r' - 2$, we have $H' = Y_{x'}$ for some $x' \in X$, as then connectivity argument finishes the proof. So without loss of generality, $\dim(Y_x \cap H) = r' - 2$. Let $z \in H \setminus Y_x$ be arbitrary. Recall that $z \in Z$ by Lemma 7.2. So by Lemma 4.1, $[z, x] \in \Theta$. By Lemma 7.1, $[x, z]$ contains Y_x. Since it also contains z, also $H \subseteq [x, z]$. As such, $[x, z]$ contains an X-point x' collinear to H, i.e., $Y_{x'} = H$. The second part of the assertion follows from Lemma 7.3(i). □

As promised, the above leads us to $k = 1$:

Corollary 7.6 *We have $k = 1$.*

Proof Let H_1 and H_2 be a pair of $(r'-1)$-spaces of Y with $\dim(H_1 \cap H_2)$ minimal, i.e., H_1 and H_2 generate Y. By Lemma 7.5, there are points $x_1, x_2 \in H$ with $H_1 = Y_{x_1}$ and $H_2 = Y_{x_2}$. According to Lemma 7.3, $\dim(H_1 \cap H_2) \geq r' - 3$ and hence $Y = \langle H_1, H_2\rangle$ has dimension at most $2(r'-1) - (r'-3) = r'+1$. Since there are points $x'_1, x'_2 \in X$ with $\dim(Y_{x'_1} \cap Y_{x'_2}) = r' - 3$ (because $|\Xi| \geq 1$), we obtain $\dim Y = r'+1$. However, we already know from Lemma 7.2 that $\dim Y = r'+k$, so we conclude that $k = 1$. □

Another corollary is the following.

Corollary 7.7 *For each r'-space Y' in Y, there is a unique $\theta \in \Theta$ with $Y(\theta) = Y'$. Moreover, if $x \in X$ has $Y_x \subseteq Y'$, then $x \in \theta$.*

Proof Take any $(r'-1)$-space H in Y'. By Lemma 7.5, we know that $H = Y_x$ for some $x \in X$. Take any point $z \in Y' \setminus H$. Then $\theta := [x, z]$ is a member of Θ with $Y(\theta) = Y'$. Let $x' \in X$ be such that $Y_{x'} \subseteq Y'$. Then $X(\theta)$ contains a point x'' with $Y_{x''} = Y_{x'}$, so by Corollary 7.4, $x' \in \langle x'', Y_{x''}\rangle \subseteq \theta$. This also shows that θ is the unique member of Θ containing Y'. □

We proceed similarly as in Section 6 and nail down the structure of (X, Z, Ξ, θ), using the projection $\rho := \rho_Y$ from (X, Z, Ξ, Θ) from Y onto a complementary subspace F, and the connection map χ (cf. Definitions 4.13 and 4.17).

Lemma 7.8 *Suppose $\rho(x_1)$ and $\rho(x_2)$ determine a singular line of $\rho(X)$, for $x_1, x_2 \in X$. Let $x'_i \in \rho^{-1}(\rho(x_i))$ be arbitrary, for $i = 1, 2$. Then:*

(i) there is a unique $\theta \in \Theta$ containing $x'_1 \cup x'_2$, and $\rho^{-1}(\rho(x_1)) \cup \rho^{-1}(\rho(x_2)) \subseteq \theta$;

(ii) there is an $x''_2 \in \rho^{-1}(\rho(x_2))$ such that $\langle x'_1, x''_2\rangle$ is an X-line.

(iii) $\{Y_x \mid \rho(x) \in \langle \rho(x_1), \rho(x_2)\rangle\}$ is the set of all $(r'-1)$-spaces through the $(r'-2)$-space $Y_{x_1} \cap Y_{x_2}$ inside the r'-space $Y(\theta)$.

Proof Recall that $\rho^{-1}(x_i) = \langle x_i, Y_{x_i}\rangle \cap X$ by Lemma 4.18. Corollary 7.4 implies that $Y_{x_1} \neq Y_{x_2}$ because $\rho(x_1) \neq \rho(x_2)$. So $Y_{x_1} \neq Y_{x_2}$. Note that $\dim Y = r' + k = r' + 1$ by Lemmas 7.2 and 7.6; and $\dim Y_{x_1} = \dim Y_{x_2} = r' - 1$. We claim that $\dim(Y_{x_1} \cap Y_{x_2}) = r' - 2$. Indeed, if so, then by Lemma 7.3 and the fact that each X-line is contained in a unique member of Θ, there is a unique $\theta \in \Theta$ containing x_1, x_2, and θ also contains $\langle x_i, Y_{x_i}\rangle \supseteq \rho^{-1}(x_i)$ for $i = 1, 2$ by Lemma 7.1. Suppose for a contradiction that $\dim(Y_{x_1} \cap Y_{x_2}) \neq r' - 2$. Then by the above and Lemma 7.3, $\dim(Y_{x_1} \cap Y_{x_2}) = r' - 3$ and hence $\langle Y_{x_1}, Y_{x_2}\rangle = Y$. We can now use a similar dimension argument as used in the third paragraph of the proof of Lemma 6.8 to show that there is an X-line containing a point $x_1^* \in \rho^{-1}(x_1)$ and a point $x_2^* \in \rho^{-1}(x_2)$. By our standing hypthesis, there is a member of Θ containing $x_1^* x_2^*$, which hence also contains x_1, x_2, a contradiction to $[x_1, x_2] \in \Xi$ (see Lemma 7.3). The claim follows, and assertion (i) is proven.

Assertion (ii) and (iii) can easily be verified inside the quadric $XY(\theta)$ containing $\rho^{-1}(\rho(x_1)) \cup \rho^{-1}(\rho(x_2))$ (see also the last paragraph of the proof of Lemma 6.8). □

Lemma 7.9 *Suppose $x_1, x_2 \in X$ are non-collinear points with $\xi = [x_1, x_2] \in \Xi$ and put $V = Y(\xi)$. Let $x'_i \in \rho^{-1}(\rho(x_i))$ for $i = 1, 2$. Then x'_1 and x'_2 are non-collinear and $\xi' := [x'_1, x'_2]$ has vertex V and $\rho(X(\xi')) = \rho(X(\xi))$.*

Proof For $i = 1, 2$, $\rho(x_i) = \rho(x'_i)$ implies, by Lemma 7.4, that $Y_{x_i} = Y_{x'_i}$. In particular, $V = Y_{x_1} \cap Y_{x_2} = Y_{x'_1} \cap Y_{x'_2}$. Lemma 7.3 tells us that x'_1 and x'_2 are non-collinear points indeed, with $\xi' := [x'_1, x'_2] \in \Xi$, and $Y(\xi') = V$ by the foregoing. To show that $\rho(X(\xi)) = \rho(X(\xi'))$, we use the map $\sigma_\xi : \rho(X(\xi)) \to \{H \mid V \subseteq H \subseteq Y, \dim H = r' - 1\} : \rho(x) \mapsto Y_x$. We claim that it is an isomorphism.

First of all, note that σ_ξ is well-defined by Lemma 4.18. We consider the residue $\mathrm{Res}_Y(V)$. Since $\dim(V) = r' - 3$ and $\dim(Y) = r' + 1$, the residue $\mathrm{Res}_Y(T)$ is isomorphic to a projective 3-space over $\mathbb{K}$, say $\Pi_V(\mathbb{K})$, in which Y_x corresponds to a line $L(x)$. Let x, x' be two points of $X(\xi)$. By Lemma 7.3, $L(x) = L(x')$ if and only if x and x' belong to the same generator of $X(\xi)$, i.e., if and only if $\rho(x) = \rho(x')$; $L(x)$ and $L(x')$ intersect in precisely a point if and only if xx' is an X-line in $X(\xi)$ and $L(x)$ and $L(x')$ are disjoint if and only if x and x' are non-collinear. Moreover, Lemma 7.8 implies that each X-line of $X(\xi)$ corresponds to a full planar point pencil in $\Pi_V(\mathbb{K})$. On the other hand, the Klein correspondence yields that the point-line geometry whose point set is the set of all lines of $\Pi_T(\mathbb{K})$ and whose lines are the planar line pencils of $\Pi_V(\mathbb{K})$, is isomorphic to the point-line geometry associated to a hyperbolic quadric Q of rank 3 in $\mathbb{P}^5(\mathbb{K})$. Since $\rho(X(\xi))$ is a quadric of the same kind (recall $r = 2$), we obtain that $\sigma(\rho(X(\xi)))$ is embedded isometrically into Q. Since both are defined over the field $\mathbb{K}$, they actually coincide, i..e, $\sigma(\rho(X(\xi))) = Q$. The claim follows. Since ξ' also has vertex V, we then have that $\mathrm{im}\sigma_\xi = \mathrm{im}\sigma_{\xi'}$ and hence $\rho(X(\xi)) = \rho(X(\xi'))$. □

We use Y to define the following point-line geometry $(\mathsf{P}, \mathsf{B})_Y$.

Definition 7.10 *Let P denote the set of $(r' - 1)$-dimensional subspaces of Y. For subspaces $S_- \subseteq S_+ \subseteq Y$ with $\dim S_- = r' - 1$ and $\dim S_+ = r'$, we define the pencil $P(S_-, S_+)$ as the set $\{P \in \mathsf{P} \mid S_- \subseteq P \subseteq S_+\}$. Then we denote by B the set $\{P(S_1, S_2) \mid S_1 \subseteq S_2 \subseteq Y, \dim S_\pm = r' - 1 \pm 1\}$.*

Lemma 7.11 *The point-line geometry $(\mathsf{P}, \mathsf{B})_Y$ is isomorphic to the (point-line truncation of the) line Grassmannian $\mathcal{G}_{r'+2,2}(\mathbb{K})$.*

Proof Since a projective space is self-dual and $\dim Y = r' + 1$, the point-line geometry $(\mathsf{P}, \mathsf{B})_Y$ (with natural incidence relation) is by definition isomorphic to the (point-line truncation of the) line Grassmannian $\mathcal{G}_{r'+2,2}(\mathbb{K})$; see also Subsection 3.2.2. □

Proposition 7.12 *The point-line geometry $\mathcal{G} := (\rho(X), \rho(\mathsf{L}))$ is an injective projection of the line Grassmannian $\mathcal{G}_{r'+2,2}(\mathbb{K})$. Moreover, we have:*

(i) for each singular r'-space S in $\mathcal{G}$, there is a unique $\theta_S \in \Theta$ with $\rho(X(\theta_S)) = S$.

(ii) for each symp Q of $\mathcal{G}$ (viewing the latter as a parapolar space), there is a unique v-space V in Y such that there is a $\xi \in \Xi$ with vertex V such that $\rho(X(\xi)) = Q$.

Proof We claim that χ induces an isomorphism between the abstract point-line geometries $(\rho(X), \rho(\mathsf{L}))$ and $(\mathsf{P}, \mathsf{B})_Y$. Indeed, the fact that $\chi : \rho(X) \to \mathsf{P} : x \mapsto \chi(x) = Y_x$ is a bijection between $\rho(X)$ and P follows immediately from Corollary 7.4 (injectivity) and Lemma 7.5 (surjectivity). The fact that a member of $\rho(\mathsf{L})$ is mapped by χ to a member of B follows from Lemma 7.8(iii). This shows the claim. So, by Lemma 7.11, $(\rho(X), \rho(\mathsf{L}))$ is as a point-line geometry isomorphic to a geometry of type $\mathsf{A}_{r'+1,2}(\mathbb{K})$. Since the latter's absolutelety universal embedding is given by the line Grassmannian $\mathcal{G}_{r'+2,2}(\mathbb{K})$, the geometry $(\rho(X), \rho(\mathsf{L}))$ is an injective projection of $\mathcal{G}_{r'+2,2}(\mathbb{K})$.

(i) Let S be a maximal singular subspace of $\rho(X)$ of dimension r' and take a line L in S. By Lemma 7.8(i), there is a unique $\theta \in \Theta$ such that $L \subseteq \rho(X(\theta))$. The properties of the line Grassmannian $\mathcal{G}_{r'+2,2}(\mathbb{K})$ imply that there is a unique r'-space through L; as such, the r'-space $\rho(X(\theta))$ coincides with S.

(ii) Let Q be any symp of $\mathcal{G}$ (so Q is a hyperbolic quadric in $\mathbb{P}^5(\mathbb{K})$ since $r = 2$). By Lemma 7.9, it suffices to show that Q coincides with $\rho(X(\xi))$ for some $\xi \in \Xi$. Let p_1 and p_2 be non-collinear points of Q and take points $x_1, x_2 \in X$ with $\rho(x_i) = p_i$. Then, since p_1 and p_2 are distinct and non-collinear, $\xi := [x_1, x_2] \in \Xi$. Now, $\rho(X(\xi))$ is a hyperbolic quadric of rank 3 in $\rho(X)$ containing the points p_1 and p_2, and since two non-collinear points determine a unique symp in $\rho(X)$, we obtain $\rho(X(\xi)) = Q$. □

7.1 Mutants of the dual line Grassmannian variety

Just like in subsection 6.4, we will show that (X, Z, Ξ, Θ) is a tweaked version a dual Line Grassmannian variety (a mutant, see Definition 7.14). For now, recall the notion of a legal projection from Definition 6.13.

Lemma 7.13 *The set X contains a subset X^* such that, with induced line set L^*, the geometry $\Omega := (X^*, \mathsf{L}^*)$ is a legal projection of $\mathcal{G}_{r'+2,2}(\mathbb{K})$, where containment gives a bijection between the r'-spaces of Ω and the set Θ. For any hyperbolic quadric Q of rank 3 of Ω, there is a unique $\xi \in \Xi$ with $\xi \cap X^* = Q$. Morever, if $F^* = \langle X^* \rangle$, then*

(i) $\bigsqcup_{x \in X^*} \langle x, Y_x \rangle \setminus Y_x = X$ and hence $\langle F^*, Y \rangle = \mathbb{P}^N(\mathbb{K})$;

(ii) Re-choose F such that $F^* \subseteq F$. Then $F^* \cap Y$ and F are complementary in F^*, and the projection ρ^* of $F^* \cap X$ from $F^* \cap Y$ on F coincides with the projection of ρ restricted to $F^* \cap X$. In particular, if $F^* \cap Y = \emptyset$, then $X^* = X \cap F^*$ and ρ is the identity on X^*.

(iii) If $r' = 4$, then $F^* \cap Y = \emptyset$.

Proof We construct a legal projection of $\mathcal{G}_{r'+2,2}(\mathbb{K})$ inside X, by choosing a set of points $X^* \subseteq X$ such that $\{Y_x \mid x \in X^*\} = \mathsf{P}$ (cf. Definition 7.10). To that end, let $\mathcal{B} := \{p_0, ..., p_{r'+1}\}$ be the set of points of a basis of Y. Put $A = \{(i, j) \mid 0 \leq i < j \leq r' + 1\}$. For each pair $(i, j) \in A$, let $H_{i,j}$ be the $(r' - 1)$-space generated by the points of $\mathcal{B} \setminus \{p_i, p_j\}$. By Lemma 7.5 and Corollary 7.4, the X-points collinear to $H_{i,j}$ are contained in the r'-space $\overline{H}_{i,j} := \langle x'_{i,j}, H_{i,j} \rangle$ for some $x'_{i,j} \in X$ with $Y_{x'_{i,j}} = H_{i,j}$. For each $i \in \{0, ..., r' + 1\}$, Lemma 7.7 yields a unique $\theta_i \in \Theta$ with $Y(\theta_i) = \langle \mathcal{B} \setminus \{p_i\} \rangle$, and by the same lemma, θ_i contains $\overline{H}_{i,j}$ for all $j \in \{0, ..., r' + 1\}$. In a similar fashion as in the proof of Lemma 6.16, we can consecutively choose points $x_{i,j} \in \overline{H}_{i,j} \cap X$, using the lexicographic order on the pairs in A, in such a way that $x_{i,j}$ is collinear to $x_{i,j'}$ for each $j' < j$ and to $x_{i',j}$ for each $i' < i$. For each $i \in \{0, ..., r'+1\}$, we define $R'_i := \langle x_{i,j} \mid 0 \leq j \leq r' \rangle$. By construction, R'_i is a singular subspace of θ_i and no point of Y is collinear to R'_i, so R'_i is an X-space of dimension r'. Observe that points $x_{i,j}$ and $x_{k,\ell}$ with $|\{i,j,k,\ell\} = 4$ are non-collinear and determine a member of Ξ because $\dim(H_{i,j} \cap H_{k,\ell}) = r' - 3$.

Claim: Each $\theta \in \Theta$ contains a unique r'-space R'_θ generated by the points $\{\theta \cap R'_i \mid 0 \leq i \leq r'+1\}$. Recall that, for each r'-space H in Y, there is a unique member $\theta_{Y'}$ of Θ with $Y(\theta_{Y'}) = H$ (cf. Corollary 7.7). Firstly, suppose θ is such that $H := Y(\theta)$ contains $H_{0,1}$. Then H meets the line p_0p_1 in a point p. If $p = p_i$, for $i \in \{0,1\}$, then $\theta_H = \theta_i$ and hence $R'_\theta = R'_i$; so suppose $p \notin \{p_0, p_1\}$. Take any $j \in \{2, ..., r'+1\}$ and consider θ_j. By definition, θ_j contains the lines p_0p_1 and $x_{0,j}x_{1j}$. It is easily verified in the quadric $XY(\theta_j)$ that the lines p_0p_1 and $x_{0,j}x_{1,j}$ are opposite, and hence there is a unique point q_j on $x_{0,j}x_{1,j}$ collinear to p. Then Y_{q_j} is a hyperplane of H, namely the one generated by the point p and the $(r'-1)$-space $x_{0,j}^\perp \cap x_{1,j}^\perp \cap H_{0,1}$. So $q_j \in \theta$ by Corollary 7.7. Observe that q_j is exactly the intersection of R_j and θ. Put $R'_\theta := \langle x_{0,1}, q_2, ..., q_{r'+1} \rangle$. We claim that R'_θ is indeed an X-space of dimension r' in θ. To see that it is singular, take $j, j' \in \{2, ..., r'+1\}$ with $j \neq j'$ and consider the points $x_{0,j}$ and $x_{1,j'}$, which determine a member of Ξ since $\dim(H_{0,j} \cap H_{1,j'}) = r' - 3$ (cf. Lemma 7.3). By Lemma 4.7, the points $x_{0j'}$ and x_{1j} also belong to $[x_{0,j}, x_{1,j'}]$. Therefore, $q_j, q'_j \in \theta \cap [x_{0,j}, x_{1,j'}]$, so by (S2), $q_jq'_j$ is a singular line. This line is contained in X as otherwise $Y_{q_j} = Y_{q_{j'}}$ (contradicting Lemma 7.3). The fact that $\dim R'_\theta = r'$ follows as usual: no point of Y is collinear to a point of R'_θ. Continuing like this with other r'-spaces H, the claim follows.

We define X^* as $\bigcup_{\theta \in \Theta} R'_\theta$. Let $x_1, x_2 \in X^*$ be distinct points. One can verify that, by the above procedure, $Y_{x_1} \neq Y_{x_2}$. This means that, if $\theta \neq \theta'$, then $R'_\theta \cap R'_{\theta'}$ is the unique point x^* in X^* with $Y_{x^*} = Y(\theta) \cap Y(\theta')$. Conversely, each point x is contained in some $\theta \in \Theta$ and hence R'_θ contains a (unique, by the foregoing) point $\tilde{x}$ with $Y_x = Y_{\tilde{x}}$ and hence $x \in \langle \tilde{x}, Y_{\tilde{x}} \rangle$. This already shows (i), ánd it shows that the map $x^* \mapsto Y_{x^*}$ is a bijection between X^* and P. We show that this extends to an isomorphims. To that end, suppose $x_1, x_2 \in X^*$ are on an X-line L. Let θ be the unique member of Θ containing L. Then by the beginning of this paragraph $x_1, x_2 \subseteq R'_\theta$ and hence $L \subseteq R'_\theta$, so $L \subseteq X^*$. Looking inside the quadric $XY(\theta)$, we see that the points of L correspond to the $(r'-1)$-space in $Y(\theta)$ containing the $(r'-2)$-space $Y_{x_1} \cap Y_{x_2}$. Moreover, if $x_1, x_2 \in X^*$ belong to the same $\theta \in \Theta$, then they are necessarily collinear: by the beginning of this paragraph, x_1, x_2 belong to the X^*-space R'_θ and hence they are on an X-line (even an X^*-line). We have shown that X^*, equipped with the set L^* of X-lines it contains, is isomorphic to $(\mathsf{P}, \mathsf{B})_Y$. Therefore (X^*, L^*) is already an injective projection of the line Grassmannian $\mathcal{G}_{r'+2,2}(\mathbb{K})$. Since each line of L^* is hence contained in a unique r'-space, it follows that each r'-space of X^* is given by R'_θ for some $\theta \in \Theta$. Next, consider a hyperbolic quadric Q of rank 3 in X^*. Take two of its non-collinear points x_1, x_2. As noted just above, $[x_1, x_2] \in \Xi$ (if $[x_1, x_2] \in \Theta$ then x_1x_2 is an X^*-line after all). By Lemma 4.7, $Q \subseteq [x_1, x_2]$. As in the proof of Proposition 6.16, we naturally obtain a pair (X^*, Ξ^*), which is a pre-DSV, so that (X^*, L^*) is a *legal* projection of $\mathcal{G}_{r'+2,2}(\mathbb{K})$.

$(i) - (ii)$ Same as in the proof of Proposition 6.16.

(iii) Finally, suppose $r' = 4$. As a consequence of the main result of [12], $\mathcal{G}_{6,2}(\mathbb{K})$ has no proper legal projections, and hence $\dim F^* = 14$. Suppose for a contradiction that $F^* \cap Y$ contains a point y. Set $X^*_y := \{x \in X^* \mid y \in Y_x\}$. Then X^*_y induces a subgeometry of Ω is isomorphic to $\mathcal{G}_{5,2}(\mathbb{K})$ (cf. Lemma 7.11); in particular $\dim\langle X^*_y \rangle = 9$. We first claim that $\langle X^*_y \rangle \cap X = X^*_y$. Suppose for a contradiction that $x \in \langle X^*_y \rangle \setminus X^*_y$. It is a well known property of $\mathcal{G}_{5,2}(\mathbb{K})$ that x then lies on a secant, i.e., there are (non-collinear) points $x_1, x_2 \in X^*_y$ with $x \in x_1x_2$. As noted above, non-collinear points of X^* determine a member of Ξ, so $[x_1, x_2] \in \Xi$. But then $x \in x_1x_2 \subseteq X([x_1, x_2])$, violating Definition 3.2. The claim follows. Consequently, $y \notin \langle X^*_y \rangle$, because otherwise, for any point $x^* \in X^*_y$, we would have that $x^*y \subseteq \langle X^*_y \rangle$, whereas x^*y contains points of $X \setminus X^*$ (since $\langle x^*, Y_x \rangle \cap X^* = \{x^*\}$ by (i)), contradicting the claim. So $\dim\langle y, X^*_y \rangle = 10$. Let $\theta \in \Theta$ be such that $y \notin Y(\theta)$. Then R'_θ is a 4-space of X^* disjoint from X^*_y. By dimension, R'_θ shares a point p

with $\langle y, X_y^* \rangle$. Let q be the unique point of yp in $\langle X_y^* \rangle$. Note that $q \notin X$ since the line yp is not singular (since $y \notin Y_p \subseteq Y(\theta)$). As in the previous claim, $q \in [x_1, x_2] \in \Xi$ for two non-collinear points $x_1, x_2 \in X_y^*$. Since $y \in Y_{x_1} \cap Y_{x_2} = Y([x_1, x_2])$, it follows that $p \in X([x_1, x_2])$, the same contradiction as before. We conclude that $Y \cap F^*$ is empty if $r' = 4$. □

Finally, we will show that X is, up to projection from a subspace in Y, a *mutant* of the dual line Grassmannian variety $\mathcal{DG}_{n+1,2}(\mathbb{K})$, which we define below. Recall that in Section 3.2.2 we constructed the dual line Grassmannian variety $\mathcal{DG}_{n+1,2}(\mathbb{K})$ by means of an n-space Y and a line Grassmannian variety $\mathsf{G} := \mathcal{G}_{n+1,2}(\mathbb{K})$ with Y and G complementary, but the construction for X, Z and Ξ, Θ do not depend on the fact that Y and G are disjoint.

Definition 7.14 Consider the dual line Grassmannian variety $\mathcal{DG}_{n+1,2}(\mathbb{K}) = (X, Z, \Xi, \Theta)$ associated to the Segre variety $\mathsf{G} := \mathcal{G}_{n+1,2}(\mathbb{K})$. Then:

- we may replace G by a legal projection of $\mathcal{G}_{n+1,2}(\mathbb{K})$
- we may choose $Y = Z$ in such a way that the projection of $\langle \mathsf{G} \rangle \cap X$ from $\langle \mathsf{G} \rangle \cap Y$ (onto a subspace of $\langle \mathsf{G} \rangle$ complementary to $\langle \mathsf{G} \rangle \cap Y$) yields an injective projection of G; the ambient projective space is afterwards restricted to $\langle \mathsf{G}, Y \rangle$.

The resulting structure is called a *mutant* of $\mathsf{DG}_{n+1,2}(\mathbb{K})$.

7.2 Conclusion

Putting everything together, we obtain the following rather general classification result. We phrase the conclusion without the standing hypothesis, to deal with the situation in full generality. Again, only now we use axiom (S3).

Theorem 7.15 *Let (X, Z, Ξ, Θ) be a pre-DSV with parameters (r, v, r', v') with $r \geq 2$, such that there is an X-line L with $|\Theta_L| = 1$. Then $r = 2$, $v = r' - 3$ and:*

- (i) *Up to projection from a v'-space $V \subseteq Y$ collinear to all points of X, we obtain that X is the point set of a mutant of the dual line Grassmannian $\mathcal{DG}_{r'+2,2}(\mathbb{K})$ with Z as $(r'+1)$-space at infinity and $\Xi \cup \Theta$ as its symps;*
- (ii) *if additionally (X, Z, Ξ, Θ) satisfies (S3), then X is the point set of a dual line Greassmannian $\mathcal{DG}_{r'+2,2}(\mathbb{K})$ with $r' = 4$ and X is projectively unique.*

Proof (i) By Lemma 7.1 it follows that there is a v'-space V in Y collinear to all points of X, and Lemma 4.14 allows us to project from V, so henceforth we assume $v' = -1$. Lemma 7.1 also shows that $r = 2$ and $v = r' - 3$.

By Proposition 7.13, X contains a subset X^* such that X^*, with induced line set L^*, is a legal projection of $\mathcal{G}_{r'+2,2}(\mathbb{K})$, and $X = \bigsqcup_{x \in X^*} \langle x, Y_x \rangle \setminus Y_x$. Therefore it suffices to show that the connection map $\chi : \Omega \to Y : x \mapsto Y_x$ satisfies the properties mentioned in the definition of the dual Line Grassmannians varieties (cf. Subsection 3.2.1). Let $\mathbb{P}$ be a projective space of dimension $r' + 1$, such that (X^*, L^*) is isomorphic to the image of $\mathbb{P}$ under the Plücker map pl (cf. Subsection 3.2.2). Consider the map $\chi' : \mathbb{P} \to Y$, taking a line $L \in \mathbb{P}$ to $\chi(\mathsf{pl}^{-1}(L)) = Y_{\mathsf{pl}^{-1}(L)}$. For ease of notation, we denote the point $\mathsf{pl}^{-1}(L)$ by x_L.

Claim: χ' induces a linear duality between $\mathbb{P}$ and Y.
Let p be any point of $\mathbb{P}$. Then $R'_p := \{x_L \mid L \text{ line of } \mathbb{P} \text{ with } p \in L\}$ is the set of points of a singular r'-space of X^*. By Lemma 7.13, there is a unique member $\theta_p \in \Theta$ containing R'_p. Then χ' induces a linear duality $x_L \mapsto Y_{x_L} = \chi'(L)$ inside θ_p, meaning in particular that $Y_{x_L} \subseteq Y(\theta_p)$ for each $x_L \in R'_p$ and that each $(r'-1)$-space in $Y(\theta_p)$ is given as Y_{x_L} for some $x_L \in R'_p$. Conversely, each hyperplane H of Y is contained in a unique member of Θ by Lemma 7.7. It follows that the map

$p \mapsto Y(\theta_p)$ is a bijection between the point set of $\mathbb{P}$ and the hyperplanes of Y (i.e., the point set of the dual of Y), preserving incidence. This means that it is a linear duality indeed: a line L of $\mathbb{P}$ goes to the r'-spaces through the $(r'-1)$-space Y_{x_L} and vice versa. The claim follows.

For each pair of non-collinear points p_1, p_2 of X, Lemmas 4.1 and 4.7 imply that the convex closure of p_1, p_2 (via lines containing X-points) coincides with $XY([p_1, p_2])$. The assertion follows.

(ii) Let $x \in X$ be arbitrary. By Axiom (S3), there are $\xi_1, \xi_2 \in \Xi$ through x such that $T_x = \langle T_x(\xi_1), T_x(\xi_2)\rangle$. As in the proof of Theorem 6.18 and with the same notation, we have that this is equivalent to $Y_x = \langle Y(\xi_1), Y(\xi_2)\rangle$ and $T^F_{\rho(x)} = \langle T^F_{\rho(x)}(\xi_1), T^F_{\rho(x)}(\xi_2)\rangle$. This time, we know $\dim T^F_{\rho(x)} = 2r' - 1$ as $\mathrm{Res}_X(x)$ is isomorphic to $\mathcal{S}_{r'-1,1}(\mathbb{K})$. Furthermore, since $r = 2$, $T^F_{\rho(x)}(\xi_1)$ and $T^F_{\rho(x)}(\xi_2)$ are just 4-spaces, and so $2r' - 1 \leq 8$. Recalling that $r' > r \geq 2$ by assumption, we deduce that $r' \in \{3, 4\}$. Since $v = r' - 3$, the $(r'-1)$-space Y_x can only be generated by the two v-spaces $Y(\xi_1)$ and $Y(\xi_2)$ if $r' = 4$ (and if $Y(\xi_1) \cap Y(\xi_2) = \emptyset$). So $r' = 4$. The variety $\mathcal{G}_{4+2,2}(\mathbb{K})$ does not admit legal projections (this follows from the main result of [12]) and $F^* \cap Y = \emptyset$ by Proposition 7.13*(iii)*. The conclusion follows as in the proof of Theorem 6.18. □

The case where $r = 2$ and $|\Theta_L| = 1$ for each X-line L hence leads us to the conclusion of Main Result 3.6*(ii)*.

8 The remaining cases

Four of the cases that occur in Table 1 remain, namely the red ones (which have in common that each X-line is contained in at least two members of Θ) and the $r = 1$ case of column 2. In Subsection 8.1, we will treat the latter case, leading to a dual Segre variety; in Subsection 8.2, we will show that the other cases do not lead to examples of DSVs.

8.1 Dual Segre variety

In this section, we deal with the $r = 1$ case of column 2 of Table 1. Due to the high similarity with the half dual Segre varieties (see Section 6, where also $r = 1$), and since this case turns out not occur as the residue of a DSV (see Proposition 8.6), we will be brief and refer to [1] for the proofs.

Standing hypothesis. Throughout this section, (X, Z, Ξ, Θ) is a pre-DSV with parameters $(1, v, r', v')$ with $\langle X, Z\rangle = \mathbb{P}^N(\mathbb{K})$ for an arbitrary field $|\mathbb{K}| > 2$ and such that $|\Theta_x| \geq 2$ for each $x \in X$ and $|\Theta_L| \in \{0, 1\}$ for each X-line L.

We only give the elements of the proof which add to the insight of the (local) structure of both the geometry and the proof. Note that we left out the phrase "no point of Y is collinear to all points of X" in the standing hypthesis, as in this case, we can prove this now.

Proposition 8.1 *Let $x \in X$ be arbitrary. Then there are precisely two members θ^x_1 and θ^x_2 of Θ containing x, and their respective vertices V^x_1 and V^x_2 are disjoint ang generate Y_x. Moreover, $\theta^x_1 \cap \theta^x_2 = \langle x, Y_x\rangle$. Each X-line through x is contained in either θ^x_1 or θ^x_2 and $L_1 \subseteq \theta^x_1$ and $L_2 \subseteq \theta^x_2$ are X-lines containing x, then L_1 and L_2 are non-collinear and $[L_1, L_2] \in \Xi$. Finally, $v = 2v' - 1 = 2r' - 3$.*

Proof This is proven on pages 139-140 of [1], Proposition 7.5.14 until Lemma 7.5.18, first for a point x contained in a 1-line (i.e., contained in a unique member of Θ), and in the end it follows by connectivity that each line is a 1-line. □

Compare θ_1^x and θ_2^x with $\pi^x \in \Pi$ and $\theta^x \in \Theta$ in Corollary 6.6. Let R_1^x be the r'-space $Z(\theta_1^x)$ and note that V_2^x (which is contained in Y_x and hence in θ_1^x) is hence contained in R_1^x. Likewise, $V_1^x \subseteq R_2^x$. An important lemma of [1] is:

Lemma 8.2 (Lemma 7.5.19 of [1]) *For any point $x \in X$, we have $Z = R_1^x \cup R_2^x$ and hence $Y = \langle Y(\theta_1^x), Y(\theta_2^x)\rangle$; moreover, given any point $x \in X$ and renumbering for x' if necessary, $R_1^x = R_1^{x'}$ and $R_2^x = R_2^{x'}$. In particular, $\dim Y = 2r' + 1$.*

Put $R_1 := R_1^x$ for any $x \in X$, likewise define R_2. The set Θ is divided in two classes: Θ_1 is the subset of Θ with $R_1 \subseteq Z(\theta)$, likewise, Θ_2 is defined (and $\Theta = \Theta_1 \cup \Theta_2$). Pages 140-142 of [1] further investigate the structure of Y. We only mention the analogue of Lemma 7.3, as this gives information about the mutual position of two X-points in terms of Y:

Lemma 8.3 (Lemma 7.5.25 of [1]) *Take two distinct points $x_1, x_2 \in X$. Putting $V_x^1 = x^\perp \cap R_2$ and $V_x^2 = x^\perp \cap R_2$ for each $x \in X$:*

1. *x_1x_2 is a singular line with a unique point in $Y \Leftrightarrow V_i^{x_1} = V_i^{x_2}$ for all $i \in \{1, 2\}$;*
2. *x_1, x_2 belong to a member of $\Theta \Leftrightarrow V_i^{x_1} = V_i^{x_2}$ for one $i \in \{1, 2\}$;*
3. *x_1, x_2 are non-collinear points with $[x_1, x_2] \in \Xi Y \Leftrightarrow V_i^{x_1} \neq V_i^{x_2}$ for all $i \in \{1, 2\}$.*

We already know form Proposition 8.1 that for $x \in X$, the subspace Y_x is generated by a v'-space in R_1 and a v'-space in R_2, but Lemma 7.5.22 of [1] shows that the subspace generated by any two such v'-spaces occurs as Y_x for some $x \in X$, moreover, if $Y_x = Y_{x'}$ then $x' \in \langle x, Y_x\rangle$. Also, Corollary 7.5.24 says that for each v'-space V of R_i, there is a unique member of Θ_j, with $\{i, j\} = \{1, 2\}$, such that V is the vertex of Θ_j. One can verify that the latter implies that each point $y \in Y$ can be put in a $\theta \in \Theta$, but not in its vertex, and hence there is a point of $X(\theta)$ not collinear to y. This confirms that we did not need to include this as an assumption in the standing hypothesis.

The analogues of Lemmas 7.9, 6.8, 7.12 and 7.13 can be found in Lemmas 7.5.26, 7.5.27, 7.5.28, 7.5.30 of [1]. We copy the conclusion (with a correction in its statement):

Theorem 8.4 (Theorem 7.5.31 of [1]) *Let (X, Z, Ξ, Θ) be a pre-DSV with with parameters $(1, v, r', v')$ such that $|\Theta_x| \geq 2$ for each $x \in X$ and $|\Theta_L| \in \{0, 1\}$ of each X-line L. Then:*

- *(i) X is the point set of a mutant of the dual Segre variety $\mathcal{DS}_{r',r'}(\mathbb{K})$ with $\langle Z\rangle$ as subspace at infinity and whose symps are given by $\Xi \cup \Theta$;*
- *(ii) if additionally (X, Z, Ξ, Θ) satisfies (S3), then X is the point set of a dual Segre variety $\mathcal{DS}_{2,2}(\mathbb{K})$ (so $r' = 2$) and X is projectively unique.*

The case where $r = 1$ and $|\Theta_x| \geq 2$ for each $x \in X$ and $|\Theta_L| \in \{0, 1\}$ for each X-line L hence leads us to the conclusion of Main Result 3.6(iii). To finish the proof, all we have to do is show that the remaining cases mentioned in Table 1 do not occur.

8.2 Non-examples

Note that for the non-examples, we consider DSVs instead of pre-DSVs, so we do use Axiom (S3). We start with the first column of Table 1.

Proposition 8.5 *There is no DSV (X, Z, Ξ, Θ) with parameters $(3, v, r', v')$.*

Proof Suppose for a contradiction that such a DSV (X, Z, Ξ, Θ) exists. By Lemma 4.14 we may suppose that there is no point of Y collinear to all points of X. Take any $x \in X$. By Proposition 5.16, the point-residue $\mathrm{Res}_X(x)$ is a pre-DSV $(X_x, Z_x, \Xi_x, \Theta_x)$ with parameters $(2, v, r'-1, v')$, where $v = (r'-1) - 3$, containing no Y_x-points collinear to all points of X_x and such that each X_x-line is contained in a unique member of Θ_x. It follows from Proposition 7.15(i) that X_x is the point-set of a mutant of the dual line Grassmannian $\mathcal{DG}_{r'+1,2}(\mathbb{K})$.

Axiom (S3) yields two members $\xi_1, \xi_2 \in \Xi$ such that $T_x = \langle T_x(\xi_1), T_x(\xi_2)\rangle$. Recalling Definition 5.7, this is equivalent to saying that $\langle X_x, Y_x\rangle$ is generated by two members $\tilde{\xi_1}$ and $\tilde{\xi_2}$ of Ξ_x (where $\tilde{\xi_i}$ is the residue of ξ_i from x, $i \in \{1,2\}$). As noted above, $v = r'-4$, so since $r-1 = 2$, we obtain $\dim(\tilde{\xi_i}) = (r'-4)+5+1 = r'+2$ for $i = 1,2$, and hence $\dim\langle X_x, Y_x\rangle \leq 2r'+5$. Moreover, projected from Y_x, we get an injective projection Ω of $\mathcal{G}_{r'+1,2}(\mathbb{K})$ (cf. Proposition 7.12), and we put $B := \dim\langle\Omega\rangle$. So $\dim\langle X_x, Y_x\rangle = B + \dim Y_x + 1 \leq 25'+5$. Since $\dim Y_x = (r'-1)+1$, we obtain $B \leq r'+4$. On the other hand, Ω contains $(r'-1)$-spaces which meet in exactly a point, so $B \geq 2r'-2$. So $2r'-2 \leq r'+4$, meaning that $r' \leq 6$. Recalling that $r' > r = 3$, we get $r' \in \{4,5,6\}$. If $r' = 4$, then $\mathcal{G}_{5,2}(\mathbb{K})$ does not admit legal projections, hence $B = 9$, contradicting $B \leq r'+4 = 8$. Likewise, if $r' \in \{5,6\}$, then Ω contains $\mathcal{G}_{6,2}(\mathbb{K})$ and since the latter contains no legal projection, we have $B \geq 14$, contradicting $B \leq r'+4 \leq 10$. This shows the proposition. $\square$

We proceed to the second column of Table 1, which relies on the dual Segre varieties and hence is not treated in full detail, but we give some hints for the proof though.

Proposition 8.6 (Lemma 7.7.7 in [1]) *There is no DSV (X, Z, Ξ, Θ) with parameters $(2, v, r', v')$ in which $|\Theta_L| \geq 2$ for each X-line L.*

Proof (For a complete proof, we refer to [1]). Suppose for a contradiction that such a DSV (X, Z, Ξ, Θ) exists. Take any $x \in X$. By Proposition 5.16, the point-reside $\mathrm{Res}_X(x) = (X_x, Z_x, \Xi_x, \Theta_x)$ is isomorphic to a mutant of the dual Segre variety $\mathcal{DS}_{r'-1,r'-1}(\mathbb{K})$. A dimension argument as in the previous proof then shows that $\dim\langle X_x, Y_x\rangle \leq 2r'+7$. Since X_x contains a legal projection of $\mathcal{S}_{r'-1,r'-1}(\mathbb{K})$ (cf. Lemma 7.5.30 in[1]) and since $r' > r = 2$, we obtain that X_x contains a copy Ω' of $\mathcal{S}_{2,2}(\mathbb{K})$ (cf. Lemma 6.15). Now, since $\dim\langle\Omega'\rangle = 8$ and $\dim Y_x = 2r'-1$, and $\dim\langle\Omega', Y_x\rangle \leq 2r'-7$, we obtain that there is a point $y \in Y \cap \langle\Omega'\rangle$. A similar argument as used in the last paragraph of the proof of Lemma 6.16 leads to a contradiction to (S2). $\square$

Finally, the third column of Table 1. Also here, we refer to [1].

Proposition 8.7 *There is no DSV (X, Z, Ξ, Θ) with parameters $(1, v, r', v')$ in which $|\Theta_x| \geq 2$ for each $x \in X$ and such that there is an X-line L for which $|\Theta_L| \geq 2$.*

Proof Subsection 7.5.1 of [1] (pages 132-138) deals with a DSV with parameters $(1, v, r', v')$ in which each X-point is contained in at least two members of Θ and where, for each X-line is either contained in no member of Θ or in at least 2, and leads to a contradiction. Hence (X, Z, Ξ, Θ) contains an X-line which is contained in a unique member of Θ. However, Proposition 7.5.14 of [1] (shown in a series of lemmas on pages 139-140 of [1], without using (S3) actually) then implies that each X-line is contained in a unique member of Θ, contradicting our assumption. $\square$

Conclusion. We treated all cases of Table 1 (see Proposition 5.16).

Acknowledgement. Many thanks to the referee, who's helpful comments increased the readabilty of this paper.

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