Maps related to polar spaces preserving an extremal Weyl distance

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Abstract

Let Ω_i and Ω_j be the sets of elements of respective types i and j of a polar space Δ of rank at least 3. We show that a permutation ρ of $\Omega_i \cup \Omega_j$ with the property that, for each $I \in \Omega_i$ and $J \in \Omega_j$, I and J generate a maximal singular subspace in Δ if and only if $\rho(I)$ and $\rho(J)$ generate a maximal singular subspace in Δ, is induced by an automorphism of Δ. Building-theoretically, this means that if ρ preserves a certain Weyl distance in the Tits-building corresponding to Δ, then it preserves all Weyl-distances.

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1 Introduction

The following situation is the central theme of some recent papers [3, 4, 5, 6, 7, 8, 10, 11, 12]: Given a polar space Δ of rank $n \geq 3$ of a certain kind (for instance unitary or symplectic), define some natural (adjacency) relation $\sim$ on the set Ω_s of singular subspaces of dimension s with $0 \leq s \leq n-1$, and determine the automorphism group of the corresponding graph $(\Omega_s, \sim)$, hoping for the automorphism group of Δ, and in case an exception occurs, determine the automorphism group. For a more detailed overview of possibilities for s and $\sim$ and the polar spaces Δ that were considered, we refer to [2]. In [2], an overarching theorem is proven (see Theorem 1.1 below), treating any polar space Δ of (finite) rank $n \geq 3$, any dimension s (more generally, type s, in the hyperbolic case) and with $\sim$ corresponding to *almost* any *Weyl distance* in the building Δ^b related to Δ. The Weyl distances that were left out correspond to 'extremal' Weyl distances in some sense (see below for a more precise statement) and these more wild cases could not be dealt with using the techniques of [2]. In the current paper we partially fill this gap by treating a subcase of these remaining cases. In order to phrase our main theorem we explain this in more detail (preliminaries and terminology regarding polar spaces can be found in Section 2).

The building Δ^b — If Δ is not hyperbolic, then Δ^b (viewed as a simplicial complex) has as vertex set the set of all singular subspaces of Δ, where the *type* of a subspace corresponds to its dimension. If Δ is a hyperbolic polar space of rank n (in which each singular subspace of dimension $n-2$ is contained in exactly two maximal singular subspaces, yielding two natural families of maximal singular subspaces), then the vertex set of Δ^b is the set of singular subspaces of the oriflamme complex of Δ, which consists of subspaces of types $\{0, ..., n-3, (n-1)', (n-1)''\}$, where the subspaces of type d with $0 \leq d \leq n-3$ are the singular subspaces of Δ of dimension d and the subspaces of types $(n-1)'$ and $(n-1)''$ are the subspaces of the two families of maximal singular subspaces of Δ, respectively. To indicate the difference between type and dimension, we sometime denote the dimension of a subspace of type t by $|t|$. We denote the type set of Δ^b by T.

Automorphisms of Δ **and** Δ^b — An automorphism ρ of Δ is a permutation of the point set that preserves collinearity and non-collinearity (ρ and ρ^{-1} preserve singular subspaces). An automorphism ρ of Δ^b is a permutation of all vertices of Δ^b such that ρ and ρ^{-1} preserve incidence. Each automorphism ρ of Δ induces an automorphism of Δ^b. The latter is type-preserving unless Δ is a hyperbolic polar space and ρ is a duality (switching the two natural families of maximal singular subspaces). Only when Δ is a triality quadric, that is, a hyperbolic polar space of rank 4 (so Δ^b has Dynkin type D_4), the building Δ^b has automorphisms (trialities and trialities composed with dualities) which do not correspond to an automorphism of Δ (since points are mapped to 3-dimensional subspaces).

The Weyl distance in Δ^b — In [1], maps between buildings preserving a single Weyl distance between the chambers were studies. In [2], the induced Weyl distance on the vertices of Δ^b is considered. Two such vertices of Δ^b are at a certain Weyl distance precisely if there are $i, j, k, \ell \in \mathsf{T} \cup \{-1\}$ such that the vertices correspond to singular subspaces U, V of Δ of types i and j, and such that the intersection $U \cap V$ has type k (with $k = -1$ if the intersection is empty) and the subspace generated by U and the projection of U on V (in symbols: $\langle U, U^\perp \cap V\rangle$) has type ℓ (cf. Lemma 2.1 of [2]).

Consider the graph which has as vertices all vertices of Δ^b with adjacency given by being at a certain Weyl distance (which we can label with $(i, j; k, \ell)$ in view of the above). When deleting the vertices without neighbors (i.e., those corresponding to vertices of types distinct from i, j), we obtain a bipartite graph if $i \neq j$, which we denote by $\Gamma^n_{i,j;k,\ell}(\Delta)$; if $i = j$, we obtain a graph on Ω_j, which we denote by $\Gamma^n_{j;k,\ell}(\Delta)$. For reasons of uniformity, we also consider a bipartite version of the latter graph, denoted by $\Gamma^n_{j,j;k,\ell}(\Delta)$, whose bipartition classes are given by two copies of Ω_j, and adjacency between those biparts is given by the Weyl distance $(j, j; k, \ell)$ as before. Note that each automorphism of $\Gamma^n_{j;k,\ell}(\Delta)$ induces an automorphism of $\Gamma^n_{j,j;k,\ell}(\Delta)$. If $\Gamma^n_{i,j;k,\ell}(\Delta)$ or its bipartite complement (changing edges and non-edges between the biparts) is either empty or a matching, then we call it trivial. A brief version of the main result of [2] reads as follows:

Theorem 1.1 (Theorem 3.5 of [2], informal statement) *Let Δ be a polar space of rank $n \geq 3$ with type set T. Suppose $i, j, \ell \in \mathsf{T}$ and $k \in \mathsf{T} \cup \{-1\}$ are such that if $|\ell| = n-1$, then also $|i| = |j| = n-1$. If $\Gamma = \Gamma^n_{i,j;k,\ell}(\Delta)$ is non-trivial, then each automorphism of Γ is induced by a (not necessarily type-preserving) automorphism of the building Δ^b related to Δ, up to two classes of exceptions where there are also automorphisms of a building naturally containing Δ^b (cf. Examples 3.2 and 3.3 of [2]).*

The above condition 'if $|\ell| = n-1$, then also $|i| = |j| = n-1$', we believe, is merely there because the proof requires a different method. Small as this case might seem, compared to the number of other cases, it is not trivial to get rid of this case, even with alternative methods. In the current paper, we tackle one of the remaining cases where $|\ell| = n-1$, requiring additionally that $|k| = |i| + |j| - n + 1$. Geometrically, this means that $I \in \Omega_i$ and $J \in \Omega_j$ are adjacent if they generate a maximal singular subspace of Δ, where the latter is of a certain type in case Δ is hyperbolic. We called this an 'extremal' Weyl distance because the projection of I on J is maximal when $|\ell| = n-1$. We show that, also here, if $\Gamma := \Gamma^n_{i,j;k,\ell}(\Delta)$ is non-trivial, then each of its automorphisms is induced by an automorphism of Δ^b (without exceptions).

Denote by $\mathsf{Aut}_c(\Gamma)$ the automorphisms of Γ preserving the biparts of Γ. We show that:

Theorem 1.2 *Let Δ be a polar space of rank $n \geq 3$ with type set T. Suppose $i, j, \ell \in \mathsf{T}$ and $k \in \mathsf{T} \cup \{-1\}$ are such that $|i| + |j| - |k| = n - 1 = |\ell|$. Then $\Gamma := \Gamma^n_{i,j;k,\ell}(\Delta)$ is non-trivial if $\min\{|i|, |j|\} < n-1$ and in that case, each element ρ of $\mathrm{Aut}_c(\Gamma)$ is induced by an automorphism $\tilde{\rho}$ of the building Δ^b related to Δ, i.e., $\rho(X) = \tilde{\rho}(X)$ for each $X \in \Omega_i \cup \Omega_j$. More specifically:*

- *(i) If Δ is not hyperbolic or $\max\{|i|, |j|\} < n-1$, then $\tilde{\rho}$ is an automorphism of Δ (including dualities, if any, but excluding trialities, if any), and conversely, each automorphism of Δ induces an element of $\mathrm{Aut}_c(\Gamma)$;*
- *(ii) If Δ is hyperbolic and $\max\{|i|, |j|\} = n - 1 \geq 3$, with $\{|i|, |j|\} = \{0, 3\}$ if $n = 4$, then $\tilde{\rho}$ is type-preserving, i.e., it is an automorphism of Δ which is not a duality. Conversely, each type-preserving automorphism of Δ^b induces an element of $\mathrm{Aut}_c(\Gamma)$;*
- *(iii) If Δ is the triality quadric (i.e., Δ is hyperbolic and $n = 4$) and $\{|i|, |j|\} = \{1, 3\}$, then $\tilde{\rho}$ is type-preserving, or it switches types 0 and t but preserves types i and j, with $\{0, t, i, j\} = \{0, 1, 3', 3''\}$. Conversely, each such $\tilde{\rho}$ induces an element of $\mathrm{Aut}_c(\Gamma)$.*

If $i = j$, or, if $i \neq j$ but Δ^b has an automorphism switching the biparts (which is the case only if Δ is the triality quadric and $\{|i|, |j|\} = \{0, 3\}$), then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}_c(\Gamma) \times 2$.

The geometric nature of our arguments allows to also consider the graphs on $\Omega_i \cup \Omega_j$ in which $I \in \Omega_i$ and $J \in \Omega_j$ are adjacent if they generate just any maximal singular subspace in case Δ is hyperbolic, a relation which does not correspond to a certain Weyl distance but which fits in the same framework nonetheless. It goes without saying that in this case we use $\mathsf{T} = \{0, 1, ..., n-2, n-1\}$ instead of $\mathsf{T} = \{0, 1, ..., n-3, (n-1)', (n-1)''\}$, and hence $\ell = n-1$ instead of $\ell \in \{(n-1)', (n-1)''\}$. This gives us the following theorem:

Theorem 1.3 *Let Δ be a hyperbolic polar space of rank $n \geq 3$. Suppose $i, j, \ell \in \{0, 1, ..., n-1\}$ and $k \in \mathsf{T} \cup \{-1\}$ are such that $i + j - k = \ell = n-1$. Then $\Gamma := \Gamma^n_{i,j;k,\ell}(\Delta)$ is non-trivial if $\min\{i, j\} < n-1$ and in that case, each element ρ of $\mathrm{Aut}_c(\Gamma)$ is induced by an automorphism $\tilde{\rho}$ of the building Δ^b related to Δ. If $i = j$, then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}_c(\Gamma) \times 2$.*

Theorems 1.2 and 1.3 imply the following result for the non-bipartite counterparts of $\Gamma^n_{i,j;k,\ell}(\Delta)$ in the case where $i = j$ (where we also include $\ell = n-1$ in the hyperbolic case, just like in the above theorem).

Corollary 1.4 *Let Δ be a polar space of rank $n \geq 3$ with type set T. Suppose $j, \ell \in \mathsf{T}$ and $k \in \mathsf{T} \cup \{-1\}$ are such that $2|j| - |k| = n-1 = |\ell|$. Then $\Gamma := \Gamma^n_{j;k,\ell}(\Delta)$ is non-trivial if $|j| < n-1$ and in that case, each element ρ of $\mathrm{Aut}(\Gamma)$ is induced by an automorphism $\tilde{\rho}$ of the building Δ^b related to Δ.*

Proof. Taking two isomorphic copies of Ω_j, it is easily seen that there is a canonical isomorphism between $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}_c(\Gamma^n_{j,j;k,\ell}(\Delta))$, and hence the result follows from Theorems 1.2 and 1.3. $\square$

Remark 1.5 Note that, if $|i| = |j| = n-1$, then $|i| + |j| - |k| = n-1 = |\ell|$ implies $|k| = n-1$, so adjacency in $\Gamma^n_{i,j;k,\ell}(\Delta)$ is given by equality, and hence the graph is trivial. So the case where $|i| = |j| = |\ell|$, which was covered by Theorem 1.1 in more generality, is naturally excluded here. If $\max\{|i|, |j|\} = n-1$, then $|i| + |j| - |k| = n-1 = |\ell|$ implies that $I \in \Omega_i$ and $J \in \Omega_j$ are adjacent vertices of $\Gamma^n_{i,j;k,\ell}(\Delta)$ precisely if $I \cap J$ is either I or J, i.e., the adjacency relation is given by (symmetrized) containment. This case is covered by Proposition 7.4. of [2]; nevertheless we mention it in our statement for clarity. For the sequel, we hence assume $\max\{|i|, |j|\} < n-1$. Observe that this implies that $|k| < \min\{|i|, |j|\}$, and that we may omit the 'absolute value signs' for $|k|, |i|, |j|$ as dimension and type coincide now.

Proof strategy — Denote by C_1 and C_2 the two biparts of $\Gamma = \Gamma^n_{i,j;k,\ell}(\Delta)$. For any $X \in C_1 \cup C_2$, we write $\mathsf{N}(X)$ for the set of its neighbors in Γ (note that $X \notin \mathsf{N}(X)$). If

$X_1, X_2 \in C_1 \cup C_2$ then $\mathsf{N}(X_1, X_2)$ denotes $\mathsf{N}(X_1) \cap \mathsf{N}(X_2)$, i.e., it is the set of all common neighbors of X_1 and X_2. Our strategy consists of investigating pairs X_1, X_2 in C_1 or in C_2 with $\mathsf{N}(\mathsf{N}(X_1, X_2)) = \{X_1, X_2\}$. It turns out that this is only the case when either X_1 and X_2 are contained in a common singular subspace, or, for some specific polar spaces Δ, also if $X_1 \cap X_2$ is a hyperplane of both X_1 and X_2 (cf. Lemma 3.10). With some additional work, we can in most cases determine whether or not two subspaces of the same bipart are collinear, which will then allow us to express the maximal singular subspaces in terms of the i-spaces or j-spaces they contain. This leads to a graph from which the automorphism group is known. In the remaining cases, we deduce a Grassmann graph from Γ, the automorphism group of which is also known.

In this approach, but also in view of the statement that no automorphism of Γ can switch the biparts if $i \neq j$, it is important that we can tell the biparts apart. In most cases this can be done geometrically, except for the case where $s = 2$, where we need to rely on further calculations. For this we rely on [9], a note concerning the graphs occurring in this paper, but containing the calculations for any finite order (s, t).

2 Preliminaries

Below we list all the notions and terminology used throughout the paper.

A pair $\Delta = (X, \mathcal{L})$ is a *point-line geometry* if X is a set and $\mathcal{L}$ is a set of subsets of X of size at least 2 covering X; the elements of X are called points and those of $\mathcal{L}$ lines. Two points p, q of X which are on a common line $\mathcal{L}$ are called *collinear*, denoted $p \perp q$; the set of points collinear to p is denoted by $p^\perp$. A *subspace* S of Δ is a subset of X such that the lines joining any two collinear points of S are contained in S. Moreover, if all pairs of points in S are collinear, the subspace is called *singular*.

A *polar space* is either a set of points X of size at least 3, or a point-line geometry $\Delta = (X, \mathcal{L})$ with the properties that, each line has at least three points, no point is collinear to all other points, every nested sequence of singular subspaces is finite, and, finally, for each point x and any line $L \in \mathcal{L}$ either one or all points of L are collinear to x.

A polar space as considered above has a well-defined finite *rank* $n \geq 1$, which is one more than the dimension of any maximal singular subspace of Δ (so it is 1 if the line set is empty). Since we assume $n \geq 3$, Δ has an *order* (s, t), which means that each line of Δ contains $s + 1$ points (possibly $s = \infty$) and through each singular subspace of dimension $n - 2$, there are $t + 1$ singular subspaces of dimension $n - 1$ (possibly $t = \infty$). If $t = 1$, then we say that Δ is *hyperbolic* or *thin*. In this case there are two natural families of maximal singular subspaces, which we will refer to as the $(n-1)'$-spaces and the $(n-1)''$-spaces, as mentioned in the introduction. Finally, we note that it follows from the classification of polar spaces of rank $n \geq 3$ that if $s < \infty$ then also $t < \infty$.

Two singular subspaces U, V of Δ are called *collinear* (denoted $U \perp V$) if they are contained in a common singular subspace and the smallest such subspace is denoted by $\langle U, V\rangle$ and referred to as the singular subspace generated or spanned by U and V (with obvious generalization to a higher number of pairwise collinear singular subspaces). The set of all points collinear to (all points of) U is denoted by $U^\perp$. The *projection* $\mathsf{proj}_V(U)$ of a singular subspace U on a singular subspace V is $V \cap U^\perp$. We have $\mathsf{proj}_V(U)$ contains $U \cap V$ and $\dim U - \dim \mathsf{proj}_U(V) = \dim V - \dim \mathsf{proj}_V(U)$. We say that U and V are *opposite* if $\mathsf{proj}_V(U) = \mathsf{proj}_U(V) = \emptyset$, in which case $\dim U = \dim V$. In particular, two non-collinear points of X are opposite. The subspace spanned by U and $\mathsf{proj}_V(U)$ is denoted by U^V (note that $\dim(U^V) = \dim(V^U)$).

If p, q are opposite points, then the set of points in $(p^\perp \cap q^\perp)^\perp$ is called the *hyperbolic line* containing p and q. All hyperbolic lines of Δ have the same size, say $h(\Delta)$, called the *hyperbolic order*. Clearly, $p, q \in (p^\perp \cap q^\perp)^\perp$ and hence $h(\Delta) \geq 2$. If $h(\Delta) > 2$, we call the hyperbolic line *proper*. Note that if Δ is a hyperbolic polar space, then $h(\Delta) = 2$.

Let Ω be the set of singular subspaces of Δ and consider a $K \in \Omega$ with $\dim(K) = k \leq n-2$. Put $X_K = \{U \in \Omega \mid K \subset U, \dim(U) = k+1\}$. First suppose $k < n-2$. Then, if M is an element of Ω of dimension $k+2$ containing K, we let M/K represent the elements of X_K contained in M. We define $\mathcal{L}_K$ as $\{M/K \mid K \subseteq M \in \Omega, \dim(M) = k+2\}$ and $\mathrm{Res}_\Delta(K) = (X_K, \mathcal{L}_K)$ is the *residue of* Δ *in* K. This is a polar space of rank $n-k-1$ of the same "kind" as Δ, e.g. the residue of a parabolic polar space is parabolic too, and likewise for hyperbolic, unitary, mixed and so on. If $\dim(K) = n-2$, then $\mathrm{Res}_\Delta(K)$ is just the set X_K, which is a polar space of rank 1. It contains at least 2 points and it contains precisely 2 if and only if Δ is hyperbolic.

3 Pairs $J_1, J_2 \in C_2$ with $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$

Let J_1, J_2 be distinct members of Ω_j. We write $S := J_1 \cap J_2$ and $D := J_1^{J_2} \cap {J_2}^{J_1}$, and let λ_1 be the type of $J_1^{J_2}$ and λ_2 the type of $J_2^{J_1}$ (note that $|\lambda_1| = |\lambda_2|$), $\alpha := |\lambda_1| - j - 1$, $s := \dim S$ and $\beta := j - (\alpha + s + 1) - 1$. Let $\mathcal{N}_1$ denote the set of all singular subspaces of type ℓ containing ${J_1}^{J_2}$, likewise we define $\mathcal{N}_2$. For $N_1 \in \mathcal{N}_1$ arbitrary, note that $N_2 = J_2^{N_1}$ is the unique element of $\mathcal{N}_2$ such that $N_1 \cap N_2$ is maximal ($N_1 \cap J_2^{N_1}$ has dimension $n - \beta - 2$).

The following lemma describes the composition of an element in $\mathsf{N}(J_1, J_2)$, if non-empty, see also Figure 1.

Lemma 3.1 *The set* $\mathsf{N}(J_1, J_2)$ *is non-empty if and only if* $k \leq s + \alpha + 1$ *and, in case* Δ *is hyperbolic and* $\ell \in \{(n-1)', (n-1)''\}$, *also if* β *is odd and either* $\dim({J_1}^{J_2}) < n-1$ *or* ${J_1}^{J_2} \in \mathcal{N}_1$.*If non-empty, then each* $I \in \mathsf{N}(J_1, J_2)$ *is generated by singular subspaces* K_0, K_1 *and* K_2 *as follows. The subspaces* K_1 *and* K_2 *are any* k*-subspaces in* $J_1 \cap D$ *and* $J_2 \cap D$*, respectively, with* $K_1 \cap S = K_2 \cap S$*. The subspace* K_0 *is any subspace complementary to both* $\langle \mathsf{proj}_{J_1}(J_2), K_2\rangle$ *and* $\langle \mathsf{proj}_{J_2}(J_1), K_1\rangle$ *in the* $(n-\beta-2)$*-space* $N_1 \cap J_2^{N_1}$*, where* N_1 *is any element of* $\mathcal{N}_1$*.*

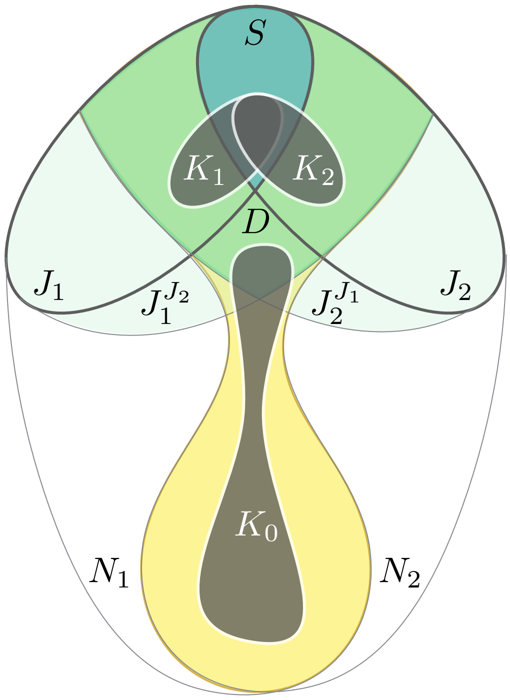

Figure 1: An element $\langle K_0, K_1, K_2\rangle$ of $\mathsf{N}(J_1, J_2)$

Proof. Suppose $I \in \mathsf{N}(J_1, J_2)$. Then for $c \in \{1, 2\}$, the subspace $I \cap J_c$ is a k-space K_c, and I and J_c generate an ℓ-space N_c. Obviously, $K_1 \cap S = K_2 \cap S$. Now $J_1 \perp I \perp J_2$ implies that $K_1 \subseteq \mathsf{proj}_{J_1}(J_2) = J_1 \cap D$ and $K_2 \subseteq \mathsf{proj}_{J_2}(J_1) = J_2 \cap D$. As such, $k \leq \dim \mathsf{proj}_{J_1}(J_2) = \dim \mathsf{proj}_{J_2}(J_1) = s + \alpha + 1$. Since $N_1 = \langle J_1, I\rangle$ has dimension $n-1$ and $J_1^{J_2}$ is collinear to J_1 and I, and thus to N_1, we have $J_1^{J_2} \subseteq N_1$ and hence $N_1 \in \mathcal{N}_1$; likewise, $N_2 = \langle I, J_2\rangle \in \mathcal{N}_2$. From $I \subseteq N_1 \cap J_2^{\perp}$ we deduce $N_2 = \langle I, J_2\rangle \subseteq \langle N_1 \cap J_2^{\perp}, J_2\rangle$, and since the dimensions coincide, $N_2 = J_2^{N_1}$. Noting that $\dim(N_1 \cap N_2) = \dim(N_1 \cap J_2^{\perp}) = n - \beta - 2$ and that N_1 and N_2 are both of type ℓ if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, we obtain that β is odd in this case and moreover, if $\dim({J_1}^{J_2}) = n-1$, then $N_1 = N_2 = {J_1}^{J_2} \in \mathcal{N}_1$.

Recall that $I \subseteq N_1 \cap J_2^{N_1} = J_1^{N_2} \cap N_2 = N_1 \cap N_2$. Let K_0 be a complement of $\langle K_1, K_2\rangle$ in I and, for $\{c, c'\} = \{1, 2\}$, let $K_c' \subseteq K_c$ be a complement of $K_{c'}$ in $\langle K_1, K_2\rangle$. Then $\langle K_0, K_c'\rangle$ is a complement of $J_{c'}$ in $N_{c'}$, hence it is also a complement of $\mathrm{proj}_{J_{c'}}(J_c)$ in $N_1 \cap N_2$, as one can see by checking that dimensions fit with these claims. Consequently K_0 is a complement of $\langle \mathrm{proj}_{J_{c'}}(J_c), K_c\rangle$ in $N_1 \cap N_2$.

Conversely, suppose $k \leq s+\alpha+1$ and β is odd in case Δ is hyperbolic. We take N_1, K_0, K_1, K_2 as described in the statement of this lemma and $N_2 = J_2^{N_1}$ (note that this is possible because of the conditions: K_1 and K_2 fit into $J_1 \cap D$ and $J_2 \cap D$, respectively, and N_1 and N_2 are of the same type). By choice of K_1 and K_2 we have that $\langle K_1, K_2 \rangle \cap J_c$ is the k-space K_c for $c \in \{1, 2\}$. Since $\dim D \leq \dim(N_1 \cap N_2)$, it follows that $\dim(\langle K_1, K_2 \rangle) \leq i$. By choice of K_0, we have $\langle K_0, K_1, K_2, J_c \rangle = N_c$ with $\{c, c'\} = \{1, 2\}$; moreover, K_0 is complementary to $\langle \mathsf{proj}_{J_c}(J_{c'}), K_{c'} \rangle$ in N_c and hence $\langle K_0, K_1, K_2 \rangle \cap J_c = \langle K_1, K_2 \rangle \cap J_c = K_c$. So $\langle K_0, K_1, K_2 \rangle \in \mathsf{N}(J_1, J_2)$ indeed. The lemma follows. $\square$

In view of the above, we say that J_1 and J_2 are ℓ-collinear (denoted $J_1 \hat{\perp} J_2$) if there is an element $N \in \mathcal{N}_1$ (so a subspace of type ℓ) with $J_1 \cup J_2 \subseteq N$. If $\ell = n-1$, this coincides with ordinary collinearity between J_1 and J_2, but if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$ then there exist collinear j-spaces J_1, J_2 with $J_1 \perp J_2$ which are not ℓ-collinear (if the singular subspace they generate has type ℓ', where $\{\ell, \ell'\} = \{(n-1)', (n-1)''\}$.

Next, we study $\mathsf{N}(\mathsf{N}(J_1, J_2))$ in case J_1 and J_2 are ℓ-collinear. Note that $\mathsf{N}(J_1, J_2)$ is non-empty in this case since $\beta = -1$ is odd, $J_1 \hat{\perp} J_2$ and $k \leq j = s+\alpha+1$, so $\mathsf{N}(\mathsf{N}(J_1, J_2))$ is a subset of Ω_j (whereas $\mathsf{N}(\emptyset) = \Omega_i \cup \Omega_j$). Clearly, $J_1, J_2 \in \mathsf{N}(\mathsf{N}(J_1, J_2))$. Our first goal is to show that $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$ provided that $J_1 \perp J_2$. We do this in a couple of lemmas.

Lemma 3.2 *Let J_1, J_2 be j-spaces with $J_1 \hat{\perp} J_2$. Then each $J \in \mathsf{N}(\mathsf{N}(J_1, J_2))$ belongs to $\langle J_1, J_2 \rangle$.*

Proof. In this case, $\mathcal{N}_1 = \mathcal{N}_2$ is the set of all ℓ-spaces containing $\langle J_1, J_2 \rangle$. By Lemma 3.1, each $I \in \mathsf{N}(J_1, J_2)$ generates a member of $\mathcal{N}_1$ together with J_1 or J_2. Take any $N \in \mathcal{N}_1$. Let p be a point contained in $N \setminus (J_1 \cup J_2)$. Lemma 3.1 implies that there is an $I \in \mathsf{N}(J_1, J_2)$ with $p \in I$ (either by choosing K_1 and K_2 such that $p \in \langle K_1, K_2 \rangle$ or by choosing K_0 such that $p \in K_0$). So, if $J \in \mathsf{N}(\mathsf{N}(J_1, J_2))$, then $J \perp I$, in particular $J \perp p$. Therefore, $J \perp \langle N \setminus (J_1 \cup J_2) \rangle = N$, and by maximality of N, we obtain $J \subseteq N$. As $N \in \mathcal{N}_1$ was arbitrary, $J \subseteq \langle J_1, J_2 \rangle$. $\square$

Lemma 3.3 *Let J_1, J_2 be j-spaces with $J_1 \hat{\perp} J_2$. Then each $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$ strictly contains $\langle J \cap J_1, J \cap J_2 \rangle$.*

Proof. Suppose for a contradiction that there is a j-space $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$ with $J = \langle J \cap J_1, J \cap J_2 \rangle$. Put $S_c := J \cap J_c$, $c \in \{1, 2\}$. Since J_1, J_2 generate $\langle J_1, J_2 \rangle$ and $J \subseteq \langle J_1, J_2 \rangle$, necessarily $\dim S \leq S_c$ for $c = \{1, 2\}$. Possibly by switching the roles of J_1 and J_2, we have $\dim S \leq \dim S_1 \leq \dim S_2$, since $J = \langle S_1, S_2 \rangle \subseteq \langle J_1, J_2 \rangle$. Note that $S \cap J = S \cap S_1 = S \cap S_2$ is a strict subspace of S_c, $c = 1, 2$, since $J_1 \neq J \neq J_2$. We use Lemma 3.1 to find an i-space $I = \langle K_1, K_2, K_0 \rangle \in \mathsf{N}(J_1, J_2)$ with $I \notin \mathsf{N}(J)$, distinguishing three cases:

- *Case 1:* $\dim S_2 \leq k$. Choose K_c such that it contains S_c, $c = 1, 2$. Then $\langle K_1, K_2 \rangle \cap J = J$ and hence $\dim(I \cap J) \geq j > k$, regardless of the choice of K_0.

- *Case 2:* $\dim S_2 > k \geq 0$. Take K_2 such that it is contained in S_2 but not contained in S and K_1 such that it shares at least a point with $S_1 \setminus S$ (which is possible since $k \geq 0$). As above, $\dim(I \cap J) \geq \dim(\langle K_1, K_2 \rangle \cap J) \geq k + 1$.

- *Case 3:* $\dim S_2 > k = -1$. Since $k = -1$, $\dim I = \dim K_0 \geq 0$. So we can select K_0 such that it shares at least a point with $J \setminus (J_1 \cup J_2) = \langle S_1, S_2 \rangle \setminus (S_1 \cup S_2)$. Then $\dim(I \cap J) \geq 0 > k$.

In each case, we obtained an $I \in \mathsf{N}(J_1, J_2)$ with $I \notin \mathsf{N}(J)$, so the hypothesis $J = \langle S_1, S_2 \rangle$ must be false. $\square$

Proposition 3.4 *Let J_1, J_2 be j-spaces with $J_1 \hat{\perp} J_2$. Then $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$.*

Firstly, as noted before, $J_1, J_2 \in \mathsf{N}(\mathsf{N}(J_1, J_2))$. We prove the other direction of this proposition in a series of lemmas. Henceforth we suppose for a contradiction that J is a j-space in $\mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$. Our aim is to find an i-space $I \in \mathsf{N}(J_1, J_2)$ with $\dim(I \cap J) \neq k$ and hence $I \notin \mathsf{N}(J)$. We use the construction of $I = \langle K_1, K_2, K_0 \rangle \in \mathsf{N}(J_1, J_2)$ as given in Lemma 3.1. By Lemma 3.2 however, we know $J \subseteq \langle J_1, J_2 \rangle$, so we first prove a refinement of Lemma 3.1 to show that we can restrict our attention to $\langle J_1, J_2 \rangle$.

Lemma 3.5 *Let J_1, J_2 be j-spaces with $J_1 \hat{\perp} J_2$, and $M = \langle J_1, J_2 \rangle$. Then, for any pair of k-spaces K_1, K_2 in J_1, J_2 with $K_1 \cap S = K_2 \cap S$ and any subspace K_0' in M complementary to $\langle J_1, K_2 \rangle$ and $\langle J_2, K_2 \rangle$ in M, the subspace $\langle K_1, K_2, K_0' \rangle$ is contained in at least one member of $\mathsf{N}(J_1, J_2)$. Conversely, for each $I \in \mathsf{N}(J_1, J_2)$, $I_M := I \cap M$ can be decomposed in this way and $\dim(I_M \cap J_c) = k$ and $\langle I_M, J_c \rangle = M$.*

Proof. By Lemma 3.1, each member I of (the non-empty set) $\mathsf{N}(J_1, J_2)$ is given as $\langle K_1, K_2, K_0 \rangle$, where K_1, K_2 are as described in the statement of this lemma, and given any ℓ-space N_1 containing M, K_0 is a subspace in N_1 complementary to $\langle J_1, K_2 \rangle$ and $\langle J_2, K_1 \rangle$ in N_1. Hence $K_0' := K_0 \cap M$ is a subspace complementary to $\langle J_1, K_2 \rangle$ and $\langle J_2, K_1 \rangle$ in M. Since $J_c \sim I$ implies $\dim(J_c \cap I) = k$ and $\langle J_c, I \rangle = N_1$, for $c \in \{1, 2\}$ and $J_c \subseteq M$, this implies that $J_c \cap I = J_c \cap I_M$ and a dimension argument gives $\langle J_c, I_M \rangle = M$. Conversely, each subspace K_0' in M complementary to $\langle J_1, K_2 \rangle$ and $\langle J_2, K_1 \rangle$ in M can be extended to a subspace K_0 complementary to $\langle J_1, K_2 \rangle$ and $\langle J_2, K_1 \rangle$ in N_1. $\square$

Notation. We keep using the notation $M = \langle J_1, J_2 \rangle$ and $I_M = \langle K_1, K_2, K_0' \rangle$, with notation as above. Recall that we assume that $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$. Put $S_c := J_c \cap J$, $c \in \{1, 2\}$ and let S_J be a subspace in J complementary to $\langle S_1, S_2 \rangle$ in J.

By Lemma 3.3, S_J is non-empty. Possibly by switching the roles of J_1 and J_2, we again have $\dim S \leq \dim S_1 \leq \dim S_2$. We first treat the case where $\dim S_2 \geq k$.

Lemma 3.6 *If* $\dim S_2 \geq k$, *then* $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$.

Proof. Suppose first that $\dim(S \cap J) \geq k$ (with J as above). Choose $K_1 = K_2 \subseteq S \cap J$. Then K_0' is non-empty and can be chosen to share at least a point with S_J, implying that $\dim(I_M \cap J) \geq k+1$. Lemma 3.5 implies that there is an $I \in \mathsf{N}(J_1, J_2)$ with $I_M \subseteq I$, and hence $\dim(I \cap J) > k$. Therefore, $J \nsim I$, a contradiction.

Next, suppose $\dim(S \cap J) < k$. Then we choose K_2 in S_2 such that it (strictly) contains $S \cap J$. If $S_1 \nsubseteq S$ then we choose K_1 such that it shares at least a point with $S_1 \setminus S$ (note that $k \geq 0$ in this case) and then again $\dim(I_M \cap J) \geq \dim(\langle K_1, K_2 \rangle \cap J) \geq k+1$. So we additionally suppose that $S_1 \subseteq S$, and since $\dim S \leq \dim S_1$, this means that $S = S_1 \subseteq S_2$. In particular, $S \cap J = S$ and, as $\dim(S \cap J) < k \leq \dim S_2$ by assumption, S is strictly contained in S_2 and hence $\langle J, J_2 \rangle \cap J_1 \subsetneq J_1$. Therefore, and since $k < j$, we can choose K_1 as any k-space of J_1 containing S but not containing $\langle J, J_2 \rangle \cap J_1$. This choice implies that $S_J \nsubseteq \langle K_1, J_2 \rangle$ (indeed, otherwise $J \subseteq \langle K_1, J_2 \rangle$ and hence $\langle J, J_2 \rangle \cap J_1 \subseteq \langle K_1, J_2 \rangle \cap J_1 = K_1$). On the other hand, since $S = S_1$, we have $\langle J, J_1 \rangle = M$, so S_J is not contained in $\langle J_1, K_2 \rangle$ (otherwise $J \subseteq \langle J_1, K_2 \rangle$ and so $M = \langle J, J_1 \rangle \subseteq \langle J_1, K_2 \rangle \subsetneq M$, the latter because $S \cap J = S \subsetneq K_2 \subsetneq J_2$). So, by the foregoing, the subspace K_0' is non-empty and can be chosen such that it intersects $S_J \setminus (\langle K_1, J_2 \rangle \cup \langle J_1, K_2 \rangle)$ non-trivially. We conclude that $\dim(I_M \cap J) > k$ and obtain the same contradiction as in the previous paragraph. $\square$

Next, we treat the case where $\dim S_2 < k$.

Lemma 3.7 *If* $\dim S_2 < k$ *and* $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$, *then* $\dim\langle S_1, S_2 \rangle \leq k$. *Moreover, there are k-spaces K_1, K_2 in J_1, J_2, respectively, with $\langle S, S_i \rangle \subseteq K_i$, and for such k-spaces,* $\dim(\langle K_1, K_2 \rangle \cap J) \leq k$.

Proof. Since $\dim S_1 \leq \dim S_2 < k$, we may choose $K_c \supset S_c$ for $c = 1, 2$. If $\dim\langle S_1, S_2 \rangle > k$ then $\dim(I_M \cap J) > k$ regardless of the choice of K_0', contradicting $J \in \mathsf{N}(J_1, J_2)$. Therefore, $\dim\langle S_1, S_2 \rangle \leq k$. Then also $\dim\langle S, S_c \rangle \leq k$, since $\dim S \leq \dim S_c$ and $S \cap S_c = S_1 \cap S_2$. Accordingly, for $c = 1, 2$ we can indeed choose K_c in such a way that $K_c \supseteq \langle S, S_c \rangle$. If $\dim(\langle K_1, K_2 \rangle \cap J) > k$ for at least one such choice, then again we obtain an I_M with $\dim(I_M \cap J) > k$, contradicting $J \in \mathsf{N}(J_1, J_2)$ once more. We conclude that $\dim(\langle K_1, K_2 \rangle \cap J) \leq k$ for any pair of such k-spaces K_1 and K_2. $\square$

Henceforth, we suppose that K_c contains $\langle S, S_c \rangle$, and we write $X := J \cap \langle K_1, K_2 \rangle$ for short. We show that, under our assumptions, $\dim X = k$.

Lemma 3.8 *If* $\dim S_2 < k$ *and* $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$*, then* $\dim X = k$*.*

Proof. By Lemma 3.7, $\dim X \leq k$, so suppose for a contradiction that $\dim X < k$. Put $x = \dim X$. Take any $I \in \mathsf{N}(J_1, J_2)$ containing $\langle K_1, K_2 \rangle$ and consider $I_M = \langle K_1, K_2, K_0' \rangle$ (cf. Lemmas 3.1 and 3.5). Since $I \sim J$ by assumption, $\dim(I_M \cap J) = k$. As K_0' is just any complement of $\langle K_1, K_2 \rangle$ in I_M, it does not affect I_M if we choose K_0' such that $K_0' \cap J$ is a complement of X (in which case $I_M \cap J = \langle X, K_0' \cap J \rangle$ and $\dim(K_0' \cap J) = k - x - 1$). Note that, since $S = K_1 \cap K_2$, we also know that $\dim K_0' = j - k - 1$. The fact that $I \sim J$ implies that I_M and J generate the $(2j - s)$-space M, and hence

$$\dim(K_0' \cap \langle J, K_1, K_2 \rangle) = (j - k - 1) + (j + (2k - s) - x) - (2j - s) = k - x - 1.$$

Comparing this to $\dim(K_0' \cap J)$, we obtain $K_0' \cap \langle J, K_1, K_2 \rangle = K_0' \cap J$.

We claim that $K_0' \cap J = K_0'$. So suppose for a contradiction that $K_0' \cap J \subsetneq K_0'$. A dimension argument yields that the subspace $\langle K_1, K_2, K_0' \cap J \rangle$ (which has dimension $(2k-s)+(k-x-1)+1$) is strictly contained in $\langle K_1, K_2, J \rangle$ (which has dimension $j + (2k - s) - x$), because $k < j$. Therefore we can form K_0' by adding to $K_0' \cap J$ a suitable (non-empty) subspace K_0'' of I_M which meets $\langle K_1, K_2, K_0' \cap J \rangle$ trivially but $\langle K_1, K_2, J \rangle$ non-trivially. With such a choice of K_0' (which does not affect the previous choice of $K_0' \cap J$) we get $K_0' \cap \langle J, K_1, K_2 \rangle \supsetneq K_0' \cap J$, a contradiction with the above. The claim follows.

The above implies that $\dim(K_0' \cap J) = \dim(K_0')$, and hence $k - x - 1 = j - k - 1$, so $2k - x = j$. Using this, it follows that $I_M = \langle K_1, K_2, K_0' \rangle$ has dimension $(2k-s)+(k-x-1)+1 = 3k-s-x = j + k - s > j$. We claim that we can choose a subspace I_M' with $\dim I_M' = \dim I_M$ such that I_M' contains J and such that $\langle J_c, I_M' \rangle = M$ for $c = 1, 2$ (and hence $\dim(J_c \cap I_M') = k$).

Recall that $\dim S_1 \leq \dim S_2 < k$. Take a subspace S_1' of dimension $\dim S_2$ with $S_1 \subseteq S_1' \subseteq K_1$ such that $S_1 \cap S = S_2 \cap S$ (this is possible since $S_1 \cap S = S_2 \cap S$). Consider $\mathrm{Res}_M \langle S_1', S_2 \rangle$, in which J_1 and J_2 correspond to subspaces J_1' and J_2' of equal dimension, and $\langle J, S_1' \rangle$ corresponds to a subspace J' which is disjoint from both J_1' and J_2'. Let Y be any subspace of $\mathrm{Res}_M \langle S_1', S_2 \rangle$ which contains J' and is complementary to J_1' and J_2'. Back in M, let $\tilde{Y}$ be the subspace containing $\langle S_1', S_2 \rangle$ which corresponds to Y in $\mathrm{Res}_M \langle S_1', S_2 \rangle$. Then, by choice of $\tilde{Y}$, we have: $J \subseteq \tilde{Y}$, $\langle J_1, \tilde{Y} \rangle = \langle J_2, \tilde{Y} \rangle = M$, $J_1 \cap \tilde{Y} = S_1'$, $J_2 \cap \tilde{Y} = S_2$, and since $\dim S_1' = \dim S_2 < k$ by assumption, we also know $\dim \tilde{Y} < \dim I_M$. Now let I_M' be any subspace of dimension $\dim I_M$ containing $\tilde{Y}$ (and hence also J). Then $M = \langle J_c, \tilde{Y} \rangle \subseteq \langle J_c, I_M' \rangle$ and hence I_M' is as claimed.

Lemma 3.5 implies that there is a member $I' \in \mathsf{N}(J_1, J_2)$ with $I_M' \subseteq I'$. However, $I' \nsim J$ since $\dim(I' \cap J) = \dim(I_M' \cap J) = j > k$, contradicting our assumption $J \in \mathsf{N}(\mathsf{N}(J_1, J_2))$. This shows the lemma. $\square$

Lemma 3.9 *If* $\dim S_2 < k$ *then* $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$*.*

Proof. Suppose for a contradiction that there is a $J \in \mathsf{N}(\mathsf{N}(J_1, J_2)) \setminus \{J_1, J_2\}$. By Lemma 3.8, we know that $X = \langle K_1, K_2\rangle \cap J$ has dimension k, for any choice of $K_1 \supseteq \langle S, S_1\rangle$ and $K_2 \supseteq \langle S, S_2\rangle$. Note that $X \subsetneq J$ since $k < j$. If neither $\langle J_1, K_2\rangle$ nor $\langle J_2, K_1\rangle$ contains J, then we can choose the $(j-k-1)$-space K'_0 in such a way that $K'_0 \cap (J \setminus X) \neq \emptyset$. With K'_0 chosen in this way we get $\dim(I_M \cap J) > k$ and we are done.

Suppose now that, for every choice of K_1 and K_2, either $\langle J_1, K_2\rangle$ or $\langle J_2, K_1\rangle$ contains J. We claim that either $\langle J_1, K_2\rangle \supseteq J$ for every k-subspace K_2 of J_2 containing $\langle S, S_2\rangle$ or $\langle J_2, K_1\rangle \supseteq J$ for every k-subspace K_1 of J_1 containing $\langle S, S_1\rangle$. Indeed, suppose that $J \not\subseteq \langle J_2, K_1\rangle$ for at least one choice of $K_1 \supseteq \langle S, S_1\rangle$, to fix ideas. Then, with K_1 chosen in that way, necessarily $J \subseteq \langle J_1, K_2\rangle$ for every k-subspace K_2 of J_2 containing $\langle S, S_2\rangle$. It follows that $\langle J_1, S_2\rangle$ contains J. Hence $\langle J_1, S_2\rangle \cap J = J$. However $\langle J_1, S_2\rangle \cap J = \langle S_1, S_2\rangle$. Therefore $J = \langle S_1, S_2\rangle$, contradicting Lemma 3.3. We conclude that $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$. □

This finishes the proof of Proposition 3.4.

We can now prove a criterion for $J_1 \hat{\perp} J_2$ (keep in mind that $\hat{\perp} = \perp$ if $\ell = n-1$, but we will sometimes use the more general relation $\hat{\perp}$ to avoid unnecessary case distinctions). We also introduce the notation $\approx$ to indicate that two j-spaces share a subspace of dimension $j-1$.

Lemma 3.10 *If $h(\Delta) > 2$ or, if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, then $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$ if and only if $J_1 \hat{\perp} J_2$. If $h(\Delta) = 2$ and, if Δ is hyperbolic, $\ell = n-1$, then $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$ if and only if $J_1 \hat{\perp} J_2$ or if $J_1 \approx J_2$ (with notation as above).*

Proof. First note that if $\mathsf{N}(J_1, J_2)$ is empty, then $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \Omega_i \cup \Omega_j$. So if $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$, then $\mathsf{N}(J_1, J_2) \neq \emptyset$ and hence, by Lemma 3.1, $k \leq s + \alpha + 1$ and in case Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, we also have that β odd, and if $\beta = -1$ then $\langle J_1, J_2\rangle$ has type ℓ.

Case 1: Suppose $\alpha \geq 0$. Suppose first that $\beta = -1$. Then by the above, $J_1 \hat{\perp} J_2$, and hence $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$ by Lemma 3.4. So assume $\beta \geq 0$. Then D is strictly contained in $J_1^{J_2}$ and hence there is a j-space J distinct from J_1 which is contained in $J_1^{J_2}$ and with $J \cap D = J_1 \cap D$ (recall that $J_1 \cap D = \mathsf{proj}_{J_1}(J_2)$). By Lemma 3.1, each $I \in \mathsf{N}(J_1, J_2)$ is collinear with $J_1^{J_2}$ and $I \cap J_1^{J_2} \subseteq D$. Therefore, $J \perp I$ and $I \cap J = I \cap J \cap D = I \cap J_1 \cap D = K_1$, i.e., $I \sim J$. We conclude that $J \in \mathsf{N}(\mathsf{N}(J_1, J_2))$.

Case 2: Suppose $\alpha = -1$. First note that this implies that $\beta \geq 0$, moreover $k \leq s$ by the first paragraph. Lemma 3.1 implies that, for each $I \in \mathsf{N}(J_1, J_2)$, $K_1 = K_2 \subseteq S$. Consider a j-space J containing S such that, in $\mathrm{Res}_\Delta(S)$, the β-spaces B_1, B_2, B corresponding to J_1, J_2, J are such that $B \subseteq (B_1^\perp \cap B_2^\perp)^\perp$. Note that B_1 and B_2 are opposite and that $B_1^\perp \cap B_2^\perp$ is disjoint from $(B_1^\perp \cap B_2^\perp)^\perp$. Since the subspace I' in $\mathrm{Res}_\Delta(S)$ corresponding to I is contained in $B_1^\perp \cap B_2^\perp$, we hence have $I \perp J$ and $J \cap I = K_1$, so $I \sim J$. Now, $(B_1^\perp \cap B_2^\perp)^\perp$ contains β-spaces other than B_1

and B_2 if and only if either $\beta > 0$, or, if $\beta = 0$ and B_1 and B_2 determine a proper hyperbolic line, i.e., $h(\Delta) > 2$. If Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, then β needs to be odd in order for $\mathsf{N}(J_1, J_2) \neq \emptyset$, so the second option does not occur in this case. The lemma follows. $\square$

4 Reduction to known graphs

Using the above, we define two new graphs on Ω_j and Ω_i, respectively.

Definition 4.1 Let $\Gamma'_j = (\Omega_j, \sim_j)$ be the graph with vertex set Ω_j, where two vertices $J_1, J_2 \in \Omega_j$ are adjacent precisely if $\mathsf{N}(\mathsf{N}(J_1, J_2)) = \{J_1, J_2\}$. Likewise, we define the graph $\Gamma'_i := (\Omega_i, \sim_i)$.

According to Lemma 3.10, $\sim_j = \hat{\perp}$ (in case $h(\Delta) > 2$ or if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$), or, $\sim_j = \perp \cup \approx$ (if $h(\Delta) = 2$ and, if Δ is hyperbolic, $\ell = n-1$; hence $\hat{\perp} = \perp$). In the latter case, the difficulty is that the two distinct adjacency relations $\perp$ and $\approx$ are somehow fused in Γ'_j. We will reduce this case to the former case by distinguishing the two types of adjacency relations.

We first show a general lemma for later purposes.

Lemma 4.2 *Suppose $J_1, J_2 \in \Omega_j$ are ℓ-collinear subspaces. Then $\mathsf{N}(J_1, J_2)$ is a (non-empty) clique in Γ'_i if and only if J_1 and J_2 generate a maximal singular subspace.*

Proof. Let M be the singular subspace generated by J_1 and J_2. By Lemma 3.1, each $I \in \mathsf{N}(J_1, J_2)$ is contained in a member of $\mathcal{N}_1 = \mathcal{N}_2$, the set of ℓ-spaces containing M. In particular, that lemma ensures us that $\mathsf{N}(J_1, J_2)$ is non-empty.

Suppose first that $\dim M = n-1$ (so $M \in \mathcal{N}_1$ since $J_1 \hat{\perp} J_2$). Then all members of $\mathsf{N}(J_1, J_2)$ are contained in M and therefore pairwise collinear and hence adjacent in Γ'_i. Next, suppose $\dim M < n-1$. Then we claim that $\mathsf{N}(J_1, J_2)$ is not a clique in Γ'_i. Suppose N, N' are distinct members of $\mathcal{N}_1$ and let $I, I' \in \mathsf{N}(J_1, J_2)$ be such that $I \subseteq N$ and $I' \subseteq N'$ with $I \nsubseteq N'$ and $I' \nsubseteq N$ (in particular, I nor I' belongs to M). It is clear that $I \not\perp I'$, so if $I \sim_i I'$ nonetheless then this means that $\sim_i$ represents $\perp \cup \approx$. So suppose $I \cap I' \subseteq N \cap N'$ has dimension $i-1$. If $\dim(N \cap N') < n-2$, then I has at least a line in $N \setminus N'$ (as $N = \langle J_1, I \rangle = \langle N \cap N', I \rangle$) and hence $\dim(I \cap I') < i-1$, a contradiction. So $\dim(N \cap N') = n-2$ for all choices of N and N', and hence $M = N \cap N'$. Then $\dim(I \cap I') = i-1$ only if the $(i-1)$-spaces $I \cap M$ and $I' \cap M$ coincide. If $k \geq 0$ then we can choose the k-spaces $K_1 = I \cap J_1$ and $K'_1 = I' \cap J_1$ distinct, so assume $k = -1$. In that case, $I \cap M$ and $I' \cap M$ can be any common complements of J_1 and J_2, which do not necessarily coincide. The claim follows. $\square$

4.1 Distinguishing the adjacency relations

Throughout this subsection we assume that adjacency in Γ'_i is given by $\hat{\perp} \cup \approx$, in other words, we suppose that $h(\Delta) = 2$ and, if Δ is hyperbolic, also that $\ell = n - 1$. In particular, $\hat{\perp} = \perp$.

Lemma 4.3 *Under the above assumptions, the following are equivalent for $J_1, J_2 \in \Omega_j$.*

(i) $J_1 \sim_j J_2$ *and* $\mathsf{N}(J_1, J_2)$ *is a (non-empty) clique in* Γ'_i

(ii) Either J_1 *and* J_2 *generate a maximal singular subspace, or,* $J_1 \not\perp J_2$ *and* $J_1 \approx J_2$, $j = n-2$ *and* $(k, i) \in \{(-1, 0), (n-3, n-2)\}$.

Proof. Recall that, according to Lemma 3.10, $J_1 \sim_j J_2$ if either $J_1 \perp J_2$ or $J_1 \not\perp J_2$ but $J_1 \approx J_2$. Suppose first that J_1 and J_2 are collinear subspaces. Then clearly $J_1 \sim_j J_2$, and $(i) \Leftrightarrow (ii)$ follows immediately from Lemma 4.2.

So suppose that J_1 and J_2 are not collinear. We show $(i) \Rightarrow (ii)$. Now $J_1 \sim_j J_2$ implies that J_1 and J_2 share a $(j-1)$-space. Note that Lemma 3.1 guarantees that $\mathsf{N}(J_1, J_2)$ is non-empty. Let $I, I' \in \mathsf{N}(J_1, J_2)$ be arbitrary. Then $I = \langle K_1, K_0 \rangle$ with $K_1 = J_1 \cap I$ and K_0 a subspace of I complementary to K_1 as usual, likewise $I' = \langle K'_1, K'_0 \rangle$. Recall from Lemma 3.1 that K_1 and K'_1 belong to S and that in $\mathrm{Res}_\Delta(S)$, K_0 and K'_0 correspond to $(i-k-1)$-spaces in $p_1^\perp \cap p_2^\perp$, where p_1, p_2 are the (opposite) points corresponding to J_1, J_2, respectively (recall that $k < i$ and $j < n-1$). If we choose K_0 and K'_0 distinct and non-collinear, then I and I' can only be adjacent in Γ'_i if $K_1 = K'_1$ and $\dim(K_0 \cap K'_0) = i - k - 2$. Given that $\mathsf{N}(J_1, J_2)$ is a clique in Γ'_i, the first fact implies that either $k = -1$ or $k = \dim S = j - 1$; the second fact, since it in particular has to hold for opposite subspaces K_0 and K'_0, implies that $i - k - 2 = -1$, or equivalently, $j = n - 2$. We conclude that $(k, i, j) \in \{(-1, 0, n-2), (n-3, n-2, n-2)\}$.

Next, we show $(ii) \Rightarrow (i)$. Firstly, $\dim(J_1 \cap J_2) = j - 1$ implies that $J_1 \sim_j J_2$. Secondly, $(k, i, j) \in \{(-1, 0, n-2), (n-3, n-2, n-2)\}$, implies that all pairs of members in $\mathsf{N}(J_1, J_2)$ share a common $(i-1)$-space, namely the empty set or S, and are hence adjacent in Γ'_i. The lemma follows. □

Remark 4.4 If $j < \frac{n-2}{2}$, then no two j-spaces can generate a maximal singular subspace and hence Lemma 4.2 tells us $\mathsf{N}(J_1, J_2)$ will never be a clique for $J_1, J_2 \in \Omega_j$ with $J_1 \perp J_2$. However, by assumption we have $i + j - k = n - 1$, so $\max\{i, j\} \geq \frac{n-2}{2}$. For ease of notation we will assume that $i \leq j$. However, a priori we do not know which bipart of Γ contains the i-spaces and which one the j-spaces if $i \neq j$. Note that so far, in all we did, the roles of i and j are interchangeable.

Lemma 4.3 hints at a different method for the cases where (k, i, j) is either $(-1, 0, n-2)$ or $(n-3, n-2, n-2)$. We treat these cases in the following more general cases, which have in common that $j = n-2$ and that we are able to recognize the bipart of Γ containing the $(n-2)$-spaces.

Case A. *Suppose $j = n-2$ and either $i < \frac{n-2}{2}$ or $i = n-2$ or Δ is a polar space of order (s, t) with $s = 2$.*

First, if $i < \frac{n-2}{2}$ then, as mentioned in Remark 4.4, two i-spaces cannot generate a maximal singular subspace. So Lemma 4.3 implies that there are no adjacent vertices I_1, I_2 in Γ'_i such that $\mathsf{N}(I_1, I_2)$ is a clique in Γ'_j (clearly, the second possibility in (ii) of that lemma is impossible, given that $i < \frac{n-2}{2} < n-2$). This allows us to detect which bipart contains the $(n-2)$-spaces. Second, if $i = n-2$ then both biparts contain $(n-2)$-spaces so either choice is fine. Third, if the order (s, t) of Δ is such that $s = 2$ and if $\frac{n-2}{2} \leq i < n-2$, then we want to tell the biparts apart by counting. Let χ_i be the degree of a vertex in Γ'_i, likewise we define χ_{n-2}. It follows from Proposition 3.6 of [9] that $\chi_i > \chi_{n-2}$, enabling us to distinguish the biparts. Worthwhile mentioning is that $|\Delta_i| \neq |\Delta_{n-2}|$, and in most cases even $|\Delta_i| > |\Delta_{n-2}|$ unless if $n = 5$, $i = 2$ and $t = 2$, as then $|\Delta_i| = |\Delta_{n-2}|$ and possibly if i is very close to $\frac{n}{2}$, it could happen that $|\Delta_i| < |\Delta_{n-2}|$. For more details we refer to [9] (in particular, Proposition 2.9).

We conclude that in all three cases we are indeed able to select a bipart of Γ containing $(n-2)$-spaces and hence we know which of the two graphs introduced in Definition 4.1 is Γ'_{n-2}.

We now determine the automorphisms of Γ'_{n-2} (of course, still under the assumption that the adjacency is given by $\perp \cup \approx$). To that end, we use the Grassmann graph $\mathsf{G}_{n-2}(\Delta)$ associated to Δ, which has the $(n-2)$-spaces as vertices, and two such vertices are adjacent whenever the $(n-2)$-spaces intersect in an $(n-3)$-space and are collinear (or equivalently, when they generate a maximal singular subspace).

Lemma 4.5 *Each automorphism of $\Gamma'_{n-2} = (\Omega_{n-2}, \perp \cup \approx)$ is induced by an automorphism of Δ and vice versa.*

Proof. Suppose that $J_1 \sim_{n-2} J_2$. Then $J_1 \perp J_2$ or $\dim(J_1 \cap J_2) = n-3$, but in the first case, $j = n-2$ implies that $\dim(J_1 \cap J_2) = n-3$ too. So Γ'_{n-2} is the graph (say $\mathsf{G}'_{n-2}(\Delta)$) in which adjacency is given by 'intersecting in a subspace of dimension $n-3$'. By Lemma 5.2 of [2], we can construct the $(n-2)$-Grassmann graph $\mathsf{G}_{n-2}(\Delta)$ from $\mathsf{G}'_{n-2}(\Delta)$. It is well known (we refer to Corollary 5.4 and Lemma 5.5 in [2] for further reading, but this is not the first occurrence by far) that the $(n-2)$-Grassmann graph $\mathsf{G}_{n-2}(\Delta)$ uniquely determines Δ if Δ is not hyperbolic, and up to duality if Δ is hyperbolic (no trialities occur if $n = 4$ since $j = 2$, so the planes are preserved). The result follows. $\square$

Next, we treat the remaining cases.

Case B. *Suppose that $i \leq j \leq n-2$ and that, if $j = n-2$, then $\frac{n-2}{2} \leq i < n-2$ and Δ is a polar space of order (s,t) with $s > 2$.*

We want to show a counterpart for Lemma 4.3 in order to recognize which i-spaces or j-spaces are collinear. There are two subcases, but the approach we use for them has an overlap.

- If $i < \frac{n-2}{2}$, then just like in the previous case, we can tell the biparts apart using Lemma 4.3 again (there are no adjacent I_1, I_2 in Γ'_i such that $\mathsf{N}(I_1, I_2)$ is a clique in Γ'_j). Note that in this case, our assumptions imply that $j < n-2$. We will apply the next lemma to the i-spaces only.

- If $i \geq \frac{n-2}{2}$, then we cannot (yet) tell the biparts apart and hence we apply the next lemma to both biparts, i.e., it is phrased for the i-spaces but we will also apply it to the j-spaces.

Note that in the following, we interchange the roles of the i-spaces and the j-spaces, but of course keep assuming $i \leq j$ (otherwise we are only changing names).

Lemma 4.6 *Suppose the conditions of Case B hold. Then $I_1, I_2 \in \Omega_i$ are collinear if and only if there are J_1, J_2 in Ω_j such that $\mathsf{N}(J_1, J_2)$ is a clique in Γ'_i containing I_1, I_2. Moreover, if $i \geq \frac{n-2}{2}$, then also $J_1, J_2 \in \Omega_j$ are collinear if and only if there are $I_1, I_2 \in \Omega_i$ such that $\mathsf{N}(I_1, I_2)$ is a clique in Γ'_j containing J_1, J_2.*

Proof. We focus on the first case and, if $i \geq \frac{n-2}{2}$, we also incorporate the second case by putting additional argument in italics, not to break with the flow of the argument too much.

Suppose that I_1 and I_2 are collinear. We apply Lemma 3.1 to I_1, I_2. Let $N \in \mathcal{N}_1$ be a maximal singular subspace containing $\langle I_1, I_2 \rangle$. We show that we can select $J_1, J_2 \in \mathsf{N}(I_1, I_2)$ such that $\langle J_1, J_2 \rangle = N$. If J_1 and J_2 are just any j-spaces generating N, then $2j - \dim(J_1 \cap J_2) = n-1 = j+i-k$ and hence $\dim(J_1 \cap J_2) = j-i+k$. So if we can choose J_1 and J_2 in N such that $J_1 \setminus J_2$ contains a subspace of dimension $j-(j-i+k)-1 = i-k-1$, we are done. *Observe that this will also work with i-spaces and j-spaces interchanged, provided that $i \geq \frac{n-2}{2}$, because we only use that two j-spaces can generate a maximal singular subspace.* We write $J_1 = \langle K_1, K_2, K'_0, K''_0 \rangle$, with notation as before: $K_c = J_1 \cap I_c$, $c = \{1,2\}$, K'_0 is a common complementary subspace of $\langle I_1, K_2 \rangle$ and $\langle I_2, K_1 \rangle$ inside $\langle I_1, I_2 \rangle$ and finally K''_0 is a subspace of N complementary to $\langle I_1, I_2 \rangle$; likewise, $J_2 = \langle \overline{K}_1, \overline{K}_2, \overline{K}'_0, \overline{K}''_0 \rangle$.

Suppose first that $\dim(I_1 \cap I_2) \leq k$ and let $K_1 = \overline{K}_1$, $K_2 = \overline{K}_2$ be such that they contain $I_1 \cap I_2$. Then $\dim K'_0 = \dim \overline{K}'_0 = i-k-1$ because $\langle K'_0, K_1 \rangle = \langle I_1, I_2 \rangle = \langle \overline{K}'_0, K_1 \rangle$. Put $K := \langle K_1, K_2 \rangle$ and $I := \langle I_1, I_2 \rangle$. We claim that we can choose K'_0 and $\overline{K}'_0$ such that in $\mathrm{Res}_I(K)$

they correspond to $(i-k-1)$-spaces which are disjoint from each other and from the two $(i-k-1)$-spaces corresponding to $\langle I_1, K_2\rangle$ and $\langle I_2, K_1\rangle$. Indeed, if $i-k-1>0$ this is clearly possible; If $i-k-1=0$, then $\mathrm{Res}_I(K)$ is a line containing s points. Since $i=k+1$ implies $j=n-2$, our assumption implies that $s>2$ and K_0' and $\overline{K}_0'$ (which are points in this case) can be chosen as required. *Note that, when interchanging the i- and j-spaces, then $i=n-2$ also implies $j=n-2$ and therefore the rest of the argument remains the same.* The claim follows. Therefore $J_1 \setminus J_2$ contains K_0', and hence $\langle J_1, J_2\rangle = N$ and $J_1, J_2 \in \mathsf{N}(I_1, I_2)$.

Next, suppose that $d := \dim(I_1 \cap I_2) > k$. We choose $K_1 = K_2$ and $\overline{K}_1 = \overline{K}_2$ such that $K_1 \cap \overline{K}_1$ is minimal, so if $d \leq 2k+1$, then $\langle K_1, \overline{K_1}\rangle = I_1 \cap I_2$ and if $d > 2k+1$, then K_1 and $\overline{K}_1$ are disjoint. Let X be a subspace of K_1 complementary to $K_1 \cap \overline{K}_1$ in K_1 and let $\overline{X}$ be defined likewise. Put $x := \dim X = \dim \overline{X}$ and note that if $d \leq 2k+1$, then $x = d-k-1$, if $d > 2k+1$, then $x = k$. Now $\dim K_0' = \dim \overline{K}_0' = i-d-1$ since K_0' is complementary to $\langle I_1, K_2\rangle = I_1$ in $\langle I_1, I_2\rangle$. Therefore $\dim\langle X, K_0'\rangle = \dim\langle \overline{X}, \overline{K}_0'\rangle$ equals $i-k-1$ if $d \leq 2k+1$ and $i-d+k$ if $d > 2k+1$. This time we try to choose K_0' and $\overline{K}_0'$ so that they correspond to disjoint $(i-d-1)$-spaces in $\mathrm{Res}_I(I_1 \cap I_2)$ (again $I = \langle I_1, I_2\rangle$), and also disjoint from the $(i-d-1)$-spaces corresponding to I_1 and I_2. As above, if $i-d-1>0$ then this always works. So suppose $i = d+1$. Then we run into trouble only if $s=2$, as then the points corresponding to K_0' and $\overline{K}_0'$ in $\mathrm{Res}_I(I_1 \cap I_2)$ necessarily coincide. Therefore, the $(i-d+x)$-spaces $\langle X, K_0'\rangle$ and $\langle \overline{X}, \overline{K}_0'\rangle$ belong to a $(d+1)$-space containing $I_1 \cap I_2$. If disjoint, they generate a subspace of dimension $2(i-d+x)+1 = 2x+3$, which only exceeds $d+1$ if $2x+1 \geq d$, which is only the case if $x=k$ and $2k+1=d$. So if $d \neq 2k+1$, we can make sure that $\langle X, K_0'\rangle$ and $\langle \overline{X}, \overline{K}_0'\rangle$ are disjoint. If $d = 2k+1$ however, i.e., if K_1 and $\overline{K_1}$ are disjoint subspaces generating $I_1 \cap I_2$, then each choice of K_0' and $\overline{K}_0'$ is such that $\langle X, K_0'\rangle$ and $\langle \overline{X}, \overline{K}_0'\rangle$ share a point. In the latter case $(J_1 \setminus J_2) \cap I$ just contains $X = K_1$, one dimension short of what we need (note that $i-k-1 = k+1$ under these assumptions). By our assumption, $s=2$ implies that both $i, j < n-2$, and hence I (which has dimension $i+1 < n-1$) is strictly contained in the maximal singular subspace N. In this case $\langle K_1, K_0', \overline{K}_1, \overline{K}_0'\rangle$ is an i-space in the $(i+1)$-space I, and in N there are at least two $(i+1)$-spaces through $\langle K_1, K_0', \overline{K}_1, \overline{K}_0'\rangle$ distinct from I, so we can take points p_0 and $\overline{p}_0$ in these respective $(i+1)$-spaces and select $K_0'' \ni p_0$ and $\overline{K}_0'' \ni \overline{p}_0$ and the remaining part (if any) of K_0'' and $\overline{K}_0''$ can be chosen as coinciding subspaces in N outside the $(i+2)$-space $\langle I, p_0, \overline{p}_0\rangle$. So, in conclusion, the have constructed (parts of the) j-spaces J_1, J_2 such that $J_1 \setminus J_2$ contains the $(i-d+x)$-space $\langle X, K_0'\rangle$. If $d \leq 2k+1$ then $i-d+x = i-k-1$ then, as explained above, this suffices to obtain $J_1, J_2 \in \mathsf{N}(I_1, I_2)$ with $\langle J_1, J_2\rangle = N$. Therefore, if $d \leq 2k+1$ we are done. If $d > 2k+1$, then then we still need a subspace of dimension $d-2k-2$ in J_1, disjoint from $\langle X, K_0'\rangle$, to obtain a subspace of dimension $(i-k-1)$ in $J_1 \setminus J_2$. The inequality $\dim N - \dim\langle I_1, I_2\rangle - 1 = (j+i-k)-(2i-d)-1 \geq d-2k-2$ is equivalent with $k \geq i-j-1$, which is trivially true since $i \leq j$. So we can choose K_0'' and $\overline{K}_0''$ in $N \setminus \langle I_1, I_2\rangle$ such that $\langle I_1, I_2, K_0''\rangle$ and $\langle I_1, I_2, \overline{K}_0''\rangle$ intersect in $\langle I_1, I_2\rangle$ only. The resulting j-spaces will then generate N, as required.

Only the one but last sentence requires some changes if we interchange the i- and j-spaces: then the inequality $\dim N - \dim\langle J_1, J_2\rangle - 1 = (j+i-k) - (2j-d) - 1 \geq d - 2k - 2$ *is equivalent with* $k \geq j - i - 1$. *Since we assume* $i \leq \frac{n-2}{2}$ *in this case, we obtain that* $n - 1 = i + j - k \leq 2i + 1$ *and hence* $j - i - 1 \leq k$ *indeed.*

Lemma 3.1 readily gives that each $I \in \mathsf{N}(J_1, J_2)$ belongs to $\langle J_1, J_2\rangle = N$ and hence $\mathsf{N}(J_1, J_2)$ is a clique in Γ'_i, containing I_1, I_2 by construction.

Conversely, suppose $\mathsf{N}(J_1, J_2)$ is a clique in Γ'_i containing I_1, I_2. Given that $j = n-2$, our assumptions imply that $\frac{n-2}{2} \leq i < n-2$, so in particular, $i \notin \{0, n-2\}$ and hence Lemma 4.3 implies that J_1 and J_2 generate a maximal singular subspace, say N. Again, by Lemma 3.1, $I_1, I_2 \subseteq N$ and hence $I_1 \perp I_2$. *If* $i \geq \frac{n-2}{2}$ *and we interchange the* i- *and* j=*spaces, then* $i = n-2$ *would imply* $j = n-2$, *contrary to our assumptions. So also here, Lemma 4.3 implies that* I_1 *and* I_2 *generate a maximal singular subspace.* □

So in Case (B), if $i < \frac{n-2}{2}$, we can determine which bipart contains the i-spaces and the previous lemma allows us to deduce the graph $(\Omega_i, \perp)$ from Γ'_i; and if $i \geq \frac{n-2}{2}$ we cannot distinguish the biparts but the previous lemma allows us to deduce $(\Omega_i, \perp)$ from Γ'_i and $(\Omega_j, \perp)$ from Γ'_j.

We treat the non-bipartite graphs with ℓ-collinearity as adjacency relation in the next subsection.

4.2 The graphs $(\Omega_x, \hat{\perp})$ with $x \in \{i, j\}$

Consider the graph on Ω_x, with $x \in \{i, j\}$, where adjacency between two x-spaces X_1 and X_2 in Ω_x is given by ℓ-collinearity. In case $h(\Delta) > 2$, or if Δ is hyperbolic and $\ell \in \{(n-1)', (n-1)''\}$, this graph coincides with the graph Γ'_x defined above and we obtain it for both $x = i$ and $x = j$; and otherwise, in Case (B) of the previous subsection, we constructed this graph from Γ'_x (cf. Lemma 4.6) for at least one of $\{i, j\}$. In Case (A) we do not need this graph, because of Lemma 4.5. In case we have both graphs $(\Omega_i, \hat{\perp})$ and $(\Omega_j, \hat{\perp})$, we want to use these to distinguish between the biparts of Γ after all (at least if $i \neq j$).

Lemma 4.7 *Each automorphism of* $(\Omega_x, \hat{\perp})$ *is induced by an automorphism of* Δ *(possibly a duality if* Δ *is hyperbolic) and vice versa. Moreover, if both* $(\Omega_i, \hat{\perp})$ *and* $(\Omega_j, \hat{\perp})$ *are given, then we can recognize which one is which if* $i \neq j$.

Proof. It is easily seen that a maximal clique of $(\Omega_x, \hat{\perp})$ consists of all x-spaces contained in a maximal singular subspace of type ℓ. So, from $(\Omega_x, \hat{\perp})$, we can construct the bipartite graph having Ω_x as one bipartition class and the set $\mathcal{M}$ of maximal cliques of $(\Omega_x, \hat{\perp})$ as the other bipartition class, where $X \in \Omega_x$ is adjacent to $C_M \in \mathcal{M}$ if $X \in C_M$. This graph is isomorphic to the graph $\mathsf{C}^n_{x,\ell}(\Delta)$, consisting of x-spaces and ℓ-spaces of Δ where adjacency is containment made symmetric. Proposition 7.4 from [2] now implies that each automorphism of $\mathsf{C}^n_{x,\ell}(\Delta)$ and hence also of $(\Omega_x, \hat{\perp})$ is induced by an automorphism of Δ (possibly a duality) and vice versa.

Now suppose we are given both $(\Omega_i, \hat{\perp})$ and $(\Omega_j, \hat{\perp})$ and that $i \neq j$. Let C^j_M be any maximal clique in Ω_j. As mentioned above, C^j_M corresponds to a maximal singular subspace M of type ℓ. By Lemma 4.2, $J_1, J_2 \in C^j_M$ generate M if and only if $\mathsf{N}(J_1, J_2)$ is a clique in Γ'_i, and moreover, such a pair of j-spaces exists since $j \geq \frac{n-2}{2}$ by Remark 4.4. So if $i < \frac{n-2}{2}$, this already suffices to distinguish the biparts, so suppose that $i \geq \frac{n-2}{2}$. By Lemma 3.1, we also know that each $I \in \mathsf{N}(J_1, J_2)$ is contained in M. Conversely, one deduces from Lemma 3.1 that each point of $M \setminus (J_1 \cup J_2)$, and even each point of M provided that $k \geq 0$, is contained in a member of $I \in \mathsf{N}(J_1, J_2)$. Unless $s = 2$ and $j = n-2$, the only subspace containing the points $M \setminus (J_1 \cup J_2)$ is M. So unless $k = -1$, $s = 2$ and $j = n-2$, we obtain that the members of $\mathsf{N}(J_1, J_2)$ generate M and hence C^i_M in $(\Omega_i, \hat{\perp})$ corresponding to M is the unique maximal clique containing $\mathsf{N}(J_1, J_2)$. However, if $k = -1$ and $j = n-2$, then $i = 0 \geq \frac{n-2}{2}$ violates our assumption that $n \geq 3$. We conclude that each maximal singular subspace of Δ of type ℓ can be expressed both in terms of the i-spaces and in terms of the j-spaces it contains (or, phrased more accurately, we can identify the maximal cliques in $(\Omega_i, \hat{\perp})$ and $(\Omega_j, \hat{\perp})$ that correspond to the same ℓ-space of Δ). Now, for any $I \in \Omega_i$, let $\mathcal{M}_I$ be the set of maximal singular subspaces of type ℓ containing I, and likewise we define $\mathcal{M}_J$ for any $J \in \Omega_j$. Then it is clear that $I \subsetneq J$ if and only if $\mathcal{M}_J \subsetneq \mathcal{M}_I$. This way, if $i \neq j$, we can recognize the biparts, as required. □

5 Conclusion

We have all it takes to complete the proofs of Theorem 1.2 and 1.3. In accordance with Remark 1.5, we may assume $i, j < n-1$.

Proposition 5.1 *Let ρ be any automorphism of $\Gamma^n_{i,j;k,\ell}(\Delta)$, where $i, j < n-1$ and $i+j-k = n-1$. Then there is an automorphism $\tilde{\rho}$ of Δ (possibly a duality if Δ is hyperbolic) such that $\rho(X) = \tilde{\rho}(X)$ for each $X \in \Omega_i \cup \Omega_j$, where possibly, if $i = j$, ρ switches the biparts. Conversely, each automorphism of Δ induces an automorphism of $\Gamma^n_{i,j;k,\ell}(\Delta)$.*

Proof. Clearly, each automorphism of Δ induces an automorphism of $\Gamma := \Gamma^n_{i,j;k,\ell}(\Delta)$, and if $i = j$, it induces two automorphisms: one that switches the biparts, and one which does not (observe that $i, j < n-1$ implies that there are only automorphisms of Δ switching the biparts if $i = j$).

Conversely, let ρ be an automorphism of Γ that does not switch the biparts. Recall that Lemma 3.10 leads us to Γ'_i and Γ'_j, where adjacency either coincides with $\hat{\perp}$ or $\perp \cap \approx$. In the latter case we obtained in Section 4.1 either the $(n-2)$-Grassmann graph (Case A) or the graph $(\Omega_x, \perp)$ for at least one x in $\{i, j\}$ (Case B). In each of these cases, Lemmas 4.5 or 4.7 yields an automorphism $\tilde{\rho}$ of Δ such that $\tilde{\rho}$ and ρ coincide on one of the biparts, say C_1, of Γ. We claim that $\tilde{\rho}$ and ρ also coincide on the other bipart of Γ. Indeed, consider the composition τ of ρ^{-1} and the automorphism of Γ induced by $\tilde{\rho}$. Then τ is an automorphism of Γ which is the identity on C_1. If τ is not the identity on C_2, the other bipart, then there are X and $\tau(X)$ in C_2 with $X \neq \tau(X)$, and for such X we have $\mathsf{N}(X) = \mathsf{N}(\tau(X))$. It is easily seen that this implies that $X^{\perp} = \tau(X)^{\perp}$, which is only possible if $X = \tau(X)$, a contradiction. So τ is the identity on Γ and hence $\tilde{\rho}$ and ρ coincide on $\Omega_i \cup \Omega_j$ indeed.

Finally, suppose that ρ is an automorphism of Γ which switches the biparts. We claim that this is only possible if $i = j$. Suppose again for ease of notation that $i \leq j$. Lemma 4.7 says that, if $(\Omega_i, \hat{\perp})$ and $(\Omega_j, \hat{\perp})$ are given, we can recognize the biparts of Γ if $i < j$, so in that case, no automorphism of Γ can interchange the biparts. We summarize the cases we distinguished in the previous section.

- Suppose $h(\Delta) > 2$ or, if Δ is hyperbolic, $\ell \in \{(n-1)', (n-1)''\}$. Then $\Gamma'_i = (\Omega_i, \hat{\perp})$ and $\Gamma'_j = (\Omega_j, \hat{\perp})$ and hence the claim follows in this case from Lemma 4.7.
- Suppose that $h(\Delta) = 2$ and that $\ell = n-1$ in case Δ is hyperbolic. We considered two cases.
  - Case (A): if $j = n-2$ and either $i < \frac{n-2}{2}$ or $i = n-2$ or Δ is a polar space of order (s, t) with $s = 2$. As explained when Case (A) was introduced, we can recognize the biparts unless $i = j = n-2$.
  - Case (B): Suppose that $i \leq j \leq n-2$ and that, if $j = n-2$, then $\frac{n-2}{2} \leq i \leq n-2$ and Δ is a polar space of order (s, t) with $s > 2$. Recall that $\hat{\perp} = \perp$ here. In this case, we constructed $(\Omega_i, \perp)$ and $(\Omega_j, \perp)$ provided that $\frac{n-2}{2} \leq i$; and we explained that, if $i < \frac{n-2}{2}$, we are able to tell the biparts apart.

  In both cases, the claim follows.

Now, if $i = j$, then consider the bipart-switching automorphism s of Γ induced by the identity on Δ. Then $\rho \circ s$ is an automorphism of Γ which does not switch the biparts, and hence by the above, $\rho \circ s$ is induced by an automorphism of Δ. It follows that ρ itself is also induced by an automorphism of Δ. $\square$

References

[1] P. Abramenko & H. Van Maldeghem, Maps between buildings that preserve a given Weyl distance, *Indag. Mathem.*, **15** (2004), 305–319.

[2] A. De Schepper and H. Van Maldeghem, Maps of polar spaces preserving a certain Weyl distance or intersection and projection properties, *Journal of Combinatorial Theory, series A* **160** (2018), 332–408.

[3] W.-l. Huang, Bounded distance preserving surjections in the geometry of matrices, *Lin. Alg. Appl.* **433** (2010), 1973–1987.

[4] W.-l. Huang, Bounded distance preserving surjections in the projective geometry of matrices, *Lin. Alg. Appl.* **435** (2011), 175–185.

[5] W.-l. Huang & H. Havlicek, Diameter preserving surjections in the geometry of matrices, *Lin. Alg. Appl.* **429** (2008), 376–386.

[6] A. Kasikova & H. Van Maldeghem, Vertex opposition in spherical buildings, *Des. Codes Cryptogr.* **68** (2013), 285–318.

[7] W. Liu, C. Ma & K. Wang, Full automorphism group of generalized unitary graphs, *Lin. Alg. Appl.* **437** (2012), 684–691.

[8] W. Liu, M. Pankov & K. Wang, Transformations of polar Grassmannians preserving certain intersecting relations, *J Alg Comb* **40** (2014), 633-646.

[9] A. Pasini, Computations regarding certain graphs associated to finite polar spaces, arXiv:2105.1261v1, 2021.

[10] M. Pankov, Grassmannians of classical buildings, *Algebra and Discrete Math. 2, World Scientific*, Singapore, 2010.

[11] M. Pankov, K. Prazmovski & M. Zynel, Geometry of polar Grassmann spaces, *Demonstratio Math.* **39** (2006), 625–637.

[12] L. Zeng, Z. Chai, R. Feng & C. Ma, Full automorphism group of the generalized symplectic graph, *Sci. China Math.* **56** (2013), 1509–1520.