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Buildings of exceptional type in buildings of type E_7

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Abstract

We investigate the possible ways in which a thick metasymplectic space Γ, that is, a Lie incidence geometry of type $\mathsf{F}_{4,1}$ (or $\mathsf{F}_{4,4}$), is embedded into the long root geometry Δ related to any building of type E_7. We provide a complete classification (if Γ is not embedded in a singular subspace). As an application we prove the uniqueness of the inclusion of the long root geometry of type E_6 in the one of type E_7; it always arises as an equator geometry. We also use the latter concept to geometrically construct one of the embeddings turning up in our classification. As a side result we obtain that all triples of pairwise opposite elements of type 7 in a building of type E_7 are projectively equivalent.

1. Introduction

Buildings were introduced by Jacques Tits as a geometric tool to study simple groups of Lie type, Chevalley groups, classical groups, simple algebraic groups, and groups of mixed type. Especially interesting and well studied are the guises of buildings obtained from the adjoint representation of the corresponding Lie algebra or algebraic group. These geometries are usually called long root geometries and they tend to turn up in many problems and situations. They show a basic common behaviour, yet with sometimes subtle differences across different (Dynkin) types. Most fascinating are the long root geometries of exceptional type ($\mathsf{E_6, E_7, E_8, F_4, G_2}$) and they are far from being well understood. Their presence in Tits' original geometric version of the so-called Freudenthal-Tits magic square [21] as the ultimate geometries at the bottom row (and most right column) provides them with many beautiful and unexpected properties. The present paper studies some specific structural properties. Namely, we classify all ("full") occurrences of any *thick* metasymplectic space (that is, a class of geometries related to *thick* buildings of type $\mathsf{F_4}$ containing as special cases all long root geometries of split buildings of that type) in the long root geometry of type $\mathsf{E_7}$. Roughly speaking, we show that *every embedded metasymplectic space arises from a group action*, that is, from *descent* in the sense of [16]; in other—casual—words, *geometric inclusion is equivalent to algebraic inclusion*.

Why metasymplectic spaces? Despite their common ground with other exceptional long root geometries, metasymplectic spaces are still the most accessible "long-root-like geometries" (that is, Lie incidence geometries which are non-strong parapolar spaces of diameter 3) of exceptional type, making them an ideal starting point. A good understanding of how the metasymplectic spaces live inside the other long root geometries of exceptional type is more than helpful to understand and unravel the latter geometries.

The metasymplectic spaces that we will consider are related to thick buildings of type $\mathsf{F_4}$. Such a building is defined by a field $\mathbb{K}$ and a quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$; for more details see Section 2.9. In order to understand our first formulation of the main result, it suffices for now to say that with every pair $(\mathbb{K}, \mathbb{A})$ as above we associate the building $\mathsf{F_4}(\mathbb{K}, \mathbb{A})$ with the diagram as in Figure 1.

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\bullet & \bullet \!\!\longrightarrow\!\! & \bullet & \bullet \\
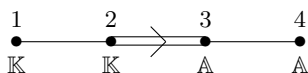\mathbb{K} & \mathbb{K} & \mathbb{A} & \mathbb{A}
\end{array}
$$

Fig. 1. The Dynkin diagram of the building $\mathsf{F_4}(\mathbb{K}, \mathbb{A})$, using Bourbaki labeling

Why long root geometries of type $\mathsf{E_7}$? There are several good reasons to start

with and limit ourselves for the moment to E_7. Obviously, the ultimate goal is to classify all occurrences of all exceptional long root geometries in one another, but this would take far too much space to accomplish at once. So limited space is one good reason, although a slightly negative one. On the positive side, a more satisfying reason is that E_7 seems to hide the most interesting and intriguing properties. In the aforementioned magic square, it turns up in the third and the fourth row, making it a rich source for a wealth of interesting properties, exhibiting many connections. The fact that it can be seen in two ways as a geometry of diameter 3 (using the nodes 1 and 7, see Figure 2) makes it a central object. For instance, it contains as a subcase the classification of full
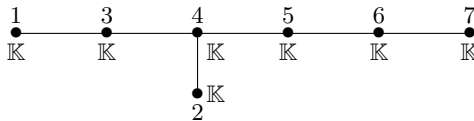
Fig. 2. The Dynkin diagram of the building $\mathsf{E}_7(\mathbb{K})$

embeddings of metasymplectic spaces in not only the long root geometries of type E_6, but also in the standard 27-dimensional module for E_6. For the latter, we already proved an existence and uniqueness result in [12]; the current paper can be seen as an extensive sequel and complement to this.

Our main result classifies, up to projective equivalence, the full embeddings of thick metasymplectic spaces in the Lie incidence geometries of type $\mathsf{E}_{7,1}$ not contained in singular subspaces. We state it below, using the terminology and the notation of the next section, see also **Theorem 5.1**.

Main Result. *Let $\mathbb{K}$ be any field, and let Δ be the long root geometry of type E_7 (the Lie incidence geometry of split type $\mathsf{E}_{7,1}$) over $\mathbb{K}$. Then the following hold.*

(i) *The long root geometry $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{K})$ has a projectively unique full embedding in Δ with the property that it is not contained in any residue of Δ. It arises as the fixed point structure of each non-trivial member of a group of collineations of Δ isomorphic to $\mathsf{PGL}_2(\mathbb{K})$.*

(ii) *The quadratic field extensions $\mathbb{L}$ (separable or not) of $\mathbb{K}$ (in its algebraic closure) are in one-to-one correspondence with the classes of projectively equivalent fully embedded metasymplectic spaces $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{L})$ in Δ, which are not contained in any residue of Δ. Each such embedding arises as the fixed point structure of each non-trivial member of a group of collineations of Δ isomorphic to $\mathbb{L}^{\times}/\mathbb{K}^{\times}$, the multiplicative group of $\mathbb{L}$ modulo the one of $\mathbb{K}$.*

(iii) *Any other full embedding of a thick metasymplectic space Γ in Δ which is not contained in a singular subspace of Δ is contained in a residue Δ' of Δ of type $\mathsf{E}_{6,1}$. This occurs if, and only if, Γ is in one of the following situations.*

    (a) *The Lie incidence geometry $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ has a projectively unique full embedding in Δ' and arises from a symplectic polarity of Δ'.*

    (b) *The subfields $\mathbb{F}$ of $\mathbb{K}$ such that $\mathbb{K}$ is a separable quadratic extension of $\mathbb{F}$ are in one-to-one correspondence with the classes of projectively equivalent fully*

embedded metasymplectic spaces $\mathsf{F}_{4,4}(\mathbb{F},\mathbb{K})$ in Δ'. Each such embedding arises from Galois descent in $\mathsf{E}_6(\mathbb{K})$ (i.e., the embedded metasymplectic space is the fixed point structure of a Galois involution of Δ' in the sense of algebraic groups).

(c) If char $\mathbb{K} = 2$ and $\mathbb{K}$ is not perfect, then the fields $\mathbb{K}'$ with $\mathbb{K}^2 \leq \mathbb{K}' < \mathbb{K}$ are in one-to-one correspondence with the classes of projectively equivalent fully embedded metasymplectic spaces $\mathsf{F}_{4,4}(\mathbb{K}',\mathbb{K})$ in Δ'. Each such embedded metasymplectic space is a canonical subspace (that is, arising from the inclusion $\mathbb{K}' \leq \mathbb{K}$) of a fully embedded metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$ in Δ' as in Case (a) above.

The case where the metasymplectic space would be contained in a singular subspace is not considered here as it is too far from an isometric embedding, obliterating the beautiful structure imposed by the presence of the exceptional long root geometry of type E_7. It would also require totally different methods; as far as we know a classification of metasymplectic spaces in projective spaces is wide open. We want to point out that Case (iii) implies a full classification of all embeddings of metasymplectic spaces in the Lie incidence geometries $\mathsf{E}_{6,1}(\mathbb{K})$ (**Corollary 5.19**) and $\mathsf{E}_{6,2}(\mathbb{K})$ (**Corollary 6.9**).

We emphasise that we include fields of characteristic 2, which may be perfect or not. The above classification heavily depends on the properties of the underlying field, in particular with respect to its behaviour regarding its subfields and its quadratic extensions. However, for each field, Cases (i) and (iii)(a) always occur. Cases (ii) and (iii)(c) never occur for quadratically closed fields (in particular, algebraically closed fields). Over the real or rational numbers, Cases (iii)(b) and (iii)(c) do not occur. Finite fields are very well-behaved concerning their subfields and quadratic extensions, and moreover, since the smallest (thick) metasymplectic space related to the finite field $\mathbb{F}_q$ has at least $q^7\sqrt{q}$ points whereas the largest singular subspace of $\mathsf{E}_{7,1}(\mathbb{F}_q)$ has less than q^7 points, no (thick) metasymplectic space can be fully embedded in a singular subspace of any finite long root geometry of type E_7. So for finite fields, the Main Result can be stated as follows (where we replace the finite field with its order when describing the Lie incidence geometries).

Main Result—Finite Fields. *Let $\mathbb{F}_q$ be the finite field with q elements. Let Δ be the long root geometry of type E_7 over $\mathbb{F}_q$. Let Γ be a fully embedded thick metasymplectic space in Δ. Then, up to projectivity, exactly one of the following occurs.*

(i) *Γ is the long root geometry $\mathsf{F}_{4,1}(q,q)$, is not contained in any residue of Δ, and arises as the fixed point structure of each non-trivial member of a group of collineations of Δ isomorphic to $\mathsf{PGL}_2(q)$.*

(ii) *Γ is the Lie incidence geometry $\mathsf{F}_{4,1}(q,q^2)$, is not contained in any residue of Δ, and arises as the fixed point structure of each non-trivial member of a group of collineations of Δ isomorphic to the cyclic group of order $q+1$.*

(iii) *Γ is contained in a residue Δ' of Δ of type $\mathsf{E}_{6,1}$ and one of the following occurs.*

(a) *Γ is the Lie incidence geometry $\mathsf{F}_{4,4}(q,q)$ and arises from a symplectic polarity of Δ'.*

(b) *q is a square, Γ is the Lie incidence geometry $\mathsf{F}_{4,4}(\sqrt{q},q)$ and arises from Galois descent in $\mathsf{E}_6(q)$ (i.e., the embedded metasymplectic space is the fixed*

point structure of a Galois involution).

Proof. First we note that (i) and $(iii)(a)$ are exactly the same as (i) and $(iii)(a)$, respectively, in our general Main Result. Regarding (ii), we note that each finite field $\mathbb{F}_q$ admits a unique quadratic extension $\mathbb{F}_{q^2}$, and then (ii) follows from (ii) of the general Main Result by observing that $\mathbb{F}_{q^2}^\times / \mathbb{F}_q^\times$ is cyclic of order $q+1$. Likewise, a finite field $\mathbb{F}_q$ admits a subfield $\mathbb{F}_r$ such that $\mathbb{F}_q$ is a quadratic separable extension of $\mathbb{F}_r$ if and only if q is a square (and then $r^2 = q$); in this case $\mathbb{F}_r$ is also unique. Now $(iii)(b)$ follows from $(iii)(b)$ of the general Main Result. Finally, finite fields are always perfect, so Case $(iii)(c)$ of the general Main Result cannot occur for finite fields. ∎

In order to obtain the classification given in the general Main Result, we first classify the point-residues, that is, we classify all fully embedded dual polar spaces of rank 3 in half spin geometries of type $\mathsf{D}_{6,6}$, but not in one of its singular subspaces (**Theorem 4.1**). This result is already interesting in its own right, given the richness of the possibilities. In fact, we expect that our methods allow for a generalization to higher rank, i.e., to classify the full embeddings of a dual polar space of rank n in the half spin geometries of type $\mathsf{D}_{2n,2n}$. This could be seen as the "classical" counterpart of our main result (being the "exceptional" case), where "classical" is used in the sense of "classical groups".

Application. As an application, showcasing the claim made earlier that a good understanding of how metasymplectic spaces live inside long root geometries of exceptional type is very helpful to understand these geometries, we deduce from our analysis and our classification that all triples of pairwise opposite points of the Lie incidence geometry $\mathsf{E}_{7,7}(\mathbb{K})$ corresponding to the 56-dimensional module for the Chevalley group of type E_7 over $\mathbb{K}$ are mutually projectively equivalent (**Corollary 6.10**). We also show, as a corollary, that the long root geometry of type E_6 is embedded in a projectively unique way in the long root geometry of type E_7 (**Proposition 6.14**). A convenient way to describe this inclusion is using the notion of an *equator geometry*, introduced systematically in [26]. In particular, Case (i) of our main result stated above will occur as an intersection of equator geometries. However, in order to have a close link with the underlying Chevalley groups, we describe and classify all embeddings using their pointwise group stabilizer, and only in the final section, we give a geometric description in terms of these equator geometries.

Structure of the paper. • In Section 2, we introduce the needed terminology on buildings, polar and parapolar spaces, Lie incidence geometries and embeddings of point-line geometries.

• In Section 3, we give an overview of all relevant properties of the parapolar spaces appearing in the present paper. This includes the various possible mutual positions of the most prominent objects of each such geometry, that is, the points, lines, symps and, if any, "paras".

• As mentioned above, we start by considering how the point-residues of the metasymplectic space Γ are embedded in the point-residues of the long root geometry Δ of type E_7. So in Section 4, we classify all full point-line embeddings of dual polar spaces Γ' of rank 3 in the half spin geometry Δ' of type $\mathsf{D}_{6,6}(\mathbb{K})$, not contained in a singular subspace, see **Theorem 4.1**. We distinguish three cases, depending on how the symps of Γ'

are embedded in Δ'. According to Lemma 3.20, a symp of Γ' either embeds isometrically in a symp of Δ' or embeds in a singular subspace of Δ'. This already puts restrictions on the nature of Γ' (the isomorphism type of the symps). If some symp of Γ' embeds isometrically in a symp of Δ', then we show that Γ' embeds isometrically in Δ' (Lemma 4.9) and this will lead to Cases (i) and (ii) of Theorem 4.1 (see **Proposition 4.35**); if not then each symp of Γ' embeds in a singular subspace of Δ', in which case Γ' is contained in a symp of Δ' (i.e., a polar space of type $\mathsf{D}_{4,1}$) (Case (iii) of Theorem 4.1). A great deal of work goes into determining the group of collineations of Δ' each element of which has Γ' as its fixed point structure. This is done piece-wise in the sense that we construct or determine such collineation groups for residues of Δ', fixing the corresponding residues of Γ', which coincide on the intersection and which we glue together using Tits' extension theorem (Proposition 4.16 of [23]).

• In Section 5 we then prove Main result 5.1 based on the results obtained in the previous section, i.e., Theorem 4.1. We show that, if there is a point p in Γ such that $\mathrm{Res}_\Gamma(p)$ embeds in $\mathrm{Res}_\Delta(p)$ as in Cases (i) or (ii) of Theorem 4.1, then all point-residues of Γ embed in the same way in Δ and the embedding of Γ in Δ is again isometric, and this leads to Cases (i) and (ii) of Theorem 5.1 (see **Propositions 5.7 and 5.10**), respectively. The case where some point-residue $\mathrm{Res}_\Gamma(p)$ of Γ embeds in a symp of $\mathrm{Res}_\Delta(p)$ leads to Case (iii) (see **Proposition 5.18**). With minor effort we then exclude the possibility that some point-residue $\mathrm{Res}_\Gamma(p)$ is embedded in a singular subspace of $\mathrm{Res}_\Delta(p)$.

• In Section 6, we make the connection with the equator geometries, showing that certain embeddings of the previous sections can be described as the intersection of equator geometries (cf. **Propositions 6.3 and 6.8**), which is a purely geometric matter. We also show the uniqueness of some equator geometries, more exactly, we show that each full point-line embedding of the long root geometry $\mathsf{E}_{6,2}(\mathbb{K})$ in the long root geometry $\mathsf{E}_{7,1}(\mathbb{K})$ arises from an equator geometry (and all such geometries are projectively equivalent), see **Proposition 6.14**. As a corollary, we obtain that that there is a projectively unique full embedding of a (thick) metasymplectic space in $\mathsf{E}_{6,2}(\mathbb{K})$ (see **Corollary 6.9**).

2. Preliminaries

We fix notation and introduce all relevant terminology.

2.1. Buildings. Buildings are numbered simplicial complexes satisfying some axioms. For the precise definition and details we refer to the literature, for instance [23]. We assume that the reader is familiar with the basic theory of abstract buildings, Coxeter groups and Dynkin diagrams [3]. We just recall some notions to fix notation.

As a numbered complex, a building is defined over a type set, which we call S in this introduction, and the cardinality $|S|$ is the *rank* of the building. Let Δ be a building over the type set S. There is a surjective mapping τ from the vertex set V of Δ to S inducing a partition of V. There is a Coxeter diagram X whose nodes bijectively correspond to the elements of S, in such a way that, for each vertex $v \in V$ of type $\tau(v) = s \in S$, the Coxeter diagram of the residue $\mathrm{Res}_\Delta(v)$, which is canonically also a building, is obtained from X

by removing the node corresponding to the vertex v and the edges incident with it and written as $\mathsf{X} \setminus \{v\}$. We will always assume that the building is thick so that the Coxeter diagram is unambiguously determined [23].

In this paper we will only be concerned with *spherical buildings*, that is, buildings for which the Coxeter diagram corresponds to a finite Coxeter group (also called *spherical Coxeter diagrams*). If the diagram is connected, the building is called *irreducible*. Recall that a *flag* is another name for a simplex. Given a flag F of type $J \subseteq S$ (that is, $J = \{\tau(v) \mid v \in F\}$), the *residue* $\mathrm{Res}_\Delta(F)$ is the building consisting of all flags F' of Δ with $F \subsetneq F'$; its natural type set is $S \setminus J$. There is an explicit list of connected spherical Coxeter diagrams, and we will use the standard symbols A_n, B_n, D_n, etc., as in [3].

The standard examples $\Delta(G)$ of spherical buildings arise from semi-simple algebraic groups G by taking as vertices the right cosets of the maximal parabolic subgroups of G, and the maximal flags (that is, the *chambers*) as the sets of such cosets containing a common right coset of the standard Borel subgroup of G (see Chapter 5 of [23]). If G is defined over an algebraically closed field, then there is an underlying simple crystallographic root system and hence a Dynkin diagram. The Coxeter diagram obtained by forgetting the orientations of the multiple edges of the Dynkin diagram is precisely the Coxeter diagram corresponding to the building $\Delta(G)$. There is a standard Bourbaki labelling of Dynkin diagrams, which we will use. It is convenient to encode the Dynkin diagrams in the notation of specific buildings and hence distinguish between types B_n and C_n, see below.

2.2. Opposition. Let Δ be a spherical building. Two chambers are *adjacent* if they only differ in one element. This defines the *chamber graph*. An important feature in spherical buildings is *opposition*. Two chambers are *opposite* if they are at maximum distance in the chamber graph. Two flags F, F' are *opposite* if for every chamber containing F there is an opposite chamber containing F', and for every chamber containing F' there is an opposite chamber containing F. For opposite flags F, F', "being nearest in the chamber graph" defines a bijection between the set of chambers containing F and those containing F'. That bijection induces an isomorphism between the buildings $\mathrm{Res}_\Delta(F)$ and $\mathrm{Res}_\Delta(F')$. Opposition defines an automorphism of the Coxeter diagram of the building, only depending on the Coxeter type. This automorphism is trivial except for the types A_n, D_{2n+1}, E_6, $n \in \mathbb{N}$, and for rank 2 buildings of odd diameter, where it is the unique involution of the diagram. For all these facts, we refer to Sections 2.39 and 3.22 of [23].

2.3. Lie incidence geometries. Let Δ be a spherical building, not necessarily irreducible. Let n be its rank, let S be its type set and let $J \subseteq S$. Then we define a point-line geometry $\Gamma = (X, \mathcal{L}, *)$ as follows. The point set X is just the set of flags of Δ of type J; the set $\mathcal{L}$ of lines are the flags of type $S \setminus \{s\}$, with $s \in J$. If F is a flag of type J and F' one of type $S \setminus \{s\}$, with $s \in J$, then $F * F'$ if $F \cup F'$ is a chamber. The geometry Γ is called a *Lie incidence geometry*. For instance, if Δ has type A_n, and $J = \{1\}$ (remember we use Bourbaki labelling), then Γ is the point-line geometry of a projective space. If X_n is the Coxeter type of Δ and Γ is defined using $J \subseteq S$ as above, then we say that Γ has *type* $\mathsf{X}_{n,J}$ and we write $\mathsf{X}_{n,j}$ if $J = \{j\}$.

By a *flag* or a *chamber* of Γ we mean a set of objects of Γ corresponding to a flag or chamber of the underling building Δ. By an *apartment* of Γ we also mean the set of objects of Γ incident with an apartment of Δ.

We now quickly review two other important classes of point-line geometries which contain many Lie incidence geometries. First we recall some notions from the theory of point-line geometries.

2.4. Abstract point-line geometries. Let $\Gamma = (X, \mathcal{L}, *)$ be a point-line geometry (X is the set of points, $\mathcal{L}$ the set of lines, and $*$ a symmetric incidence relation). We will not consider geometries with repeated lines, so henceforth we view $\mathcal{L}$ as a subset of the power set of X, and $*$, which is then just inclusion made symmetric, is not mentioned explicitly (and so we write $\Gamma = (X, \mathcal{L})$). We will also always assume that there are at least two lines, to exclude trivial cases.

Points $x, y \in X$ contained in a common line are called *collinear*, denoted $x \perp y$; the set of all points collinear to x is denoted by $x^{\perp}$. We will always deal with situations where every point is contained in at least one line, so $x \in x^{\perp}$. The *collinearity graph* of Γ is the graph on X with collinearity as adjacency relation. The *distance* δ between two points $p, q \in X$ (denoted $\delta_{\Gamma}(p, q)$, or $\delta(p, q)$ if no confusion is possible) is the distance between p and q in the collinearity graph, where $\delta(p, q) = \infty$ if there is no such path. If $\delta := \delta(p, q)$ is finite, then a *geodesic path* or a *shortest path* between p and q is a path between them in the collinearity graph of length δ. The *diameter* of Γ (denoted $\mathrm{Diam}\,\Gamma$) is the diameter of the collinearity graph. We say that Γ is *connected* if every pair of vertices is at finite distance from one another. The point-line geometry Γ is called a *partial linear space* if each pair of distinct points is contained in at most one line.

A *subspace* of Γ is a subset S of X such that, if $x, y \in S$ are collinear and distinct, then all lines containing both x and y are contained in S. A subspace S is called *convex* if, for any pair of points $\{p, q\} \subseteq S$, every point occurring in a shortest path between p and q in the collinearity graph is contained in S; it is *singular* if $\delta(p, q) \leq 1$ for all $p, q \in S$. If S is a set of pairwise collinear points, then $\langle S \rangle$ denotes the singular subspace generated by S (this is the intersection of all singular subspaces containing S); if S consists of two distinct collinear points p and q, then, if $\langle S \rangle$ is a line, it is sometimes briefly denoted by pq. The intersection of all convex subspaces of Γ containing a given subset $S \subseteq X$ is called the *convex closure* of S. Two singular subspaces S_1 and S_2 are called *collinear* if $S_1 \cup S_2$ is a set of pairwise collinear points, and if so, we write $\langle S_1, S_2 \rangle$ instead of $\langle S_1 \cup S_2 \rangle$. A proper subspace H is called a *geometric hyperplane* if each line of Γ has either one or all its points contained in H.

An *isomorphism* from $\Gamma = (X, \mathcal{L})$ to the point-line geometry $\Gamma' = (X', \mathcal{L}')$ is a bijection $\beta : X \to X'$ inducing a bijection $\mathcal{L} \to \mathcal{L}'$. A *collineation* of Γ is an isomorphism of Γ to itself.

2.5. Embeddings. Consider two point-line geometries $\Gamma = (X', \mathcal{L}')$ and $\Delta = (X, \mathcal{L})$. We say that Γ is *embedded* in Δ if $X' \subseteq X$ and for each $L' \in \mathcal{L}'$, there is a line $L \in \mathcal{L}$ with L' (viewed as subset of X') contained in L (viewed as a subset of X). The embedding is called *full* if $\mathcal{L}' \subseteq \mathcal{L}$, i.e., $L' \subseteq X'$ coincides with $L \subseteq X$ in the foregoing. A non-full

embedding is called a *lax embedding*. Collinearity in Γ (resp. Δ) will be denoted by $\perp_\Gamma$ (resp. $\perp_\Delta$). Note that, if Γ is fully or laxly embedded in Δ, then $\delta_\Delta(p, q) \leq \delta_\Gamma(p, q)$ for all points $p, q \in X'$. An embedding is *point-isometric* if, for all points $p, q \in X'$, $\delta_\Gamma(p, q) = \delta_\Delta(p, q)$; in particular, $\perp_\Gamma$ and $\perp_\Delta$ coincide on $\Gamma \times \Gamma$.

2.6. Polar spaces. Polar spaces are the Lie incidence geometries of Coxeter types $\mathsf{B}_{n,1}$ and $\mathsf{D}_{n,1}$. Abstractly, a *polar space* $\Gamma = (X, \mathcal{L})$ is a point-line geometry satisfying the following four axioms, due to Buekenhout and Shult, which simplify the axiom system in [23].

(PS1) Every line contains at least three points, i.e., every line is *thick*.

(PS2) No point is collinear to every other point.

(PS3) Every nested sequence of singular subspaces is finite.

(PS4) The set of points incident with a given arbitrary line L and collinear to a given arbitrary point p is either a singleton or coincides with L.

We will assume that the reader is familiar with the basic theory of polar spaces, see for instance [4]. Let us recall that every polar space, as defined above, is a partial linear space and has a unique *rank*, given by the length of the longest nested sequence of singular subspaces (including the empty set); the rank is always assumed to be finite (by (PS3)) and at least 2 since we always have a sequence $\emptyset \subseteq \{p\} \subseteq L$, for a line $L \in \mathcal{L}$ and a point $p \in L$.

Now let $\Gamma = (X, \mathcal{L})$ be a polar space of rank $r \geq 2$. It is well known that the maximal singular subspaces are mutually isomorphic projective spaces of dimension $r - 1$ (and so two arbitrary points of Γ are contained in at most one line). Moreover, there is a (not necessarily finite) constant t such that every singular subspace of dimension $r - 2$ is contained in exactly $t + 1$ maximal singular subspaces. If $t = 1$, then we say that Γ is of *hyperbolic type*, or is a *hyperbolic* polar space. A hyperbolic polar space is isomorphic to one of the following.

$r = 2$ $\mathcal{L}$ consists of two disjoint systems of lines, each covering the point set, such that two lines intersect nontrivially (hence in exactly one point) if, and only if, they belong to different systems. Such a polar space is also sometimes referred to as a *grid*. A typical example is a ruled nondegenerate quadric in a projective 3-space.

$r = 3$ X is the set of lines of a 3-dimensional projective space $\mathsf{PG}(3, \mathbb{L})$ over a non-commutative skew field $\mathbb{L}$. The members of $\mathcal{L}$ are the (full) planar line pencils in $\mathsf{PG}(3, \mathbb{L})$.

$r \geq 3$ X is the point set of a nondegenerate hyperbolic quadric Q in $\mathsf{PG}(2r - 1, \mathbb{K})$, $\mathbb{K}$ a (commutative) field. The lines are the lines of $\mathsf{PG}(2r-1, \mathbb{K})$ entirely contained in Q. Note that a standard equation for Q is given by $X_{-1}X_1 + X_{-2}X_2 + \cdots + X_{-r}X_r = 0$.

A maximal singular subspace of a hyperbolic polar space is also called a *generator*. The family of generators of each hyperbolic polar space of rank r is the disjoint union of two systems of generators, called the *natural systems* such that two generators intersect in a singular subspace of odd codimension in each of them if, and only if, they belong to different systems (the *codimension* of a subspace U in a projective space W is just $\dim W - \dim U$).

Polar spaces of rank $n \geq 3$ are in one-to-one correspondence with buildings of Coxeter types B_n and D_n as follows. Given a polar space not of hyperbolic type, the vertices of type i of the corresponding building, $1 \leq i \leq n$, are the nonempty singular subspaces of projective dimension $i - 1$, and the simplices (or flags) are the nested sequences of singular subspaces. If the polar space is of hyperbolic type, then we have to consider the *oriflamme complex*. Its vertices are the singular subspaces of dimension distinct from $n - 2$ and its maximal simplices are the nested maximal sequences of singular subspaces where we replace the singular subspace U of dimension $n - 2$ by the unique generator W containing U and distinct from the generator already in the sequence. It follows that the two natural systems of generators correspond to two different types of vertices of the corresponding building, which is of type D_n. We will always distinguish between the two types of generators by referring to $(n-1)$-spaces for one type, and $(n-1)'$-spaces for the other type.

We call a collineation of a hyperbolic polar space *type-preserving* if it stabilizes the two natural systems of generators; hence if it induces a type-preserving automorphism of the corresponding spherical building. A collineation which is not type-preserving is called *type-interchanging*.

A polar space will be called *orthogonal* if it arises from a non-degenerate quadric in a (not necessarily finite dimensional) projective space of finite Witt index at least 2. A standard equation for such a quadric, say with Witt index n, is given by

$$X_{-1}X_1 + X_{-2}X_2 + \cdots + X_{-n}X_n = f, \tag{2.1}$$

where f is an anisotropic quadratic form in the remaining coordinate variables (that is, a quadratic form the null space of which is the trivial subspace consisting of only the zero vector), sometimes called the *anisotropic kernel*. In case f is trivial, the polar space is of hyperbolic type. The intersection of all tangent hyperplanes to a quadric will be called the *nucleus* of the quadric. Note that the nucleus is always trivial if the characteristic is distinct from 2.

We will use some notions of the theory of buildings in polar spaces. For instance, two subspaces are called *opposite* if no point of their union is collinear to every point of this union; in particular two points are opposite if, and only if, they are not collinear and two maximal singular subspaces are opposite if, and only if, they are disjoint.

2.7. Parapolar spaces. Parapolar spaces are point-line geometries that are designed to be Lie incidence geometries. A standard reference is [17], see also [7]. A point-line geometry $\Gamma = (X, \mathcal{L})$ is a *parapolar space* if it satisfies the following axioms.

(PPS1) There is line L and a point p such that no point of L is collinear to p.

(PPS2) The geometry is connected.

(PPS3) Let x, y be two points at distance 2. Then either there is a unique point collinear to both, or the convex closure of $\{x, y\}$ is a polar space. Such polar spaces are called *symplecta*, or *symps* for short.

(PPS4) Each line is contained in a symplecton.

A pair $\{x, y\}$ of points with $x^\perp \cap y^\perp = \{z\}$ is called *special* and we denote $z = x \bowtie y$; we also say that *x is special to y*. A pair of points $\{x, y\}$ at distance 2 from one another and

contained in a (necessarily unique) symp ξ is called *symplectic* and we write $x \perp y$ and $\xi = x \Diamond y$; we also say that *x is symplectic to y*. A parapolar space without special pairs of points is called *strong*. Due to (PPS4) and the fact that symps are convex subspaces isomorphic to polar spaces, each parapolar space is automatically a partial linear space and, by (PPS1), it is not a polar space. Note that the symps are not required to all have the same rank. A *para* is a proper convex subspace of Γ, whose points and lines form a parapolar space themselves. The set of symps of a para is a subset of the set of symps of Γ.

Most Lie incidence geometries are parapolar spaces, in particular, if, with the notation of 2.1 and 2.3, $|J| = 1$, then we either have a projective space, a polar space, or a parapolar space. We assume that the reader is familiar with some basic properties of parapolar spaces such as the facts that the intersections of symps are singular subspaces, and also that the set of points collinear to a given point x and belonging to a symp $\xi \not\ni x$ is a singular subspace.

Here, too, we use some notions from the theory of spherical buildings, such as opposition and apartments. There is no general rule in parapolar spaces to call two elements opposite, but we will indicate this in the examples of Lie incidence geometries given in Section 3. Note also that the automorphism (or collineation) group of a parapolar space defined from a spherical building, coincides with the automorphism group of the spherical building fixing the type of the points of the parapolar space. We will view apartments as sets of flags, along with their natural incidence.

2.8. Some special types of Lie incidence geometries.
In the literature, some types of Lie incidence geometries are better known under a more specific name. We review those that are relevant for the present paper.

Lie incidence geometries of types $\mathsf{F}_{4,1}$ and $\mathsf{F}_{4,4}$ are the main examples of *metasymplectic spaces*. We define them and review their properties we need in Section 3.3. They are non-strong parapolar spaces. In general, metasymplectic spaces also arise from non-thick buildings of type F_4; however, we will only consider metasymplectic spaces defined by thick buildings of type F_4. The metasymplectic spaces left out this way are the line Grassmannians of polar spaces of rank 4.

Lie incidence geometries of type $\mathsf{B}_{2,1}$ (or, equivalently, $\mathsf{B}_{2,2}$), are *(thick) generalized quadrangles*, that is, polar spaces of rank 2 not of hyperbolic type.

Lie incidence geometries of type $\mathsf{B}_{n,n}$, $n \geq 3$, are called *dual polar spaces (of rank n)* and Lie incidence geometries of type $\mathsf{D}_{n,n}$, $n \geq 5$, are called *half spin geometries (of rank n)*. All these are strong parapolar spaces. Their symps are thick generalized quadrangles and polar spaces of hyperbolic type and rank 4 (that is, Lie incidence geometries of type $\mathsf{D}_{4,1}$), respectively.

Long root geometries are special Lie incidence geometries related to split irreducible spherical buildings. The original definition takes as point set the set of elation groups corresponding to the long roots of the underlying root system and as set of lines the family of sets consisting of such elations groups, each maximal relative to the property that their union forms a group [20]. It turns out that long root geometries thus defined

are just the Lie incidence geometries of type $\mathsf{X}_{n,J}$, where $J \subseteq S$ is the set of types corresponding to the roots of a fundamental system not perpendicular to the longest root. Explicitly, but restricting to rank at least 3, they are the Lie incidence geometries of types $\mathsf{A}_{n,\{1,n\}}$, $\mathsf{B}_{n,2}$, $\mathsf{C}_{n,1}$, $\mathsf{D}_{n,2}$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, $\mathsf{E}_{8,8}$ and $\mathsf{F}_{4,1}$. These geometries all share some intriguing properties, and they are so to speak the prototypes of non-strong parapolar spaces. A lot of information about long root geometries can be found in Shult's book [17].

2.9. Notation for buildings and Lie incidence geometries. Let Δ be a (thick) spherical building with type set S, $|S| \geq 2$. If Δ is irreducible, and its diagram X_n is simply laced, then Δ is unambiguously defined by a (skew)field $\mathbb{K}$ [23]. We denote Δ by $\mathsf{X}_n(\mathbb{K})$. The Lie incidence geometry of type $\mathsf{X}_{n,J}$, $J \subseteq S$, is denoted by $\mathsf{X}_{n,J}(\mathbb{K})$. For each connected Dynkin diagram X_n, $n \geq 1$, and each field $\mathbb{K}$ there is a unique split building $\mathsf{X}_n(\mathbb{K})$ obtained from the algebraic group of type X_n over the algebraic closure of $\mathbb{K}$ by split Galois descent [22, 23]. In these cases we use the natural Bourbaki labelling of the diagram and the type set is just $\{1, 2, \ldots, n\}$. We now introduce notation for all buildings of type F_4 and their residues.

Recall the Dynkin diagram of type F_4, see Figure 2. Let $\mathbb{K}$ be a field and let $\mathbb{A}$ be a quadratic alternative division algebra over $\mathbb{K}$. If $\mathbb{A}$ is not an inseparable field extension, then by Tits' classification [23] there is a unique building of type F_4 over the type set $\{1, 2, 3, 4\}$ whose residues of type $\{1, 2\}$ are projective planes over $\mathbb{K}$, whose residues of type $\{3, 4\}$ are projective planes over $\mathbb{A}$, and whose residues of type $\{1, 2, 3\}$ are the buildings, denoted by $\mathsf{B}_3(\mathbb{K}, \mathbb{A})$, associated to an orthogonal polar space of rank 3 over the field $\mathbb{K}$ whose anisotropic kernel is given by the norm form of $\mathbb{A}$. We denote that building by $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$. Note that, if $\mathbb{A} = \mathbb{K}$, then $\mathsf{F}_4(\mathbb{K}, \mathbb{A}) = \mathsf{F}_4(\mathbb{K})$ (as introduced in the previous paragraph). The residues of type $\{2, 3, 4\}$ are the buildings associated to polar spaces of rank 3 defined by an alternating form, or some σ-quadratic form, $\sigma \neq \mathrm{id}$, with trivial anisotropic kernel, or the nonembeddable polar space defined by $\mathbb{A}$. We denote this building by $\mathsf{C}_3(\mathbb{A}, \mathbb{K})$. If $\mathbb{A} = \mathbb{K}$, then $\mathsf{B}_3(\mathbb{K}, \mathbb{K}) = \mathsf{B}_3(\mathbb{K})$ and $\mathsf{C}_3(\mathbb{K}, \mathbb{K}) = \mathsf{C}_3(\mathbb{K})$. The point-residues in $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$ are orthogonal quadrangles which we denote by $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{A})$. The associated building is $\mathsf{B}_2(\mathbb{K}, \mathbb{A}) = \mathsf{C}_2(\mathbb{A}, \mathbb{K})$ and $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{A})$ is dual to $\mathsf{C}_{2,1}(\mathbb{A}, \mathbb{K})$.

Now suppose char $\mathbb{K} = 2$ and let $\mathbb{K}'$ be a(n inseparable) field extension of $\mathbb{K}$ with $\mathbb{K}'^2 \leq \mathbb{K} \leq \mathbb{K}'$. View $\mathbb{K}'$ as a vector space over $\mathbb{K}$ and let L be a sub(vector)space of $\mathbb{K}'$ containing $\mathbb{K}$. Since $\mathbb{K}'^2 \subseteq \mathbb{K}$, squaring in L defines an anisotropic quadratic form on L with values in $\mathbb{K}$, and hence defines, for each integer $n \geq 2$ via Equation 2.1, a unique orthogonal polar space of rank n the associated building of which we denote by $\mathsf{B}_n(\mathbb{K}, L)$. The corresponding polar space $\mathsf{B}_{n,1}(\mathbb{K}, L)$ is a quadric, which we sometimes refer to as an *inseparable quadric* (since it is related to the inseparable field extension $\mathbb{K}'$ of $\mathbb{K}$). Note that $\mathsf{B}_3(\mathbb{K}, \mathbb{K}) = \mathsf{B}_3(\mathbb{K})$, as defined above. We define the building $\mathsf{C}_n(\mathbb{K}', L)$ as $\mathsf{B}_n(\mathbb{K}'^2, L)$. Then $\mathsf{B}_n(\mathbb{K}, L)$ is the same as $\mathsf{C}_n(\mathbb{K}, L^2)$.

For fields $\mathbb{K}$ and $\mathbb{K}'$ as in the previous paragraph, $\mathsf{C}_n(\mathbb{K}', \mathbb{K}) \cong \mathsf{B}_n(\mathbb{K}'^2, \mathbb{K})$ and $\mathsf{B}_n(\mathbb{K}, \mathbb{K}') \cong \mathsf{C}_n(\mathbb{K}, \mathbb{K}'^2)$. For $n = 2$ we also have that $\mathsf{C}_{2,1}(\mathbb{K}, \mathbb{K}') \cong \mathsf{B}_{2,2}(\mathbb{K}', \mathbb{K})$. Now we define $\mathsf{F}_4(\mathbb{K}, \mathbb{K}')$ as the building of type F_4 with type set $\{1, 2, 3, 4\}$ over the Dynkin diagram F_4 (and with Bourbaki labelling) and residues of type $\{1, 2, 3\}$ isomorphic to

$\mathsf{B}_3(\mathbb{K}, \mathbb{K}')$, and hence residues of type $\{2, 3, 4\}$ isomorphic to $\mathsf{C}_3(\mathbb{K}', \mathbb{K})$. We occasionally refer to this building and its related metasymplectic spaces as *of inseparable type*.

This definition implies that, for fields $\mathbb{K}'^2 \leq \mathbb{K} \leq \mathbb{K}'$ of characteristic 2, we have $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{K}') \cong \mathsf{F}_{4,4}(\mathbb{K}'^2, \mathbb{K})$ and $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K}') \cong \mathsf{F}_{4,1}(\mathbb{K}'^2, \mathbb{K})$.

Note that by Tits' classification [23], and with the notation just introduced, any building of type F_4 is isomorphic to $\mathsf{F}_4(\mathbb{K}, \mathbb{A})$, for some quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$ (including inseparable field extensions in characteristic 2).

2.10. Projectivities. The main results of the present paper classify embeddings of subgeometries "up to projective equivalence" in either $\mathsf{D}_{6,6}(\mathbb{K})$ or $\mathsf{E}_{7,1}(\mathbb{K})$. In order to define this, we first need the definition of a projectivity. The *group of projectivities* of a Lie incidence geometry Δ containing projective planes as (not necessarily maximal) singular subspaces, is the group of collineations of Δ generated by all collineations each of which pointwise fixes some line or elementwise fixes a full line pencil of a singular plane. This amounts to the universal Chevalley group of respective type. A *projectivity* of Δ is a member of the group of projectivities of Δ. Then two embeddings Γ' and Γ'' in Δ are *projectively equivalent* if there exists a projectivity of Δ mapping Γ' bijectively to Γ''. A projectivity will also be referred to as a linear automorphism.

2.11. General notation. For a field $\mathbb{K}$, we denote by $\mathbb{K}^\times$ its multiplicative group. A *duality* σ of a Lie incidence geometry is an automorphism of the underlying spherical building that does not preserve types. For a duality, it is possible that an object is incident with its image without coinciding with it. Such objects will be called *absolute (with respect to the duality)*. If σ has order 2, then each pair of absolute objects $\{x, x^\sigma\}$ defines a σ-fixed flag. In this case, considering the fixed point structure is equivalent to considering only the absolute elements of certain types. For instance, a projective space of odd dimension $2n + 1$ over a field $\mathbb{K}$ admits a symplectic polarity, and the absolute elements of dimension at most n form the building $\mathsf{C}_n(\mathbb{K})$, the related polar space $\mathsf{C}_{n,1}(\mathbb{K})$ being fully embedded in $\mathsf{PG}(2n + 1, \mathbb{K}) = \mathsf{A}_{2n+1,1}(\mathbb{K})$.

Finally, when a proof contains several numbered claims, we indicate the end of the proof of each claim with a little white square □ at the end of the line for clarity.

3. Distances in the parapolar spaces under consideration

In the next four sections we review some incidence and distance-related properties of specific parapolar spaces. In all the facts we state, all possibilities do occur, and we shall not mention this each time (since we do not need it in our proofs).

In order to shape the unexperienced reader's intuition, we shall demonstrate many properties on the "thin version" of the corresponding geometry, that is, the case where lines have exactly two points. These are in fact the apartments, and, as is well known, many properties of apartments carry over without any change to the whole building. In particular, the possibilities of the different mutual positions of two flags can be read off the

apartment since two flags are always contained in a common apartment, and apartments are convex (once again see [23], more exactly Section 3 and Proposition 3.18 therein).

Also, some of the geometries admit a representation (full embedding) in a projective space that can be described in an explicit algebraic way (using "forms"). This is not the case for the long root geometries. However, these geometries have been defined as an intersection of quadrics, see [1] for the case of $\mathsf{E}_{7,1}$. The half spin geometries can be defined using the algebraic theory of spinors (see [5]); in particular this provides an explicit construction of the half spin geometry of type $\mathsf{D}_{6,6}$. Finally, there is a four form for the 56-dimensional module for groups of type E_7 defining an embedding of the Lie incidence geometries of type $\mathsf{E}_{7,7}$ in 55-dimensional projective space, see [9].

3.1. Dual polar spaces of rank 3. Let Δ be a dual (thick) polar space of rank 3 and let Δ^* be the corresponding (thick) polar space of rank 3. The points of Δ correspond to the singular planes of Δ^* and the lines of Δ correspond to the lines of Δ^*. We will view Δ as a strong parapolar space of diameter 3. Its symps are generalised quadrangles and correspond to the residues of points in Δ^*. The maximal singular subspaces of Δ are lines.

The following facts can easily be derived by arguing in Δ^*. Note that two points x, y of Δ are collinear when the respective corresponding singular planes π_x, π_y in Δ^* intersect in a line of Δ^*; they are at distance 2 when $\pi_x \cap \pi_y$ is a point of Δ^* and, finally, at distance 3 when π_x and π_y are opposite in Δ^*.

The "thin version" of a dual polar space of rank 3 is a cube. The points are the vertices, the lines are the edges and the symps are the faces. We illustrate each fact on the cube.

FACT 3.1 (Point-point relations). *Let x and y be two points of Δ. Then $\delta_\Delta(x, y) \leq 3$ (and distance 3 occurs and corresponds to opposite points) and if $\delta_\Delta(x, y) = 2$, then x and y are contained in a unique symp $x \Diamond y$.*

Indeed, on the cube points at distance 3 correspond to opposite vertices; points at distance 2 correspond to vertices not on an edge, but contained in a unique face.

FACT 3.2 (Point-symp relations). *Let p be a point and Σ a symp of Δ with $p \notin \Sigma$. Then Σ contains a unique point q collinear to p; moreover, $\delta(p, x) = 2$ for all $x \in (q^\perp \setminus \{q\}) \cap \Sigma$ and $\delta(p, q) = 3$ for all $x \in \Sigma \setminus q^\perp$.*

Indeed, this can also easily be seen on the cube: a vertex off a face is on an edge with a unique vertex of the face.

FACT 3.3 (Symp-symp relations). *Let Σ and Σ' be distinct symps of Δ. Then $\Sigma \cap \Sigma'$ is either empty or a line.*

Indeed, two distinct faces of a cube are either opposite and disjoint, or meet in an edge.

3.2. The half spin geometry of type D_6. Let Δ be the half spin geometry $\mathsf{D}_{6,6}(\mathbb{K})$, for some field $\mathbb{K}$ and let Δ^* be the corresponding polar space $\mathsf{D}_{6,1}(\mathbb{K})$. Recall that we refer to the two natural families of generators of Δ^* by 5-spaces and 5'-spaces (objects of type 6 and 5, respectively, of the underlying spherical building of type $\mathsf{D}_6(\mathbb{K})$). The points of Δ correspond to the 5-spaces of Δ^* and the lines of Δ correspond to the singular 3-spaces

of Δ^* (objects of type 4 of the underlying spherical building). We will view Δ as a strong parapolar space of diameter 3. Its symps are polar spaces isomorphic to $\mathsf{D}_{4,1}(\mathbb{K})$ and correspond to the residues of lines in Δ^*. The objects corresponding to the point-residues in Δ^* are half spin geometries $\mathsf{D}_{5,5}(\mathbb{K})$ and these are *paras* of Δ (which are parapolar spaces with diameter 2). A maximal singular subspace of Δ either corresponds to a $5'$-space of Δ^* (in case it has dimension 5), or to a plane of Δ^* (in case it has dimension 3). Any singular 4-space of Δ is the intersection of a unique singular 5-space and a unique para.

The following facts can easily be derived by arguing in Δ^*. Note that two points x, y of Δ are collinear when the respective corresponding 5-spaces U_x, U_y in Δ^* intersect in a 3-space of Δ^*; they are at distance 2 when $U_x \cap U_y$ is a line of Δ^* and, finally, at distance 3 when U_x and U_y are opposite in Δ^*.

The "thin version" of a half spin geometry is a "halved hypercube". In the case of D_6, it is the halved 6-cube, that is, the halved hypercube on 2^6 vertices. It can be defined as the ordinary 5-cube where one adds edges between every pair of vertices at distance 2 (see Proposition 4.2.18 in [2]). The points of the geometry are the vertices of the halved 6-cube, the lines are the edges, the planes are the triangles, the solids the 4-cliques, the 4-dimensional singular spaces the 5-cliques, the 5-dimensional singular spaces the 6-cliques (given by all vertices adjacent to a fixed vertex of the ordinary 5-cube), the symps the halved 4-cubes and the paras the halved 5-cubes contained in the halved 6-cube.

For the convenience of the reader we mention that an explicit algebraic representation of the halved 6-cube can be given by the binary strings of length 5, adjacent when differing in at most two positions. Maximal 4-cliques are given by all strings sharing 4 fixed positions with a given string; maximal 6-cliques by all strings differing in at most one position from a given string; a symp is given either by the set of all strings agreeing in (at least) two fixed positions with a given string (type 1 for further reference), or by the set of strings agreeing in at least one fixed position and in total exactly in an odd number of positions with a given string (type 2). Finally, the paras are given either by the strings agreeing in (at least) one fixed position with a given string, or by the set of strings containing either an even number of zeros, or an odd number of zeros.

FACT 3.4 (Point-point relations). *Let x and y be two points of Δ. Then $\delta_\Delta(x, y) \le 3$ (and distance 3 occurs and corresponds to opposite points) and if $\delta_\Delta(x, y) = 2$, then x and y are contained in a unique symp $x \Diamond y$.*

Indeed, two strings can always be transformed into each other by altering at most three times at most two positions. Two strings differing in exactly three positions are clearly contained in a type 1 symp; two strings differing in exactly four positions in a type 2 string.

FACT 3.5 (Point-symp relations). *Let p be a point and Σ a symp of Δ with $p \notin \Sigma$. Then precisely one of the following occurs.*

 (i) *$p^\perp \cap \Sigma$ is a unique point q. In this case, $\delta(p, x) = 2$ for all $x \in \Sigma \cap (q^\perp \setminus \{q\})$ and $\delta(p, x) = 3$ for all $x \in \Sigma \setminus q^\perp$.*
 (ii) *$p^\perp \cap \Sigma$ is a 3-space U of Σ. In this case, $\delta(p, x) = 2$ for all $x \in \Sigma \setminus U$; moreover, there is a unique para containing p and Σ.*

Indeed, we illustrate this in case the symp is determined by the strings 00000 and 11110. Clearly, each string can be transformed into a string with an even number of zeros in the first four positions, and 0 in the last position, by altering at most two digits. This transformation is unique if the string already has an even number of zeros in the first four positions (and 1 in the last); in all other cases there are precisely four possibilities, which moreover mutually differ in exactly two positions.

FACT 3.6 (Symp-symp relations). *Let Σ and Σ' be distinct symps of Δ. Then precisely one of the following occurs.*

(i) $\Sigma \cap \Sigma'$ is a 3-space and Σ and Σ' are contained in a unique common para.
(ii) $\Sigma \cap \Sigma' = \emptyset$ and Σ and Σ' are contained in a unique common para.
(iii) $\Sigma \cap \Sigma'$ is a line, and they are not contained in a common para.
(iv) $\Sigma \cap \Sigma' = \emptyset$, they are not contained in a common para, and there is a unique symp Σ'' intersecting both Σ and Σ' in 3-spaces.
(v) $\Sigma \cap \Sigma' = \emptyset$, they are not contained in a common para, and each point of Σ is collinear to a unique point of Σ' and vice versa.

This is rather tedious to check on the halved 5-cube, but nevertheless can be done by the interested reader.

3.3. Parapolar spaces of type $\mathsf{F_4}$. Consider any (thick) building of type $\mathsf{F_4}$ and let Γ be the Lie incidence geometry of either type $\mathsf{F_{4,1}}$ or $\mathsf{F_{4,4}}$ derived from it. These are the *(thick) metasymplectic space* we will be dealing with. Such a space is a non-strong parapolar space with symps of rank 3, not of hyperbolic type (the types that do occur are listed in 2.9). The basic properties of Γ are the following, stated as facts (and we refer to [6]). All these facts can be checked inside a "thin version" of a metasymplectic space. We present and use one model of such an apartment in the proof of Theorem 5.6 below. Here, we content ourselves with providing a picture of this thin version in Figure 3. All symps can be obtained by considering the convex closure of two "outer" vertices in this picture; we gave one example in grey. The geodesics (paths of length 3) between the two marked opposite vertices are drawn in bold; each subpath of length 2 of these paths connects a pair of special vertices (this readily follows from Fact 3.11). The use of black en grey is merely for clarity and has no mathematical reason.

FACT 3.7. *The lines, planes and symps through a given point p, endowed with the natural incidence relation, form a dual polar space $\mathrm{Res}_\Gamma(p)$ of rank 3, where the points of the corresponding polar space are the symps through p, the lines are the planes of Γ through p, and the planes are the lines of Γ through p.*

The geometry $\mathrm{Res}_\Gamma(p)$ is usually called the *point-residue geometry* of Γ. Its isomorphism class does not depend on p, it only depends on the isomorphism class of Γ (for the types that occur, we again refer to 2.9).

FACT 3.8 (Point-point relations). *Let x and y be two points of Γ. Then $\delta_\Gamma(x, y) \leq 3$ (and distance 3 occurs and corresponds to opposite points) and if $\delta_\Gamma(x, y) = 2$, then either x and y are contained in a unique symp $x \diamondsuit y$, or there is a unique point $x \bowtie y$ collinear to both x and y.*
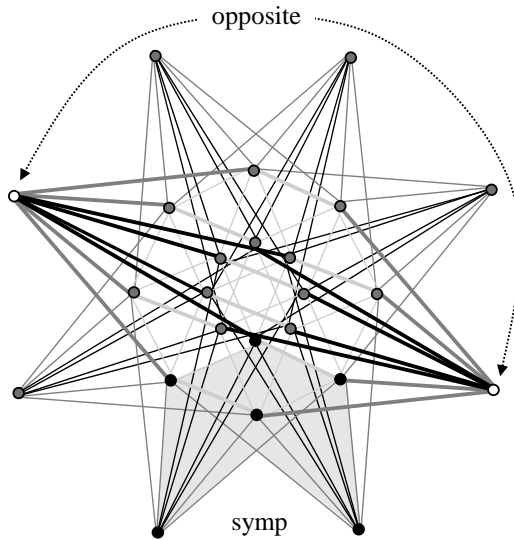
Fig. 3. A thin version of a metasymplectic space

FACT 3.9 (Symp-symp relations). *The intersection of two symps is either empty, or a point, or a plane.*

FACT 3.10 (Point-symp relations). *Let p be a point and Σ a symp of Γ with $p \notin \Sigma$. Then one of the following occurs:*

   (i) $p^\perp \cap \Sigma$ is line L. In this case, p and x are symplectic for all $x \in \Sigma \cap (L^\perp \setminus L)$ (and $L \subseteq p \Diamond x$), and p and x are special for all $x \in \Sigma \setminus L^\perp$ (and $p \bowtie x \in L$). We say that p and Σ are <u>close</u>;

   (ii) $p^\perp \cap \Sigma$ is empty, but there is a unique point u of Σ symplectic to p (so $\Sigma \cap (p \Diamond u) = \{u\}$). Then x and p are special for all $x \in \Sigma \cap (u^\perp \setminus \{u\})$ (and $x \bowtie p \notin \Sigma$), and x and p are opposite if $x \in \Sigma \setminus u^\perp$. We say that p and Σ are <u>far</u>.

As a consequence, one can easily deduce the following fact which we will frequently use (and which in fact holds for all long root geometries related to buildings).

FACT 3.11. *If $p \perp\!\!\!\perp r \perp q$, then p is never opposite q.*

3.4. Non-strong parapolar spaces of type $\mathsf{E}_{7,1}$. Let Δ be the long root geometry $\mathsf{E}_{7,1}(\mathbb{K})$, for some field $\mathbb{K}$. We view Δ as a parapolar space, which has diameter 3 and is non-strong. The elements of the corresponding building of types $1, 2, 3, 4, 5, 6, 7$, are the *points*, 6-*spaces*, *lines*, *planes*, 4-*spaces*, *symps* and *paras*, respectively. The symps are isomorphic to polar spaces $\mathsf{D}_{5,1}(\mathbb{K})$, and the paras are strong parapolar spaces $\mathsf{E}_{6,1}(\mathbb{K})$. The other types are singular (projective) subspaces of Δ. Besides those, Δ also contains singular subspaces of dimension 5, which do not correspond to a type but each of them is the intersection of a unique para and a unique element of type 2 (i.e. a 6-space). The

4-dimensional subspaces contained in those 5-spaces are also singular subspaces of Δ not corresponding to a single type of Δ and those are referred to as $4'$-spaces.

One can deduce the possible mutual position of points, symps and paras, etc., by considering an appropriate model of an apartment of a building of type E_7. Such models are given in [26]. One of them is worked out in Section 6.3.2 at the end of this paper. For the moment, we limit ourselves to mentioning that the root system of type E_7 provides such a model: the points of the geometry are the roots; two points are collinear if the corresponding roots form an angle of 60 degrees; two points are special if the corresponding roots form an angle of 120 degrees; two points are symplectic if the corresponding roots are perpendicular; two points are opposite if the corresponding roots are opposite. The symps are the subsystems of type D_5. The paras those of type E_6.

The relations between two points are the same as in Fact 3.8.

FACT 3.12 (Point-symp relations). *If p is a point and Σ a symp of Δ with $p \notin \Sigma$, then precisely one of the following occurs.*

(i) *p is symplectic to a unique point $q \in \Sigma$. In this case, p and x are special for all $x \in \Sigma \cap (q^\perp \setminus \{q\})$ (and $p \bowtie x \notin \Sigma$), and p and x are opposite for all $x \in \Sigma \setminus q^\perp$.*

(ii) *p is collinear to a 4-space U of Σ; also p and Σ are contained in a unique para Π. In this case, p and x are symplectic if $x \in \Sigma \setminus U$.*

(iii) *p is symplectic to each point of a 4-space U of Σ; in this case, p and x are special if $x \in \Sigma \setminus U$ (and $p \bowtie x \notin \Sigma$).*

(iv) *there is a unique line $L \subseteq \Sigma$ with $p \perp L$. In this case, p and x are symplectic if $x \in \Sigma \cap (L^\perp \setminus L)$ and p and x are special if $x \in \Sigma \setminus L^\perp$ (and $p \bowtie x \in L$).*

(v) *p is symplectic to all points of Σ. In this case, p and Σ are contained in a unique para Π, in which they are Π-opposite.*

In Cases (i), (iii) and (iv), the point p and the symp Σ are not contained in a common para.

FACT 3.13 (Point-para relations). *If p is a point and Π a para of Δ with $p \notin \Pi$, then precisely one of the following occurs.*

(i) *p is collinear to a unique 5-space W in Π. In this case, p is said to be <u>close</u> to Π. The point p is symplectic or special to the points of $\Pi \setminus W$; it is special to $x \in \Pi$ precisely when x is collinear to a unique point of W.*

(ii) *p is not collinear to any point of Π, but it is contained in a unique para Π' that intersects Π in a symp. In this case, p and Π are said to be <u>far</u> from each other. The point p and the symp $\Pi \cap \Pi'$ are opposite in Π'.*

FACT 3.14 (Para-para relations). *If Π and Π' are distinct paras, then precisely one of the following occurs.*

(i) *$\Pi \cap \Pi'$ is a symp;*

(ii) *$\Pi \cap \Pi'$ is a point;*

(iii) *$\Pi \cap \Pi' = \emptyset$ and each point $x \in \Pi$ is far from Π'. Let Σ_x be the unique symp of Π' contained in a para with x, unique by Fact 3.13(ii). Then each point of $\Pi' \setminus \Sigma_x$ collinear to a point of Σ_x is special to x, and each point in Π' which is Π'-opposite Σ_x is at distance 3 from x. The correspondence $\Pi \longrightarrow \Pi' : x \mapsto \Sigma_x$ induces a*

duality.

FACT 3.15. *Let Π be a para of Δ and let W, W' be two Π-opposite singular 5-spaces of Π. Let U and U' be the unique singular 6-spaces containing W and W', respectively. Then every point $u \in W$ is collinear to a unique point $\theta(u)$ of W'. Let $u \in U$ and $u' \in U'$. Then*

(*i*) *$u \perp u'$ if, and only if, $u \in W$ and $u' = \theta(u)$,*
(*ii*) *$u \perp\!\!\!\perp u'$ if, and only if, $u \in W$, $u' \in W'$ and $u' \neq \theta(u)$,*
(*iii*) *$u \bowtie u'$ if, and only if, either $u \in U \setminus W$ and $u' \in W'$, or $u \in W$ and $u' \in U' \setminus W'$,*
(*iv*) *u is opposite u' if, and only if, $u \in U \setminus W$ and $u' \in U' \setminus W'$.*

Conversely, let W and W' be two singular 5-spaces such that each point of W is collinear to a unique point of W'. Then W and W' are contained in a unique para Π, where they are Π-opposite.

We record the following property of Δ (which in fact holds for all long root geometries related to spherical buildings):

FACT 3.16. *Let $p \perp x \perp y \perp q$ be a path in Δ with (p, y) and (q, x) special. Then p and q are opposite, i.e., $\delta(p, q) = 3$.*

Proof. We extend the path in Δ with a further point $z \perp q$ such that z is opposite x. Then xy is the projection of zq onto $\mathrm{Res}_\Delta(x)$. The hypothesis implies that xp and xy are opposite in $\mathrm{Res}_\Delta(x)$. Hence 3.28 and 3.29 of [23] imply that the lines xp and zq are opposite. Again 3.28. and 3.29 of [23] imply that non-opposition between these lines is a bijection. Hence the only point on xp not opposite q is x. But then p is opposite q. ∎

3.5. Strong parapolar spaces of type $\mathsf{E}_{7,7}$. We will also need the Lie incidence geometry $\mathsf{E}_{7,7}(\mathbb{K})$, since this parapolar space is slightly less complicated than the long root geometry since it also has diameter 3 but has no special pairs, and there are no paras.

We shall use the notation Δ^* for the point-line geometry $\mathsf{E}_{7,7}(\mathbb{K})$ obtained from $\Delta = \mathsf{E}_{7,1}(\mathbb{K})$ by taking as points the paras of Δ and as lines the symps of Δ, with obvious incidence relation. We refer to Δ^* as the *dual* of Δ.

Then Δ^* is a strong parapolar space of diameter 3; points at distance 3 are called *opposite*. A maximal singular subspace has either dimension 5 (in this case occurring as an intersection of two symps and corresponding to a type 3 element in the Dynkin diagram) or dimension 6 (type 2 in the Dynkin diagram). The 5-dimensional subspaces of a 6-space will be called $5'$-*spaces*. They do not correspond to a single node of the Dynkin diagram, but rather to a flag of type $\{1, 2\}$. Each symp of Δ^* is isomorphic to the polar space $\mathsf{D}_{6,1}(\mathbb{K})$ (the residue of an element of type 1 in the underlying spherical building). Furthermore, the lines, planes, 3-dimensional singular subspaces and 4-dimensional subspaces correspond to types 6, 5, 4 and $\{2, 3\}$ in the Dynkin diagram.

We now review the point-symp and symp-symp relations. As in the previous sections, they can be deduced by considering an appropriate model of an apartment (the "thin version") of a building of type E_7, as given in [26]. Basically, such a model is given by the *Gosset graph*, which has on its turn many descriptions and constructions. One of them is as

the 1-skeleton of the 3_{21} polytope (see [2], pages 103 and 104). A traditional construction runs as follows. The 56 vertices are the pairs from the respective 8-sets $\{1, 2, \ldots, 8\}$ and $\{1', 2', \ldots, 8'\}$. Two pairs from the same set are adjacent if they intersect in precisely one element; two pairs $\{a, b\}$ and $\{c', d'\}$ from different sets are adjacent if $\{a, b\}$ and $\{c, d\}$ are disjoint. The symps correspond to cross-polytopes of size 12 (so-called *hexacrosses* or 6-*orthoplexes*) contained in the Gosset graph. There are 126 such, and 56 of these are determined by an ordered pair (i, j) with $i, j \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, $i \neq j$, and induced on the vertices $\{i, k\}$ and $\{j', k'\}$, $k \notin \{i, j\}$, whereas the other 70 are determined by a 4-set $\{i, j, k, \ell\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ and are induced on the vertices $\{s, t\} \subseteq \{i, j, k, \ell\}$, $s \neq t$, and $\{u', v'\} \subseteq \{1', 2', 3', 4', 5', 6', 7', 8'\} \setminus \{i', j', k', \ell'\}$, $u \neq v$.

Armed with this description of the thin version, one can verify the following facts.

FACT 3.17 (Point-symp relations). *If p is a point and Σ a symp of Δ^* with $p \notin \Sigma$, then precisely one of the following occurs.*

(i) *p is collinear to a unique point $q \in \Sigma$. In this case, p and x are symplectic if $x \in \Sigma \cap (q^\perp \setminus \{q\})$ and $\delta(p, x) = 3$ for $x \in \Sigma \setminus q^\perp$.*

(ii) *p is collinear to a $5'$-space U of Σ. In this case, x and p are symplectic if $x \in \Sigma \setminus U$.*

This fact implies that on each line L, there is at least one point symplectic to a given point p (unique when L contains at least one point opposite p). We will use this without reference.

FACT 3.18 (Symp-symp relations). *If Σ and Σ' are two symps of Δ^*, then precisely one of the following occurs.*

(i) *$\Sigma = \Sigma'$;*

(ii) *$\Sigma \cap \Sigma'$ is a 5-space.*

(iii) *$\Sigma \cap \Sigma'$ is a line L. Then points $x \in \Sigma \setminus L$ and $x' \in \Sigma' \setminus L$ are never collinear and $\delta(x, x') = 3$ if, and only if, $x^\perp \cap L$ is disjoint from $x'^\perp \cap L$.*

(iv) *$\Sigma \cap \Sigma' = \emptyset$ and there is a unique symp Σ'' intersecting Σ in a 5-space U and intersecting Σ' in a 5-space U', with U and U' opposite in Σ''.*

(v) *$\Sigma \cap \Sigma' = \emptyset$ and every point of Σ is collinear to a unique point of Σ'. In this situation, Σ and Σ' are <u>opposite</u>.*

3.6. The embeddings of polar spaces in parapolar spaces.
Recall that polar spaces are assumed to have thick lines.

LEMMA 3.19. *Let $\Sigma = (X, \mathcal{L})$ and $\Sigma' = (X', \mathcal{L}')$ be two polar spaces of rank at least 2. Suppose that Σ' is fully embedded in Σ. Then either*

(i) *X' is contained in a singular subspace of Σ; or*

(ii) *Σ' is isometrically embedded in Σ.*

If in the latter case, Σ is a quadric embedded in a (not necessarily finite-dimensional) projective space $\mathbb{P}$, then Σ' arises as the intersection of Σ with a subspace of $\mathbb{P}$.

Proof. If every pair of points of Σ' is collinear in Σ, then clearly Case (i) holds. So suppose that there is a pair $\{x, y\} \subseteq X'$ of points opposite in Σ. We will denote collinearity in Σ and Σ' by $\perp_\Sigma$ and $\perp_{\Sigma'}$, respectively. In order to show that Case (ii) holds, we only need

to show that every pair of opposite points of Σ' is also opposite in Σ. We prove some claims.

Claim 1: Every pair of points in $x^{\perp_{\Sigma'}}$ which is opposite in Σ', is opposite in Σ.

Let $u, v \in x^{\perp_{\Sigma'}}$ and assume that u is opposite v in Σ', but $u \perp_\Sigma v$. Our assumption implies that x, u, v generate a plane π_x of Σ, in which $L_u := ux$ and $L_v := vx$ are two non-collinear lines of Σ'. Let M_u and M_v be the unique lines of Σ' through y meeting L_u and L_v, respectively. Then M_u, M_v are contained in a plane π_y of Σ, with $x \notin \pi_x \cap \pi_y = K \in \mathcal{L}$. Select a point $z \in L_u \setminus K$, with $z \neq x$. In Σ', there is a unique point $w \in M_v$ collinear to z, and necessarily $w \notin K$. This implies that z is also collinear to w in Σ. As $z, w \notin K$, this means that the planes π_x and π_y are contained in a singular 3-space of Σ, contradicting the fact that x and y are not collinear. □

Claim 2: Every point opposite x in Σ' is also opposite x in Σ.

Take any $y' \in X'$ be opposite x in Σ'. Select $u, v \in x^{\perp_{\Sigma'}} \cap y'^{\perp_{\Sigma'}}$ so that u is not collinear to v in Σ'. By Claim 1, u and v are also opposite in Σ. Using the pair u, v in the role of x, y in Claim 1, we obtain that x and y', which belong to $u^{\perp_{\Sigma'}}$ and are opposite in Σ', are also opposite in Σ. □

Claim 3: Every pair of points opposite in Σ' is opposite in Σ.

By Claim 1, we may assume that at least one of a, b, say a, is opposite x in Σ'. By Claim 2, a and x are also opposite in Σ. Again by Claim 2, and with a in the role of x, it follows that a and b are opposite in Σ indeed. □

Since the mutual positions of singular subspaces of polar spaces are determined by the collinearity behaviour of their points, we conclude that Σ' is isometrically embedded in Σ, completing the proof of the first part of the lemma.

For the final assertion, suppose that Σ is a quadric in $\mathsf{PG}(V)$ in which Σ' embeds isometrically, i.e., the collinearity relation of Σ' coincides with that of Σ on X', so we denote both relations by $\perp$. Let U be the subspace of $\mathsf{PG}(V)$ generated by X'. We show that $U \cap X' = U \cap X$.

For each point $x' \in X'$, we denote by $H_{x'}$ the subspace generated by the set of lines of Σ' through x', and by $T_{x'}$ the tangent hyperplane of Σ at x'. Noting that all points of $T_{x'} \cap X'$ belong to $H_{x'}$ and that there is at least one point of X' opposite x', we obtain that $T_{x'} \cap U$ is a hyperplane of U that coincides with $H_{x'}$. Assume for a contradiction that U contains a point $x \in X \setminus X'$. Then $x^\perp \cap X'$ either coincides with X', or is a geometric hyperplane of Σ'. Either way, $x^\perp \cap X'$ contains two opposite points u, v. We now consider the polar space $\Sigma'_{u,v}$ given by $u^\perp \cap v^\perp \cap X'$, spanning the subspace $H_u \cap H_v$, which is fully embedded in the polar space $\Sigma_{u,v}$ given by $u^\perp \cap v^\perp$. Since $H_u = T_u \cap U$ and $H_v = T_v \cap U$, the point x belong to $H_u \cap H_v$, but by assumption not to $\Sigma'_{u,v}$.

Repeated use of the above argument shows that we may assume that Σ' has rank 2. Then, Theorem 8.5.13 of [24] implies that X' is the point set of a quadric itself. Consider the plane $\pi = \langle u, v, x \rangle$. Since v is opposite u, both H_u and H_v meet π in just a line. Hence $\pi \cap X'$ is a possibly degenerate conic through u and v, contained in $\pi \cap X$, which is the union of the non-collinear lines xu and xv. This implies that $\pi \cap X' = \pi \cap X$, contradicting $x \notin X'$. ∎

LEMMA 3.20. *Let Σ be a polar space fully embedded in a parapolar space Δ. Then either*

Σ *is completely contained in a singular subspace of* Δ, *or* Σ *is fully and isometrically embedded in a unique symp of* Δ.

Proof. If all points of Σ are pairwise Δ-collinear, then Σ is contained in a singular subspace of Δ. So suppose that there exist two points $x, y \in \Sigma$ such that $\delta_\Sigma(x, y) = \delta_\Delta(x, y) = 2$. Every point $z \in x^{\perp_\Sigma} \cap y^{\perp_\Sigma}$ belongs to $x^{\perp_\Delta} \cap y^{\perp_\Delta}$, hence $z \in x \Diamond y$. Using the same argument as in Claim 1 of the proof of Lemma 3.19, one shows that, if points $u, v \in x^{\perp_\Sigma}$ are Σ-opposite, then they are also opposite in $x \Diamond y$. Now let $z \in \Sigma$ be arbitrary, but opposite x. Then we can find two points $u, v \in x^{\perp_\Sigma} \cap z^{\perp_\Sigma}$ which are Σ-opposite. Hence z belongs to $u \Diamond v = x \Diamond y$. We obtain that Σ is fully embedded in $x \Diamond y$. It then follows from Lemma 3.19 that Σ embeds isometrically in $x \Diamond y$. Since any pair of Δ-symplectic points determines a unique Δ-symp by definition, the uniqueness follows. ∎

4. Dual polar spaces of rank 3 in half spin geometries $\mathsf{D}_{6,6}(\mathbb{K})$

In this section, we prove the following theorem.

THEOREM 4.1. *Let* Γ *be a dual polar space of rank* 3 *fully embedded in the half spin geometry* $\Delta = \mathsf{D}_{6,6}(\mathbb{K})$, *for some field* $\mathbb{K}$, *but not contained in a singular subspace of* Δ. *Then precisely one of the following occurs.*

(i) $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$, *for some (separable or inseparable) quadratic extension* $\mathbb{L}$ *of* $\mathbb{K}$, *and is the fixed point structure of each nontrivial element of a subgroup of collineations of* Δ *isomorphic to the factor group* $\mathbb{L}^\times / \mathbb{K}^\times$; *the quadratic extensions* $\mathbb{L}$ *of* $\mathbb{K}$ *in the algebraic closure of* $\mathbb{K}$ *are in one-to-one correspondence with the classes of projectively equivalent fully embedded dual polar spaces* $\mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$ *in* Δ *(and each such embedding is isometric).*

(ii) $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{K})$ *and arises as the fixed point structure of a subgroup of collineations of* Γ *isomorphic to* $\mathsf{PGL}_2(\mathbb{K})$; *this embedding is isometric and projectively unique.*

(iii) $\Gamma \cong \mathsf{B}_{3,3}(\mathbb{F}, \mathbb{K})$, *for some subfield* $\mathbb{F}$ *of* $\mathbb{K}$, *with either* $\mathbb{F} = \mathbb{K}$, *or* $\mathbb{K}$ *a quadratic Galois extension of* $\mathbb{F}$ *(i.e., a separable quadratic extension), or* char $\mathbb{K} = 2$ *and* $\mathbb{K}^2 \leq \mathbb{F} \leq \mathbb{K}$. *In the latter case* Γ *is contained in a unique fully embedded dual polar space of* Δ *isomorphic to* $\mathsf{B}_{3,3}(\mathbb{K})$. *In all cases* Γ *is contained in a symp* $Q \cong \mathsf{D}_{4,1}(\mathbb{K})$ *of* Δ. *If* $\mathbb{F} = \mathbb{K}$ *or if* $\mathbb{K}$ *is a quadratic Galois extension of* $\mathbb{F}$, *then* Γ *is the fixed point structure of an involutory duality of* Q. *If* $\mathbb{F} = \mathbb{K}$, *then the embedding in* Q *is projectively unique; the subfields* $\mathbb{F}$ *of* $\mathbb{K}$ *such that* $\mathbb{K}$ *is a separable quadratic extension of* $\mathbb{F}$ *are in one-to-one correspondence with the classes of projectively equivalent fully embedded dual polar spaces* $\mathsf{B}_{3,3}(\mathbb{F}, \mathbb{K})$ *in* Q; *if* char $\mathbb{K} = 2$, *then the subfields* $\mathbb{F}$ *of* $\mathbb{K}$ *such that* $\mathbb{K}^2 \leq \mathbb{F} \leq \mathbb{K}$ *are in one-to-one correspondence with the classes of projectively equivalent fully embedded dual polar spaces* $\mathsf{B}_{3,3}(\mathbb{F}, \mathbb{K})$ *in* Q. *Each class of projectively equivalent fully embedded dual polar spaces in* Q *induces exactly two classes of projectively equivalent fully embedded dual polar spaces in* Δ.

Throughout Section 4, let Γ be any dual polar space of rank 3, fully embedded in the half spin geometry $\Delta = \mathsf{D}_{6,6}(\mathbb{K})$, but not contained in a singular subspace of Δ. The proof

of Theorem 4.1 evolves around the possible ways to embed symps of Γ in Δ. This will lead to a case distinction as explained in the next subsection. The proof of Theorem 4.1 will be completed in Subsection 4.5.

4.1. The embedding of a symp of Γ — case distinction. Let Σ be a symp of Γ. By Lemma 3.20, Σ is either isometrically embedded in a unique symp Ω of Δ, or it is contained in a singular subspace of Δ. In the first case, there are two possibilities to consider. To that end, consider a representation of Ω in projective 7-space $\mathsf{PG}(7, \mathbb{K})$. It follows from Lemma 3.19 that Σ is either contained in a singular subspace of Ω, contrary to our assumption, or it arises as the intersection of Ω with a subspace W of $\mathsf{PG}(7, \mathbb{K})$. Note that $\dim W \geq 3$ for it contains opposite lines of Σ, in fact $\dim W \geq 4$ since Σ is not a hyperbolic quadrangle; lastly, $\dim W \leq 5$ for otherwise W, and hence Σ, contains a singular plane of Ω. If $\dim W = 5$, Σ is isomorphic to an orthogonal quadrangle $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{L})$ where $\mathbb{L}$ is a quadratic extension of $\mathbb{K}$; if $\dim W = 4$, Σ is isomorphic to an orthogonal quadrangle $\mathsf{B}_{2,1}(\mathbb{K})$. We will hence distinguish three cases (we show in Lemma 4.2 that Cases I and II are mutually exclusive cases indeed):

  (I) There is a symp Σ of Γ isomorphic to an orthogonal quadrangle $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{L})$ where $\mathbb{L}$ is a quadratic extension of $\mathbb{K}$, embedding isometrically in a symp Ω of Δ;

  (II) There is a symp Σ of Γ isomorphic to an orthogonal quadrangle $\mathsf{B}_{2,1}(\mathbb{K})$, embedding isometrically in a symp Ω of Δ;

  (III) Each symp of Γ is contained in a singular subspace of Δ.

LEMMA 4.2. *If some Γ-symp Σ is isometrically embedded in some Δ-symp, then either $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$, for some quadratic field extension $\mathbb{L}$ of $\mathbb{K}$, or $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{K})$. In the first (respectively, second) case, all symps of Γ that embed isometrically in Δ do so as described in Case I (respectively, Case II).*

Proof. As mentioned above, $\Sigma = W \cap \Omega$ for a certain symp Ω of Δ (represented in $\mathsf{PG}(7, \mathbb{K})$) and a subspace W of $\mathsf{PG}(7, \mathbb{K})$ of dimension 4 or 5. Suppose first that $\dim W = 5$. Then, in the corresponding polar space Γ^*, we consider the point-residue at Σ, which is dual to $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{L})$, for a quadratic field extension $\mathbb{L}$ of $\mathbb{K}$, that is, isomorphic to $\mathsf{C}_{2,1}(\mathbb{L}, \mathbb{K})$. Since a polar space is determined by its point-residues (see [23], Corollary 8.8), $\Gamma^* \cong \mathsf{C}_{3,1}(\mathbb{L}, \mathbb{K})$, and hence all symps of Γ which embed isometrically in Δ, do so as described in Case I. Similarly, if $\dim W = 4$, then $\Gamma^* \cong \mathsf{C}_{3,1}(\mathbb{K})$ and all symps of Γ which embed isometrically in Δ, do so as described in Case II. ∎

Before starting a case-by-case discussion, we make a few observations, using the following notation.

NOTATION 4.3. Throughout this section, we assume that Σ is a Γ-symp which is isometrically embedded in some Δ-symp Ω. We fix a point p of Σ and denote by Π the set of lines in Σ through p. Let U_p be the 5-space of Δ^* corresponding to p, in which we consider the 5′-spaces of Δ^* (or, equivalently, the 5-spaces of Δ) incident with U_p as the points (remember that a 5-space and a 5′-space of Δ^* are incident if they intersect in a 4-space). The 3-space of U_p corresponding to Ω is denoted by S_Ω.

We first determine the structure of $\mathrm{Res}_\Sigma(p)$ in Cases I and II.

DEFINITION 4.4. A *regulus* in a projective space $\mathsf{PG}(n, \mathbb{K})$, $n \geq 3$, is the set of generators of either of the natural types of a hyperbolic quadric in a 3-dimensional subspace. A *line spread* of a projective space is a partition of the point set of that space into lines. A line spread $\mathcal{S}$ of a projective space $\mathsf{PG}(n, \mathbb{K})$, $n \geq 3$, is called *regular* if for every line L of $\mathsf{PG}(n, \mathbb{K})$ not belonging to $\mathcal{S}$, the set of lines of $\mathcal{S}$ intersecting L nontrivially is a regulus.

Viewed in W, Π has the structure of a cone with vertex p over a (non-ruled) quadric Q in a subspace W' of W of codimension 2 (Q is isomorphic to the quadric $\mathsf{B}_{1,1}(\mathbb{K}, \mathbb{L})$ in Case I and to the conic $\mathsf{B}_{1,1}(\mathbb{K})$ in Case II), with $W' \cap \mathrm{Res}_\Omega(p) = Q$. Viewed in U_p, Π corresponds to a set of mutually skew lines in S_Ω (and we also denote this set of lines by Π). Any two lines are skew since they are not incident with a common 5-space of Δ.

LEMMA 4.5. *Let Σ be a symp of Γ through p which is isometrically embedded in Ω. Then, with the above notation, the set Π of lines of Γ through p and in Σ, corresponds in U_p to a regular spread of S_Ω in Case I and to a regulus of S_Ω in Case II.*

Proof. Let q be any point of S_Ω. Then q corresponds to a 5-space V_q of Δ meeting Ω in a $3'$-space V'_q through p (the 3-spaces of the other kind of Ω correspond to the maximal singular 3-spaces of Δ). The intersection $W'_q := V'_q \cap W$ is a singular subspace of Σ since $\Sigma = W \cap \Omega$ and V'_q is a singular subspace of Ω. Since Σ does not contain singular planes, W'_q is at most a line. We continue depending on $\dim W$.

Case I Here, $\dim W = 5$ and hence W'_q is a line, i.e., an element of Π. The corresponding line in $S_\Omega \subseteq U_p$ contains the point q. Since $q \in S_\Omega$ was arbitrary, Π is a line spread of S_Ω. We claim that this spread is regular. Indeed, consider three lines L_1, L_2, L_3 of Π, viewed inside Ω. Then L_1, L_2 and L_3 generate a non-singular 3-dimensional space V in W, intersecting Q in a conic C. In the 7-space $\langle \Omega \rangle$, let $\overline{V}$ be the 3-dimensional subspace of $\langle \Omega \rangle$ corresponding to V under the polarity associated to Ω. Observe that $\overline{V} \cap V = \{p\}$ and that $\overline{V}$ meets Ω in a cone with vertex p and base a conic C' (it cannot contain singular planes). Now suppose that the line in S_Ω corresponding to L_1 contains the point $q \in S_\Omega$, equivalently, $L_1 \subseteq W'_q$. Then the $3'$-space V'_q contains L_1 and meets $L_2^\perp \cap L_3^\perp$ in a unique line $M_q \ni p$. Note that M_q is then collinear to L_1, L_2, L_3 and hence to all lines $L_c := pc$, where c is a point of the conic C. We now translate this to S_Ω, and for ease of notation we will refer to the lines in S_Ω corresponding to L_1, L_2, L_3 and M_q by the same notation. Then M_q is the unique transversal of L_1, L_2, L_3 containing q. Set $q_i := M_q \cap L_i$, $i = 2, 3$. Then V'_{q_i} is the unique $3'$-space of Ω through the plane $\langle M_q, L_i \rangle$, $i = 2, 3$. Let $q' \in S_\Omega$ be an arbitrary point of $M_q \setminus \{q\}$. Then $V'_{q'}$ is a $3'$-space of Ω inside the 5-dimensional subspace $M_q^\perp$ and hence meets $\langle C \rangle \subseteq M_q^\perp$ in a point $c' \in C$ (as $V'_{q'}$ is singular). Thus, the spread line $L_{c'}$ in S_Ω meets M_q in the point q'. Since $q \in L_1$ was arbitrary, we obtain that the lines L_1, L_2, L_3 determine the regulus $\{L_c \mid c \in C\}$ in the spread Π of S_Ω.

Case II Here, $\dim W = 4$ and hence Π is a set of pairwise disjoint lines in S_Ω not covering all points of S_Ω. Note that this time, Q is just a conic, so, as explained in the previous item, Q corresponds to a regulus in S_Ω. We conclude that Π is

the line set of this regulus.

This completes the proof of the lemma. ∎

Our next goal is to show that in both Cases I and II every symp of Γ is isometrically embedded in some symp of Δ. Therefore we now consider the situation that there is a symp $\Sigma' \neq \Sigma$ of Γ through p that spans a singular subspace U of Δ. Let Π' be the set of lines of Σ' containing p. As before, Π' has the structure of a cone with vertex p over a (non-ruled) quadric Q'.

Given the existence of a symp Σ through p which is isometrically embedded in Ω, we have the following possibilities for Q'. They can be deduced from the classification of (embeddable) polar spaces (see [23], Chapter 8). Firstly, we note that Q' is isomorphic to $\mathsf{B}_{1,1}(\mathbb{K}, \mathbb{L})$ in Case I and to $\mathsf{B}_{1,1}(\mathbb{K})$ in Case II. In both cases, Q' is embedded in a subspace U' of U of dimension $\dim U - 2$ in a canonical way.

Case I If $\mathbb{L}$ is a separable field extension of $\mathbb{K}$, then Q' only admits a standard embedding in a 3-space as a non-ruled quadric, say $\mathcal{Q}_3$. If $\mathbb{L}$ is an inseparable field extension (so $\operatorname{char} \mathbb{K} = 2$), then Q' admits standard embeddings in dimensions 3, 2 and 1. In the first case Q' is embedded in a 3-space as a non-ruled quadric $\mathcal{Q}_3$ as in the separable case, now having a line N as nucleus though; in the second case, Q' is embedded in a plane as a conic $\mathcal{Q}_2$, which is the projection of $\mathcal{Q}_3$ from a point of N (this could be called an "inseparable Hermitian curve", see [14]), and it has a point n as nucleus (n is the projection of N); in the third case Q' is embedded as a Baer subline $\mathcal{Q}_1$ in a line as the projection of $\mathcal{Q}_2$ from n, more precisely $\mathcal{Q}_1$ is a projective subline $\mathsf{PG}(1, \mathbb{L}^2)$ of $\mathsf{PG}(1, \mathbb{K})$. Note that in the latter case we have a full line pencil if, and only if, $\mathbb{K} = \mathbb{L}^2$. Also note that an inseparable Hermitian curve does not contain full lines.

Case II If $\operatorname{char} \mathbb{K} \neq 2$, then Q' only admits its standard embedding as a conic $\mathcal{C}_2$ in a plane. If $\operatorname{char} \mathbb{K} = 2$, then Q' admits embeddings in dimensions 2 and 1. In the first case, Q' is embedded in the plane as a conic $\mathcal{C}_2$ as above, though now having a point n as a nucleus; in the second case, Q' is embedded as a Baer subline $\mathcal{C}_1$ in a line by projecting $\mathcal{C}_2$ from the point n, we obtain the projective subline $\mathsf{PG}(1, \mathbb{K}^2)$ (which coincides with $\mathsf{PG}(1, \mathbb{K})$ if $\mathbb{K}$ is perfect).

Of course, the above analysis is independent of the fact that there is a Γ-symp containing p isometrically embedded in Δ. However, it does depend on the fact that *some* Γ-symp is isometrically embedded in Δ as the arguments use the knowledge of the possible isomorphism types of the Γ-symps. This analysis now implies the following lemma (where we keep as much as possible the same notation as above, except for the point p by virtue of what we just noted).

LEMMA 4.6. *Assume that Γ is fully embedded in Δ in such a way that some Γ-symp is isometrically embedded in some Δ-symp. Let Σ' be a Γ-symp which is embedded in a singular subspace U of Δ. Let $r \in \Sigma'$ be arbitrary and let U_r be the 5-space of Δ^* corresponding to r. Then the set Π' of lines of Γ through r and in Σ', corresponds in U_r to a cone with some vertex V, where V corresponds to some 5-space in Δ, and, with the*

above notation, with base isomorphic to $\mathcal{Q}_i$ in an i-space, $i \in \{1, 2, 3\}$, in Case I, and to $\mathcal{C}_i$ in an i-space, $i \in \{1, 2\}$, in Case II.

Proof. Since Σ' contains disjoint lines and the singular subspaces of Δ have dimension at most 5, $\dim U \in \{3, 4, 5\}$. Moreover, $r^\perp \cap \Sigma'$ is embedded in a hyperplane of U, which in all cases is contained in some maximal singular 5-space V of Δ (if U is not a maximal singular 3-space then U is already contained in some singular 5-space; otherwise V is the unique singular 5-space of Δ intersecting U in $p^\perp \cap \Sigma'$). As such, when viewing Π' in U_r, we obtain a set of lines through the point V, and hence Π' has, in U_r, the structure of a cone with vertex V. To determine the base, we note that it is isomorphic to $\Pi' \cap \mathrm{Res}_\Delta(V)$. Since r is contained in each member of Π', this equals $\Pi' \cap \mathrm{Res}_\Delta(\{r, V\})$, and hence it also coincides with $\Pi' \cap \mathrm{Res}_\Delta(r)$, which we determined above (for p, but as we noted, this is independent of p). The lemma is proved. ∎

For further reference, we shall call the cones in U_r of the previous lemma *perp cones*.

4.2. Case I: some symp of Γ is contained in a symp of Δ as a 5-dimensional quadrangle.
Our strategy is to use Proposition 4.16 of [23] to show that Γ arises from Δ as the fixed point structure of a subgroup of the collineation group of Δ, unique up to conjugation. Towards that goal, we have to show that Γ is isometrically embedded in Δ, and exhibit an apartment of Δ intersecting Γ in an apartment of Γ. In fact, we combine these two assertions and show that every apartment of Γ can be obtained from an apartment of Δ by restriction to Γ. Our strategy is as follows: We first show that Γ is isometrically embedded in Δ (Lemma 4.9). Then we show how this implies that opposite chambers of Γ correspond to opposite flags in Δ. Lastly, we show that all flags of the apartment in Γ defined by two opposite chambers C, C' are contained in every apartment of Δ containing the flags C, C' (Corollary 4.10).

Hypotheses. Throughout Section 4.2 we assume that we are in Case I of the previous section, i.e., there is a symp Σ of Γ isomorphic to $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{L})$, for some quadratic field extension $\mathbb{L}$ of $\mathbb{K}$, that is isometrically embedded in some symp Ω of Δ (occurring as the intersection of Ω, when viewed in $\mathsf{PG}(7, \mathbb{K})$), with a 5-space of $\mathsf{PG}(7, \mathbb{K})$. Note that $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{L}) \cong \mathsf{C}_{2,2}(\mathbb{L}, \mathbb{K})$ by 2.9. Also, Lemma 4.2 implies that Γ is the dual polar space $\mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$. Finally, if $\mathbb{L}$ is an inseparable extension of $\mathbb{K}$, then Γ is of inseparable type.

4.2.1. The embedding of Γ in Δ is isometric.
We first show that each symp of Γ embeds isometrically in a symp of Δ. Let p be any point of Σ and let U_p be the 5-space of Δ^* corresponding to p. We take a look at the residue of p in Γ. Recall that the lines of Γ through p correspond to a set $\mathcal{L}_p$ of lines in U_p; and, according to Lemmas 4.5 and 4.6, those in a symp of Γ through p corresponds to a regular spread in a 3-space of U_p if it is isometrically embedded, or to a perp cone if it is contained in a singular subspace. This gives rise to the following representation of $\mathrm{Res}_\Gamma(p) \cong \mathsf{PG}(2, \mathbb{L})$ in U_p. The points of $\mathsf{PG}(2, \mathbb{L})$ are the members of $\mathcal{L}_p$; the lines of $\mathsf{PG}(2, \mathbb{L})$, which we shall from now on call "blocks" to avoid confusion with lines of U_p, are the spreads and perp cones mentioned above. Blocks corresponding to spreads will be referred to as *spread-blocks*.

LEMMA 4.7. *Each symp of Γ through p is isometrically embedded in a symp of Δ.*

Proof. The block of $\mathsf{PG}(2,\mathbb{L})$ corresponding to Σ is a regular spread $\mathcal{S}$.

Claim 1: There is a second spread-block.

Indeed, if not, then consider any line L of $\mathcal{S}$, and any perp cone containing L. Since each perp cone contains at least three lines, we can select a line $M \neq L$ belonging to that perp cone and not meeting every line of $\mathcal{S}$. Hence there is a line $K \in \mathcal{S}$ disjoint from M. But then the block containing M and K is a spread, a contradiction. □

Hence there is a second spread-block $\mathcal{S}'$. Let L be the line common to $\mathcal{S}$ and $\mathcal{S}'$.

Claim 2: The 3-spaces $\langle \mathcal{S} \rangle$ and $\langle \mathcal{S}' \rangle$ intersect each other in precisely L.

Suppose for a contradiction that $\langle \mathcal{S} \rangle \cap \langle \mathcal{S}' \rangle$ contains a plane α. We may assume that $L \subseteq \alpha$. For each point $x \in \alpha \setminus L$, there exist unique lines M, M' of $\mathcal{S}, \mathcal{S}'$, respectively, containing x. Note that M and M' are not contained in α, as they are disjoint from L, so in particular, $M \neq M'$. Hence x is the vertex of a cone corresponding to a block B_x. Fix such x and let y be any other point of $\alpha \setminus L$. Then the block B_x has a line in common with B_y, which is necessarily the line xy. Varying y over $\alpha \setminus (L \cup \{x\})$, we see that B_x contains all lines of α through x. The only perp-cones containing such a full line pencil are the planar line pencils themselves, hence B_x is a planar line pencil, contradicting the fact that it also contains the lines M, M' not contained in α. □

Claim 3: All blocks are spread-blocks.

Consider any block B not containing L. Set $\{M\} = B \cap \mathcal{S}$ and $\{M'\} = B \cap \mathcal{S}'$. Then $M \cap M' = \emptyset$ by Claim 2 and so B is a spread-block. Note that $\langle M, M' \rangle \cap L = \emptyset$, as $\langle L, M, M' \rangle = \langle \mathcal{S}, \mathcal{S}' \rangle = U_p$, and $\dim\langle L, M, M' \rangle = \dim\langle M, M' \rangle + \dim L - \dim(\langle M, M' \rangle \cap L)$. Hence every block containing L contains a member of B disjoint from L and is therefore a spread-block. Claim 3 is proved. □

So all blocks are of spread-type, i.e., every symp of Γ through p is isometrically embedded in a symp of Δ. ■

We now show that no symp of Γ is contained in a singular subspace of Δ.

LEMMA 4.8. *If the dual polar space $\Gamma = \mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$ is fully embedded in the half spin parapolar space $\Delta = \mathsf{D}_{6,6}(\mathbb{K})$ such that at least one symp of Γ is isometrically embedded in a symp of Δ, then each symp of Γ is isometrically embedded in a symp of Δ.*

Proof. Let p be a point of Γ contained in a Γ-symp which is isometrically embedded in a Δ-symp. By Lemma 4.7, each Γ-symp through p is isometrically embedded in a Δ-symp. Hence every point q collinear to p is contained in a Γ-symp that is isometrically embedded in a Δ-symp. Connectivity of Γ implies that each Γ-symp is isometrically embedded in a Δ-symp. ■

LEMMA 4.9. *The embedding of Γ in Δ is isometric.*

Proof. Take any two points p, q of Γ.

(i) Suppose first that $\delta_\Gamma(p, q) = 1$. Then the very definition of embedding yields $\delta_\Delta(p, q) = 1$.

(ii) Next, suppose that $\delta_\Gamma(p, q) = 2$. Then p and q determine a unique symp of Γ. Lemma 4.8 and our assumption that at least one symp of Γ is isometrically embedded in a symp of Δ implies that $\delta_\Delta(p, q) = 2$.

(*iii*) Finally, suppose that $\delta_\Gamma(p, q) = 3$. Let (p, x, y, q) be a path of length 3 between p and q. Set $L := yq$ and let Σ be the Γ-symp containing p and y. In U_y, the symp Σ corresponds to a spread-block $\mathcal{S}$ and L corresponds to a line disjoint from $\langle \mathcal{S} \rangle$. In the dual of U_y (hence in the polar space Δ^*), the symp Σ corresponds to a line σ disjoint from the 3-space W_L corresponding to L. Now, $W_L = U_y \cap U_q$, and $\sigma = U_y \cap U_p$. It now easily follows that $U_p \cap U_q = \emptyset$ since a point in $U_p \cap U_q$ would be collinear to σ and to W_L and hence has to belong to $\langle \sigma, W_L \rangle = U_y$, a contradiction. ∎

COROLLARY 4.10. *Two opposite chambers of Γ are opposite flags in Δ. Moreover, all elements of the unique apartment of Γ containing two opposite chambers of Γ are contained in the convex Δ-closure of these chambers as flags of Δ.*

Proof. By the existence and properties of projections in buildings, see Section 3.19 of [23], two lines are opposite whenever "not being opposite" induces a bijection between their point sets. This immediately implies that Γ being isometrically embedded in Δ implies that opposite lines of Γ are also opposite in Δ.

Now let $\{p, L, Q\}$ and $\{p', L', Q'\}$ be two opposite chambers of Γ. We consider the corresponding flags in Δ^*: let $U_p, U_{p'}$ be the 5-spaces of Δ^* corresponding to p, p', respectively; $S_L, S_{L'}$ be the 3-spaces of Δ^* corresponding to L, L', respectively; and $L_Q, L_{Q'}$ be the lines of Δ^* corresponding to the symps of Δ containing Q and Q' (see Lemma 4.8), respectively. Then U_p and $U_{p'}$ are opposite, by Lemma 4.9 (note that opposition is the same in Δ and Δ^*). By the first paragraph above, S_L and $S_{L'}$ are opposite, too. In Γ, there is a unique line K in Q in a common symp R' with L' (note that $K \neq L$). Hence there is a 3-space S_K of Δ^* containing L_Q and intersecting $S_{L'}$ in the line $L_{R'}$ corresponding to the symp R'. Also in Δ^*, S_K is the unique 3-space containing L_Q and meeting $S_{L'}$ in the line $L_{R'}$, since S_L and $S_{L'}$ are opposite. So $L_{R'}$ is also the unique line in $S_{L'}$ that is Δ^*-collinear to L_Q. Clearly, $L_{R'} \neq L_{Q'}$, and since, by Lemma 4.5 (considering the dual of $U_{p'}$), both are members of a line spread of $S_{L'}$, $L_{R'} \cap L_{Q'} = \emptyset$, from which we deduce that L_Q is opposite $L_{Q'}$. This already shows the first assertion.

Let L_R and $S_{K'}$ be defined likewise, i.e., L_R is the unique line in S_L that is Δ^*-collinear to $L_{Q'}$ and $S_{K'} = \langle L_R, L_{Q'} \rangle$, where R corresponds to a Γ-symp and K' to a line of Q'. Since S_L and $S_{L'}$ are opposite, we deduce as before that L_R and $L_{R'}$ are opposite. Let L_P (resp. $L_{P'}$) be the unique line obtained by intersecting U_p (resp. $U_{p'}$) with the unique maximal singular subspace of Δ^* containing $S_{L'}$ (resp. S_L) and intersecting U_p (resp. $U_{p'}$) in a line. Then L_P (resp. $L_{P'}$) corresponds to the unique symp P (resp. P') of Γ containing p (resp. p') and a point of L' (resp. L).

We obtain a set of six points of Γ^*, namely $\mathcal{T} = \{P, Q, R, P', Q', R'\}$, with the property that each point in $\mathcal{T}$ is opposite a unique point of $\mathcal{T}$. As such, the set of all singular subspaces of Γ^* generated by subsets of $\mathcal{T}$ forms an apartment $\mathcal{A}$ of Γ^*. As above, it is easily checked that all elements of $\mathcal{A}$ are obtained by projecting, intersecting and joining elements of $\mathcal{T}$ in Γ^*. This means that shortest paths in Γ^* correspond to shortest paths in Δ^*, and hence in Δ. Note that, to establish this, we only used the fact that opposite points of Γ are also opposite points of Δ. This completes the proof of the second assertion. ∎

In the next subsection, we construct automorphisms of Δ fixing only Γ.

4.2.2. Collineations of residues of Δ with residues of Γ as fixed point structure.

PROPOSITION 4.11. *Let $Q \cong \mathsf{D}_{3,1}(\mathbb{K})$ be the Klein quadric in $\mathsf{PG}(5, \mathbb{K})$ and let $\mathcal{O}$ be the intersection of Q and a 3-space S such that $\mathcal{O}$ is a non-degenerate quadric of Witt index 1. Let $G \leq \mathsf{PGL}_6(\mathbb{K})$ be the group of (linear) collineations of $\mathsf{PG}(5, \mathbb{K})$ stabilising Q, preserving each of the two natural systems of generators of Q, and pointwise fixing $\mathcal{O}$. Then the fixed point set of each nontrivial member of G is precisely $\mathcal{O}$, and G acts sharply transitively on the set of generators of Q of each type through each point of $\mathcal{O}$ (and hence has size $|\mathbb{K}| + 1$). If the characteristic of $\mathbb{K}$ is not equal to 2, then G contains a unique involution; if $\mathcal{O}$ is an inseparable quadric, then G is an elementary abelian 2-group; in the other cases G does not contain any involution. Also, G acts freely on the set of lines and the set of planes of Q. As an abstract group, G is isomorphic to the factor group $\mathbb{L}^\times / \mathbb{K}^\times$, where $\mathbb{L}$ is the quadratic extension of $\mathbb{K}$ defined by $\mathcal{O}$.*

Proof. We will use the Klein correspondence between the Klein quadric and the (lines of the) 3-space $\mathsf{PG}(3, \mathbb{K})$. All this is well known, but to fix our notation: We label the coordinates of $\mathsf{PG}(3, \mathbb{K})$ using indices defined by (X_0, X_1, X_2, X_3) and those of $\mathsf{PG}(5, \mathbb{K})$ using indices defined by

$$(X_{01}, X_{02}, X_{03}, X_{12}, X_{31}, X_{23}).$$

A line of $\mathsf{PG}(3, \mathbb{K})$ containing the distinct points (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3) defines the point $(p_{01}, p_{02}, p_{03}, p_{12}, p_{31}, p_{23})$ of $\mathsf{PG}(5, \mathbb{K})$, where $p_{ij} = x_i y_j - x_j y_i$, for all $ij \in \{01, 02, 03, 12, 31, 23\}$. Varying the line over all lines of $\mathsf{PG}(3, \mathbb{K})$, one obtains the Klein quadric with equation $X_{01}X_{23} + X_{02}X_{31} + X_{03}X_{12} = 0$.

Now the lines in $\mathsf{PG}(3, \mathbb{K})$ corresponding to the points of $\mathcal{O}$ form a regular line spread $\mathcal{S}$. Since reguli in $\mathsf{PG}(3, \mathbb{K})$ are pairwise projectively equivalent and each of them is determined by any three of its members, we may assume that $\mathcal{S}$ contains the lines

$$L_1 = \langle (1, 0, 0, 0), (0, 0, 0, 1) \rangle, \; L_2 = \langle (0, 1, 0, 0), (0, 0, 1, 0) \rangle, \; L_3 = \langle (1, 1, 0, 0), (0, 0, 1, 1) \rangle.$$

Hence $\mathcal{O}$ contains the points $(0, 0, 1, 0, 0, 0)$, $(0, 0, 0, 1, 0, 0)$ and $(0, 1, 1, 1, -1, 0)$, which generate a plane π.

We write S as the intersection of two hyperplanes of $\mathsf{PG}(5, \mathbb{K})$. Since a generic hyperplane through π has equation $aX_{01} + bX_{02} + bX_{31} + cX_{23} = 0$, $a, b, c \in \mathbb{K}$ not all zero, we may assume without loss of generality that one of them has equation $X_{01} = \beta X_{23}$, $\beta \in \mathbb{K}$. If both have such an equation, then S has equations $X_{01} = X_{23} = 0$, which does not intersect Q in an elliptic quadric. Hence the second hyperplane has an equation of the form $X_{02} + X_{31} = \alpha X_{23}$, $\alpha \in \mathbb{K}$. As such, a set of equations for $\mathcal{O}$ is now given by

$$X_{01} = \beta X_{23}; \; X_{02} = \alpha X_{23} - X_{31}; \; X_{03}X_{12} = X_{31}^2 - \alpha X_{23}X_{31} - \beta X_{23}^2.$$

where the quadratic polynomial $q(x) = x^2 - \alpha x - \beta$ is irreducible over $\mathbb{K}$, in particular, $\beta \neq 0$. Note that $\mathcal{O}$ is an inseparable quadric if, and only if, char $\mathbb{K} = 2$ and $\alpha = 0$. Consider the point $p_4 \in S \cap Q \setminus \pi$ given by the coordinates $(\beta, \alpha, \beta, -1, 0, 1)$. This point corresponds to the line $L_4 = \langle (\beta, 0, 1, 0), (-\alpha, 1, 0, 1) \rangle$ in $\mathsf{PG}(3, \mathbb{K})$. We determine all linear collineations of $\mathsf{PG}(3, \mathbb{K})$ fixing the lines L_1, L_2, L_3, L_4. A straightforward calculation

reveals that the generic form of the matrix of a linear collineation fixing L_1, L_2, L_3 is

$$\begin{bmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ c & 0 & 0 & d \end{bmatrix},$$

for all $a, b, c, d \in \mathbb{K}$ with $ad - bc \neq 0$; and we denote it by $\varphi_{a,b,c,d}$. We note in passing that each of them fixes every line of the regulus containing L_1, L_2, L_3. We have

$$\varphi_{a,b,c,d}(\beta, 0, 1, 0) = (a\beta, b, d, c\beta) \text{ and } \varphi_{a,b,c,d}(-\alpha, 1, 0, 1) = (-a\alpha + b, a, c, -c\alpha + d),$$

and (recalling that $\beta \neq 0$) these points belong to L_4 if, and only if, $b = c\beta$ and $a = d - c\alpha$. Note that $\varphi_{d-c\alpha,c\beta,c,d}$ induces the identity in $\mathsf{PG}(3, \mathbb{K})$ if (and only if) $c = 0$. As such we may assume that $c = 1$ and denote the corresponding collineation briefly by φ_d. Hence φ_d has matrix

$$M_d = \begin{bmatrix} d - \alpha & 0 & 0 & \beta \\ 0 & d - \alpha & \beta & 0 \\ 0 & 1 & d & 0 \\ 1 & 0 & 0 & d \end{bmatrix}.$$

Denote by θ_d the corresponding (linear) collineation of $\mathsf{PG}(5, \mathbb{K})$ (via the Klein correspondence). Then, by our remark above that φ_d fixes each member of the regulus containing L_1, L_2, L_3, the collineation θ_d pointwise fixes the conic $\mathcal{O} \cap \pi$ in π, and hence it fixes π pointwise. Also, θ_d fixes the point $p_4 \in \mathcal{O} \setminus \pi$. Hence θ_d fixes each line through p_4 in S, and since each such line contains at most one other point of $\mathcal{O}$, and each point of $\mathcal{O}$ is on at least one such line, it follows that θ_d fixes $\mathcal{O}$ pointwise. Therefore, φ_d fixes every member of $\mathcal{S}$. Observe that no nontrivial collineation of $\mathsf{PG}(3, \mathbb{K})$ fixing every member of $\mathcal{S}$ fixes a point of $\mathsf{PG}(3, \mathbb{K})$. Indeed, otherwise we could have taken $(1, 0, 0, 0)$ for that fixed point, yielding $c = 0$ in the above, leading to the identity.

The set of all φ_d, $d \in \mathbb{K}$, together with the identity, forms the group G^* consisting of all collineations of $\mathsf{PG}(3, \mathbb{K})$ that fix each member of $\mathcal{S}$ (and no non-trivial element of G^* fixes any line of $\mathsf{PG}(3, \mathbb{K})$ not in $\mathcal{S}$). This group acts sharply transitively on the set of points of each member of $\mathcal{S}$. Indeed, transitivity follows from varying d and the sharpness from our remark above that no φ_d fixes any point of $\mathsf{PG}(3, \mathbb{K})$. Since the points of $\mathsf{PG}(3, \mathbb{K})$ on a member of $\mathcal{S}$ correspond to generators of Q of one natural system through a point of $\mathcal{O}$, it follows that the group G consisting of the identity and all θ_d, $d \in \mathbb{K}$, is the unique group enjoying the transitivity property of the statement. This also implies that G acts freely on the set of planes of Q, and hence also on the set of lines of Q.

We now determine the elements $d \in \mathbb{K}$ such that φ_d is an involution. Note that the square of φ_d has matrix

$$\begin{bmatrix} (d-\alpha)^2 + \beta & 0 & 0 & (2d-\alpha)\beta \\ 0 & (d-\alpha)^2 + \beta & (2d-\alpha)\beta & 0 \\ 0 & 2d-\alpha & \beta + d^2 & 0 \\ 2d-\alpha & 0 & 0 & \beta + d^2 \end{bmatrix},$$

which is a non-zero scalar matrix if, and only if, $2d - \alpha = 0$ (noting that, in case $2d = \alpha$,

we have $ad - bc = -d^2 - \beta \neq 0$). Now $2d - \alpha$ is always zero in the inseparable case (i.e., char $\mathbb{K} = 2$ and $\alpha = 0$), never zero if char $\mathbb{K} = 2$ and $\alpha \neq 0$, and is zero precisely if $d = \frac{1}{2}\alpha$ if char $\mathbb{K} \neq 2$.

Let $\mathbb{L}$ be the (quadratic) field extension of $\mathbb{K}$ with respect to the irreducible polynomial $x^2 + \alpha x - \beta$. Let $\zeta \in \mathbb{L}$ be such that $\zeta^2 + \alpha\zeta - \beta = 0$. Then every element of $\mathbb{L}$ can be written as $a + b\zeta$ for unique $a, b \in \mathbb{K}$. One verifies that the mapping

$$\mathbb{L}^\times \to G : a + b\zeta \mapsto \begin{cases} \theta_{a/b} & \text{if } b \neq 0, \\ \text{id} & \text{if } b = 0, \end{cases}$$

is a group epimorphism with kernel $\mathbb{K}^\times$. ∎

There are two interesting corollaries.

COROLLARY 4.12. *Let $Q \cong \mathsf{D}_{n+1,1}(\mathbb{K})$ be a hyperbolic quadric in $\mathsf{PG}(2n + 1, \mathbb{K})$, $n \geq 2$, and let $\mathcal{O}$ be the intersection of Q and a $(2n - 1)$-space S such that $\mathcal{O} \cong \mathsf{B}_{n-1,1}(\mathbb{K}, \mathbb{L})$, $\mathbb{L}$ a quadratic extension of $\mathbb{K}$, is a quadric of Witt index $n - 1$. Let $G \leq \mathsf{PGL}_{2n+2}(\mathbb{K})$ be the group of (linear) type preserving collineations of $\mathsf{PG}(2n + 1, \mathbb{K})$ stabilizing Q, and pointwise fixing $\mathcal{O}$. Then the fixed point set of each nontrivial member of G is precisely $\mathcal{O}$ and G acts sharply transitively on the set of generators of Q of given type through each singular $(n - 2)$-space of $\mathcal{O}$ (and hence has size $|\mathbb{K}| + 1$). Also, G acts freely on the set of $(n - 1)$-spaces and generators of Q. As an abstract group, G is isomorphic to the factor group $\mathbb{L}^\times / \mathbb{K}^\times$.*

Proof. We assume $n \geq 3$, as Proposition 4.11 is the case $n = 2$. Let U, U' be a pair of opposite subspaces of $\mathcal{O}$ of dimension $n - 3$. Then, taking perps in Q, we see that $\mathcal{O}$ induces a quadric $\mathcal{O}'$ of Witt index 1 in the Klein quadric $Q' = U^\perp \cap U'^\perp$. Hence, by Proposition 4.11, the stabilizer G^* of Q' in $\mathsf{PGL}_6(\mathbb{K})$, naturally acting on $\langle Q' \rangle$, which preserves each of the natural systems of generators of Q' and fixes $\mathcal{O}'$ pointwise, acts sharply transitively on the set of planes of Q' of either system containing a fixed point of $\mathcal{O}'$. We can now extend each element of G^* to an element of $\mathsf{PGL}_{2n+2}(\mathbb{K})$ by defining it as the identity on the whole of $\langle \mathcal{O} \rangle$ (this extension exists and is unique because $\langle \mathcal{O}, Q' \rangle = \langle Q \rangle$ and $\langle \mathcal{O} \rangle \cap \langle Q' \rangle = \langle \mathcal{O}' \rangle$), and thus obtain a group G, which consists of members of $\mathsf{PGL}_{2n+2}(\mathbb{K})$ fixing $\mathcal{O}$ pointwise and preserving the natural systems of generators of Q. Now clearly, G acts sharply transitively on the set of generators of Q of fixed type through the $(n - 2)$-space W generated by U (or U') and any point of $\mathcal{O}'$. Whenever an $(n - 1)$-space W' of $\mathcal{O}$ is opposite W, it is seen (by projection) that G acts sharply transitively on the generators of respective fixed type through W'. If W'' is an $(n - 1)$-space of $\mathcal{O}$ not opposite W, then we can find an $(n - 1)$-space W' opposite both W, W'' and apply the argument twice. In view of Proposition 4.11 it remains to show that any type preserving collineation θ of Q pointwise fixing $\mathcal{O}$ and fixing a generator V of Q, is the identity. But each point x of $V \setminus \mathcal{O}$ is the projection of an $(n - 2)$-space of $\mathcal{O}$ (look in $x^\perp$), and hence is fixed by θ. It follows that θ is the identity. ∎

Before we mention the second corollary, we have to state a result that probably belongs to folklore. We provide a proof for completeness' sake. Recall Definition 4.4.

LEMMA 4.13. *Let $\mathcal{S}$ be a regular line spread of the finite dimensional projective space* $\mathsf{PG}(n, \mathbb{K})$, $n \geq 4$. *Then n is odd and the set of lines of $\mathcal{S}$ contained in the span S of any finite number of members of $\mathcal{S}$ induces a regular spread in S. Also, any two regular spreads in distinct 3-spaces thus obtained are mutually isomorphic via a projectivity.*

Proof. Let $T \subseteq \mathcal{S}$ be arbitrary but finite. Using induction on $|T|$, we show that $\mathcal{S}$ induces a regular spread in $\langle T \rangle$ and that $\dim \langle T \rangle$ is odd. The result is trivial for $|T| = 1$, so suppose $|T| > 1$. Take any $L \in T$. If $\langle T \setminus \{L\} \rangle = \langle T \rangle$, then the induction hypothesis proves the claim, so we assume that $\langle T \setminus \{L\} \rangle$ is a proper subspace of $\langle T \rangle$. The former is odd-dimensional by the induction hypothesis and contains a line spread induced by $\mathcal{S}$. It follows that L is disjoint from $\langle T \setminus \{L\} \rangle$ and so $\dim \langle T \rangle = \dim \langle T \setminus \{L\} \rangle + 2$ is odd. Let M be an arbitrary line of $\langle T \setminus \{L\} \rangle$ belonging to $\mathcal{S}$. Then, $S = \langle L, M \rangle$ is a 3-space. Each point $p \in S \setminus (L \cup M)$ lies on a unique line K meeting M and L, and, by regularity of $\mathcal{S}$, the set of lines of $\mathcal{S}$ meeting K in a point is a regulus containing L and M, and is contained in S. So $\mathcal{S}$ induces regular spreads in these 3-spaces. Since every point q of $\langle T \rangle$ belongs to such a 3-space, $\mathcal{S}$ induces a regular spread in $\langle T \rangle$. As $\mathsf{PG}(n, \mathbb{K})$ is generated by a finite number of lines of $\mathcal{S}$, it follows that n is odd.

Now let S_1 and S_2 be two 3-spaces in which $\mathcal{S}$ induces respective regular spreads $\mathcal{S}_1$ and $\mathcal{S}_2$. By possibly considering the 3-space generated by a member of $\mathcal{S}_1$ and one of $\mathcal{S}_2$, we may assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ share a unique member L of $\mathcal{S}$, in which case it is clear that $S_1 \cap S_2 = L$ too. So if $F = \langle S_1, S_2 \rangle$, then $\dim F = 5$. By the foregoing, $\mathcal{S}$ determines a regular spread in F too, so there is a line $M \in \mathcal{S}$ in $F \setminus (S_1 \cup S_2)$. Consider an arbitrary member $L_1 \in \mathcal{S}_1 \setminus \{L\}$. Then $\langle M, L_1 \rangle$ intersects S_2 in a line L_2. Since $\langle M, L_1 \rangle$ contains two members of $\mathcal{S}$, the latter induces a spread in $\langle M, L_1 \rangle$. Hence the spread member on any point $x \in L_2$ belongs to $\langle M, L_1 \rangle$; likewise it belongs to S_1. Consequently, $L_2 \in \mathcal{S}_2$. Hence $\mathcal{S}_2$ is the projection of $\mathcal{S}_1$ from M, and both spreads are isomorphic via a perspectivity.

The lemma now follows. ■

COROLLARY 4.14. *Let $\mathcal{S}$ be a regular line spread of $\mathsf{PG}(2n+1, \mathbb{K})$, $n \geq 1$. Then the elementwise stabilizer G of $\mathcal{S}$ in $\mathsf{PGL}_{2n+2}(\mathbb{K})$ acts sharply transitively on the points of each member of $\mathcal{S}$. As an abstract group G is isomorphic to the factor group $\mathbb{L}^\times / \mathbb{K}^\times$, where $\mathbb{L}$ is the quadratic extension of $\mathbb{K}$ defined by the anisotropic kernel of the quadric in a 3-dimensional projective space over $\mathbb{K}$ obtained from $\mathcal{S}$ by applying the Klein correspondence to the regular spread induced by $\mathcal{S}$ in a 3-space containing at least two elements of $\mathcal{S}$.*

Proof. We prove the statement for $n = 2$. If $n = 1$, this is the content of Proposition 4.11. since under the Klein correspondence, $\mathcal{S}$ corresponds to a non-degenerate quadric of Witt index 1 on the Klein quadric. For $n > 2$ an obvious induction (proceeding analogously as in the case $n = 2$) does the trick.

Let S be a 3-subspace of $\mathsf{PG}(5, \mathbb{K})$ containing at least two members of $\mathcal{S}$. Then, since $\mathcal{S}$ is regular, by Lemma 4.13, $\mathcal{S}$ induces a regular spread $\mathcal{S}'$ in S. Under the Klein correspondence, $\mathcal{S}'$ corresponds to a non-degenerate quadric of Witt index 1 in the Klein quadric Q. Applying Proposition 4.11, we obtain a subgroup G_1 of $\mathsf{PGL}_6(\mathbb{K})$ stablizing Q. It action on S, via the Plücker map, stabilizes $\mathcal{S}'$ elementwise, and acts sharply transitively on each member of $\mathcal{S}'$. Now consider a 3-space T of $\mathsf{PG}(5, \mathbb{K})$ containing

exactly one member L_0 of $\mathcal{S}'$ and at least one member L_1 of $\mathcal{S} \setminus \mathcal{S}'$. Take a line $L \in \mathcal{S}$ not contained in $S \cup T$. Let φ be the projectivity with centre L mapping S to T. As in the proof of Lemma 4.13, φ maps $\mathcal{S}'$ to a spread $\mathcal{S}''$ of T, and $\mathcal{S}'' \subseteq \mathcal{S}$. Moreover, the group $G_2 = G_1^\varphi$ fixes each member of $\mathcal{S}''$, acts sharply transitively on each member of $\mathcal{S}''$, and has the same action on L_0 as G_1 (because φ fixes L_0 pointwise). Hence we can extend each member g of G_1 uniquely to the whole of $\mathsf{PG}(5, \mathbb{K})$, fixing every member of $\mathcal{S}' \cup \mathcal{S}''$ (using g^φ), obtaining a group of collineations G. Now every line $M \in \mathcal{S} \setminus (\mathcal{S}' \cup \mathcal{S}'')$ is the intersection of two 3-spaces spanned by members of $\mathcal{S}' \cup \mathcal{S}''$, and hence is also fixed by G.

Clearly, $G \cong G_1$ and so G is isomorphic to $\mathbb{L}^\times / \mathbb{K}^\times$, with $\mathbb{L}$ as in Proposition 4.11. Moreover, G is the full elementwise stabilizer of $\mathcal{S}$ as otherwise some non-trivial elementwise stabilizer of $\mathcal{S}$ fixes some point of $\mathsf{PG}(5, \mathbb{K})$, which we may assume to be contained in S. But this contradicts Proposition 4.11. ∎

REMARK 4.15. Note that the explicit form of G in Proposition 4.11 shows the existence of the projective groups G mentioned in Corollaries 4.12 and 4.14, given the fact that the base field $\mathbb{K}$ admits the corresponding quadratic extension. But we can say more:

LEMMA 4.16. *Let $Q \cong \mathsf{D}_{3,1}(\mathbb{K})$ be the Klein quadric in $\mathsf{PG}(5, \mathbb{K})$ and let H be its stabilizer in $\mathsf{PGL}_6(\mathbb{K})$. Then the H-equivalence classes of non-degenerate quadrics of Witt index 1 in Q obtained by intersecting Q with 3-subspaces of $\mathsf{PG}(5, \mathbb{K})$, is in bijective correspondence with the quadratic field extensions $\mathbb{L}$ of $\mathbb{K}$ in its algebraic closure. Also, two such quadrics are projectively equivalent if, and only if, their pointwise stabilizers in the group of linear type-preserving collineations of Q are conjugate.*

Proof. By Proposition 4.11 and Remark 4.15, we only have to show that projectively equivalent quadrics give rise to the same field extensions, and projectively inequivalent ones to different field extensions. This is an elementary, though tedious, calculation left to the reader. ∎

This is everything we need concerning collineations of the residues of Δ whose fixed point structure is isomorphic to Γ.

The next result completes the analysis of Case I and corresponds to (i) of Theorem 4.1.

For a chamber C of a building Ω, the complex $E_2(C)$ is the restriction of Ω to all the rank 2 residues of C, that is, the residues of flags $F \subseteq C$ with $|C \setminus F| = 2$.

PROPOSITION 4.17. *Let Γ be a dual polar space of rank 3 isomorphic to $\mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$, for some quadratic field extension $\mathbb{L}$ of the field $\mathbb{K}$, fully embedded in the half spin geometry $\Delta \cong \mathsf{D}_{6,6}(\mathbb{K})$, such that at least one Γ-symp is isometrically embedded in some Δ-symp. Then Γ is isometrically embedded and it is the fixed point structure of each non-trivial element of a group G of collineations of Δ, abstractly isomorphic to $\mathbb{L}^\times / \mathbb{K}^\times$, with the following properties.*

(a) Each collineation of Δ pointwise fixing Γ belongs to G.

(b) Inclusion is a bijective correspondence between the symps of Γ and the symps of Δ that are stabilized by each element of G.

(c) For each symp Ω of Δ stabilized by G, the group G acts sharply transitively on the set of paras of Δ containing Ω.

(d) If char $\mathbb{K} \neq 2$, or the polar space corresponding to Γ is an inseparable quadric, then G contains an involution.

(e) If $\Gamma' \cong \Gamma$ is also fully embedded in Δ, then there exists a projectivity of Δ mapping Γ to Γ'.

Conversely, for every quadratic field extension $\mathbb{L}$ of $\mathbb{K}$, the dual polar space $\mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$ is fully embedded in Δ.

Proof. Fix a chamber C in Δ containing a chamber D of Γ. Let $C = \{p, L, U, \Omega, V, W\}$ and $D = \{p, L, \Sigma\}$, with p a point of Γ, L a line of Γ, Σ a symp of Γ containing L and isometrically embedded in the symp Ω of Δ, and U, V are maximal singular subspaces of dimension 5 and 3 of Δ, respectively, and W a para of Δ. We have

$$p \in L \subseteq V \subseteq \Omega \subseteq W, \quad \dim(U \cap V) = 2.$$

It follows that $U \cap \Sigma$ is a 3-space and $U \cap W$ a 4-space. Corollaries 4.12 and 4.14 imply the existence of unique groups G_1, G_2, G_3 of collineations of $\mathrm{Res}_\Delta(\Omega)$, $\mathrm{Res}_\Delta(p)$, and $\mathrm{Res}_\Delta(\{p, \Omega\})$, respectively, each nontrivial member of which has fixed point structure consisting of the elements of Γ in the respective corresponding residue. Each of the groups G_1, G_2, G_3 acts sharply transitively on the set of maximal singular subspaces of dimension 3 of Δ incident with L, with L and Σ, and with L, respectively. Let $g_3 \in G_3$ be arbitrary but nontrivial. Then there exist unique $g_1 \in G_1$ and $g_2 \in G_2$ with $g_1(V) = g_2(V) = g_3(V)$. Moreover, by the sharp transitivity of the actions mentioned in Corollaries 4.12 and 4.14, and the uniqueness of G_1, G_2, G_3, the action of g_i and g_j coincides over their common domain, for all $i, j \in \{1, 2, 3\}$. Now we define $g : E_2(C) \to E_2(g_3(C))$, where we tacitly assume that $g_3(L) = L$ and $g_3(\Omega) = \Omega$, as follows. Let $F \subseteq C$ be a flag of corank 2, that is, $|F| = 6 - 2 = 4$. If $p \in F$, then we let g coincide with g_2 on $\mathrm{Res}_\Delta(F)$; if $\Sigma \in F$, then we let g coincide with g_1 on $\mathrm{Res}_\Delta(F)$ (if $p, \Omega \in F$, then g_1 and g_2 coincide over $\mathrm{Res}_\Delta(F)$ and so this is well defined); finally, if $F = \{L, U, V, W\}$, then we already defined $g(p') = g_1(p')$, for all $p' \in L$, and $g(\Omega') = g_2(\Omega')$, for each symp Ω' of Δ with $V \subseteq \Omega' \subseteq W$. Since g_1 and g_2 are automorphisms of residues in Δ, it follows easily that g preserves adjacency of chambers.

Now we pick a chamber D' in Γ opposite D (opposite in Γ). We project C onto D' and obtain a chamber C_0' of Δ. In $\mathrm{Res}_\Delta(D')$, we select a chamber C' opposite C_0'. Then, by Proposition 3.29 of [23], C and C' are opposite chambers of Δ. Hence C and C' are contained in a unique apartment $\mathcal{A}$ of Δ, which, by Corollary 4.10, contains all elements of an apartment of Γ. Note that $\mathcal{A}$ contains the projection C_0 of C' onto D, and C_0 is opposite C_0'. It follows that $\mathcal{A}$ is determined by D, D', C, C_0. We now define the appartement $\mathcal{A}'$ as the unique apartment containing $D, D', g_3(C), g_3(C_0)$. The elementwise action of g on $\mathcal{A}$ is defined as the unique isomorphism $\mathcal{A} \to \mathcal{A}'$ mapping C to $g_3(C)$. Since $g_3 \in \mathrm{AutRes}\Delta(D)$, we see that $g(C_0) = g_3(C_0)$ and moreover, we claim that the actions of g on $E_2(C)$ and $\mathcal{A}$ are compatible, i.e., they agree on the intersection. Indeed, it suffices to show that g_1 and g_2 map elements of their domain in $\mathcal{A}$ to elements of $\mathcal{A}'$, such that the distance of such an element to C is the same as the distance of its image to $g_3(C)$. Consider g_1, the arguments for g_2 being similar. Since both $\mathcal{A}$ and $\mathcal{A}'$ contain two opposite chambers D and D' of Γ, they contain a Γ-apartment $\mathcal{B}$. The

intersection of $\mathcal{B}$ and Σ is a Σ-apartment $\mathcal{B}_\Sigma$. Let $\mathcal{A}_\Sigma$, $\mathcal{A}'_\Sigma$ be the intersections of Σ and $\mathcal{A}$, $\mathcal{A}'$, respectively. Then, as above, $\mathcal{A}_\Sigma$ $(\mathcal{A}'_\Sigma)$ is completely determined by the flags $\{p, L\}$, its opposite in $\mathcal{B}_\Sigma$, $C \setminus \{\Sigma\}$ and $C_0 \setminus \{\Sigma\}$ $(g_3(C) \setminus \{\Sigma\}$ and $g_3(C_0) \setminus \{\Sigma\})$. Hence it follows that g_1 maps $\mathcal{A}_\Sigma$ to $\mathcal{A}'_\Sigma$ and $C \setminus \{\Sigma\}$ to $g_3(C) \setminus \{\Sigma\}$, showing the claim.

We can now apply Proposition 4.16 of [23]. It follows that g extends uniquely to an automorphism of Δ, which we also denote by g, fixing $(E_2(C) \cup \mathcal{A}) \cap \Gamma$. This implies, since Γ is convex in Δ (by Corollary 4.10), that g fixes Γ pointwise. Now, g does not fix anything outside Γ since a larger fixed point building would induce a larger fixed point structure on $E_2(C)$ than $E_2(C) \cap \Gamma$. By the arbitrariness of g, the whole group G_3 (equivalently, G_1, G_2) extends uniquely to a group G of linear automorphisms, i.e., projectivities, of Δ each element of which fixes Γ and no more.

Hence Γ is the fixed point structure of the group G, and the assertions (a) to (d) follow. As for (e) and the last part, the existence, we note that, for a given building Δ, we can define the groups G_1, G_2, G_3 as above by Remark 4.15. We can then repeat the application of Proposition 4.16 of [23] that we did above to obtain a group G of collineations of Δ with fixed point building of desired type in view of the fixed point set in $E_2(C)$. Uniqueness (and hence (e)) then follows from Lemma 4.16. The only difference now is that we have to make a good choice for the flag D, since we do not dispose of Γ, but rather are constructing it. The choice of D must be so that the action of g on $E_2(C)$ is compatible with the action of the unique isomorphism $\mathcal{A} \to \mathcal{A}'$ (using above's notation). The arguments above show that this is true if $\mathcal{A}$ intersects the residues $\mathrm{Res}_\Delta(\Sigma)$ and $\mathrm{Res}_\Delta(p)$ in apartments of the respective fixed point buildings. This can be accomplished as follows.

In Σ we select a flag $\{q, M\}$ fixed by g_1, with q a point not collinear to p, and M a line not intersecting L. Let K be the line through p meeting M and let L' be the line through q meeting L. We select a line R through p which is opposite Σ in the building $\mathrm{Res}_\Delta(p)$ and which is fixed by g_2. Hence L, K, R form a triangle in the projective plane which is the fixed point structure of g_2 in $\mathrm{Res}_\Delta(p)$. Let Σ' be the symp containing K and R. Then we consider an apartment $\mathcal{A}$ of Δ containing the flags $\{p, R, \Sigma'\}$ and $\{q, L', \Sigma\}$. Clearly, $\mathcal{A}$ contains L. Inside $\mathcal{A}$, there is a unique flag D' opposite D. Now we have everything in place to apply Proposition 4.16 of [23] just like we did this above.

Finally, the embedding of Γ in Δ is isometric by Corollary 4.10. ∎

COROLLARY 4.18. *Let Γ be a dual polar space of rank 3 isomorphic to $\mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$, for some quadratic field extension $\mathbb{L}$ of the field $\mathbb{K}$, fully embedded in the half spin geometry $\Delta \cong \mathsf{D}_{6,6}(\mathbb{K})$, such that at least one Γ-symp is isometrically embedded in some Δ-symp. Then the set of lines of the corresponding polar space $\Delta^* \cong \mathsf{D}_{6,1}(\mathbb{K})$ which corresponds to the set of symps of Γ partitions the point set of Δ^*. Hence no member of the pointwise stabilizer of Γ in the automorphism group of Δ maps a point of Δ^* to an opposite point.*

Proof. Let $\mathcal{L}_\Gamma$ be the set of lines of Δ^* which correspond to symps of Γ (cf. Lemma 4.8), or equivalently, to the points of Γ^*. We have to show that every point p of Δ^* is contained in a member of $\mathcal{L}_\Gamma$. By Lemma 4.5, the set of members of $\mathcal{L}_\Gamma$ contained in a generator U of Δ^*, with U a 5-space of Δ^* corresponding to a point of Γ, form a regular spread of U. So we may assume $p \notin U$. Then p is collinear to a hyperplane H of U. Since the spread

is regular, H contains a 3-space W of Δ^* corresponding to a line of Γ. The fullness now implies that the unique generator U' of Δ^* containing p and W corresponds to a point of Γ. Now the fact that the members of $\mathcal{L}_\Gamma$ in U' form a spread shows the assertion. ∎

4.3. Case II: some symp Σ of Γ is contained in a symp Ω of Δ as a 4-dimensional quadrangle. As explained before, we now assume that Γ contains a symp Σ isomorphic to $\mathsf{B}_{2,1}(\mathbb{K})$ that is isometrically embedded in some symp Ω of Δ (occurring as the intersection of Ω, when viewing Ω in $\mathsf{PG}(7,\mathbb{K})$, with a 4-space of $\mathsf{PG}(7,\mathbb{K})$). By Lemma 4.2, $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{K})$. If char $\mathbb{K} \neq 2$, then $\mathsf{B}_{2,1}(\mathbb{K})$ admits only one (full) embedding in some projective space over $\mathbb{K}$, up to projectivity, and this is in projective 4-space; If char $\mathbb{K} = 2$, then $\mathsf{B}_{2,1}(\mathbb{K})$ also embeds in projective 3-space as the sub-quadrangle $\mathsf{C}_{2,1}(\mathbb{K},\mathbb{K}^2)$ of $\mathsf{C}_{2,1}(\mathbb{K})$. In the latter case, the lines of Σ through a point form a cone over a subline $\mathsf{PG}(1,\mathbb{K}^2)$ of $\mathsf{PG}(1,\mathbb{K})$ (if $\mathbb{K}$ is perfect, this is just a full planar line pencil).

Let p be a point of Σ and let U_p be the 5-space of Δ^* corresponding to p, in which we consider the $5'$-spaces of Δ^* incident with U_p as the points (i.e., the $5'$-space of Δ^* intersecting U_p in a 4-space).

4.3.1. The embedding of $\mathrm{Res}_\Gamma(p)$ — case distinction. We view the embedding of $\mathrm{Res}_\Gamma(p)$ in $\mathrm{Res}_\Delta(p)$ as follows: The set $\mathcal{L}_p$ of lines of Γ through p is identified with a set of lines in U_p, also denoted by $\mathcal{L}_p$. By Lemmas 4.5 and 4.6, a symp Σ' through p is identified either with a regulus of such lines in a 3-space $S_{\Omega'}$ of U_p (when Σ' is embedded isometrically in a symp Ω' of Δ), or with a perp cone, that is, a cone in U_p of such lines over a conic, or a partial planar line pencil in U_p over a subline $\mathsf{PG}(1,\mathbb{K}^2)$ of $\mathsf{PG}(1,\mathbb{K})$ (when Σ' is embedded in a singular subspace of Δ). As before, we will refer to these subsets of $\mathcal{L}_p$ as *blocks*, and we denote the set of all blocks by $\mathcal{B}_p$. Then $(\mathcal{L}_p, \mathcal{B}_p)$ is a projective plane $\mathfrak{P}_p$ isomorphic to $\mathsf{PG}(2,\mathbb{K})$. It is also convenient to refer to the cones over a conic briefly as *cones*, and the cones over a projective subline will be referred to as *(planar, partial) line pencils*.

By assumption, Σ is a symp of Γ that embeds isometrically in a symp of Δ, so $\mathcal{B}_p$ contains at least one regulus. Hence $3 \leq d_p := \dim\langle \mathcal{L}_p \rangle \leq 5$. We review the three possibilities. We start with the case $d_p = 5$, which, opposed to the others, leads to examples, and in which case we will show $d_q = d_p$ for all points q of Γ (cf. Corollary 4.21).

4.3.2. The case $d_p = 5$.

LEMMA 4.19. *If $d_p = 5$, then all blocks of $\mathfrak{P}_p$ are reguli, each pair of which generates U_p, and the configuration $\mathfrak{P}_p$ in U_p is projectively unique in U_p. Also, the subgroup of $\mathsf{PGL}_6(\mathbb{K})$ of collineations of U_p fixing every member of $\mathcal{L}_p$ is isomorphic to $\mathsf{PGL}_2(\mathbb{K})$ and acts sharply triply transitively on the points of each element of $\mathcal{L}_p$.*

Proof. Let $R \in \mathcal{B}_p$ be a regulus (which exists by assumption), spanning some solid S. Since $d_p = 5$, there is a line $L \in \mathcal{L}_p$ such that $L \nsubseteq S$. Then L intersects at most one member of R and hence every block through L, except for at most one, is a regulus in $\langle S, L \rangle$. It is then easily deduced that $\mathfrak{P}_p$ lies in $\langle S, L \rangle$. As $d_p = 5$, this means that L does not intersect S. This implies that all blocks through L are reguli, and consequently, all members of $\mathcal{L}_p$ outside R are outside S. We obtain that all blocks are reguli.

We can choose coordinates in such a way that R contains the lines

$$L_1 = \langle (1,0,0,0,0,0), (0,1,0,0,0,0) \rangle,$$
$$L_2 = \langle (0,0,1,0,0,0), (0,0,0,1,0,0) \rangle,$$
$$L_3 = \langle (1,0,1,0,0,0), (0,1,0,1,0,0) \rangle.$$

Let L_4 be a line not in R. Then we can assign it the coordinates such that

$$L_4 = \langle (0,0,0,0,1,0), (0,0,0,0,0,1) \rangle.$$

Moreover, in $\langle L_1, L_4 \rangle$ we may choose the unit point $(1,1,0,0,1,1)$ so that the line

$$L_5 = \langle (1,0,0,0,1,0), (0,1,0,0,0,1) \rangle$$

belongs to $\mathcal{L}_p$. Let R' be the regulus of $\mathcal{B}_p$ containing L_1 and L_4. Let $L \in \mathcal{L}_p$ be arbitrary but not contained in $R \cup R'$. Then $\langle L_4, L \rangle$ intersects $\langle R \rangle$ in a member L' of $\mathcal{L}_p$ (as $\mathfrak{P}_p$ is a projective plane), and likewise $\langle L_5, L \rangle$ intersects $\langle R \rangle$ in a member $L'' \in \mathcal{L}_p$. So $L = \langle L_4, L' \rangle \cap \langle L_5, L'' \rangle$. Varying L' and L'' over R, we thus obtain all members of $\mathcal{L}_p \setminus (R \cup R')$. It now follows without too much effort that also the elements of R' are determined (argue with R and an arbitrary $R'' \in \mathcal{B}_p$ containing L_1 but different from R'). Hence $\mathcal{L}_p$ is completely determined and so $\mathfrak{P}_p$ is projectively unique.

We continue assigning coordinates. Elementary calculations reveal that a each member of $\mathcal{L}_p$ is generated by the points $(x,0,y,0,z,0)$ and $(0,x,0,y,0,z)$ for some $(x,y,z) \in \mathbb{K}^3 \setminus \{(0,0,0)\}$. A calculation similar to the one performed in the proof of Proposition 4.11 now shows that the pointwise stabilizer of $\mathcal{B}_p$ in $\mathsf{PGL}_6(\mathbb{K})$ is given by the set of matrices

$$\begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{bmatrix},$$

with $a, b, c, d \in \mathbb{K}$ such that $ad - bc \neq 0$. Since $\mathsf{PGL}_2(\mathbb{K})$ acts sharply 3-transitively in its standard representation, this completes the proof of the lemma. ∎

REMARK 4.20. The union of all members of the set $\mathcal{L}_p$ is the Segre variety $\mathcal{S}_{1,2}(\mathbb{K})$. In general, the *Segre variety* $\mathcal{S}_{n-1,m-1}(\mathbb{K})$, with $n, m > 1$, is the set of points of the projective space $\mathsf{PG}(nm-1, \mathbb{K})$ projectively equivalent to the set

$$\{(x_i y_j)_{1 \leq i \leq n, 1 \leq j \leq m} \mid (x_1, ..., x_n) \in \mathbb{K}^n \setminus \{(0, ..., 0)\}, (y_1, ..., y_m) \in \mathbb{K}^m \setminus \{(0, ..., 0)\}\}.$$

For fixed $(x_i)_{1 \leq i \leq n}$ distinct from $(0, ..., 0)$, or fixed $(y_j)_{1 \leq j \leq m}$ distinct from $(0, ..., 0)$, the sets

$$\{(x_i y_j)_{1 \leq i \leq n, 1 \leq j \leq m} \mid (y_1, ..., y_m) \in \mathbb{K}^m \setminus \{(0, ..., 0)\}\}$$

and

$$\{(x_i y_j)_{1 \leq i \leq n, 1 \leq j \leq m} \mid (x_1, ..., x_n) \in \mathbb{K}^n \setminus \{(0, ..., 0)\}\}$$

are called *generators* of $\mathcal{S}_{n-1,m-1}(\mathbb{K})$.

Again we have two interesting and useful corollaries.

COROLLARY 4.21. *If $d_p = 5$, then each symp of Γ is isometrically embedded in a symp of Δ. Also, for every point q of Γ, $d_q = 5$.*

Proof. Take a point q of Γ collinear to p. Let B_1 and B_2 be distinct blocks of $\mathfrak{P}_q$ containing the line L of $\mathcal{L}_q$ corresponding to pq, viewed as subsets of $\mathcal{L}_q$ in the 5-space U_q corresponding to q. Note that B_1 and B_2 are reguli because the respective corresponding symps Σ_1 and Σ_2 contain p and hence are embedded isometrically in Δ by Lemma 4.19. Suppose that $\dim\langle B_1, B_2\rangle < 5$, i.e., that $\langle B_1, B_2\rangle$ is contained in a 4-space S of U_q. Recalling the structure of U_q (see Notation 4.3), S corresponds to a para of Δ, which contains the symps Σ_1 and Σ_2. However, this para would then also correspond to a 4-space S' of U_p containing the blocks B_1' and B_2' of U_p corresponding to Σ_1 and Σ_2, respectively, contradicting the fact that $\langle B_1', B_2'\rangle = U_p$ by Lemma 4.19. So $\dim\langle B_1, B_2\rangle = d_q = 5$. Since moreover Σ_1 and Σ_2 are symps through q which are embedded isometrically, we may apply Lemma 4.19 to q, from which we conclude that all symps of Γ containing q are embedded isometrically. A connectivity argument now completes the proof of the corollary. ∎

COROLLARY 4.22. *Let $Q \cong \mathsf{D}_{n+1,1}(\mathbb{K})$ be a hyperbolic quadric in $\mathsf{PG}(2n+1, \mathbb{K})$, $n \geq 2$, and let Q' be the intersection of Q with a $(2n-2)$-space S such that $Q' \cong \mathsf{B}_{n-1,1}(\mathbb{K})$ is the parabolic quadric of Witt index $n-1$. Let $G \leq \mathsf{PGL}_{2n+2}(\mathbb{K})$ be the group of (linear) type preserving collineations of $\mathsf{PG}(2n+1, \mathbb{K})$ stabilizing Q, and pointwise fixing Q'. Then $G \cong \mathsf{PGL}_2(\mathbb{K})$ and acts sharply 3-transitively on the set of generators of Q of given type through each singular $(n-2)$-space of Q' (and hence has size $|\mathbb{K}|(|\mathbb{K}|^2 - 1)$). Also, G acts freely on the set of $(n-1)$-spaces and generators of Q.*

Proof. The linear stabilizer G of the set of lines of one set of generators of a hyperbolic quadric in $\mathsf{PG}(3, \mathbb{K})$ is isomorphic to $\mathsf{PGL}_2(\mathbb{K})$ (which also follows from the proof of Proposition 4.11—there the current G is the stabilizer of L_1, L_2, L_3). Translated through the Klein correspondence to quadrics in $\mathsf{PG}(5, \mathbb{K})$, we obtain exactly the statement of the corollary for $n = 2$. The case $n > 2$ then follows completely analogously as in the proof of Corollary 4.12. ∎

We will not need the next corollary, but we mention it for completeness. It probably belongs to folklore. Its proof is a straightforward extension of the proof of Lemma 4.19.

COROLLARY 4.23. *Let $\mathcal{P}$ be a generating set of lines in $\mathsf{PG}(2n+1, \mathbb{K})$, in bijective correspondence with the set of points of $\mathsf{PG}(n, \mathbb{K})$, and such that the points on a line of $\mathsf{PG}(n, \mathbb{K})$ correspond to a set of lines in a 3-space of $\mathsf{PG}(2n+1, \mathbb{K})$, and we assume that at least one of these sets is a regulus. Then $\mathcal{P}$ is the set of maximal singular subspaces of dimension 1 of the Segre variety $\mathcal{S}_{1,n}(\mathbb{K})$ and $\mathsf{PGL}_2(\mathbb{K})$ is the elementwise stabilizer of $\mathcal{P}$ in $\mathsf{PGL}_{2n+2}(\mathbb{K})$. ∎*

The proof of the fact that Γ is convex in Δ and that apartments of Γ are contained in apartments of Δ is now completely similar to Case 1, see Section 4.2.1. Then the following proposition is also proved similarly to Proposition 4.17, so we will not repeat the proof.

PROPOSITION 4.24. *Let Γ be a dual polar space of rank 3 isomorphic to $\mathsf{C}_{3,3}(\mathbb{K})$, fully embedded in the half spin geometry $\Delta \cong \mathsf{D}_{6,6}(\mathbb{K})$, such that at least one Γ-symp is iso-*

metrically embedded in some Δ-symp, and such that for some Γ-point p of such a symp $d_p = 5$ (with the same notation as above). Then Γ is isometrically embedded in Δ and it is the fixed point structure of each non-trivial element of a group G of collineations of Δ, abstractly isomorphic to $\mathsf{PGL}_2(\mathbb{K})$, with the following properties.

(a) Each collineation of Δ pointwise fixing Γ belongs to G.

(b) Inclusion is a bijective correspondence between the symps of Γ and the symps of Δ that are stabilized by each element of G.

(c) For each symp Σ of Δ stabilized by G, the group G acts on the set of paras of Δ containing Σ in its natural sharply 3-transitive action.

(d) If $\Gamma' \cong \Gamma$ is also fully embedded in Δ, then there exists a projectivity of Δ mapping Γ to Γ'.

Conversely, the dual polar space $\mathsf{C}_{3,3}(\mathbb{K})$ is always fully embedded in Δ. ∎

4.3.3. The case $d_p \in \{3, 4\}$. We first note that it follows from Lemmas 7.6 and 7.7 of [12] that, if $d_p \leq 4$, then some block of $\mathfrak{P}_p$ is not a regulus. Recall that we also assume that some block of $\mathfrak{P}_p$ is a regulus. Blocks corresponding to reguli, quadratic cones and partial planar line pencils will be briefly referred to as *reguli, cones* and *pencils*, respectively. Also, throughout, set $D_p = \langle \mathcal{L}_p \rangle \subseteq U_p$ (recall $d_p = \dim D_p$).

LEMMA 4.25. *If $d_p = 4$, then there are no pencils in $\mathcal{B}_p$. Also, there exists a (unique) line T in D_p enjoying the following properties:*

$-$ *T is the transversal of each regulus in $\mathcal{B}_p$,*

$-$ *$T \in \mathcal{L}_p$*

$-$ *T is contained in each cone in $\mathcal{B}_p$, and*

$-$ *each point of T is the vertex of a unique cone in $\mathcal{B}_p$.*

Proof. Let R be a regulus of $\mathcal{B}_p$ and let C be a cone or a pencil of $\mathcal{B}_p$ with vertex v. Then $R \cap C$ is a line L. It is easily verified that each line of $\mathcal{L}_p$ is contained in $\langle R, C \rangle$. Since $d_p = 4$, this means that some line $M \in C$ is not contained in the 3-space $\langle R \rangle$. So, if L' is any line of $R \setminus \{L\}$, then M and L' are contained in a regulus R' of $\mathcal{B}_p$. Select any line $K \in R' \setminus \{M, L'\}$. Then K intersects $\langle R \rangle$ in a unique point u. We claim that $u \in T \setminus \{v\}$, where T is the transversal of R though v. Note that $u \in \langle R' \rangle \cap \langle R \rangle = \langle v, L' \rangle$. So, if u lies on a member of R, then it necessarily belongs to T. Hence we assume that this is not the case. Now let L'' be any line of R distinct from L' and L. The regulus $R'' \in \mathcal{B}_p$ through L'' and K has a line M'' in common with C, with $M'' \notin \langle R \rangle$. Hence the same argument as before now shows that $u \in \langle v, L'' \rangle$. Since $T = \langle v, L' \rangle \cap \langle v, L'' \rangle$, the claim follows.

Now K intersects a line of R and so u is the vertex of a cone or a pencil C' of $\mathcal{B}_p$. Since $u \neq v$, we have $C' \cap C = T$. Hence $T \in \mathcal{L}_p$. So, if C were a pencil, then $\langle C \rangle = \langle L, T \rangle \subseteq \langle R \rangle$, a contradiction. Consequently, for each point t on T, the unique line L_t of R and T determine a cone. Since each pair of cones has to have a line in common, there cannot be more cones than these. The lemma follows. ∎

LEMMA 4.26. *If $d_p = 3$, then there are no cones in $\mathcal{B}_p$. Also, there exists a (unique) line T in D_p enjoying the following properties:*

$-$ *T is the transversal of each regulus in $\mathcal{B}_p$,*

$-$ *$T \in \mathcal{L}_p$,*

− T is contained in each pencil of $\mathcal{B}_p$ and
− each point of T is the vertex of a unique pencil in $\mathcal{B}_p$.

Proof. We first prove that there is no cone in $\mathcal{B}_p$. Suppose for a contradiction that there is a cone C. By assumption, there is also a regulus R in $\mathcal{B}_p$. Denote by L the common line of C and R. The line L corresponds to a line L' of Γ through p. The 3-space D_p of U_p corresponds to a symp Q of Δ which contains all Γ-lines through p. This in particular implies that the Γ-symps through $L' \ni p$ that correspond to reguli in $\mathfrak{P}_p$ embed isometrically in Q. Now let $q \in L'$ with $q \neq p$. If $\mathcal{L}_q$ were not contained in a 3-space, then $\mathcal{L}_q$ is contained in a 4-space (because $\mathcal{B}_q$ contains a regulus and a cone), and then Lemma 4.25 implies that there are at least two reguli in $\mathcal{B}_q$ containing the line of $\mathcal{L}_q$ corresponding to L'. Since these reguli span the 4-space containing $\mathcal{L}_q$, they correspond to distinct symps, contradicting the fact that the corresponding reguli of $\mathcal{B}_p$ both define Q. Consequently, all Γ-lines through q, in particular those contained in the Γ-symp Σ_C corresponding to C, are also contained in Q. But the lines of Σ_C containing p or q generate the singular 4-space of Δ containing Σ_C, contradicting the fact that the maximal singular subspaces of Q are 3-dimensional. Hence there are no cones in $\mathcal{B}_p$.

Now let π be a pencil in $\mathcal{B}_p$, and denote by L the intersection of π with R. Let x be the vertex of π and let α be the plane spanned by π. Let $T_x(R)$ denote the tangent hyperplane of R at x, i.e., the tangent plane of the underlying hyperbolic quadric at x.

Claim 1: $\alpha = T_x(R)$.

Suppose the contrary. Then α contains a transversal T of R with $x \notin T$, and hence each line of π intersects T. Consider two such lines L_1 and L_2, with $L_1 \neq L \neq L_2$. As the point $x_i = L_i \cap T$ is contained in a line M_i of R, there is a pencil $\pi_i \in \mathcal{B}_p$ having x_i as vertex, $i = 1, 2$. Obviously, T is the unique line $\pi_1 \cap \pi_2$, so $T \in \mathcal{L}_p$. As such, $\pi_1 \subseteq \alpha$ as it contains the two lines T and L_1 of α. However, $M_1 \in \pi_1$ but $M_1 \not\subseteq \alpha$, a contradiction. □

We denote the transversal of R through x by T.

Claim 2: $T \in \pi$.

Select lines $L' \in \pi \setminus \{L\}$ and $M' \in R \setminus \{L\}$. We may assume $L' \neq T$ as otherwise there is nothing to prove. There is a regulus R' in $\mathcal{B}_p$ defined by the pair L', M'. Let T' be the transversal of R' through x. Claim 1 applied to π and R' implies that T' lies in α. Since $T' \subseteq \alpha$ intersects M', it contains $M' \cap \alpha \in T$, and since $M' \cap \alpha \neq x$, we conclude $T' = T$. Now, for any point y of T not on $L' \cup M'$, the respective lines belonging to R and R' through it are distinct and hence y is the vertex of a pencil $\pi_y \in \mathcal{B}_p$; so $T = \pi \cap \pi_y$ and the claim follows. □

It follows from Claim 2 that each point on T occurs as the vertex of a pencil, and these pencils are all members of $\mathcal{B}_p$ joining T and line belonging to R. Every other member of $\mathcal{B}_p$ contains a line of each such pencil distinct from T and hence is a regulus with transversal T. This proves the lemma. ∎

So, the set of lines of Γ through p contains a unique line through which each Γ-symp is contained in a singular d_p-subspace. In general, we refer to a Γ-line as an *S-line* if it has the property that each Γ-symp it contains is embedded in a singular subspace of Δ. Likewise, every Γ-symp contained in a singular subspace of Δ will be referred to as an

S-symp. Note that in these definitions, we do not necessarily assume that the Γ-line or the Γ-symp contains p. In fact, in the sequel, we will sometimes let p vary among points q which are also contained in at least one Γ-symp that embeds isometrically in a Δ-symp:

COROLLARY 4.27. *For each point q of Γ contained in at least one Γ-symp that embeds isometrically in a Δ-symp, $d_q \in \{3, 4\}$. Moreover, there is a unique S-line containing q, and this line is contained in every S-symp through q; every other Γ-line through q is contained in exactly one S-symp. Finally, each S-symp through q embeds in a singular subspace of dimension d_q.*

Proof. If $d_q = 5$ then Corollary 4.21 together with our assumption implies that $d_p = 5$, a contradiction. So $d_q \in \{3, 4\}$ indeed. As such, the corollary follows from Lemma 4.25 (if $d_q = 4$) or Lemma 4.26 (if $d_q = 3$) with q in the role of p (the unique S-line containing q corresponding to the line T). ∎

LEMMA 4.28. *The set H of all S-symps is a geometric hyperplane of the polar space Γ^*. Also, H is the union of all S-lines (considered as lines of Γ^*), i.e., a Γ-symp is an S-symp if and only if it contains an S-line.*

Proof. Let L be a Γ-line which is contained in at least one Γ-symp Σ which is not an S-symp. Considering any point $q \in L$, Corollary 4.27 implies that L is contained in a unique S-symp. We conclude that, through each line L of Γ, either all Γ-symps through L are S-symps or exactly one Γ-symp through L is an S-symp. Since by assumption there is at least one Γ-symp which is not an S-symp, H is a geometric hyperplane of Γ^*.

By the definition of S-line, each Γ-symp containing an S-line is an S-symp. For the converse statement, let Σ be an S-symp. We show that Σ contains an S-line. If all Γ-symps containing a line of Σ are S-symps, then all lines of Σ are S-lines. So we may assume there exists a Γ-symp Σ' that embeds isometrically in a Δ-symp and that intersects Σ in a line M. Pick $r \in M$. By Corollary 4.27, there is a unique S-line containing r and this line belongs to Σ. ∎

By Corollary 1.3(ii) of [8], a geometric hyperplane H of the polar space $\Gamma^* \cong \mathsf{C}_{3,1}(\mathbb{K})$ admits at most one *deep point* x, that is, a point with the property that all lines of Γ^* through x are contained in H.

Notation. As in the previous lemma, we let H be the set of all S-symps and we denote by Σ_H the symp corresponding to the unique deep point of H, if it exists.

LEMMA 4.29. *Every S-symp, except possibly Σ_H, spans a singular d_p-space in Δ.*

Proof. Let $\Sigma \notin H$ be any Γ-symp through p. By Lemmas 4.25 and 4.26, each S-symp through p generates a singular d_p-space. Take any Γ-symp $\Sigma' \notin H$ intersecting Σ in some Γ-line M. Then M contains a point q which is Γ-collinear to p. Since the Γ-line pq is contained in $\Sigma \notin H$, Corollary 4.27 applied to both p and q implies that there is a unique S-symp through the line pq, which embeds in a singular subspace of dimension $d_p = d_q$. This in particular shows that Σ' plays the same role as Σ. By the connectivity of the complement of H in Γ^* (see Lemma 1.1(i) in [8]), it thus follows that each S-symp containing some line which is not an S-line, spans a d_p-space in Δ. This holds for all members of H, except for Σ_H. ∎

LEMMA 4.30. *Let Ξ be the subspace of Δ generated by the union of all Γ-symps which do not belong to H. Then, if either $(|\mathbb{K}|, d_p) \neq (2, 4)$ or Ξ is Δ-convex, then $\Gamma \subseteq \Xi$.*

Proof. First suppose that Ξ is convex. Let x be any point of Γ. If x belongs to a Γ-symp not in H, then trivially $x \in \Xi$. So suppose the contrary and let Σ be any Γ-symp not in H. In Γ^*, x corresponds to a singular plane π_x and Σ corresponds to a point $p_\Sigma \notin \pi_x$, and hence there is a unique plane π_y containing p_Σ and intersecting π_x in a line. The plane π_y corresponds in Γ to a point $y \in \Sigma$ which is Γ-collinear to x. Put $L = xy$. Note that our assumption on x implies that each symp through L is an S-symp and, recalling that $\Sigma \notin H$, Corollary 4.27 implies that L is the unique S-line through y. Select distinct Γ-symps Σ_1 and Σ_2 containing L (note that they are distinct from Σ_H as there is only one S-line through y). Pick a line $M_i \subseteq \Sigma_i$ containing y and distinct from $L = \Sigma_1 \cap \Sigma_2$, $i = 1, 2$ and consider the unique Γ-symp Σ' determined by M_1 and M_2. Since each S-symps through y contains L by Corollary 4.27 and Σ' does not contain L (otherwise $\Sigma_1 = \Sigma' = \Sigma_2$), we have $\Sigma' \notin H$. Hence, if $z_i \in M_i \setminus \{y\}$, $i = 1, 2$, then $\delta_\Delta(z_1, z_2) = 2$. However, z_1 and z_2 are both Δ-collinear to x. Consequently, x belongs to the Δ-convex closure of $\{z_1, z_2\}$, while $\{z_1, z_2\} \subseteq \Xi$. Hence $x \in \Xi$, proving $\Gamma \subseteq \Xi$.

Now suppose $(|\mathbb{K}|, d_p) \neq (2, 4)$. Let Σ be an S-symp distinct from Σ_H. Then, by Lemma 4.28, there exist two Γ-collinear points $x, y \in \Sigma$ such that the line xy is not an S-line. Hence there are Γ-symps $\Sigma_x, \Sigma_y \notin H$ containing x and y, respectively. Then all lines through x, except for the unique S-line L_x through x, are contained in Γ-symps not belonging to H. Consider any S-symp Σ'_x through L_x. Then we claim that the singular subspace of Δ generated by the set T_x of all Γ-lines through x in Σ'_x, except for L_x, contains L_x. Indeed, if $d_p = 3$, then, by Lemma 4.6, T_x spans a plane containing L_x; if $d_p = 4$ and $|\mathbb{K}| > 2$, then, again by Lemma 4.6, T spans a 3-space containing L_x. Hence all Γ-lines through x are contained in Ξ. Likewise, all Γ-lines through y are contained in Ξ. It follows that Σ is contained in Ξ since $x^{\perp_\Gamma} \cap \Sigma$ and $y^{\perp_\Gamma} \cap \Sigma$ generate distinct hyperplanes of the singular subspace of Δ generated by Σ. Now also Σ_H is contained in Ξ since every point of Σ_H is contained in some other Γ-symp, too. ∎

LEMMA 4.31. *If $d_p = 3$, then Γ is contained in some Δ-symp Q.*

Proof. Consider a Γ-symp $\Sigma \notin H$. Let Q be the unique Δ-symp containing Σ. Take any Γ-symp $\Sigma' \notin H$ intersecting Σ in a Γ-line. Then, by Lemmas 4.28 and 4.29, we may suppose $p \in \Sigma \cap \Sigma'$. Recalling that $D_p = \langle \mathcal{L}_p \rangle$, the Γ-symp Σ corresponds to a regulus in D_p, and D_p corresponds to Q. Since $\mathcal{L}_p$ is contained in D_p, all Γ-lines through p are contained in Q, and so is Σ'. By connectivity of the complement of H in Γ^*, all Γ-symps not in H are contained in Q. Hence, since Q is a subspace, Lemma 4.30 implies that also each S-symp is contained in (a singular subspace of) Q. ∎

LEMMA 4.32. *If $d_p = 4$, then Γ is contained in some Δ-para Π.*

Proof. Consider any pair of Γ-symps $\Sigma, \Sigma' \notin H$ intersecting in a Γ-line. By Lemmas 4.28 and 4.29, we may again suppose $p \in \Sigma \cap \Sigma'$. Let Σ_1, Σ_2 be any pair of distinct Γ-symps containing p. It follows from Lemma 4.25 that the reguli or cones corresponding to Σ_1, Σ_2 span the 4-space D_p of U_p. In other words, the unique para Π of Δ corresponding to D_p is determined by the symps Σ_1 and Σ_2. If $\Sigma'' \notin H$ is any Γ-symp sharing a Γ-line with Σ,

then there is a point $q \in \Sigma \cap \Sigma''$ that is Γ-collinear to p. Since Π is in particular determined by any pair of Γ-symps through pq, each Γ-symp containing q and not contained in H has its Γ-lines through p in Π and is hence contained in Π; so $\Sigma'' \subseteq \Pi$. Again, connectivity of the complement of H in Γ^* implies that all Γ-symps not in H are contained in Π. Since Π is convex, it now follows from Lemma 4.30 that Γ is contained in Π. ∎

LEMMA 4.33. *The case $d_p = 3$ cannot occur.*

Proof. We know from Lemma 4.31 that Γ is embedded in a symp Q. This embedding induces a projective embedding in the projective space $\mathsf{PG}(7, \mathbb{K})$, possibly already in a subspace. By [10] and [11], it follows that Γ is a projection of the universal embedding $\mathcal{E}$ of $\mathsf{C}_{3,3}(\mathbb{K})$ in $\mathsf{PG}(13, \mathbb{K})$. Now, recalling that char $\mathbb{K} = 2$ since $d_p = 3$, the embedding $\mathcal{E}$ has a 5-dimensional nucleus space N, which is the intersection of all tangent hyperplanes, or also the set of nuclei of all its symps (note that each nucleus is a point). This nucleus space can hence be identified with Γ^*. Lemma 4.28 implies that Γ is the projection of $\mathcal{E}$ from a space W that intersects N in a 4-space. Hence N is projected onto a point n, which is the common nucleus of all Γ-symps that are isometrically embedded in Q. Hence n belongs to the tangent spaces of all Γ-symps not in H at each of their points. This implies that all Γ-symps not in H are contained in the perp hyperplane H_n of n with respect to the symplectic polarity in which Q is embedded (as a hyperbolic quadric). Since $n \notin Q$, H_n intersects Q in a non-degenerate quadric Q' isomorphic to $\mathsf{B}_{3,1}(\mathbb{K})$.

Now Lemma 4.30 implies that every member of H, except possibly Σ_H, is contained in Q'. But every such member spans a singular 3-space of Δ and Q' does not contain singular 3-spaces. ∎

LEMMA 4.34. *The case $d_p = 4$ does not occur.*

Proof. By Lemma 4.32, Γ is contained in a unique para Π, which is a half spin geometry $\mathsf{D}_{5,5}(\mathbb{K})$. Since H is a geometric hyperplane of Γ^*, there are three S-symps $\Sigma, \Sigma_1, \Sigma_2$ distinct from Σ_H with $\Sigma \cap \Sigma_i$ a line L_i, $i = 1, 2$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$. By Lemma 4.29, the singular subspaces W, W_1, W_2 of Δ spanned by $\Sigma, \Sigma_1, \Sigma_2$, respectively, are 4-dimensional, hence maximal singular 4-spaces in $\mathsf{D}_{5,5}(\mathbb{K})$. Now W and W_i are distinct since they contain lines of a common Γ-symp which does not belong to H, and such lines are not collinear in Δ. Since $W \cap W_i$ contains a line, it is a line (as two distinct 4-spaces of $\mathsf{D}_{5,5}(\mathbb{K})$ are either disjoint or intersect in a line). Since $W \cap W_1 \neq W \cap W_2$ (as $L_1 \cap L_2 = \emptyset$), we also have $W_1 \neq W_2$; even $W_1 \cap W_2 = \emptyset$ (as otherwise the line $W_1 \cap W_2$ would be collinear to the disjoint lines $W_1 \cap W$ and $W_2 \cap W$). So, in the polar space $\Pi^* = \mathsf{D}_{5,1}(\mathbb{K})$, W_i corresponds to a maximal singular $4'$-space W_i^*, $i = 1, 2$, and there is a point w with $W_1^* \cap W_2^* = \{w\}$. Now, let V_i be any 4-space intersecting W_i in a 3-space, but not containing w, $i = 1, 2$. Then $V_1 \cap V_2$ is a unique point, and hence V_1 and V_2 represent non-collinear points of Π. The point w corresponds to a symp Q_w of Π, intersecting W_i in a 3-space S_i, $i = 1, 2$. No point of $W_1 \setminus S_1$ is collinear to any point of $W_2 \setminus S_2$ (such points correspond to V_1 and V_2 above). Now Σ_1 generates W_1 and every point of Σ_1 is Γ-collinear to a unique point of Σ_2. So $\Sigma_2 \subseteq S_2$, which contradicts the fact that Σ_2 generates W_2. This contradiction shows that the case $d_p = 4$ does not occur. ∎

Taking everything together, we obtain (i) and (ii) of Theorem 4.1.

PROPOSITION 4.35. *Let Γ be a dual polar space of rank 3 fully embedded in the half spin geometry $\Delta = \mathsf{D}_{6,6}(\mathbb{K})$, for some field $\mathbb{K}$, such that at least one Γ-symp is isometrically embedded in some Δ-symp. Then precisely one of the following occurs.*

(i) *$\Gamma \cong \mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$, for some (separable or inseparable) quadratic extension $\mathbb{L}$ of $\mathbb{K}$, and is the fixed point structure of each non-trivial element of a subgroup of collineations of Δ isomorphic to the factor group $\mathbb{L}^{\times}/\mathbb{K}^{\times}$; the quadratic extensions $\mathbb{L}$ of $\mathbb{K}$ are in one-to-one correspondence with the classes of projectively equivalent fully embedded dual polar spaces $\mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$ in Δ (and each such embedding is isometric).*

(ii) *$\Gamma \cong \mathsf{C}_{3,3}(\mathbb{K})$ and arises as the fixed point structure of a subgroup of collineations of Γ isomorphic to $\mathsf{PGL}_2(\mathbb{K})$; this embedding is isometric and projectively unique.*

Proof. Lemma 4.2 yields $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$, for some quadratic field extension $\mathbb{L}$ of $\mathbb{K}$, or $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{K})$. If $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$, then Proposition 4.17 leads to (i). If $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{K})$, then Proposition 4.24 and Lemmas 4.33 and 4.34 lead to (ii). ∎

REMARK 4.36. A necessary condition for a full embedding of one parapolar space Ω into another parapolar space Ω' to be isometric is obviously that at least one symp of Ω embeds isometrically in some symp of Ω'. In the situation of Ω being a dual polar space of rank 3 and Ω' the half spin geometry $\mathsf{D}_{6,6}(\mathbb{K})$, Proposition 4.35 shows that this (local) condition is also sufficient.

4.4. Case III: each symp of Γ is contained in a singular subspace of Δ, but Γ itself is not.

In this case we assume that for all points $p, q \in \Gamma$ with $p \perp\!\!\!\perp_\Gamma q$, we have $p \perp_\Delta q$, but that there exists a pair of points in Γ which is not Δ-collinear. Our first aim is to show that Γ is contained in a symp of Δ.

LEMMA 4.37. *There is a unique symp Q of Δ containing Γ. Also, two points of Γ are opposite in Γ if, and only if, they are not collinear in Δ, hence Q-opposite in Q.*

Proof. Let p, q be any pair of points of Γ which are not Δ-collinear. Our assumption implies that such a pair exists, and it moreover implies that p and q are opposite in Γ, since all points of Γ which are at distance at most 2 in Γ, are collinear in Δ. Observe that any Γ-line L through p contains a unique point x which is Γ-symplectic, and hence Δ-collinear, to q. This means that p and q determine a unique symp Q of Δ, which contains $p^{\perp_\Gamma}$ and likewise also $q^{\perp_\Gamma}$.

Take any point $r \in p^{\perp_\Gamma} \setminus q^{\perp\!\!\!\perp_\Gamma}$. Then r is also Γ-opposite q, and hence Δ-symplectic to q (if q would be Δ-collinear to r, then it would also be Δ-collinear to p as the line pr contains a unique point Γ-symplectic, and hence also Δ-collinear, to q). It follows that p and r play the same role: the symp determined by r and q hence coincides with Q and contains $r^{\perp_\Gamma}$. As can be verified by the reader, the opposite geometry of q in Γ is connected, meaning that each pair of points of Γ opposite q can be connected by a path (not necessarily a shortest path in Γ), all vertices of which are points of Γ opposite q and all edges of which are Γ-lines having all but one of their points opposite q. As such, each Γ-point r that is Γ-opposite q is Δ-symplectic to q and satisfies $r^{\perp_\Gamma} \subseteq Q$. Now each point of Γ which is not opposite q, is Γ-collinear to either q or to a point of Γ opposite q, and as such belongs to Q. So $\Gamma \subseteq Q$ indeed. Moreover, we showed that for all points r Γ-opposite

q, the pair $\{r, q\}$ plays the same role as $\{p, q\}$. Repeating the argument with $\{r, q\}$ and now varying q amongst the points Γ-opposite r, also the second assertion follows. ∎

This lemma now has the remarkable consequence that each symp of Γ is contained in a singular 3-space, because the symp Q has Witt index 4. We could now invoke the classification of (fully) embedded generalized quadrangles in 3-dimensional projective space by Dienst [13] to nail down the possible isomorphism classes of Γ. However, we are interested in also determining the projective equivalence classes of full embeddings of Γ in Δ, so we take a little detour and perform the former classification in the guise of the rank 2 analogue of the latter one (see Lemma 4.39), which is thus slightly more efficient, although less direct.

A second consequence of Lemma 4.37 is that each symp Σ of Γ uniquely defines a maximal singular subspace of Q, which we denote by $\langle \Sigma \rangle$. Note that Σ is embedded in the absolute geometry of a non-degenerate polarity of $\langle \Sigma \rangle$ and hence, for each point $p \in \Sigma$, the Σ-perp of p (i.e., the union of all lines of Σ through p) generates a (hyper)plane of $\langle \Sigma \rangle$. In the next lemma, oppositeness of symps in Γ is to be understood as in Subsection 2.2 (so two symps are opposite in Γ if and only if the corresponding points of Γ^* are not collinear). We also remark that Σ could be a proper subgeometry of the absolute geometry of the above polarity (this happens for instance if the polarity is symplectic and $\Sigma \cong \mathsf{C}_2(\mathbb{K}, \mathbb{K}')$, for some field $\mathbb{K}'$ with $\mathbb{K}^2 \leq \mathbb{K}' \leq \mathbb{K}$ and char $\mathbb{K} = 2$).

LEMMA 4.38. *The mapping $\Sigma \mapsto \langle \Sigma \rangle$ is a monomorphism from the polar space Γ^* to the half spin geometry related to one system of generators of Q, and hence defines a (not necessarily full) embedding of Γ^* in the triality quadric $Q^* \cong \mathsf{D}_{4,1}(\mathbb{K})$. Moreover, if Σ and Σ' are two opposite symps of Γ, then $\langle \Sigma \rangle$ and $\langle \Sigma' \rangle$ are opposite maximal singular subspaces of Q. Finally, for a point $p \in \Gamma$, $p^{\perp_\Gamma}$ generates a singular 3-space of Q, which meets each generator $\langle \Sigma \rangle$, with Σ a symp through p, in the plane generated by the Σ-perp of p.*

Proof. Let Σ and Σ' be two distinct symps of Γ, sharing a Γ-line L. Obviously, $\langle \Sigma \rangle$ and $\langle \Sigma' \rangle$ have at least one line in common. Since $\Sigma \cup \Sigma'$ contains point pairs at Γ-distance 3, we deduce that $\langle \Sigma \rangle \neq \langle \Sigma' \rangle$. Suppose now that $\langle \Sigma \rangle \cap \langle \Sigma' \rangle$ is a plane π. Then π contains the Σ-perp of at most one point of L, and the same holds for the Σ'-perp. Consequently, since L has at least three points, there exist a point $x \in L$ and points $y \in \Sigma \setminus \pi$ and $y' \in \Sigma' \setminus \pi$ with $y \perp_\Gamma x \perp_\Gamma y'$. Then $\delta_\Gamma(y, y') = 2$ and hence $y \perp_\Delta y'$. But then y is Δ-collinear to both π and y' and hence to $\langle \Sigma' \rangle$, a contradiction. We conclude that $\langle \Sigma \rangle \cap \langle \Sigma' \rangle = L$, so $\langle \Sigma \rangle$ and $\langle \Sigma' \rangle$ have the same type in Q and lines of Γ^* correspond to lines of Q. Connectivity now implies that all 3-spaces $\langle \Sigma \rangle$, with Σ running through the set of all symps of Γ, are of the same type. Hence lines of Γ^* correspond with lines of the corresponding half spin geometry.

Secondly, let Σ and Σ' be two opposite symps of Γ. If $\langle \Sigma \rangle$ and $\langle \Sigma' \rangle$ are not opposite maximal singular subspaces of Q, then by the previous paragraph they intersect in a line, say L (again, $\langle \Sigma \rangle \neq \langle \Sigma' \rangle$ since $\Sigma \cup \Sigma'$ contains pairs of Γ-opposite points). Since the polarity in $\langle \Sigma \rangle$, for which Σ is embedded in its absolute geometry, is non-degenerate, there exists a point $x \in \Sigma$ such that L is not contained in the span U of the Σ-perp of x.

According to Fact 3.2, the unique point x' of Σ' which is Γ-collinear to x, is Γ-symplectic to all points of $x^{\perp_\Sigma} \setminus \{x\}$ and hence x' is Δ-collinear to U. As such, x' is Δ-collinear to L and U, implying that $x' \in L$. But then x' is Δ-collinear to all points of Σ, a contradiction to Lemma 4.37.

Finally, let p be a point of Γ. First note that the Γ-distance between any two points of $p^{\perp_\Gamma}$ is at most 2. Hence $p^{\perp_\Gamma}$ generates a singular subspace V of Q. For any symp Σ through p, the subspace V contains the plane generated by the Σ-perp of p. As this holds for each symp through p, this is only possible if V is a 3-space of the other kind.

The lemma is proved. ∎

Hence we are left with classifying polar spaces $\Gamma^* =: (\mathcal{P}^*, \mathcal{L}^*)$ admitting a lax embedding in the triality quadric $Q^* \cong \mathsf{D}_{4,1}(\mathbb{K})$ with the additional property

(∗) *For each $L \in \mathcal{L}^*$, containment is a bijective correspondence between the planes of Γ^* containing L and the generators of Q^* of one (a priori fixed) natural system through L.*

Moreover, by the last assertion of Lemma 4.38, this embedding is *polarized*, that is, the points collinear to a given point do not generate the whole ambient projective space.

We first study this problem in rank 2.

LEMMA 4.39. *Let $\Gamma_0^* = (\mathcal{P}_0^*, \mathcal{L}_0^*)$ be a generalized quadrangle laxly embedded in the Klein quadric $Q_0^* \cong \mathsf{D}_{3,1}(\mathbb{K})$ with the additional property*

(∗∗) *For each $p \in \mathcal{P}_0^*$, containment is a bijective correspondence between the lines of Γ_0^* containing p and the generators of Q_0^* of one (a priori fixed) natural system containing p.*

Then we have one of the following possibilities:
 (i) *The embedding is full and $\Gamma_0^* \cong \mathsf{B}_{2,1}(\mathbb{K})$.*
 (ii) *There is a subfield $\mathbb{F} \leq \mathbb{K}$ such that $\mathbb{K}$ is a separable quadratic extension of $\mathbb{F}$, and $\Gamma_0^* \cong \mathsf{B}_{2,1}(\mathbb{F}, \mathbb{K})$. Moreover the embedding induces the natural embedding $\mathsf{PG}(1, \mathbb{F}) \leq \mathsf{PG}(1, \mathbb{K})$ of projective lines on the lines of Q_0^* which are also lines of Γ_0^*.*
 (iii) *$\operatorname{char} \mathbb{K} = 2$, there is a vector space L over $\mathbb{K}^2$ containing $\mathbb{K}^2$ and contained in $\mathbb{K}$ (as a subspace) and a generalized quadrangle Γ_1^* fully embedded in Q_0^*, with Γ_0^* an ideal subquadrangle of Γ_1^*, i.e., each line of Γ_1^* meeting Γ_0^* in at least one point, meets Γ_0^* in a line, inducing the natural inclusion $\Gamma_0^* \cong \mathsf{B}_{2,1}(L, \mathbb{K}) \leq \mathsf{B}_{2,1}(\mathbb{K}) \cong \Gamma_1^*$.*

Proof. If we apply inverse Klein correspondence, then, denoting the dual of Γ_0^* with Γ_0, we see that Γ_0 is fully embedded in $\mathsf{PG}(3, \mathbb{K})$. The lemma now follows directly from the classification of fully embedded generalized quadrangles in $\mathsf{PG}(3, \mathbb{K})$ (see [13] and also Chapter 8 of [24]). ∎

LEMMA 4.40. *Let $\Gamma^* = (\mathcal{P}^*, \mathcal{L}^*)$ be a polar space of rank 3 laxly embedded in the triality quadric $Q^* \cong \mathsf{D}_{4,1}(\mathbb{K})$ satisfying (∗). Then we have one of the following possibilities:*
 (i) *$\Gamma^* \cong \mathsf{B}_{3,1}(\mathbb{K})$, the embedding is full and projectively unique. Also, there is a unique collineation θ of Q^* with fixed point set $\mathcal{P}^*$ and θ is an involutive type-interchanging projectivity.*

(ii) *There is a (proper) subfield $\mathbb{F} \leq \mathbb{K}$ such that $\mathbb{K}$ is a separable quadratic extension of $\mathbb{F}$, and $\Gamma^* \cong \mathsf{B}_{3,1}(\mathbb{F}, \mathbb{K})$. Moreover there is a unique collineation θ of Q^* with fixed point set $\mathcal{P}^*$, and θ is a type-interchanging involution, but not a projectivity. The set of such subfields $\mathbb{F}$ of $\mathbb{K}$ is in one-to-one correspondence with the classes of projectively equivalent laxly embedded polar spaces $\mathsf{B}_{3,1}(\mathbb{F}, \mathbb{K})$ satisfying $(*)$.*

(iii) *char $\mathbb{K} = 2$, there is a subfield $\mathbb{K}'$ of $\mathbb{K}$ containing $\mathbb{K}^2$ and a polar space Γ^{**} of rank 3 fully embedded in Q^*, with Γ^* an ideal sub-polar space of Γ^{**}, i.e., each plane of Γ^{**} containing a line of Γ^* meets Γ^* in a plane, inducing the natural inclusion $\Gamma^* \cong \mathsf{B}_{3,1}(\mathbb{K}', \mathbb{K}) \leq \mathsf{B}_{3,1}(\mathbb{K}) \cong \Gamma^{**}$. The unique involution from (i) fixing exactly all points of Γ^{**} is the unique involution fixing all points of Γ^*. The set of such subfields $\mathbb{K}'$ is in one-to-one correspondence with the classes of projectively equivalent laxly embedded polar spaces $\mathsf{B}_{3,1}(\mathbb{K}', \mathbb{K})$ satisfying $(*)$.*

Proof. Let Γ_0^* be a point-residue of Γ^*. Then Γ_0^* is embedded in the Klein quadric $\mathsf{D}_{3,1}(\mathbb{K})$ and satisfies $(**)$ of Lemma 4.39. Hence we may apply that lemma and conclude that we have one of the following possibilities.

(i) The embedding is full and $\Gamma \cong \mathsf{B}_{3,1}(\mathbb{K})$. Using Lemma 3.19 we readily deduce that $\mathcal{P}^*$ is the intersection of Q^* with a hyperplane H of $\mathsf{PG}(6, \mathbb{K})$. It already follows that the embedding is projectively unique. We may let Q^* have (standard) equation $X_{-1}X_1 + X_{-2}X_2 + X_{-3}X_3 + X_{-4}X_4 = 0$ in $\mathsf{PG}(7, \mathbb{K})$, after suitable coordinatization. Without loss of generality we may assume that H has equation $X_{-4} = X_4$. A general collineation θ of $\mathsf{PG}(7, \mathbb{K})$, necessarily linear, fixing $Q^* \cap H$ pointwise, and hence also fixing H pointwise is given by $X_i' = X_i + a_i X_{-4} - a_i X_4$, $a_i \in \mathbb{K}$, $i \in \{-3, -2, -1, 1, 2, 3\}$, and

$$\left\{ \begin{array}{rcl} X_{-4}' &=& aX_{-4} + (1-a)X_4 \\ X_4' &=& (1-b)X_{-4} + bX_4 \end{array} \right. \quad a, b \in \mathbb{K}.$$

Expressing that θ leaves the equation of Q^* invariant, we obtain $a_i = 0$, for all $i \in \{-3, -2, -1, 1, 2, 3\}$, and

$$a(1-b) = (1-a)b = 0, \quad ab + (1-a)(1-b) = 1.$$

The latter implies that either $a = b = 1$ (and we have the identity), or $a = b = 0$, and we have a unique involutive projectivity, which is clearly not type preserving.

(ii) There is a subfield $\mathbb{F} \leq \mathbb{K}$ such that $\mathbb{K}$ is a separable quadratic field extension of $\mathbb{F}$, and $\Gamma^* \cong \mathsf{B}_{3,1}(\mathbb{F}, \mathbb{K})$. Since we have a polarized lax embedding, it follows from Theorem 1 of [19] that Γ^* is fully embedded in $\mathsf{PG}(7, \mathbb{F})$ obtained from $\mathsf{PG}(7, \mathbb{K})$ by restricting $\mathbb{K}$ to $\mathbb{F}$. Clearly the only collineation θ of $\mathsf{PG}(7, \mathbb{K})$ fixing all points of Γ^* (and hence of $\mathsf{PG}(7, \mathbb{F})$, since it must clearly be linear over $\mathbb{F}$) is, after obvious coordinatization, the Galois involution, which is semi-linear but not linear (and hence not a projectivity), and which is not type-preserving. It follows that the classes of projectively equivalent laxly embedded polar spaces $\mathsf{B}_{3,1}(\mathbb{F}, \mathbb{K})$ satisfying $(*)$ are in one-to-one correspondence with the Galois involutions of $\mathbb{K}$, that is, with the subfields $\mathbb{F}$ such that $\mathbb{K}$ is a quadratic Galois extension of $\mathbb{F}$.

(iii) We have char $\mathbb{K} = 2$ and each point-residue of Γ^* is isomorphic to $\mathsf{B}_{2,1}(L, \mathbb{K})$, where

L is a vector space over $\mathbb{K}^2$ containing $\mathbb{K}^2$ and contained in $\mathbb{K}$ as a subspace. Since the planes of Γ^* embed in $\mathsf{PG}(2, \mathbb{K})$ and are coordinatized by L, we see that L is a subfield $\mathbb{K}'$ of $\mathbb{K}$. Hence $\Gamma^* \cong \mathsf{B}_{3,1}(\mathbb{K}', \mathbb{K})$. Let A be an apartment of Γ^*. Then A generates a 5-space U of $\mathsf{PG}(7, \mathbb{K})$. Let π be a plane of Γ^* through one of the lines, say M, of A, not contained in A. Then A and π generate a 6-space H. We now prove that Γ^* is contained in H and $H \cap Q^*$ is a parabolic quadric. We start with the latter. If $H \cap Q^*$ is not parabolic, then it is a cone, say with vertex p, over $U \cap Q^*$. Clearly, p does not belong to any plane of A, and it does not belong to π. Hence there are at least three subspaces in Q of dimension 3 through the plane $\langle p, M \rangle$, a contradiction. Suppose now for a contradiction that Γ^* is not contained in H. Then it generates $\mathsf{PG}(7, \mathbb{K})$ linearly. But, by Lemma 4.39(iii), every point-residue of Γ^* is contained in a 5-space of $\langle Q^* \rangle$. Hence the geometry of points of Γ^* opposite p cannot be connected, a contradiction. Now the assertions follow (including the last claim of (iii)).

The lemma is proved. ∎

PROPOSITION 4.41. *Let Γ be a dual polar space of rank 3 fully embedded in the half spin geometry $\Delta = \mathsf{D}_{6,6}(\mathbb{K})$, for some field $\mathbb{K}$, such that each Γ-symp is embedded in a singular subspace of Δ, but Γ is not contained in a singular subspace of Δ. Then there is a subfield $\mathbb{F}$ of $\mathbb{K}$ such that $\Gamma \cong \mathsf{B}_3(\mathbb{F}, \mathbb{K})$ and Γ is embedded in a symp $Q \cong \mathsf{D}_{4,1}(\mathbb{K})$ of Δ. More precisely, exactly one of the following occurs.*

  (i) *$\mathbb{F} = \mathbb{K}$, so $\Gamma \cong \mathsf{B}_{3,3}(\mathbb{K})$, and there is a unique collineation θ of Q with fixed point set Γ, and θ is a type-interchanging involutive projectivity.*

 (ii) *$\mathbb{K}$ is a separable quadratic extension of $\mathbb{F}$, and there is a unique collineation θ of Q with fixed point set Γ, where θ is a type-interchanging involution, but not a projectivity.*

(iii) *$\operatorname{char} \mathbb{K} = 2$ and $\mathbb{K}^2 \leq \mathbb{F} \leq \mathbb{K}$ and Γ is fully embedded in a dual polar space $\Gamma' \cong \mathsf{B}_{3,3}(\mathbb{K})$ which is also fully embedded in Δ. The unique involution from (i) fixing exactly all points of Γ' is the unique involution fixing all points of Γ. The subfields $\mathbb{F}$ of $\mathbb{K}$ such that $\mathbb{K}^2 \leq \mathbb{F} \leq \mathbb{K}$ are in one-to-one correspondence with the classes of projectively equivalent fully embedded dual polar spaces $\mathsf{B}_{3,3}(\mathbb{F}, \mathbb{K})$ in Q. Each class of projectively equivalent fully embedded dual polar spaces in Q induces exactly two classes of projectively equivalent fully embedded dual polar spaces in Δ.*

Proof. By Lemmas 4.37 and 4.38, there is a unique symp Q of Δ in which Γ embeds, and the embedding is such that the dual $\Gamma^* = (\mathcal{P}^*, \mathcal{L}^*)$ is laxly embedded in $Q^* \cong \mathsf{D}_{4,1}(\mathbb{K})$ with the additional property $(*)$. By dualising, the proposition follows from Lemma 4.40. ∎

4.5. Proof of Theorem 4.1. We can now finish the proof of Theorem 4.1. Indeed, if at least one Γ-symp is isometrically embedded in some Δ-symp, then (i) and (ii) of Theorem 4.1 hold, as follows from Proposition 4.35. If each Γ-symp is contained in a singular subspace of Δ, then (iii) of Theorem 4.1 holds as follows from Proposition 4.41. The claims in (iii) about the projective uniqueness in Q follow from Lemma 4.40. The last claim in (iii) follows from the fact that all symps of Δ are mutually projectively

equivalent and that the collineation group of a symp induced by the stabilizer of the symp in the collineation group of Δ is the full type-preserving collineation group of the symp and hence has index 2 in the full collineation group the symp. This clearly covers all cases. □

5. Parapolar spaces of type $\mathsf{E}_{7,1}$

We now use the results of the previous section to classify the full embeddings of meta-symplectic spaces in the long root geometry $\mathsf{E}_{7,1}(\mathbb{K})$.

THEOREM 5.1. *Let the (thick) metasymplectic space* Γ *be fully embedded in the parapolar space* $\Delta = \mathsf{E}_{7,1}(\mathbb{K})$, *but not in a singular subspace of* Δ. *Then precisely one of the following occurs.*

(i) $\Gamma \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{L})$, $\mathbb{L}$ *a quadratic extension of* $\mathbb{K}$, *and is the fixed point structure of each non-trivial element of a collineation group of* Δ *isomorphic to* $\mathbb{L}^\times / \mathbb{K}^\times$; *the classes of projectively equivalent embeddings of this type are in one-to-one correspondence with the quadratic extensions* $\mathbb{L}$ *of* $\mathbb{K}$ *in the algebraic closure of* $\mathbb{K}$. *Here,* Γ *is not contained in a para, and hence* Γ *is not contained in any residue of the building underlying* Δ.

(ii) $\Gamma \cong \mathsf{F}_{4,1}(\mathbb{K})$ *and is the fixed point structure of each non-trivial element of a group of collineations of* Δ *isomorphic to* $\mathsf{PGL}_2(\mathbb{K})$ *acting in its natural sharply 3-transitive action on the paras containing an arbitrary but fixed symp of* Γ; *for each field* $\mathbb{K}$ *there is a projectively unique such embedding. Here,* Γ *is not contained in a para, and hence* Γ *is not contained in any residue of the building underlying* Δ.

(iii) $\Gamma \cong \mathsf{F}_{4,4}(\mathbb{F}, \mathbb{K})$, $\mathbb{F} \le \mathbb{K}$, *with either* $\mathbb{F} = \mathbb{K}$, *or* $\mathbb{K}$ *a quadratic Galois extension of* $\mathbb{F}$ *(i.e., a separable quadratic extension), or* $\operatorname{char} \mathbb{K} = 2$ *and* $\mathbb{K}^2 \le \mathbb{F} \le \mathbb{K}$. *In the latter case,* Γ *is contained in a metasymplectic space isomorphic to* $\mathsf{F}_{4,4}(\mathbb{K})$ *which is fully embedded in* Δ. *In all cases,* Γ *is contained in a para* $\Pi \cong \mathsf{E}_{6,1}(\mathbb{K})$. *If* $\mathbb{F} = \mathbb{K}$ *or if* $\mathbb{K}$ *is a Galois extension of* $\mathbb{F}$, *then* Γ *is the absolute structure of a polarity of the para* Π. *The classes of projectively equivalent embeddings of these types are in one-to-one correspondence with the subfields* $\mathbb{F}$ *of* $\mathbb{K}$ *with the said properties.*

5.1. Case distinction. Let Γ be any metasymplectic space, fully embedded in the parapolar space Δ isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$, for some field $\mathbb{K}$, but not contained in a singular subspace of Δ. Then, for any point $p \in \Gamma$, the point-residue $\operatorname{Res}_\Gamma(p)$, which is a dual polar space of rank 3, is fully embedded in $\operatorname{Res}_\Delta(p)$, which is a parapolar space isomorphic to $\mathsf{D}_{6,6}(\mathbb{K})$. According to Theorem 4.1, the following are the only possibilties:

Case I: There exists a point-residue $\operatorname{Res}_\Gamma(p)$ such that (i) of Theorem 4.1 holds in $\operatorname{Res}_\Delta(p)$. Such a point p is called *of type* (i).

Case II: There exists a point-residue $\operatorname{Res}_\Gamma(p)$ such that (ii) of Theorem 4.1 holds in $\operatorname{Res}_\Delta(p)$. Such a point p is called *of type* (ii).

Case III: There exists a point-residue $\mathrm{Res}_\Gamma(p)$ such that (iii) of Theorem 4.1 holds in $\mathrm{Res}_\Delta(p)$. Such a point p is called *of type* (iii).

Case O: Each point-residue of Γ is embedded in a singular subspace of the corresponding point-residue of Δ. A point p for which $\mathrm{Res}_\Gamma(p)$ is contained in a singular subspace of $\mathrm{Res}_\Delta(p)$ is called *of type* (o).

PROPOSITION 5.2. *Either each point of Γ has type (i), or each point of Γ has type (ii), or each point of Γ has either type (iii) or (o).*

Proof. Take any point p in Γ and any Γ-line L through p. Let U be the projective 5-space corresponding to $\mathrm{Res}_\Delta(L)$, which is a line Grassmannian of type $\mathsf{A}_{5,2}(\mathbb{K})$, so that we can view the point set of $\mathrm{Res}_\Gamma(L)$ as a line set $\mathcal{P}_L$ in U. We claim that:

(i) If p has type (i), then $\mathcal{P}_L$ is a regular line spread of U.

(ii) If p has type (ii), then $\mathcal{P}_L$ arises from a Segre variety $\mathcal{S}_{1,2}(\mathbb{K})$ in U in the sense that the elements of $\mathcal{P}_L$ are precisely the lines between the disjoint planes of $\mathcal{S}_{1,2}(\mathbb{K})$.

(iii) If p has type (iii), then $\mathcal{P}_L$ is either a subplane of a point-residue of a 3-space of U or a subplane of a plane of U.

(o) If p has type (o), then the lines in $\mathcal{P}_L$ either contain a common point of U or are contained in a plane of U.

Indeed, in the first two cases, the structure of $\mathcal{P}_L$ in U is, with Notation 4.3 applied to $\mathrm{Res}_\Gamma(p)$, the same as the structure of $\mathfrak{P}_q$ in U_q, if q is the point of $\mathrm{Res}_\Gamma(p)$ corresponding to L. Lemma 4.38 yields (iii). Case (o) is clear as the points and planes of U correspond to singular subspaces of $\mathrm{Res}_\Delta(p)$.

We deduce that, if a point p of Γ has type (i), then all points on a Γ-line L through p also have type (i). By connectivity of Γ, all points have type (i). The same thing holds for type (ii): If some point of Γ has type (ii), then all have type (ii).

This shows the proposition. ∎

We first handle Cases I and II.

5.2. Case I: Each point of Γ has type (i).

We head off by noting that $\Gamma \cong \mathsf{F}_{4,1}(\mathbb{K},\mathbb{L})$ and each symp Σ of Γ is isomorphic to $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{L})$, with $\mathbb{L}$ a quadratic extension of $\mathbb{K}$ (possibly inseparable) and is isometrically contained in a unique symp Q of Δ (by Lemma 3.20 and the fact that the point-residue is not contained in a singular subspaces). If we view Q as a hyperbolic quadric in $\mathsf{PG}(9,\mathbb{K})$, then Lemma 3.19 implies that Σ arises from taking the intersection of Q with a subspace, which is of dimension 7 since we know that any of its point-residues is obtained as the intersection of the point-residue of Q with a 5-dimensional subspace. It then follows from Corollary 4.12 that there is a group of type preserving collineations of Q acting simply transitively on the singular 4-spaces of Q containing a singular plane of Σ, each non-trivial element of the group has Σ as its fixed point structure. We can now argue similarly as in the first paragraph of the proof of Proposition 4.17 to obtain:

PROPOSITION 5.3. *Let $\Gamma \cong \mathsf{F}_{4,1}(\mathbb{K},\mathbb{L})$, $\mathbb{L}$ a quadratic extension of $\mathbb{K}$, be fully embedded in the parapolar space $\Delta \cong \mathsf{E}_{7,1}(\mathbb{K})$, such that some symp of Γ is isometrically embedded*

in some symp of Δ. *Let* D *be a chamber of* Γ. *Let* $p \in D$ *and* $\Sigma \in D$ *be the point and the symp of* D, *respectively. Then there exists a group* $G^* \cong \mathbb{L}^\times / \mathbb{K}^\times$ *of permutations of* $\mathrm{Res}_\Delta(\Sigma) \cup \mathrm{Res}_\Delta(p)$ *inducing collineations in both* $\mathrm{Res}_\Delta(\Sigma)$ *and* $\mathrm{Res}_\Delta(p)$, *each non-trivial element of which fixes each element of* Γ *in its domain; also,* G^* *acts sharply transitively on the paras of* Δ *containing* Σ. ∎

In order to apply Proposition 4.16 of [23], we only need to show that the embedding of Γ into Δ is isometric and convex.

LEMMA 5.4. *The embedding of* Γ *in* Δ *is isometric.*

Proof. Let p, q be distinct points of Γ. If $p \perp_\Gamma q$, then by definition, $p \perp_\Delta q$. Since symps of Γ embed isometrically in Δ in this case, Γ-symplectic points are Δ-symplectic. So next, suppose that $\{p, q\}$ is special in Γ. Set $x = p \bowtie_\Gamma q$. Then, in $\mathrm{Res}_\Gamma(x)$, xp and xq are opposite. By Lemma 4.9 and the fact that for the embedding of $\mathrm{Res}_\Gamma(x)$ into $\mathrm{Res}_\Delta(x)$, Theorem 4.1(ii) holds, we conclude that xp and xq are also at distance 3 from each other in $\mathrm{Res}_\Delta(x)$. Consequently $\{p, q\}$ is a special pair in Δ. Finally, suppose p and q are opposite in Γ. Then there is a path $p \perp x \perp y \perp q$ with (p, y) and (x, q) special pairs in Γ. The previous case implies that this path is also a path in Δ with (p, y) and (x, q) special pairs. Then Lemma 3.16 completes the proof of the lemma. ∎

We stick with our assumptions on Γ and Δ, and with the definition of Δ^*.

LEMMA 5.5. *Let* D *and* D' *be opposite chambers of* Γ. *Then* D *and* D' *correspond to opposite flags of* Δ. *Also, if* p *and* q *are opposite points in* Γ, *then the projection operator from* $\mathrm{Res}_\Delta(p)$ *to* $\mathrm{Res}_\Delta(q)$ *maps* $\mathrm{Res}_\Gamma(p)$ *to* $\mathrm{Res}_\Gamma(q)$.

Proof. We already showed in Lemma 5.4 that opposite points of Γ are also Δ-opposite. Note that lines and planes of Γ correspond to lines and planes of Δ, respectively, both containing points of Γ only, by the fullness of the embedding. Since, by the existence and properties of projections in buildings, see Section 3.19 of [23], opposition of lines and of planes is completely determined by the opposition of the points they contain, we deduce that Γ-opposite lines and planes are also Δ-opposite.

We dualise the situation and argue in Δ^*. Now, points, lines, planes and symps of Γ correspond to symps (isomorphic to $\mathsf{D}_{6,1}(\mathbb{K})$), maximal singular 5-spaces, singular 3-spaces and lines of Δ^*. So in Δ^* we have, by the above, that D and D' correspond to flags $\{Q, U, S, L\}$ and $\{Q', U', S', L'\}$, respectively for Δ^*-opposite symps Q and Q', Δ^*-opposite 5-spaces U, U', Δ^*-opposite 3-spaces S and S' and lines L, L'. We have to show that L and L' are Δ^*-opposite, given that they arise from symps of Γ which are Γ-opposite.

Since Q and Q' are Δ^*-opposite, we know that Δ^*-collinearity induces an isomorphism θ from Q to Q'. Let M be any line of Q corresponding to a symp Σ_M of Γ and let M' be its image under θ in Q'. We claim that M' corresponds to a Γ-symp $\Sigma_{M'}$ and the latter is the Γ-projection of Σ_M onto the Γ-point p' corresponding to Q'. Indeed, M and M' define a unique Δ^*-symp $P \supseteq M \cup M'$. Since Q and Q' are Δ^*-opposite, $P \cap Q = M$, $P \cap Q' = M'$ and P is the unique Δ^*-symp containing M and intersecting Q' non-trivially. Since in Γ, there is also a unique symp Σ containing p' and intersecting Σ_M nontrivially,

say in the point p, we necessarily have that $\Sigma = \Sigma_{M'}$ corresponds to M' and p to P. The claim follows. Now take $L = M$ and set $L'' := M'$. Then in Γ, the symps corresponding to L' and L'' are opposite in the residue of p', by 3.28 and 3.29 of [23]. Since point-residues of Γ embed isometrically in the corresponding residue of Δ and hence also Δ^*, L' and L'' are Δ^*-opposite in Q'. Using 3.28 and 3.29 of [23] again, we conclude that L and L' are Δ^*-opposite.

By the above, θ maps 3-spaces and 5-spaces containing a regular line spread (in the sense of the proof of Proposition 5.2(i)) to 3-spaces and 5-spaces, respectively, containing a regular line spread. This shows the second assertion and the lemma. ∎

We are now ready to show the convexity of Γ.

LEMMA 5.6. *The embedding of Γ is convex in Δ, i.e., Γ is a convex subspace of Δ.*

Proof. Since an apartment of Γ is unambiguously determined by its points, it suffices to show that any apartment $\mathcal{A}$ of Δ containing two opposite chambers D, D' of Γ, contains every point of the unique apartment $\mathcal{B}$ of Γ containing those two chambers. We again consider the dual Δ^* and use the same notation as in the proof of Lemma 5.5, namely, $D = \{Q, U, S, L\}$ and $D' = \{Q', U', S', L'\}$. The last assertion of Lemma 5.5 implies that the projections of D onto Q' and D' onto Q belong to $\mathcal{A} \cap \mathcal{B}$. Since these projections are Q'-opposite and Q-opposite the flags $\{U', S', L'\}$ and $\{U, S, L\}$, respectively, and the embedding of each point-residue of Γ is convex in the corresponding point-residue of Δ, we already obtain that $\mathcal{B} \cap (\mathrm{Res}_\Gamma(Q) \cup \mathrm{Res}_\Gamma(Q')) \subseteq \mathcal{A}$. Now we can treat U as an arbitrary 5-space in Q belonging to Γ (where it corresponds to a line). In both Γ^* and Δ^* there exist unique symps, which we denote by P, P' in both cases, such that

1. $Q \cap P = U$,
2. $W = P \cap P'$ is a 5-space of Δ^* corresponding to a line of Γ, and
3. $P' \cap Q'$ is the projection of U onto Q' (projection in both Γ or Γ^* and Δ or Δ^*).

Consequently, the Δ^*-symps P and P' correspond to Γ^*-symps.

Repeating the above argument with U' and Q, we obtain symps R, R' of Δ^* corresponding to symps of Γ^* such that $R' \cap Q' = U'$, $W' = R' \cap R$ is a 5-space belonging to Γ (where, again, it corresponds to a line) and $R \cap Q$ is a 5-space belonging to Γ opposite U in Q. Since apartments are convex, all objects found thus far belong to both $\mathcal{B}$ and $\mathcal{A}$. One verifies that the symps P and R' correspond to opposite points in both Γ and Δ and so Δ^*-collinearity between them is an isomorphism again, mapping elements of Γ to elements of Γ. It follows that U' is mapped to W and W' to U. Hence $\mathcal{A}$ contains all elements of $\mathcal{B}$ incident with W and W'. These contain in total six points of $\mathcal{B}$. By varying U in Q, we now claim that all points of $\mathcal{B}$ are obtained this way.

This can be done explicitly in an apartment of type F_4. A concrete model is given by the 48 roots of the root system of type F_4. The short roots represent the points of the apartment, whereas the long roots represent the symps. A short root is incident with the symp defined by a long root if, and only if, these two roots make an angle of 45 degrees. This uniquely defines the structure of the apartment. More concretely, this implies that two short roots making an angle of 60 degrees are collinear, making an angle of 90 degrees are symplectic, making an angle of 120 degrees are special, and making an angle of 180

degrees are opposite. Dually, long roots making an angle of 60 degrees represent symps intersecting in a plane, etc. Considering Γ^*, the symps Q and Q' correspond to opposite long roots α and $-\alpha$. The symp P corresponds, as U varies, to all eight long roots making an angle of 60 degrees with α. The points of W in Γ^* then correspond to all short roots making an angle of 45 degrees with one of those eight long roots. Hence we obtain all short roots making an angle of at most 90 degrees with α.

Explicitly, let the short roots be given by $\pm e_i$, $i = 1, 2, 3, 4$, and all sign combinations in $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$, with $\{e_1, e_2, e_3, e_4\}$ an orthonormal basis of $\mathbb{R}^4$. The long roots are given by all sign combinations of $\pm e_i \pm e_j$, with $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$. If $\alpha = e_1 + e_2$, then the eight roots making an angle of 60 degrees with α are $e_i \pm e_j$, $i \in \{1, 2\}$, $j \in \{3, 4\}$. The 18 short roots making an angle of 45 degrees with one of those are

$$e_1, e_2, \pm e_3, \pm e_4, \frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4), \pm \frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4).$$

Together with the 6 points in Q', this accounts for all points of $\mathcal{B}$. Hence $\mathcal{B}$ is contained in $\mathcal{A}$ and the lemma is proved. ∎

We can now prove the existence and uniqueness of the embedding.

PROPOSITION 5.7. *Let $\Gamma \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{L})$, $\mathbb{L}$ a quadratic extension of $\mathbb{K}$, be fully embedded in the parapolar space $\Delta \cong \mathsf{E}_{7,1}(\mathbb{K})$, such that some symp of Γ is isometrically embedded in some symp of Δ. Then Γ is the fixed point structure of each non-trivial element of a group $G \cong \mathbb{L}^\times / \mathbb{K}^\times$ of projectivities of Δ, acting simply transitively on the set of paras containing an arbitrary fixed symp of Δ. If $\operatorname{char} \mathbb{K} \neq 2$, or Γ is of inseparable type, then Γ is also the fixed point set of an involutive projectivity of Δ. Also, the group of all projectivities of $\mathsf{E}_{7,1}(\mathbb{K})$ contains, for each quadratic extension $\mathbb{L}$ of $\mathbb{K}$, a unique such group, up to conjugacy. Also, Γ is not contained in a para, and hence Γ is not contained in any residue of the building underlying Δ.*

Proof. The proof is completely similar to the proof of Proposition 4.17, given all the ingredients in the foregoing lemmas, in particular Proposition 5.3 and Lemma 5.6. The fact that Γ is not contained in a para follows from the said transitivity of G. ∎

5.3. Case II: Each point of Γ has type (ii)**.** In this case, each symp of Γ is contained in a symp of Δ as a 6-dimensional quadric of Witt index 3, hence a parabolic quadric $\mathsf{B}_{3,1}(\mathbb{K})$. This implies that Γ necessarily arises from the split building $\mathsf{F}_4(\mathbb{K})$ (i.e., $\mathsf{F}_4(\mathbb{K}, \mathbb{K})$) as the parapolar space $\mathsf{F}_{4,1}(\mathbb{K})$.

Just like we deduced Proposition 5.3, we can now deduce the following proposition, using Lemma 4.19 and Corollary 4.22.

PROPOSITION 5.8. *Let $\Gamma \cong \mathsf{F}_{4,1}(\mathbb{K})$ be fully embedded in the parapolar space $\Delta \cong \mathsf{E}_{7,1}(\mathbb{K})$ such that some symp of Γ is isometrically embedded in some symp of Δ. Let D be a chamber of Γ. Let p be the point and Σ be the symp of D. Then there exists a group G^* of permutations of $\operatorname{Res}_\Delta(\Sigma) \cup \operatorname{Res}_\Delta(p)$ inducing collineations in both $\operatorname{Res}_\Delta(\Sigma)$ and $\operatorname{Res}_\Delta(p)$, fixing each element of Γ in its domain and acting sharply 3-transitively on the paras of Δ containing Σ.* ∎

In order to apply Proposition 4.16 of [23], we only again need to show that the embedding of Γ into Δ is isometric and convex. The proof of the latter is completely similar to the corresponding proof in Case I.

Now the proofs of Lemmas 5.5 and 5.6 can be copied to show the analogous results for Case II. This means that we have:

LEMMA 5.9. *Opposite chambers of Γ correspond to opposite flags of Δ. Also, the embedding of Γ in Δ is isometric and convex.* ∎

We deduce, similarly as Proposition 5.7:

PROPOSITION 5.10. *Let $\Gamma \cong \mathsf{F}_{4,1}(\mathbb{K})$ be fully embedded in the parapolar space $\Delta \cong \mathsf{E}_{7,1}(\mathbb{K})$, but not in a singular subspace of Δ, and such that some symp of Γ is isometrically embedded in some symp of Δ. Then Γ is the fixed point structure of a group $G \cong \mathsf{PGL}_2(\mathbb{K})$ of projectivities of Δ, acting sharply 3-transitively on the set of paras of Δ containing an arbitrary, but fixed, symp of Δ, equivalently to its natural action on the projective line $\mathsf{PG}(1,\mathbb{K})$. Also, the full group of projectivities of each parapolar space $\mathsf{E}_{7,1}(\mathbb{K})$ contains such a subgroup (with the specified fixed point strucure), unique up to conjugation. Here, Γ is not contained in a para, and hence Γ is not contained in any residue of the building underlying Δ.* ∎

REMARK 5.11. Referring to Remark 4.36, we can make a similar remark here: Necessary and sufficient for a fully embedded metasymplectic space Γ in $\mathsf{E}_{7,1}(\mathbb{K})$ to be isometrically embedded is that some symp of Γ is isometrically embedded in (a symp of) $\mathsf{E}_{7,1}(\mathbb{K})$.

5.4. Case III: some point of Γ has type (iii).

Let p be any point of Γ of type (iii). Recall that $\mathrm{Res}_\Gamma(p)$ is contained in a unique symp of $\mathrm{Res}_\Delta(p)$ isomorphic to $\mathsf{D}_{4,1}(\mathbb{K})$ (cf. Lemma 4.37). Hence:

(a) There is a unique Δ-symp containing $p^{\perp_\Gamma}$. We denote this symp by Q_p and refer to its 4-dimensional singular subspaces as $4'$-*spaces* if they are contained in 6-dimensional singular subspaces of Δ, and as 4-*spaces* if they are maximal singular subspaces of Δ.

We summarise what we additionally deduce from Lemma 4.38:

(b) A Γ-symp Σ through p generates a 5-dimensional singular subspace of Δ, which we will denote by W_Σ. In particular we have that Σ is either symplectic, mixed or Hermitian.

(c) $W_\Sigma \cap Q_p$ is the $4'$-space of Q_p generated by the Σ-perp of p. Note that all points of Σ contained in $W_\Sigma \cap Q_p$ are collinear to p.

(d) For each Γ-line L through p, the (singular) subspace generated by all Γ-planes through L generates a 4-space of Q_p. We denote this 4-space by V_L.

We start by showing that this holds for all points of Γ.

LEMMA 5.12. *If one point of Γ has type (iii) then all points of Γ have type (iii).*

Proof. Suppose for a contradiction that some point q of Γ does not have type (iii), so by Proposition 5.2, it has type (o). By connectivity we may assume that p and q are collinear in Γ. By Lemma 4.38, all Γ-planes through pq generate a 4-space W of Q_p. As q is of type (o) and as W is a maximal singular subspace of Δ, $q^{\perp_\Gamma}$ generates W. In particular, each

symp through q is contained in W, which is impossible as the symps have Witt index 3 and require at least a subspace of dimension 5. ∎

Next, we investigate what happens to distances in Γ.

LEMMA 5.13. *Let x, y be distinct points of Γ. If x and y are collinear or symplectic in Γ, then $x \perp_\Delta y$; if x and y are special or opposite in Γ, then $x \perp\!\!\!\perp_\Delta y$.*

Proof. The first part of the statement follows immediately from our assumptions. Now let x, y be special in Γ and let $p \in \Gamma$ be the unique point that is Γ-collinear to x and y. In $\mathrm{Res}_\Gamma(p)$, the lines px and py of Γ are opposite points, so by Lemma 4.37, x and y are non-collinear points in Δ. Finally, suppose x, y are Γ-opposite. Then there are points u, v with $x \perp\!\!\!\perp_\Gamma u \perp\!\!\!\perp_\Gamma y \perp\!\!\!\perp_\Gamma v \perp\!\!\!\perp_\Gamma x$. Since Γ-symplectic points are Δ-collinear in Δ, either $x \perp_\Delta y$ or $x \perp\!\!\!\perp_\Delta y$.

Suppose for a contradiction that $x \perp_\Delta y$. Let L be a Γ-line through x and let p be its unique point Γ-special to y. By the above, y and p are Δ-symplectic, and, as $y \perp_\Delta x \perp_\Delta p$, we obtain that the Δ-symp determined by y and p coincides with Q_x. Now let Σ be any symp of Γ through x. Then the Σ-perp of x generates a 4-space V of Q_x. As $y \in Q_x$, we see that $y^{\perp_\Delta} \cap V$ is at least a 3-space, which hence contains a Γ-line M through p. Now M contains a unique point m special to y in Γ and $m \perp_\Delta y$, contradicting the foregoing. We conclude that $x \perp\!\!\!\perp_\Delta y$. ∎

As a consequence, we can show that the 4-space V_L associated to a Γ-line L can also be obtained as the intersection of Q_p and Q_q for any pair of distinct points $p, q \in L$.

LEMMA 5.14. *Let $p, q \in \Gamma$, $p \neq q$, be Γ-collinear. Then $V_{pq} = Q_p \cap Q_q$.*

Proof. Put $L = pq$. By definition, $V_L \subseteq Q_p \cap Q_q$. So suppose for a contradiction that $Q_p = Q_q$. Select points $x \in p^{\perp_\Gamma}$ and $y \in q^{\perp_\Gamma}$ with x and y Γ-opposite. Note that x is Γ-opposite all points of $py \setminus \{p\}$ and Γ-special to p, so by Lemma 5.13, x is Δ-symplectic to all points of py. As $Q_p = Q_q$, both x and py belong to Q_p, implying that x is Δ-collinear to some point of py after all, a contradiction. We conclude that $Q_p \cap Q_q = V_L$. ∎

LEMMA 5.15. *There is a unique para Π in Δ containing all points of Γ.*

Proof. Take any point $p \in \Gamma$ and consider an arbitrary Γ-symp Σ through p. Recall that the Δ-symp Q_p and the 5-space W_Σ share a $4'$-space V (generated by the Σ-perp of p), and hence there is a unique para Π containing $Q_p \cup W_\Sigma$.

Claim 1: Π contains $W_{\Sigma'}$ for all Γ-symps Σ' through p.
Let Σ' be any symp in Γ through p. Firstly, let $\Sigma \cap \Sigma' = \{p\}$ and suppose for a contradiction that $W_{\Sigma'} \not\subseteq \Pi$. Then the singular 5-space $W_{\Sigma'}$ intersects Π in the $4'$-space V' generated by the Σ'-perp of p. Take a point $a \in \Sigma' \setminus \Pi$. By the point-para relations, a is Δ-collinear to the unique 5-space $\overline{V}'$ of Π through V'. Let x be a point in Σ not Γ-collinear to p. Then $x^{\perp_\Delta} \cap Q_p = V$ and hence $x^{\perp_\Delta} \cap V' = \{p\}$, so in particular, $\dim(x^{\perp_\Delta} \cap \overline{V}') \leq 1$. Again by the point-para relations, it follows that x and a are special in Δ. However, viewed in Γ, x and a are Γ-opposite, so by Lemma 5.13, they are Δ-symplectic, a contradiction. So $W_{\Sigma'} \subseteq \Pi$. Secondly, if $\Sigma' \cap \Sigma$ is a plane, then we take a second symp Σ'' with $\Sigma \cap \Sigma'' = \Sigma' \cap \Sigma'' = \{p\}$. By the above, the unique para containing Q_p and $W_{\Sigma''}$ is

Π, so repeating the argument with Σ'' in the role of Σ, implies that $W_{\Sigma'}$ is contained in Π as well.   □

Claim 2: Π *contains all* Γ-*points which are* Γ-*opposite* p.

Take any Γ-point q that is Γ-opposite p. Consider two Γ-symps Σ and Σ' with $\Sigma \cap \Sigma' = \{p\}$. Then q is Γ-symplectic to unique points $w \in \Sigma$ and $w' \in \Sigma'$, and w and w' are Γ-opposite. By Lemma 5.13, q is Δ-collinear to the Δ-symplectic pair of points w, w', so q is contained in the unique Δ-symp Q defined by w, w'. By Claim 1, $w, w' \in \Pi$ and therefore $q \in Q \subseteq \Pi$.

□

As Π contains Q_p, it in particular contains $p^{\perp_\Gamma}$. Claims 1 and 2, respectively, imply that Π also contains $p^{\perp\!\perp_\Gamma}$ and $p^{\mathrm{opp}_\Gamma} \subseteq \Pi$. Since every point Γ-special to p is contained in a line which contains at least two points Γ-opposite p, we obtain $\Gamma \subseteq \Pi$. This completes the proof of the lemma. ∎

Let $c = \{p, L, \pi, \Sigma\}$ be a chamber of Γ, where p is a Γ-point, L a Γ-line, π a Γ-plane and Σ a Γ-symp. By definition of Q_p and V_L, the set $c_\Pi := \{p, W_\Sigma, L, \pi, V_L, Q_p\}$ is a chamber of Π. Suppose that c and d are opposite chambers in Γ, i.e., for each element of c there is an element opposite to it in Γ. Our goal is to show that c_Π and d_Π are opposite chambers in Π. Let us first give an overview of the opposition relation of Γ and of Π in terms of point-relations. For Γ we have:

(i) two Γ-lines L and M are Γ-opposite if each point of L is Γ-special to a unique point of M and Γ-opposite the others and vice versa; so "being Γ-special" induces a bijection between L and M;

(ii) two Γ-planes α and β are Γ-opposite if each point of α is Γ-special to each point of a unique line of β and Γ-opposite the others and vice versa; so "being Γ-special" induces a duality between α and β;

(iii) two Γ-symps Σ and Σ' are Γ-opposite if each point of Σ is far from Σ' and vice versa; the map taking a point of Σ to the unique point of Σ' that is Γ-symplectic to it induces a collineation between Σ and Σ'.

In Π, the opposition relation (see Subsection 2.2; note that it is not type-preserving) behaves as follows:

(i) a point p of Π and a symp Q of Π are Π-opposite if p is *far* from Q, that is, if $p^{\perp_\Delta} \cap Q = \emptyset$;

(ii) a line L of Π and a 4-space V of Π are Π-opposite if for each point $p \in L$, the set $p^{\perp_\Delta} \cap V$ is empty;

(iii) two Δ-planes π and π' of Π are Π-opposite if (and only if) for each point $p \in \pi$, the set $p^{\perp_\Delta} \cap \pi'$ is empty;

(iv) two 5-spaces W and W' of Π are Π-opposite if (and only if) each point of W is Δ-collinear to a unique point of W'; so "being Π-collinear" induces a collineation between Π-opposite 5-spaces.

We can now relate these opposition relations as follows. In a para Π of Δ, a point p and a symp Q are called *close* if $p \notin Q$ and they are not far. This is equivalent to $p^{\perp_\Delta} \cap Q$ is a 4-dimensional singular space (called a $4'$-space).

LEMMA 5.16. *Let* p, q *be points of* Γ, L, M *lines of* Γ, α, β *planes of* Γ *and* Σ, Σ' *symps of* Γ. *Then*

(*i*) *p and q are* Γ-*opposite* ⇔ *p is* Π-*opposite* Q_q ⇔ *q is* Π-*opposite* Q_p;

(*ii*) *L and M are* Γ-*opposite* ⇔ *L is* Π-*opposite* V_M ⇔ *M is* Π-*opposite* V_L;

(*iii*) *α and β are* Γ-*opposite* ⇔ *α and β are* Π-*opposite;*

(*iv*) *Σ and Σ′ are* Γ-*opposite* ⇔ W_Σ *and* $W_{\Sigma'}$ *are* Π-*opposite.*

Proof. (*i*) Suppose that p and q are Γ-opposite. We show that q is far from Q_p in Π. Suppose for a contradiction that it is not. Observe that $q^{\perp\Delta} \cap p^{\perp\Gamma}$ is empty because it is contained in $(q^{\perp\Gamma} \cup q^{\perp\!\!\perp\Gamma}) \cap p^{\perp\Gamma}$, which is empty since p and q are Γ-opposite. In particular, $q \notin Q_p$ and hence q is close to Q_p; set $V := q^{\perp\Delta} \cap Q_p$. Consider a Γ-symp Σ and let V' be the $4'$-space of Q_p generated by the Σ-perp of p. Then $V \cap V'$ is either a point or a plane, and since it is disjoint from $p^{\perp\Gamma}$ as mentioned above, it has to be a point, say x. As q is Γ-far from Σ (because $p \in \Sigma$), there is a unique point $w \in \Sigma$ with $q \perp\!\!\!\perp_\Gamma w$. Then $w \perp\!\!\!\perp_\Gamma p$ and hence $w \notin Q_p$ (recall that $\Sigma \cap Q_p = p^{\perp\Sigma}$), so in particular, $w \notin V'$ and hence $w \neq x$. It is well known (see Fact 4.10 of [12]) that the relations between points and 5-spaces in Π now imply that q is Δ-collinear to a 3-space U of W_Σ. However, U then meets V' in a plane, contradicting the fact that $V \cap V' = q^{\perp\Delta} \cap V' = \{x\}$. We conclude that q is Π-opposite Q_p. Analogously one obtains that p is Δ-opposite Q_q.

Conversely, suppose that q is far from Q_p, so $q^{\perp\Delta} \cap Q_p$ is empty. So if p and q are not Γ-opposite, then they are Γ-special. However, in that case, as Q_p contains $p^{\perp\Gamma}$, the point $p \bowtie q$ belongs to Q_p, contradicting $q^{\perp\Delta} \cap Q_p = \emptyset$. So p and q are Γ-opposite indeed.

(*ii*) Suppose that L and M are Γ-opposite. Take $p \in L$. Then there is a point q on M such that p and q are Γ-opposite. Note that Q_q contains V_M since it contains $q^{\perp\Gamma}$. By (*i*), p is far from Q_q, so $p^{\perp\Delta} \cap Q_q$ is empty; in particular, $p^{\perp\Delta} \cap V_M$ is also empty. As $p \in L$ was arbitrary, we conclude that L and V_M are Π-opposite.

Conversely, suppose that L is Π-opposite V_M, i.e., for each point $p \in L$, the intersection $p^{\perp\Delta} \cap V_M$ is empty. Then $M \subseteq V_M$ implies that each point on M is either Γ-special or Γ-opposite p. Suppose, for a contradiction, that there are at least two points $q, r \in M$ which are Γ-special to p. Then the points $q' := p \bowtie q$ and $r' := p \bowtie r$ belong to $Q_q \setminus V_M$ and $Q_r \setminus V_M$, respectively. As such, $q' \perp\!\!\!\perp_\Delta r'$ and the unique Δ-symp Q determined by q' and r' has at least a line M' in V_M. This however means that $p^{\perp\Delta}$ contains a point of M' and hence of V_M, a contradiction.

(*iii*) Suppose that α and β are Γ-opposite. This implies that $p^{\perp\Delta} \cap \beta = \emptyset$, for each point p of α. So α and β are Π-opposite indeed.

Conversely, suppose that α and β are Π-opposite. Then arbitrary points $p \in \alpha$ and $q \in \beta$ are either Γ-special or Γ-opposite. Consider a Γ-symp Σ through β. Since β contains no points Γ-collinear or Γ-symplectic to p, we obtain that p is far from Σ. The unique point w of Σ that is Γ-symplectic to p is collinear to a line of β, so p is Γ-special to the points of a unique line of β and Γ-opposite the other points.

(*iv*) Suppose that Σ and Σ' are opposite in Γ, i.e., each point $p \in \Sigma$ is far from Σ' and hence "being Γ-symplectic" induces a collineation between Σ and Σ'. Suppose for a contradiction that W_Σ and $W_{\Sigma'}$ are not Π-opposite. If $W_\Sigma \cap W_{\Sigma'}$ is non-empty, then $W_\Sigma \cap W_{\Sigma'} = \{w\}$, a singleton, and each point $p \in \Sigma$ is Γ-symplectic to the same point w, a contradiction. Likewise, if W_Σ and $W_{\Sigma'}$ are disjoint, then there is a 5-space W of Δ such that $W_\Sigma \cap W$ and $W_{\Sigma'} \cap W$ are planes π and π', respectively; and then each

point of Σ is Γ-symplectic to each point in π', which again violates the fact that "being Γ-symplectic" defines a collineation between Σ and Σ'.

Conversely, suppose that W_Σ and $W_{\Sigma'}$ are opposite 5-spaces of Π. Then each point $p \in W_\Sigma$ is Δ-collinear to a unique point of $W_{\Sigma'}$, so if $p \in \Sigma$ then p is far from Σ' for otherwise it would be Δ-collinear to a line of $\Sigma' \subseteq W_{\Sigma'}$. ∎

Recall that for each chamber $c = \{p, L, \pi, \Sigma\}$ of Γ, we write c_Π for the associated chamber $\{p, W_\Sigma, L, \pi, V_L, Q_p\}$ of Π.

LEMMA 5.17. *Let c and d be opposite chambers of Γ. Then c_Π and d_Π are opposite chambers of Π. If $\mathcal{A}$ is the unique apartment of Γ determined by c and d, and $\mathcal{A}_\Pi$ the unique apartment of Π determined by c_Π and d_Π, then the map taking a chamber c of Γ to the chamber c_Π of Π yields an embedding of $\mathcal{A}$ into $\mathcal{A}_\Pi$.*

Proof. The fact that c_Π and d_Π are opposite follows from Lemma 5.16. For the second statement, we have to show that, for each Γ-chamber e on a shortest path in Γ between c and d, the chamber e_Π is on a shortest path in Π between c_Π and d_Π. It suffices to show this for the chambers e which are adjacent to c. By symmetry, it then holds for chambers e' of Γ adjacent to d. By choosing such Γ-opposite chambers e and e', we can continue like that in $\mathcal{A}$ and reach all pairs of Γ-opposite chambers.

So suppose first that e is point-adjacent to c; say $c = (p, L, \pi, \Sigma)$ (with the usual notation), and $e = (q, L, \pi, \Sigma)$, where p, q are distinct points of L. Then e is the unique point of the $\{2,3,4\}$-residue R of Γ containing c that is not Γ-opposite d. Now let $e' = (r, W_\Sigma, L, \pi, V_L, Q)$ be the projection of $d_\Pi = (p', W_{\Sigma'}, L', \pi', V_{L'}, Q_{p'})$ onto the $J = \{2,3,4,5\}$-residue R' of Π containing c_Π. Then (r, Q) is the unique point-symp pair of Π such that $(r, W_\Sigma, L, \pi, V_L, Q)$ is a chamber of R' not opposite d. Since all point-symp pairs of R' are incident, r is the unique Δ point on L not opposite $Q_{p'}$ and Q is the unique Δ-symp incident with both V_L and W_Σ that is not opposite p'. Since q is not Γ-opposite p', Lemma 5.16 says that q is not Π-opposite $Q_{p'}$ and Q_q is not Π-opposite p', from which it follows that $r = q$ and $Q = Q_q$. So $e' = e_\Pi$ indeed. The other cases follow in a similar way from Lemma 5.16. ∎

We claim that there is a unique polarity ρ of Π such that its restriction to Σ takes Σ-points p to Q_p (and hence Σ-lines L to V_L) having X' as a subset of its absolute points.

PROPOSITION 5.18. *Let $\Gamma = (X', \mathcal{L}')$ be a metasymplectic space fully embedded in a parapolar space $\Delta = (X, \mathcal{L})$ isomorphic to $\mathsf{E}_{7,1}(\mathbb{K})$ such that the symps of Γ are contained in singular subspaces of Δ but Γ is not contained in a singular subspace of Δ. Then one of the following holds.*
(a) $\Gamma \cong \mathsf{F}_{4,4}(\mathbb{K})$;
(b) $\Gamma \cong \mathsf{F}_{4,4}(\mathbb{F}, \mathbb{K})$, where $\mathbb{K}$ is a separable quadratic extension of $\mathbb{F}$;
(c) $\Gamma \cong \mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$, where $\operatorname{char} \mathbb{K} = 2$ and $\mathbb{K}'$ is a proper subfield of $\mathbb{K}$ with $\mathbb{K}^2 \leq \mathbb{K}' \leq \mathbb{K}$.
Moreover, Γ is contained in a unique para $\Pi \cong \mathsf{E}_{6,1}(\mathbb{K})$ of Δ and there is a unique polarity ρ of Π such that each element of Γ is absolute under ρ. In Cases (a) and (b), ρ has no other absolute elements than those of Γ and their images under ρ. In Case (c), the absolute geometry of ρ in Π is isomorphic to $\mathsf{F}_{4,4}(\mathbb{K})$.

Proof. The first part of the proposition follows immediately from the fact that the symps of Γ are contained in singular 5-spaces of Δ. The fact that Γ is contained in a unique para Π was proved in Lemma 5.15.

To show that Γ is contained in the absolute geometry of a unique polarity ρ of Π, we consider a chamber c of Γ and its associated chamber c_Π of Π, and an apartment $\mathcal{A}$ of Γ containing c, and its associated apartment $\mathcal{A}_\Pi$ of Π (cf. Lemma 5.17). We apply the extension theorem of Tits (cf. Proposition 4.16 of [23]), which says that each adjacency preserving map ρ on $E_2^\Pi(c_\Pi) \cup \mathcal{A}_\Pi$ extends to a unique automorphism of Π.

Suppose $c_\Pi = (p, W_\Sigma, L, \pi, V_L, Q_p)$, with the usual notation.

- Consider $R_{1,6} := \mathrm{Res}_\Pi(p, Q_p)$ (which is isomorphic to the quadric $\mathsf{D}_{4,1}(\mathbb{K})$). According to Lemma 4.40, there is a unique polarity $\rho_{1,6}$ of $\mathrm{Res}_\Pi(p, Q_p)$ whose absolute geometry contains $\mathrm{Res}_\Gamma(p)$. Moreover, the absolute geometry of $\rho_{1,6}$ is exactly $\mathrm{Res}_\Gamma(p)$ if Γ is of type (a) or (b) and it is the unique $\mathsf{B}_3(\mathbb{K})$ containing $\mathrm{Res}_\Gamma(p)$ if Γ is of type (c).

- Next, consider $R_2 := \mathrm{Res}_\Pi(W_\Sigma)$. The embedding of $\Gamma_2 := \mathrm{Res}_\Gamma(\Sigma)$ in $\mathrm{Res}_\Pi(W_\Sigma)$ is nothing more than the standard embedding of Σ in W_Σ, so there is a unique polarity ρ_2 of W_Σ whose absolute geometry contains Γ_2. Moreover, the absolute geometry of ρ_2 is exactly Γ_2 if Γ is of type (a) or (b) and it is the unique polar space $\mathsf{C}_{3,1}(\mathbb{K})$ containing Γ_2 if Γ is of type (c).

- The polarities $\rho_{1,6}$ and ρ_2 coincide on $R_{1,2,6} := \mathrm{Res}_\Pi(p, W_\Sigma, Q_p)$, since both induce a polarity on $R_{1,2,6}$ (which is isomorphic to the projective space $\mathsf{A}_{3,1}(\mathbb{K})$) with absolute geometry containing the generalised quadrangle $\mathrm{Res}_\Gamma(p, \Sigma)$ and this polarity is unique.

- All rank 2 residues R in Π of flags contained in c_Π are contained either in $R_{1,6}$ or in R_2, or correspond to digons. In the latter case, we may define ρ_R by the action of $\rho_{1,6}$ and ρ_2 on the rank 1 residues of R contained in $R_{1,6}$ and R_2, respectively; and then by definition and by the previous item, the action of ρ_R coincides with that of $\rho_{1,6}$ and ρ_2.

- Let $\rho_{\mathcal{A}}$ be the unique non-type preserving automorphism of $\mathcal{A}_\Pi$ fixing c_Π (as a chamber; element-wise p and Q_p are interchanged, so are L and V_L, whereas π and W_Σ are fixed). Now consider the sub-apartment $\mathcal{A}_\Gamma := \{d_\Pi \mid d \text{ a chamber of } \Gamma\}$ of $\mathcal{A}_\Pi$ which is by definition isomorphic to $\mathcal{A}$. Then $\rho_{\mathcal{A}}$ is the unique type-preserving automorphism of $\mathcal{A}_\Gamma$ which fixes c_Π, so $\rho_{\mathcal{A}}$ is the identity on $\mathcal{A}$. Moreover, for each rank 2 residue R of Π containing c, it is easily verified that $\rho_{\mathcal{A}}$ coincides with ρ_R on $\mathcal{A} \cap R$.

We conclude that there is a unique duality ρ of Π such that ρ coincides with ρ_R on each rank 2 residue R of Π containing c_Π and with $\rho_{\mathcal{A}}$ on $\mathcal{A}_\Pi$. We claim that ρ is the unique polarity of Π whose absolute geometry contains Γ; moreover, ρ has no other absolute elements other than those of Γ and their images if Γ is of type (a) or (b) and ρ fixes the unique metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K})$ containing Γ if Γ is of type (c).

By definition, ρ^2 is the identity on $E^\Pi(c) \cup \mathcal{A}_\Pi$. An application of the extension theorem again implies that ρ^2 is the identity on Π. Likewise, by definition, ρ is the identity on $E^\Gamma(c) \cup \mathcal{A}$ and applying the extension theorem in Γ, we obtain that ρ is the identity on Γ. If ρ has more absolute elements than the flags related to Γ, then it would also have more absolute elements in the rank 2 residues of Π containing c, which is not the case if Γ is of type (a) or (b). If Γ is of type (c), we consider the unique metasymplectic space

$\Gamma' \cong \mathsf{F}_{4,4}(\mathbb{K})$ containing Γ, which is indeed fixed by ρ as can be seen by again applying the extension theorem, but now to Γ' (and Γ' is embedded of type (a)). ∎

COROLLARY 5.19. *Let* $\Gamma = (X', \mathcal{L}')$ *be a metasymplectic space fully embedded in a parapolar space* $\Pi = (X, \mathcal{L})$ *isomorphic to* $\mathsf{E}_{6,1}(\mathbb{K})$ *such that* Γ *is not contained in a singular subspace of* Δ. *Then one of the following holds.*

(a) $\Gamma \cong \mathsf{F}_{4,4}(\mathbb{K})$;

(b) $\Gamma \cong \mathsf{F}_{4,4}(\mathbb{F}, \mathbb{K})$, *where* $\mathbb{K}$ *is a separable quadratic extension of* $\mathbb{F}$;

(c) $\Gamma \cong \mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$, *where* char $\mathbb{K} = 2$ *and* $\mathbb{K}'$ *is a proper subfield of* $\mathbb{K}$ *with* $\mathbb{K}^2 \leq \mathbb{K}' \leq \mathbb{K}$. *Moreover, there is a unique polarity* ρ *of* Π *such that each element of* Γ *is absolute under* ρ. *In Cases* (a) *and* (b), ρ *has no absolute elements other than those of* Γ *and their images under* ρ. *In Case* (c), *the absolute geometry of* ρ *in* Π *is isomorphic to* $\mathsf{F}_{4,4}(\mathbb{K})$. ∎

5.5. Case O: some point has type (o). We show that this is impossible.

LEMMA 5.20. *No point has type* (o).

Proof. In view of Proposition 5.2 and Lemma 5.12, all points have type (o). We claim that Γ is contained in a singular subspace of Δ, contradicting our assumptions.

Indeed, let Σ be any symp of Γ. For each point $p \in \Sigma$, our assumption implies that $p^{\perp_\Gamma} \cap \Sigma$ is contained in a singular subspace. Hence Lemma 3.19 implies that Σ is also contained in a singular subspace $\langle \Sigma \rangle$ of Δ. Since symps of Γ are embedded polar spaces of rank 3, $\langle \Sigma \rangle$ has dimension at least 6 and so Σ is contained in a unique maximal singular 6-space W_Σ.

Now let Σ' be a symp of Γ intersecting Σ in a plane π. Let $x \in \Sigma$ and $x' \in \Sigma'$ be arbitrary. Then there is a point $p \in \pi$ collinear to both x and x'. Since p has also type (o), $x \perp_\Delta x'$. Hence $\Sigma \cup \Sigma'$ is contained in a maximal singular subspace, which necessarily coincides with both W_Σ and $W_{\Sigma'}$. By connectivity, we now see that each symp is contained in W_Σ, our wanted contradiction. ∎

5.6. Proof of Theorem 5.1. Proposition 5.2 and Lemma 5.20 implies that only points with types (i), (ii) and (iii) can appear. Then Theorem 5.1(i) and and (ii) of the Main Result (including existence) follow from Proposition 5.7, and Theorem 5.1(ii) and (i) of the Main Result follow from Proposition 5.10. That the possibilities mentioned in Theorem 5.1 and (iii) of the Main Result are the only ones, follows from Proposition 5.18. The existence part of (iii) in the Main Result follows from the following considerations.

(1) The Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$ contains, up to conjugacy in $\mathrm{Aut}\,\mathsf{E}_{6,1}(\mathbb{K})$, a unique polarity with a fully embedded metasymplectic space isomorphic to $\mathsf{F}_{4,4}(\mathbb{K})$ as corresponding absolute geometry, as follows from Theorems 1 and 2 of [12]. This can also be deduced from the proof of Proposition 5.18.

(2) If $\mathbb{K}$ is a separable quadratic extension of a subfield $\mathbb{F}$, then the Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$ contains, up to conjugacy in the group of projectivities of $\mathsf{E}_{6,1}(\mathbb{K})$, a unique polarity with a fully embedded metasymplectic space isomorphic to $\mathsf{F}_{4,4}(\mathbb{F}, \mathbb{K})$ as corresponding absolute geometry, as follows from [22]. This can also be deduced from the proof of Proposition 5.18. Consequently. if $\mathbb{F}'$ is another subfield of $\mathbb{K}$ such that $\mathbb{K}$ is a separable quadratic extension of $\mathbb{F}'$, then $\mathbb{F}$ and $\mathbb{F}'$ are conjugate in the

full automorphism group of $\mathbb{K}$ if, and only if, the corresponding fully embedded meta-symplectic spaces can be mapped onto each other by some member of $\mathrm{Aut}\,\mathsf{E}_{6,1}(\mathbb{K})$.

(3) If $\operatorname{char}\mathbb{K} = 2$ and $\mathbb{K}'$ is a subfield of $\mathbb{K}$ containing $\mathbb{K}^2$, then $\mathsf{F}_{4,4}(\mathbb{K}',\mathbb{K})$ is fully embedded in $\mathsf{F}_{4,4}(\mathbb{K})$ (as follows from the construction of $\mathsf{F}_{4,4}(\mathbb{K}',\mathbb{K})$, see also Section 10.3 of [23]).

6. Equator geometries

6.1. Intersections of equator geometries. Generally speaking, an *equator set* of a Lie incidence geometry Δ is the set of points lying at equal distance from two given opposite flags F, F' along a shortest path connecting these flags. There are various ways to furnish this set with lines so that it becomes a point-line geometry, the standard way is to just consider the lines completely contained in it (and we shall always do it in this way). The point-line geometries thus obtained are again Lie-incidence geometries, they are related to the building $\mathrm{Res}_\Delta(F)$. We shall define some specific equator geometries below. For more examples and some theory, see [26].

We begin with $\Delta = \mathsf{D}_{6,6}(\mathbb{K})$. Let Δ^* be the corresponding polar space $\mathsf{D}_{6,1}(\mathbb{K})$. Recall that we refer to the maximal singular subspaces of Δ^* that correspond to the points of Δ as 5-spaces; and to the others as 5′-spaces.

DEFINITION 6.1. Let W, W' be two opposite 5′-spaces of Δ^* (equivalently, two opposite 5-spaces V, V' of Δ). The point set of the *equator geometry* $E(W, W')$ is given by the set of 5-spaces of Δ^* (equivalently, points of Δ) intersecting both W and W' in planes of Δ^* (equivalently, collinear to simultaneously a plane of V and a plane of V'), where a typical line consists of all 5-spaces of $E(W, W')$ pairwise intersecting in a common 3-space of Δ^* (equivalently, equipped with the lines of Δ entirely contained in it). The elements W and W' are called the *poles* of $E(W, W')$; they are unique as a pair. In fact Lemma 6.2(i) tells us that they are the unique 5′-spaces of Δ^* intersecting each member (i.e., 5-space) of $E(W, W')$ in a plane.

As a point-line geometry, the equator geometry $E(W, W')$ defined above is isomorphic to $\mathsf{A}_{5,3}(\mathbb{K})$, as can be seen by identifying each 5-space U of $E(W, W')$ with the plane $U \cap W$; noting that each plane of W arises this way, and noting that a line of $E(W, W')$ is given by the set of 5-spaces containing a 3-space which meets W (and also W') in a line of Δ^* (then these 5-spaces pairwise intersect in that 3-space). We call $E(W, W')$ an equator geometry *of type* $\mathsf{A}_{5,3}$. We will prove in Theorem 6.13 that the full embedding defined by such an equator geometry is the unique full embedding of $\mathsf{A}_{5,3}(\mathbb{K})$ into $\mathsf{D}_{6,6}(\mathbb{K})$.

Let $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{K})$ be fully and isometrically embedded in Δ; then (ii) of Theorem 4.1 applies. We now show that Γ also arises as the intersection of at least two equator geometries of type $\mathsf{A}_{5,3}$. The basic properties are summarised in the following lemma.

LEMMA 6.2. *Let W, W' and W'' be three pairwise opposite 5′-spaces of Δ^*. Then*

(i) *each point of $W \cup W' \cup W''$ is contained in a unique line of Δ^* intersecting each of W, W', W'' nontrivially.*

Let $\mathcal{P}^*$ be the set of all such lines, and let $\mathcal{L}^*$ be the set of all 3-spaces of Δ^* intersecting each of W, W', W'' in a line. Then

  (ii) $\Gamma^* = (\mathcal{P}^*, \mathcal{L}^*)$, *with incidence induced by containment, is isomorphic to the polar space* $\mathsf{C}_{3,1}(\mathbb{K})$,

 (iii) *each point on each member of $\mathcal{P}^*$ is contained in a unique $5'$-space intersecting every member of $\mathcal{P}^*$ nontrivially, and*

 (iv) *the subset of $E(W, W')$ consisting of all 5-spaces of Δ^* intersecting W'' in a plane of Δ^* is the point set in Δ (with induced lines) of a fully embedded dual polar space $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{K})$, all of whose symps are isometrically embedded in symps of Δ.*

Proof. Let x be an arbitrary point of W. Then x is collinear to a unique 4-space F' of W', which is in turn collinear to a unique point x'' of W'', which is itself collinear to a unique 4-space F of W. Let β be the map taking x to x'' and σ the map taking x to F. Being the composition of an even/odd number of dualities (in the sense of projective space, that is, incidence preserving bijections that map points to hyperplanes and vice versa), β is a collineation from W to W'' and σ is a duality of W to itself.

Claim 1: σ is the restriction of a symplectic polarity ρ of W to the point set of W.

By [18] it suffices to show that each point $x \in W$ is absolute, i.e., x is incident with its image $\sigma(x)$. Since x and $\beta(x)$ are both collinear to F', with $x, \beta(x) \notin F'$, and as there is only one 5-space in Δ^* containing F', we obtain that x and $\beta(x)$ are collinear. It then follows immediately that $F = \sigma(x)$ contains x. □

Observe that $\langle x, \beta(x) \rangle \subseteq \langle x, F' \rangle$ also implies that $\langle x, \beta(x) \rangle$ is the unique singular line through x meeting W, W', W'' in a point each. Interchanging the roles of W, W', W'' freely, this shows (i).

Claim 2: A line $L \subseteq W$ is totally singular with respect to ρ, i.e., $L \subseteq \rho(L) = \bigcap_{x \in L} \sigma(x)$, if, and only if, it is contained in a 3-space intersecting each of W, W', W'' in a line.

Let L be a line of W totally singular with respect to the symplectic polarity ρ. Then L and $L'' := \beta(L)$ are contained in the same $5'$-space which intersects W' in a 3-space. Hence L and L'' span a singular 3-space intersecting W' in a line L'.

Conversely, suppose S is a 3-space intersecting W, W' and W'' in lines M, M', M'', respectively. Then, for each $x \in M$, there is a unique line L_x in S intersecting both M' and M'' in some point. The line L_x intersects both W' and W'' nontrivially, and so by (i), it coincides with $\langle x, \beta(x) \rangle$. Consequently $\beta(x) \in M''$ and, since each point of M is collinear to $\beta(x)$, we conclude $M \subseteq \sigma(x)$, implying that M is totally singular with respect to ρ. □

Assertion (ii) follows easily. We now show (iii). Consider Δ^* as a hyperbolic quadric in $\mathsf{PG}(11, \mathbb{K})$. Then W, W', W'' define a unique Segre variety $\mathcal{S}_{1,5}(\mathbb{K})$. By the above, its 1-dimensional generators are precisely the members of $\mathcal{P}^*$. Hence, the 5-dimensional generators of $\mathcal{S}_{1,5}(\mathbb{K})$ (and we have one such through each point of each member of $\mathcal{P}^*$) are the unique 5-spaces of $\mathsf{PG}(11, \mathbb{K})$ intersecting all of the members of $\mathcal{P}^*$. Since all points of such generators belong to Δ^* and since they are pairwise disjoint, each 5-dimensional generator of $\mathcal{S}_{1,5}(\mathbb{K})$ is a $5'$-space of Δ^*.

Claim 3: A plane $\pi \subseteq W$ is totally singular with respect to ρ if, and only if, it is

contained in a 5-space intersecting each of W, W', W'' in a plane.

This is proven similarly to Claim 2.   □

The 5-spaces intersecting W, W', W'' in planes correspond to planes of Γ^* and so we can see the dual polar space Γ corresponding to Γ^* fully embedded in the equator geometry $E(W, W')$ and hence also in Δ. Since two 5-spaces corresponding to points of Γ at distance 2 intersect in a line, we see that these points are also at distance 2 in Δ. Hence (iv) follows. ■

PROPOSITION 6.3. *Let $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{K})$ be fully and isometrically embedded in the half spin geometry $\Delta \cong \mathsf{D}_{6,6}(\mathbb{K})$. Let $\mathcal{L}_\Gamma$ be the set of lines of Δ^* corresponding to the symps of Γ, and let $L \in \mathcal{L}_\Gamma$ be arbitrary. Then, for each Δ^*-point $z \in L$, there is a unique $5'$-space $W_z \ni z$ in Δ^* intersecting each line of $\mathcal{L}_\Gamma$. Set $\Phi_\Gamma = \{W_z : z \in L\}$. Let $G \leq \mathrm{Aut}\Delta$ be the pointwise stabilizer of Γ. Then the following statements hold.*

(i) $\Gamma = E(W_z, W_{z'}) \cap E(W_{z''}, W_{z'''})$ *for each quadruple z, z', z'', z''' of Δ^*-points on L, with $z \neq z'$, $z'' \neq z'''$ and $\{z, z'\} \neq \{z'', z'''\}$; moreover each equator geometry of type $\mathsf{A}_{5,3}$ containing Γ coincides with $E(W_z, W_{z'})$, for some distinct $z, z' \in L$, and Φ_Γ is the set of poles of these equator geometries.*

(ii) $\mathcal{L}_\Gamma$ *and Φ_Γ are the 1-dimensional and 5-dimensional generators of a Segre variety $\mathcal{S}_{1,5}(\mathbb{K})$, respectively.*

(iii) $G \cong \mathsf{PGL}_2(\mathbb{K})$ *and acts naturally sharply 3-transitively on L and ditto on Φ_Γ.*

Proof. By uniqueness, up to a projectivity of Δ, of Γ in Δ (cf. Proposition 4.24), we infer from Lemma 6.2(iv) that there are distinct $5'$-spaces W, W', W'' of Δ^* intersecting each member of $\mathcal{L}_\Gamma$ in a point; for such triple of $5'$-spaces we have $\Gamma = E(W, W') \cap E(W, W'')$. Hence (i) follows if $|\{W_z, W_{z'}, W_{z''}, W_{z'''}\}| = 3$. Also, the existence and uniqueness of W_z given z now follows from Lemma 6.2(iii).

Now let W''' be a fourth $5'$-space intersecting all members of $\mathcal{L}_\Gamma$ nontrivially. Then again we have $\Gamma \subseteq E(W'', W''')$. If some 5-space of Δ^* belongs to $E(W, W') \cap E(W'', W''')$, then it intersects each of W, W', W'', W''' in a plane, and hence it belongs to $E(W, W'')$. So we have $\Gamma \subseteq E(W, W') \cap E(W'', W''') \subseteq E(W, W') \cap E(W, W'') = \Gamma$, whence (i).

Now (ii) follows from Lemma 6.2(iii) and (iii) follows from Proposition 4.24 noting that paras of Δ are points of Δ^* and every point on L uniquely defines a member of Φ_Γ. ■

A perhaps surprising corollary is that we can easily classify all full embeddings of dual polar spaces of rank 3 in the Lie incidence geometry $\mathsf{A}_{5,3}(\mathbb{K})$.

COROLLARY 6.4. *There is a projectively unique fully embedded dual polar space of rank 3 in the Lie incidence geometry $\mathsf{A}_{5,3}(\mathbb{K})$. It arises from a symplectic polarity in the associated building $\mathsf{A}_5(\mathbb{K})$. The embedding is isometric and the dual polar space is isomorphic to $\mathsf{C}_{3,3}(\mathbb{K})$. It is the fixed point structure of an involutory collineation of $\mathsf{A}_{5,3}(\mathbb{K})$.*

Proof. Let Γ be a fully embedded dual polar space of rank 3 in $\Omega = \mathsf{A}_{5,3}(\mathbb{K})$. Embed Ω as an equator geometry in $\Delta = \mathsf{D}_{6,6}(\mathbb{K})$. Then Γ is a fully embedded dual polar space of rank 3 in Δ and we can go through the list given in Theorem 4.1. The symps of $\mathsf{C}_{3,3}(\mathbb{L}, \mathbb{K})$ are isomorphic to $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{L})$ and cannot be fully embedded in the symps of Ω (due to Lemma 3.19); also Γ is not contained in a symp of Δ, as in this case, the point set of Γ is not contained in a proper subspace of that symp, which would be the case if Γ were

contained in an equator geometry. So the only possibility is that $\Gamma \cong \mathsf{C}_{3,3}(\mathbb{K})$ and that Γ is isometrically embedded in Δ, and hence also in Ω. Suppose Γ' is a second embedded dual polar space of rank 3 in Ω. By Proposition 4.35(ii), there is a projectivity θ of Δ mapping Γ' to Γ, and by Proposition 6.3(iii) there is a projectivity θ' of Δ pointwise fixing Γ and mapping Ω^θ to Ω. Hence the composition (from left to right) $\theta\theta'$ stabilizes Ω and maps Γ' to Γ. The corollary is now clear. $\blacksquare$

Combining Theorem 4.1(ii), Lemma 6.2 and Lemma 6.3, we readily obtain the following result (which can also be shown directly, and which also holds for higher even rank):

COROLLARY 6.5. *All triples of pairwise opposite maximal singular subspaces of the polar space* $\mathsf{D}_{6,1}(\mathbb{K})$ *are projectively equivalent.* $\blacksquare$

We now intend to prove the analogue of Proposition 6.3 for metasymplectic spaces isomorphic to $\mathsf{F}_{4,1}(\mathbb{K})$ fully and isometrically embedded in $\mathsf{E}_{7,1}(\mathbb{K})$. First we prove the analogue of Lemma 6.2.

We set $\Delta = \mathsf{E}_{7,1}(\mathbb{K})$ and Δ^* the corresponding strong parapolar space $\mathsf{E}_{7,7}(\mathbb{K})$. We define the appropriate equator geometry.

DEFINITION 6.6. Let Π, Π' be opposite paras of Δ (equivalently, two opposite points p, p' of Δ^*). The *equator geometry* $E(\Pi, \Pi')$ is given by the set of points of Δ close to both Π and Π', i.e., collinear to simultaneously a 5-space of Π and a 5-space of Π' (equivalently, the set of symps Σ of Δ^* meeting $p^{\perp_{\Delta^*}}$ and $p'^{\perp_{\Delta^*}}$ in (Σ-opposite) $5'$-spaces of Δ^*), equipped with the lines of Δ entirely contained in it (equivalently, a typical line consists of all symps sharing a common maximal singular 5-space of Δ^*, which necessarily meets both $p^{\perp_{\Delta^*}}$ and $p'^{\perp_{\Delta^*}}$ in a plane). The paras Π and Π' are called the *poles* of $E(\Pi, \Pi')$. It will follow from Lemma 6.7 that they are unique.

As a point-line geometry, the equator geometry $E(\Pi, \Pi')$ defined above is isomorphic to $\mathsf{E}_{6,2}(\mathbb{K})$, as can be seen by identifying each point of $E(\Pi, \Pi')$ with the 5-space of Π it is collinear to; noting that each 5-space of Π arises this way, and noting that a typical line corresponds to the set of such 5-spaces sharing a common plane of Π. We call $E(\Pi, \Pi')$ an equator geometry *of type* $\mathsf{E}_{6,2}$ (see [26]). We will prove in Proposition 6.14 that the embedding defined by such an equator geometry is, up to projectivity, the unique full embedding of $\mathsf{E}_{6,2}(\mathbb{K})$ into $\mathsf{E}_{7,1}(\mathbb{K})$. Note that the set of equator geometries of type $\mathsf{E}_{6,2}$ form a single orbit under the automorphism group of Δ since the latter acts transitively on pairs of opposite paras (as follows from the BN-pair property of the corresponding Chevalley group, see [23]). .

Note that, if p and p' are opposite points of Δ^*, then the union of the symps of Δ^* that share a 5-space with both $p^{\perp_{\Delta^*}}$ and $p'^{\perp_{\Delta^*}}$, is the union of lines L of Δ^* with the property that p and p' are collinear to unique (distinct) points of L. In general, the distance between a point p and a line L in a point-line geometry Ω is defined as $\min\{\delta_\Omega(p, x) \mid x \in L\}$.

LEMMA 6.7. *Let* p, p' *and* p'' *be three pairwise opposite points of* Δ^*. *Let* $\mathcal{P}^*$ *be the set of lines of* Δ^* *at distance 1 from each of* p, p', p''. *Then*

(i) $\mathcal{M}_p := p^\perp \cap \bigcup \mathcal{P}^*$ *is the point set of a metasymplectic space fully embedded in Δ^* and isomorphic to $\mathsf{F}_{4,4}(\mathbb{K})$. The same holds for similarly defined $\mathcal{M}_{p'}$ and $\mathcal{M}_{p''}$.*

Let $\mathcal{L}^*$ be the set of all 3-spaces of Δ^* intersecting each of $\mathcal{M}_p, \mathcal{M}_{p'}, \mathcal{M}_{p''}$ in a line. Then

(ii) $\Gamma^* = (\mathcal{P}^*, \mathcal{L}^*)$, *with incidence induced by containment, is isomorphic to the metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K})$,*

(iii) *each point on each member of $\mathcal{P}^*$ is collinear to a unique point of Δ^* which lies at distance 1 from every member of $\mathcal{P}^*$, and*

(iv) *the set of all symps of Δ^* intersecting each of $\mathcal{M}_p, \mathcal{M}_{p'}, \mathcal{M}_{p''}$ in a $5'$-space, is the point set in Δ of an embedded metasymplectic space $\Gamma \cong \mathsf{F}_{4,1}(\mathbb{K})$ all of whose symps are isometrically embedded in symps of Δ; we also have that Γ is contained in $E(\Pi_p, \Pi_{p'})$, where Π_x is the para of Δ corresponding to the point x of Δ^*, $x \in \{p, p'\}$.*

Proof. For opposite points x, y of Δ^*, denote by ρ_y^x the projection from $\mathrm{Res}_{\Delta^*}(x)$ onto $\mathrm{Res}_{\Delta^*}(y)$. Note that this projection maps a line L of Δ^* containing x to the unique symp of Δ^* containing y and intersecting L in a point, and conversely, a symp Σ of Δ^* containing x is mapped to the line yz of Δ^* where z is the unique point of Σ collinear to y. Then the composition (from left to right) $\sigma := \rho_{p''}^p \rho_{p'}^{p''} \rho_p^{p'}$ is a duality of $\Omega := \mathrm{Res}_{\Delta^*}(p)$. We view Ω as a parapolar space isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$. In order to show that σ is a symplectic polarity it suffices, according to Main Result 2.1 of [25], to show that no point of Ω (which is a line of Δ^* through p) is mapped onto a neighbouring symp of Ω (which is a symp of Δ^* containing p; a point and a symp of Ω are *neighbouring* if the point is collinear to a $4'$-space of the symp), and some point of Ω is mapped onto an incident symp of Ω.

Consider an arbitrary line L containing p. Let q be the unique point on L symplectic to p''. Then $\Sigma'' := \rho_{p''}^p(L)$ is just $p'' \lozenge q$. Since p' is opposite p'', Fact 3.17 yields a unique point $q' \in \Sigma''$ collinear to p'. Then $\rho_{p'}^{p''}(\Sigma'') = p'q'$. Finally, let x' be the unique point on $p'q'$ symplectic to p. Then $\Sigma := \sigma(L)$ is nothing other than $p \lozenge x'$.

(a) Suppose first that $q' = x'$. Then, Fact 3.17 implies that q' and q are collinear, $q \in p \lozenge q' = \Sigma$ and hence $L \subseteq \Sigma$, i.e., the element L of Ω is absolute for Σ.

(b) Next, suppose $q' \neq x'$. Then p and q' are opposite and hence Fact 3.17 subsequently implies that q' and q are symplectic, that q and x' are opposite in Δ^*, and that q is collinear to exactly the point p of Σ. Hence the elements L and Σ of Ω are not neighbouring.

Note that in Case (a), p' and q are symplectic, whereas in Case (b), p' and q are opposite. To show that σ admits an absolute point, it hence suffices to find a point q collinear to p and symplectic to p' and p''. To that end, take a singular plane π through p and note that $p'^{\perp\!\!\!\perp} \cap p''^{\perp\!\!\!\perp} \cap \pi$ is non-empty (since $p'^{\perp\!\!\!\perp} \cap \pi$ and $p''^{\perp\!\!\!\perp} \cap \pi$ are lines of π).

In particular we have shown that every line L containing p and absolute with respect to σ is concurrent with a member of $\mathcal{P}^*$ (with the above notation this is the line qq'). Conversely, let $M \in \mathcal{P}^*$ be arbitrary and denote by q, q', q'' the points of M collinear to p, p', p'', respectively. As deduced above and using that $x' = q'$ in this case, it follows that $\sigma(pq) = p \lozenge q'$. No other member of $\mathcal{P}^*$ can be concurrent with pq since $\sigma(pq)$ contains only one point q' collinear to p'.

Now let π be an absolute plane with respect to σ containing pq, i.e., $\sigma(\pi)$ is a singular 5-space containing π. This is equivalent to the fact that for each line L' through p in π, we have $\pi \subseteq \sigma(L')$. We use the same notation as above; note that we are in Case (a). Then π is contained in $\sigma(pq)$, which implies that q' and q'' are collinear to a common line $K \subseteq \pi$. This implies that each point $\tilde{q}$ on K is at distance 2 from both p' and p'' and so there is a unique member $M_{\tilde{q}}$ of $\mathcal{P}^*$ containing $\tilde{q}$, and hence $K \subseteq \mathcal{M}_p$. Let M_K be the union of the lines $M_{\tilde{q}}$, with $\tilde{q} \in K$. We claim that M_K generates a singular 3-space S of Δ^* intersecting each of $\mathcal{M}_p$, $\mathcal{M}_{p'}$ and $\mathcal{M}_{p''}$ in a unique line. Denote by K' the set of points $\tilde{q}' := M_{\tilde{q}} \cap p'^{\perp}$. Noting that $\rho_{p'}^{p''}(p'' \Diamond \tilde{q}) = p' \tilde{q}'$, it follows that K' is a line in a singular plane through p' (consisting of points symplectic to both p and p''). Since q' is collinear to K, it follows likewise that all points $\tilde{q}'$ on K' are collinear to K and hence K and K' generate a singular 3-space S, which coincides with $\langle M_K \rangle$. Interchanging the roles of p' and p'', we conclude that S also contains a line collinear to p'' and symplectic to p and p', meeting each line $M_{\tilde{q}}$ in a point. It follows that the line set $\{M_{\tilde{q}} \mid \tilde{q} \in K\}$ is the line set of a regulus in S and hence $p^{\perp} \cap S = K$ indeed, likewise for p' and p''. The claim follows.

Now, if some 3-space S intersects each of $\mathcal{M}_p, \mathcal{M}_{p'}, \mathcal{M}_{p''}$ in lines K, K' and K'', respectively, then the plane $\langle p, K \rangle$ is absolute with respect to σ, since for each point $q \in K$, we have $\sigma(pq) = p \Diamond q'$, where $q' \in K'$, so $K \subseteq \sigma(pq)$ indeed. So we have shown that $\mathcal{M}_p$ corresponds to the set of absolute points of a symplectic polarity in Ω and $\mathcal{L}^*$ corresponds to the set of absolute lines of that same symplectic polarity in Ω. Since the absolute geometry of a symplectic polarity in $\mathsf{E}_{6,1}(\mathbb{K})$ is isomorphic to $\mathsf{F}_{4,4}(\mathbb{K})$ (see for instance [12] or [25]), (i) and (ii) follow.

Now we claim that each 6-space U through p fixed under σ is incident with a unique symp Σ_U intersecting both $\mathcal{M}_{p'}$ and $\mathcal{M}_{p''}$ in 5'-spaces. Indeed, let $W = U \cap \mathcal{M}_p$. Pick arbitrary disjoint planes π_i in W, $i = 1, 2$, such that $\langle p, \pi_i \rangle$ is fixed by σ. Then, as above, one shows that π_i is contained in a 5-space T_i intersecting both $\mathcal{M}_{p'}$ and $\mathcal{M}_{p''}$ in planes π_i' and π_i'', $i = 1, 2$, respectively. Pick a point x in π_1 and a point y in π_2'. Then the existence of T_1, T_2 and the 6-spaces $U' := \rho_{p'}^p(U)$ and $U'' := \rho_{p''}^p(U)$ implies that the symp $x \Diamond y$ contains all of W, W', W'', where $W' = U' \cap \mathcal{M}_{p'}$ and $W'' = U'' \cap \mathcal{M}_{p''}$. Setting $\Sigma_U = x \Diamond y$ proves the claim, as it follows from the construction that there is no other such symp.

It is now also clear that $\mathcal{P}^*$ induces an isomorphism between the metasymplectic spaces defined by $\mathcal{M}_p$, $\mathcal{M}_{p'}$ and $\mathcal{M}_{p''}$. It follows that the set of symps Σ_U, with U running through all 6-spaces containing p and fixed under σ, defines a fully embedded metasymplectic space $\Gamma \cong \mathsf{F}_{4,1}(\mathbb{K})$ in Δ and (iv) follows from the fact that the symps of Γ correspond to lines of Δ^*, that is, symps of Δ. Now (iii) follows from the action of the pointwise stabilizer of Γ in $\mathrm{Aut}\,\Delta$, see Proposition 5.10. ∎

We now have the analogue of Proposition 6.3. The proof is very similar, in view of the similar Lemmas 6.2 and 6.7, and is left to the reader.

PROPOSITION 6.8. *Let* $\Gamma \cong \mathsf{F}_{4,1}(\mathbb{K})$ *be fully and isometrically embedded in* $\Delta \cong \mathsf{E}_{7,1}(\mathbb{K})$. *Let* $\mathcal{L}_{\Gamma}$ *be the set of the lines of* Δ^* *corresponding to the symps of* Γ, *and let* $L \in \mathcal{L}_{\Gamma}$ *be*

arbitrary. Then, for each Δ^*-point $z \in L$, there is a unique Δ^*-point p_z collinear to z and at distance 1 from each member of $\mathcal{L}_\Gamma$. Let Π_z be the para of Δ corresponding to the point p_z of Δ^* and set $\Phi_\Gamma := \{\Pi_z : z \in L\}$. Let $G \leq \mathrm{Aut}\Delta$ be the pointwise stabilizer of Γ. Then the following statements hold.

- *(i)* $\Gamma = E(\Pi_z, \Pi_{z'}) \cap E(\Pi_{z''}, \Pi_{z'''})$ *for each quadruple z, z', z'', z''' of Δ^*-points on L, with $z \neq z'$, $z'' \neq z'''$ and $\{z, z'\} \neq \{z'', z'''\}$; moreover each equator geometry of type $\mathsf{E}_{6,2}$ containing Γ coincides with $E(\Pi_z, \Pi_{z'})$, for some distinct $z, z' \in L$, and Φ_Γ is the set of poles of these equator geometries.*

- *(ii)* *Consider Δ^* in its natural embedding in $\mathsf{PG}(55, \mathbb{K})$. For each $z \in L$, set $\mathcal{M}_{p_z} := p_z^\perp \cap \bigcup \mathcal{L}_\Gamma$. Then $\mathcal{L}_\Gamma$ and $\{\langle \mathcal{M}_{p_z} \rangle \mid z \in L\}$ are the 1-dimensional and 25-dimensional, respectively, generators of a Segre variety $\mathcal{S}_{1,25}(\mathbb{K})$.*

- *(iii)* $G \cong \mathsf{PGL}_2(\mathbb{K})$ *and acts naturally sharply 3-transitively on L and ditto on Φ_Γ.*

∎

Completely similar to Corollaries 6.4 and 6.5, we now also have:

COROLLARY 6.9. *There is a projectively unique fully embedded metasymplectic space in the Lie incidence geometry $\mathsf{E}_{6,2}(\mathbb{K})$. It arises from a symplectic polarity in the associated building $\mathsf{E}_6(\mathbb{K})$. The embedding is isometric and the metasymplectic space is isomorphic to $\mathsf{F}_{4,1}(\mathbb{K})$. It is the fixed point structure of an involutory automorphism of $\mathsf{E}_{6,2}(\mathbb{K})$.* ∎

COROLLARY 6.10. *All triples of pairwise opposite points of the parapolar space $\mathsf{E}_{7,7}(\mathbb{K})$ are projectively equivalent.* ∎

6.2. Uniqueness of equator geometries. The main goal of this section is to show the uniqueness, up to a projectivity, of the full embedding of $\mathsf{E}_{6,2}(\mathbb{K})$ in $\mathsf{E}_{7,1}(\mathbb{K})$, implying that it is always embedded as an equator geometry. We accomplish this using a small induction process, proving the analogues for point and line residues. We set some notation. Let Γ_0 be the point-line geometry arising from the Segre variety $\mathcal{S}_{2,2}(\mathbb{K})$ by taking as point set the point set of $\mathcal{S}_{2,2}(\mathbb{K})$ and as set of lines the set of projective lines contained in it. We denote Γ_0 also by $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$. Set $\Delta_0 := \mathsf{A}_{5,2}(\mathbb{K})$, $\Gamma_1 := \mathsf{A}_{5,3}(\mathbb{K})$, $\Delta_1 := \mathsf{D}_{6,6}(\mathbb{K})$, $\Gamma_2 := \mathsf{E}_{6,2}(\mathbb{K})$ and $\Delta_2 := \mathsf{E}_{7,1}(\mathbb{K})$ and suppose that Γ_i is fully embedded in Δ_i, $i = 0, 1, 2$. We will inductively show that Γ_i admits a projectively unique such embedding in Δ_i, $i = 0, 1, 2$, under the assumption that, for $i \in \{0, 1\}$, no symp of Γ_i is embedded in a singular subspace of Δ_i (note that $\Gamma_0, \Delta_0, \Gamma_1$ and Γ_2 are indeed parapolar spaces with symps isomorphic to $\mathsf{D}_{2,1}(\mathbb{K})$, $\mathsf{D}_{3,1}(\mathbb{K})$, $\mathsf{D}_{3,1}(\mathbb{K})$ and $\mathsf{D}_{4,1}(\mathbb{K})$, respectively). We will also show that the subgroup of $\mathrm{Aut}\Delta_i$ pointwise fixing Γ_i is isomorphic to $\mathbb{K}^\times$ and acts sharply transitively on well defined objects of Δ_i incident with a well defined object of Γ_i. This is made more precise case-by-case below.

We start by showing that each full embedding of Γ_i in Δ_i is isometric if $i \in \{1, 2\}$.

LEMMA 6.11. *Let $i \in \{1, 2\}$. Each full embedding of Γ_i in Δ_i such that, for $i = 1$, no symp of Γ_1 is contained in a singular subspace of Δ_1, is isometric; in particular each symp Σ_i of Γ_i is contained in a unique symp Q_i of Δ_i (and $\Sigma_i = p^{\perp_{\Delta_i}} \cap q^{\perp_{\Delta_i}}$ for opposite points $p, q \in Q_i \setminus \Sigma_i$).*

Proof. Note that the symps of Γ_2, being $\mathsf{D}_{4,1}(\mathbb{K})$, cannot be embedded in singular subspaces of Δ_2 for dimension reasons. So, by Lemma 3.20, each symp of Γ_i embeds isometrically in a unique symp of Δ_i.

Let x, y be an arbitrary pair of Γ_i-opposite points of Γ_i. Since $\mathrm{Aut}\Gamma_i$ acts transitively on the set of pairs of opposite points of Γ_i, Corollaries 6.4 and 6.9 yield a dual polar space $\Gamma_1' \cong \mathsf{C}_{3,3}(\mathbb{K})$ for $i = 1$ and a metasymplectic space $\Gamma_2' \cong \mathsf{F}_{4,1}(\mathbb{K})$ for $i = 2$ fully and isometrically embedded in Γ_i and containing x and y. Hence the symps of Γ_i' embed isometrically into symps of Γ_i, and consequently also in symps of Δ_i. It follows from our classification in Theorems 4.1 and 5.1 that Γ_i' now embeds isometrically in Δ_i and hence x and y are Δ_i-opposite. If there is a pair p, q of points of Γ_i that are Γ_i-special but not Δ_i-special (and hence Δ_i-collinear or Δ_i-symplectic), then by Fact 3.16 and 3.11, there is also a pair x, y of points of Γ_i that are Γ_i-opposite but not Δ_i-opposite, a contradiction. Likewise, Γ_i-symplectic points are also Δ_i-symplectic.

Since symps of Γ_i are isomorphic to $\mathsf{D}_{2+i,1}(\mathbb{K})$ and those of Δ_i to $\mathsf{D}_{3+i,1}(\mathbb{K})$, the last statement also follows. $\blacksquare$

We now start with uniqueness in case of $\Gamma_0 \subseteq \Delta_0$, assuming that no symp of the former is contained in a singular subspace of the latter.

LEMMA 6.12. *Suppose that the point-line geometry* $\Gamma_0 = (X_0, \mathcal{L}_0) \cong \mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$ *is fully embedded in* $\Delta_0 = (Y_0, \mathcal{K}_0) \cong \mathsf{A}_{5,2}(\mathbb{K})$ *so that no symp of* Γ_0 *is contained in a singular subspace of* Δ_0. *Then in the corresponding Lie incidence geometry* $\Delta_0^* := \mathsf{A}_{5,1}(\mathbb{K}) \cong \mathsf{PG}(5, \mathbb{K})$, *the set* X_0 *is just the set of all lines* L *joining two skew planes* α *and* α' *of* Δ_0^*. *The group of collineations of* Δ_0 *pointwise fixing* Γ_0 *is isomorphic to* $\mathbb{K}^\times$ *and acts sharply transitively on the points of* $L \setminus (\alpha \cup \alpha')$.

Proof. We can identify Y_0 with the set of all lines of the projective 5-space Δ_0^*; an arbitrary symp of Δ_0 is then the set of all lines of a 3-dimensional subspace of Δ_0^*. On the other hand, by definition, we can identify X_0 with the set of lines in a projective 5-space Δ_0^{**} over $\mathbb{K}$ that meet two skew planes π and π'; a symp Σ of Γ_0 corresponds to the set of lines meeting certain (skew) lines $L \subseteq \pi$ and $L' \subseteq \pi'$. The fact that Γ_0 is fully embedded in Δ_0 is equivalent to the existence of an injective map $\sigma : X_0 \to Y_0$ which takes planar line pencils consisting of members of X_0 to planar line pencils consisting of members of Y_0. Since symps of Γ_0 are embedded isometrically in symps of Δ_0 and since each pair of members of X_0 is contained in a symp of Γ_0, intersecting members of X_0 correspond to intersecting members of Y_0 and vice versa. In other words, σ preserves concurrency and non-concurrency.

Let p be any point of $\pi \cup \pi'$ and consider the set $X_0(p) \subseteq X_0$ of lines containing p (that is, all lines through p meeting the other plane). Let $\sigma(X_0(p))$ denote the set $\{\sigma(L) \mid L \in X_0(p)\}$.

Claim 1. $\sigma(X_0(p))$ *is the set of lines incident with some point and some* 3-*space of* Δ_0^*.

Note that $X_0(p)$, endowed with the planar point pencils as lines, has the structure of a projective plane. Since σ is a morphism between $X_0(p)$ and $\sigma(X_0(p))$ (it preserves concurrency and non-concurrency), $\sigma(X_0(p))$ is either the set of lines in a plane α of Δ_0^*

or the set of all lines incident with some point and some 3-space of Δ_0^*. Suppose for a contradiction that the first option occurs. Let P and P' be two planar line pencils in $X_0(p)$. Then there is a line $L \in X_0$ intersecting precisely one line M of P and no line of P'. In $\sigma(X_0(p))$ however, $\sigma(L) \cap \sigma(M)$ is a point of α through which there is at least one line $\sigma(M')$ of $\sigma(P')$, contradicting the fact that σ preserves non-concurrency. $\quad\square$

By Claim 1, σ induces a map $\tilde{\sigma}$ on the points of $\pi \cup \pi'$ by defining $\tilde{\sigma}(p)$ as the unique point contained in all lines of $\sigma(X_0(p))$.

Claim 2. $\tilde{\sigma}$ is injective.

If p and q are distinct points of $\pi \cup \pi'$, then there are lines L_p and L_q of X_0 containing p and q respectively, which are skew. Therefore, $\sigma(L_p)$ and $\sigma(L_q)$ are also skew, so $\tilde{\sigma}(p) \neq \tilde{\sigma}(q)$.
$\quad\square$

Let L be any line contained in $\pi \cup \pi'$ and set $\tilde{\sigma}(L) := \{\tilde{\sigma}(p) \mid p \in L\}$.

Claim 3. $\tilde{\sigma}(L)$ is the set of points of a line of Δ_0^*.

Without loss of generality, $L \subseteq \pi$. Let L' be any line in π' and consider the symp Σ of Γ_0 determined by L and L'. Since the embedding of Σ into Δ_0 is isometric and full, there are two skew lines M and M' in Δ_0^* such that $\sigma(\Sigma)$ is the set of all lines meeting M and M' . Suppose that, for two points $p_1, p_2 \in L$, $\tilde{\sigma}(p_1) \in M$ and $\tilde{\sigma}(p_2) \in M'$. Then $\sigma(X_0(p_1))$ and $\sigma(X_0(p_2))$ share the line $\langle \tilde{\sigma}(p_1), \tilde{\sigma}(p_2) \rangle$, whereas $X_0(p_1) \cap X_0(p_2)$ is empty, a contradiction. So, by possibly interchanging M and M', we may assume $\tilde{\sigma}(L) = M$ and $\tilde{\sigma}(L') = M'$.

Claim 4. $\tilde{\sigma}(\pi \cup \pi')$ is the set of points of two disjoint projective planes of Δ_0^*

Let L and M be two lines of π intersecting in a point p. Then $\tilde{\sigma}(L)$ and $\tilde{\sigma}(M)$ are, by Claims 2 and 3, two (full) lines of Δ_0^* meeting each other in the point $\tilde{\sigma}(p)$. Let x be any point of $\langle \tilde{\sigma}(L), \tilde{\sigma}(M) \rangle \setminus (\tilde{\sigma}(L) \cup \tilde{\sigma}(M))$. Then there is a line K' through x meeting $\tilde{\sigma}(L)$ and $\tilde{\sigma}(M)$ in respective points $\tilde{\sigma}(x_L)$ and $\tilde{\sigma}(x_M)$. But then $K' = \tilde{\sigma}(K)$ for $K = \langle x_L, x_M \rangle \subseteq \pi$ and again by Claim 3, $x \in \tilde{\sigma}(K)$. So $\tilde{\sigma}(\pi)$ is the set of all points of a plane α of Δ_0^*; likewise, $\tilde{\sigma}(\pi')$ is the set of all points of a plane α' of Δ_0^*. Since $\pi \cap \pi' = \emptyset$, Claim 2 implies that $\alpha \cap \alpha' = \emptyset$. $\quad\square$

We conclude that the set of lines $\sigma(X_0) \subseteq Y_0$ coincides with the set of lines in Δ_0^* meeting the skew planes α and α', proving the first part of the lemma. The second part follows from the observation that the group of collineations of Δ_0 pointwise fixing Γ_0 corresponds to the group of collineations of Δ_0^* pointwise fixing $\alpha \cup \alpha'$. $\quad\blacksquare$

LEMMA 6.13. *Suppose that $\Gamma_1 = (X_1, \mathcal{L}_1) \cong \mathsf{A}_{5,3}(\mathbb{K})$ is fully embedded in $\Delta_1 \cong \mathsf{D}_{6,6}(\mathbb{K})$. Then Γ_1 arises as the equator geometry $E(W, W')$, for a unique pair of opposite 5-spaces W, W' of Δ_1. Also, for each symp Σ of Γ_1, the pointwise stabiliser of Γ_1 in $\mathrm{Aut}\,\Delta_1$ acts sharply transitively on the set of paras of Δ_1 incident with Σ but not incident with W or W'.*

Proof. Considering the absolute elements of a symplectic polarity of a projective 5-space, one obtains a geometry $\Gamma' = (X', \mathcal{L}')$ isomorphic to the dual polar space $\mathsf{C}_{3,3}(\mathbb{K})$ that is fully and isometrically embedded in Γ_1. By Lemma 6.11, Γ' is also fully and isometrically embedded in Δ_1. Once more, we switch to the dual point of view. Let $\mathcal{L}_{\Gamma'}$ denote the set of lines of $\Delta_1^* \cong \mathsf{D}_{6,1}(\mathbb{K})$ corresponding to the symps of Γ'. According to Proposition 6.3, $\mathcal{L}_{\Gamma'}$ is contained in the equator geometry $E(W, W')$ of type $\mathsf{A}_{5,3}$ for each pair of distinct

(and hence disjoint) $5'$-spaces W, W' of Δ_1^* with the property that W and W' both intersect each element of $\mathcal{L}_{\Gamma'}$. Equivalently, it follows from Definition 6.1 that there is a collineation $\beta : W \to W'$ such that the elements of $\mathcal{L}_{\Gamma'}$ are given by the lines $\langle x, \beta(x) \rangle$, with x a Δ_1^*-point in W. We show that we may choose W, W' such that $E(W, W') = \Gamma_1$.

Let V be any element of X_1. Then $\mathrm{Res}_{\Delta_1}(V)$ can be identified with a certain 5-space V^* of Δ_1^*. By Lemma 6.12, there are two disjoint planes α_V and α_V' in V^* such that the set of points of Γ_1^* incident with V corresponds to the set of lines $\mathcal{L}(V^*)$ of V^* meeting both α_V and α_V'. If $V \in X'$, the set of members of $\mathcal{L}_{\Gamma'}$ incident with V is the subset $\mathcal{L}'(V^*)$ of $\mathcal{L}(V^*)$ given by $\langle x, \beta(x) \rangle$ for each point $x \in V^* \cap W$. So $W \cap V^*$ and $W' \cap V^*$ are planes in V^* with the property that they meet each member of $\mathcal{L}'(V^*)$. The planes α_V and α_V' also have this property and hence, as can be deduced from Proposition 6.3, α_V and α_V' are contained in respective $5'$-spaces W_V and W_V' of Δ_1^* with $\mathcal{L}_{\Gamma'} \subseteq E(W_V, W_V')$.

Claim 1: For each 5-space $U \in X'$, we have $\{W_V, W_V'\} = \{W_U, W_U'\}$.
Let $U \in X'$ be arbitrary. By connectivity of Γ', we may assume that V and U correspond to collinear points of Γ'. Since all embeddings under consideration are isometric, we have, with the notation introduced in the previous paragraph, that V^* and U^* are 5-spaces of Δ_1^* such that $U^* \cap V^*$ is a 3-space and $\mathcal{L}(V^*) \cap \mathcal{L}(U^*)$ is the set of lines meeting two skew lines L and L', where necessarily (possibly by interchanging the roles of L and L' and/or those of α_U and α_U'), we have $L \subseteq \alpha_U \cap \alpha_V$ and $L' \subseteq \alpha_U' \cap \alpha_V'$. As such, $W_V \cap W_U$ contains L and as the $5'$-spaces meeting all lines of $\mathcal{L}_{\Gamma'}$ are either disjoint or equal, $W_V = W_U$; likewise $W_V' = W_U'$. □

Henceforth we assume that W and W' are the $5'$-spaces of Δ_1^* such that, for each $V \in X'$, they meet V^* in the planes α_V and α_V', respectively.

Claim 2: For each 5-space $Z \in X_1 \setminus X'$, we have $\alpha_Z \cup \alpha_Z' \subseteq W \cup W'$; hence we may assume without loss $\alpha_Z \subseteq W$ and $\alpha_Z' \subseteq W'$.
Let $Z \in X_1 \setminus X'$ be arbitrary. Recall that $W \cap Z^*$ is a plane, non-isotropic under the symplectic polarity corresponding to Γ'. Then there are isotropic planes in W which meet this plane in distinct lines, and as $\Gamma' \subseteq \Gamma_1$ is an isometric embedding, there are $U_0, V_0 \in X'$ with $U_0 \perp_{\Gamma_1} Z \perp_{\Gamma_1} V_0$ and $U_0 \perp\!\!\!\perp_{\Gamma_1} V_0$. As in the previous claim and up to changing the roles of α_Z and α_Z', the plane α_Z meets the planes α_{V_0} and α_{U_0} (which necessarily intersect each other in a unique point) in lines, and hence $\alpha_Z \subseteq W$; likewise $\alpha_Z' \subseteq W'$. □

We conclude that $\Gamma_1 \subseteq E(W, W')$. Since Γ_1 and $E(W, W')$ are both isomorphic to $\mathsf{A}_{5,3}(\mathbb{K})$, and the former is fully embedded in the latter, equality holds.

The statement about the group follows from the fact that the said group is the pointwise stabilizer of two opposite maximal singular subspaces of a polar space isomorphic to $\mathsf{D}_{6,1}(\mathbb{K})$. ∎

Finally we have:

PROPOSITION 6.14. *Suppose that $\Gamma_2 = (X_2, \mathcal{L}_2) \cong \mathsf{E}_{6,2}(\mathbb{K})$ is fully embedded in $\Delta_2 = (Y_2, \mathcal{K}_2) \cong \mathsf{E}_{7,1}(\mathbb{K})$. Then Γ_2 arises as the equator geometry $E(\Pi, \Pi')$, for a unique pair of opposite paras Π, Π' of Δ_2. Also, for each symp Σ_2 of Γ_2, the pointwise stabiliser of Γ_2 in $\mathrm{Aut}\Delta_2$ acts sharply transitively on the set of paras of Δ_2 containing Σ_2, and intersecting both Π and Π' in unique points.*

Proof. In this proof, we denote by $\perp$ and $\bowtie$ the collinearity relation and the special relation, respectively, in Δ_2. When restricted to Γ_2, we add a $\circ$. So, for instance, for a point $x \in X_2$, $x^{\bowtie^\circ}$ is the set of all points of Γ_2-special to x (they are automatically Δ_2-special by Lemma 6.11).

Consider two Γ_2-opposite points $p, q \in X_2$. Then, by Lemma 6.11, p and q are also Δ_2-opposite. Consider $p^\perp \cap q^\bowtie$, endowed with the lines contained in it. This is a point-line geometry Δ_1 isomorphic to $\mathrm{Res}_{\Delta_2}(p) \cong \mathsf{D}_{6,6}(\mathbb{K})$. If we restrict Δ_1 to the points of Γ_2 contained in it, then we obtain a point-line embedded geometry Γ_1 isomorphic to $\mathrm{Res}_{\Gamma_2}(p) \cong \mathsf{A}_{5,3}(\mathbb{K})$. Hence, by Lemma 6.13, there are unique opposite 5-spaces W_p and W_p' in Δ_1 such that $\Gamma_1 = E(W_p, W_p')$. Likewise, there are unique opposite 5-spaces W_q and W_q' in $q^\perp \cap p^\bowtie$ such that the point set of Γ_2 contained in $q^\perp \cap p^\bowtie$ coincides with $E(W_q, W_q')$. Since collinearity is an isomorphism between $p^\perp \cap q^\bowtie$ and $q^\perp \cap p^\bowtie$, the uniqueness of W_p, W_p', W_q and W_q' implies that $W_p \cup W_p'$ corresponds to $W_q \cup W_q'$ under that isomorphism. Then Fact 3.15 yields paras Π and Π' containing W_p, W_q, and W_p', W_q', respectively.

We claim that Π and Π' are opposite. Indeed, if not, then, by Fact 3.14, they have at least one point z in common. In Π, this yields a point $z_p \in W_p$ collinear to z. Now, let z_p' be a point of W_p' which is Δ_1-opposite z_p. Then z_p and z_p' are special in Δ_2 and $z_p \bowtie z_p' = p$. Let z_q' be the unique point of W_q' collinear to z_p'. Then also z_q' and p are special and hence, by Fact 3.16, z_p is opposite z_q'. Fact 3.13 implies that z_p is far from Π'. However, $z_p \perp z \in \Pi'$ means that z_p is close to Π'. This contradiction shows the claim. It follows that $\Gamma_1 \subseteq E(\Pi, \Pi')$ as each point of Γ_1 is collinear to a plane of W_p and also with a plane of W_p', so by Fact 3.13, it is collinear to a 5-space of Π and also with one of Π'. This, in turn, implies that $p^{\perp^\circ} \cup q^{\perp^\circ} \subseteq E(\Pi, \Pi')$. Moreover, we claim that $p^{\perp^\circ} = p^\perp \cap E(\Pi, \Pi')$ and $q^{\perp^\circ} = q^\perp \cap E(\Pi, \Pi')$. Indeed, let us show the former. We already know $p^{\perp^\circ} \subseteq p^\perp \cap E(\Pi, \Pi')$, so let $p_* \in p^\perp \cap E(\Pi, \Pi')$ be arbitrary. Then $p_*^\perp \cap \Pi$ is a singular 5-space W_*. Since $p \perp p_*$, we have $W \cap W_*$ is a plane α, and so the unique point on the line pp_* not opposite q is also collinear to α and hence belongs to Γ_1. Consequently $p_* \in p^{\perp^\circ}$. The claim is proved.

Let p' be a point of Γ_2 collinear to p. First suppose that $p' \notin q^\bowtie$. Then p' is opposite q. Therefore there is a bijection between the sets of Δ_2-lines, Γ_2-lines and $E(\Pi, \Pi')$-lines, respectively, through q and those through p', given by "Containing respective points that are collinear" (we denote it in both directions by β). Since Γ_2-collinearity is the restriction to Γ_2 of Δ_2-collinearity, and the same thing holds for $E(\Pi, \Pi')$, a line L through p' belongs to Γ_2 if, and only if, $\beta(L)$ belongs to Γ_2. Since $q^{\perp^\circ} = q^\perp \cap E(\Pi, \Pi')$, this happens if, and only if, L belongs to $E(\Pi, \Pi')$. Hence $p'^{\perp^\circ} = p'^\perp \cap E(\Pi, \Pi')$. Now, if $q' \in q^\perp$ is a point of Γ_2 opposite p', the same thing holds for q', namely, $q'^{\perp^\circ} = q'^\perp \cap E(\Pi, \Pi')$. So, in case $p' \in p^{\perp^\circ} \cap q^\bowtie$, we may replace q by a point $q' \in q^{\perp^\circ} \setminus p^\bowtie$ opposite p' and again obtain $p'^{\perp^\circ} = p'^\perp \cap E(\Pi, \Pi')$. Hence the set of Γ_2-points not opposite p coincides with the set of $E(\Pi, \Pi')$-points not opposite p. We can now interchange the roles of p and each point $p' \in p^{\perp^\circ}$, and possibly also of q and some $q' \in q^{\perp^\circ}$ opposite p' (if p' and q are not opposite). Then the set of Γ_2-points not opposite p' coincides with the set of $E(\Pi, \Pi')$-points not opposite p'. Since every point of Γ_2 is not Γ_2-opposite at least one

point Γ_2-collinear to p, we conclude that $\Gamma_2 = E(\Pi, \Pi')$.

The statement about the group follows from the fact that the said group is the linewise stabilizer of two opposite points of a parapolar space isomorphic to $\mathsf{E}_{7,7}(\mathbb{K})$. ∎

6.3. Some additional notes on equator geometries.

6.3.1. Origin and generalities. The notion of "equator geometry" arose first in the context of Lie incidence geometries of type $\mathsf{F}_{4,4}$, see Proposition 6.26 of [15]. It follows from the proof of that proposition that in non-strong parapolar spaces of diameter 3 the set of points symplectic to two given opposite point forms an interesting geometry. In the case of type $\mathsf{F}_{4,4}$, this was further exploited by the authors in [12]. Equator geometries were introduced in full generality in the survey paper [26]. The basic idea is always to take two opposite flags and consider points at equal fixed distance from those lying on a geodesic—that is, a shortest path in the 1-skeleton of the simplicial complex—joining the two flags. Not every type of flags gives rise to such an equator geometry since points are not necessarily part of a geodesic joining two opposite flags of certain types, or are not at equal distance from the end-flags. For instance, a simple example are two opposite lines in a generalized quadrangle: only lines appear in the middle of a geodesic joining these to lines.

However, in long root geometries, this idea seems to work quite well for many types of flags, in particular for most types of vertices. The resulting equator geometry is in such a case isomorphic to the long root geometry related to the residue of one of the vertices. For example, in $\mathsf{E}_{7,1}(\mathbb{K})$, the equator geometry of two opposite vertices of type 7 (which we called paras above) is the long root geometry $\mathsf{E}_{6,2}(\mathbb{K})$, see Definition 6.6.

The simplest case occurs when the two opposite vertices are two opposite points. In this case we obtain the foundational situation in which the equator geometry is defined by all points symplectic to the two given points. In the classical cases B_n and D_n, the presence of symps of different rank gives rise to the inhomogeneous situation that the equator geometry is disconnected (reflecting the fact that the diagram of the residue of a point of the long root geometry is also disconnected); but especially for the exceptional types, we obtain a nice long root geometry as equator geometry. The purpose of this section is to highlight two beautiful features of these kind of equator geometries.

6.3.2. Decomposition of an apartment. Equator geometries provide adequate descriptions of apartments. To illustrate this, we take, to fix the ideas, the example of main interest for the current paper, namely type E_7. Set $\Delta = \mathsf{E}_{7,1}(\mathbb{K})$, for some field $\mathbb{K}$. Consider two opposite points p and q of Δ. Let $E(p,q)$ be the set of points of Δ symplectic to both p and q, and endow this set with the lines of Δ completely contained in it. This gives $E(p,q)$ the structure of a point-line geometry.

PROPOSITION 6.15. *With the above notation, $E(p,q)$ is isomorphic to $\mathsf{D}_{6,2}(\mathbb{K})$.*

Sketch of the proof. By the very definition of $E(p,q)$, there is a bijective correspondence between the symps through p and the points of $E(p,q)$ (in each symp through p there is a unique point symplectic to q; each point symplectic to p is contained in a unique symp with p). Note that the symps through p correspond to the points of the long root geometry

of $\mathrm{Res}_\Delta(p)$. We claim that the lines of $E(p,q)$ correspond to the lines of the long root geometry of $\mathrm{Res}_\Delta(p)$, which correspond to flags of Δ consisting of a maximal singular subspace of dimension 4 and a para, both through p. Suppose two points $x,y \in E(p,q)$ are collinear. Due to Fact 3.12, the point x is collinear to a maximal singular subspace of $p\lozenge y$; subsequently we deduce that $p\lozenge x$ and $p\lozenge y$ have a maximal singular subspace U of dimension 4 in common. Then it can be deduced that every point z on the line xy belongs to $E(p,q)$ and that $p\lozenge z$ contains U; moreover from Fact 3.13 we infer that the symps $p\lozenge x$, $p\lozenge y$ and $p\lozenge z$ are contained in a common para. Conversely, similar arguments show that the symps through p sharing a fixed maximal singular subspace and contained in a fixed common para correspond to the points of a line of $E(p,q)$. All this implies that the lines of $E(p,q)$ correspond to the lines of the long root geometry of $\mathrm{Res}_\Delta(p)$. The claim and the proposition follows. ■

Now let the *trace geometry* $T(p,q)$ be the set of points of Δ collinear to p and special to q. Clearly, $T(p,q)$, endowed with all lines of Δ it contains, is a point-line geometry isomorphic to $\mathrm{Res}_\Delta(p) \cong \mathsf{D}_{6,6}(\mathbb{K})$. We have the following property ($T(q,p)$ is defined in the obvious way).

PROPOSITION 6.16. *Collinearity defines a natural duality between $T(p,q)$ and $E(p,q)$; collinearity defines an isomorphism between $T(p,q)$ and $T(q,p)$.*

Sketch of the proof. The first assertion follows from the proof of Proposition 6.15 and the observation that, if $x \in E(p,q)$, then precisely the points of the symp induced by $p\lozenge x$ in $T(p,q)$ are collinear to x. The second assertion follows straight from Proposition 3.28 of [23]. ■

We can now apply the previous propositions to the "thin version" (i.e., apartment) of a building of type E_7 to obtain a model of an apartment where points are objects of type 1. It suffices to establish models of apartments of types $\mathsf{D}_{6,6}$ and $\mathsf{D}_{6,2}$ and suitably connect them, following Proposition 6.16. This results in the following description of the corresponding graph Γ. Let ∞ and ∞' be two vertices (playing the role of the points p,q, respectively), let $S_\infty = \{-6,-5,\ldots,-1,1,2,\ldots,6\}$ and $S_{\infty'} = \{-6',-5',\ldots,-1',1',2',\ldots,6'\}$ be two 12-sets and let the vertices of Γ adjacent to ∞ be the 6-subsets of S_∞ containing no pair of elements with the same absolute value and containing an even number of negative numbers. Two such subsets are adjacent if they have exactly four elements in common (the resulting geometry is the thin version of $T(p,q)$). Similarly the neighbourhood of ∞' is defined using S'_∞. Finally, let also each pair $\{k,\ell\}$, with $k,\ell \in \{-6,-5,\ldots,-1,1,2,\ldots,6\}$, $k^2 \neq \ell^2$, be a vertex of Γ (then there are $2 + 2 \cdot 2^5 + 60 = 126$ vertices in total). Two such pairs $\{k,\ell\}$ and $\{u,v\}$ are adjacent if $|\{k,\ell,u,v\}| = |\{|k|,|\ell|,|u|,|v|\}| = 3$ (this time, we obtain the thin version of $E(p,q)$). Following Proposition 6.16 we declare the vertices $\{i_1,i_2,i_3,i_4,i_5,i_6\}$ and $\{i'_1,i'_2,i'_3,i'_4,i'_5,i'_6\}$ adjacent (where $\{i_1,\ldots,i_6\} \subseteq S_\infty$, with the given restrictions to be a vertex of Γ). Also, the same vertices $\{i_1,\ldots,i_6\}$ and $\{i'_1,\ldots,i'_6\}$ are adjacent to each vertex $\{i_k,i_\ell\}$, for $\{k,\ell\} \subseteq \{1,2,3,4,5,6\}$, $k \neq \ell$. The graph Γ, viewed as a point-line geometry (vertices are the points, edges the lines), is a thin version of a long root geometry of type $\mathsf{E}_{7,1}$. The symps are the subgraphs isomorphic to a complete graph on ten vertices minus a matching. For instance, the vertices ∞, $\{1,2\}$ and

every eligible 6-subset of S_∞ containing $\{1, 2\}$ induce such a subgraph and hence form a symp. The facts of Section 3.4 can now also be checked on this model.

The construction of Γ in the previous paragraph immediately provides the so-called *distance distribution diagram*, see [2]:
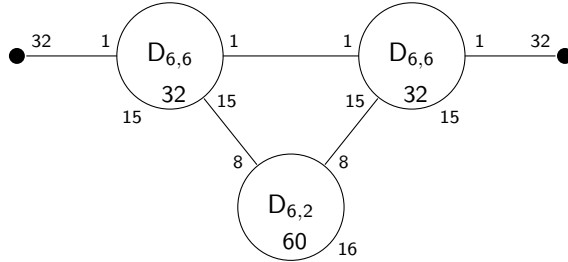
6.3.3. Orthogonality. Comparing the models of the thin versions of the long root geometries of type $\mathsf{E}_{7,1}$ given in Section 3.4 using root systems and Section 6.3.2 using an equator geometry and two trace geometries, we conclude that the roots corresponding to the vertices of $E(p, q)$ are perpendicular to the roots corresponding to p and q (since perpendicular roots correspond to symplectic pairs of points). This implies that the subgroup $A(p, q) \leq \operatorname{Aut} \mathsf{E}_{7,1}(\mathbb{K})$ generated by the two root groups with respective centres p and q commutes with the subgroup generated by all root groups with centre inside $E(p, q)$. The former is a group of type A_1; the latter has type D_6. Hence we have given geometric evidence of a (maximal) subgroup of type $\mathsf{A}_1 \times \mathsf{D}_6$ inside an arbitrary algebraic group of type E_7.

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