OVERVIEW

(generalised) Veronese varieties

a little bit of algebra

point-line geometries with 3 axioms
OVERVIEW

(generalised) Veronese varieties

description

a little bit of algebra

examples

point-line geometries with 3 axioms
EXAMPLE 1

Let $K$ be a field. The Veronese variety $\mathcal{V}(K)$ is defined as follows.
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line $\to$ ?

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- line $\mapsto$ conic in a plane

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- each two points of $\text{im}(\rho)$ belong to a conic-plane
- the conic-planes intersect in points of $\text{im}(\rho)$
- the tangent space of a point is a plane

the span of all tangent lines through a point is the tangent space of that point
**EXAMPLE 1**

**Axiomatic description** of the Veronese variety $\mathcal{V}(K)$

$X = $ spanning point set of $\text{PG}(N,K)$

$\Xi = $ family of planes $\xi$ in $\text{PG}(N,K)$ s.th. $\xi \cap X$ is an conic in $\xi$

such that the following properties are satisfied

- each two points of $X$ belong to a plane of $\Xi$
- two planes of $\Xi$ intersect in points of $X$
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- The tangent space of a point of $X$ is a plane

Mazzocca, Melone (1984)

If $K=F(q)$ with $q$ odd, and $N=5$:

$(X, \Xi)$ is projectively unique and hence equivalent with $\mathcal{V}(K)$. 
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Hirschfeld, Thas (1991)

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such that the following **properties** are satisfied

- each two **points** of $X$ belong to a **plane** of $\Xi$
- two **planes** of $\Xi$ intersect in **points** of $X$
- the tangent space of a **point** of $X$ is a plane

---

Thas, Van Maldeghem (2004)

If $K = F(q)$ and we use **ovals** instead of **conics**:

$(X, \Xi)$ is **projectively unique** and hence equivalent with $\mathcal{V}(K)$. 
**EXAMPLE 1**

Axiomatic description of the Veronese variety \( \mathcal{V}(K) \)

- \( X = \) spanning point set of \( \text{PG}(N,K) \)
- \( \mathcal{E} = \) family of planes \( \xi \) in \( \text{PG}(N,K) \) s.th. \( \xi \cap X \) is an conic in \( \xi \)

such that the following properties are satisfied

- each two points of \( X \) belong to a plane of \( \mathcal{E} \)
- two planes of \( \mathcal{E} \) intersect in points of \( X \)
- the tangent space of a point of \( X \) is a plane

---

Schillewaert, Van Maldeghem (2013)

For any field \( K \) (and still using ovals):

\( (X, \mathcal{E}) \) is projectively unique and hence equivalent with \( \mathcal{V}(K) \).
EXAMPLE 1

Axiomatic description of the Veronese variety \( \mathcal{Y}(K) \)

\( X \) = spanning point set of \( \text{PG}(N,K) \)
\( \Xi \) = family of planes \( \xi \) in \( \text{PG}(N,K) \) s.th. \( \xi \cap X \) is an conic in \( \xi \)

such that the following properties are satisfied

**MM1**

each two points of \( X \) belong to a plane of \( \Xi \)

**MM2**

two planes of \( \Xi \) intersect in points of \( X \)

**MM3**

the tangent space of a point of \( X \) is a plane

Schillewaert, Van Maldeghem (2013)

For any field \( K \) (and still using ovals):

\( (X, \Xi) \) is projectively unique and hence equivalent with \( \mathcal{Y}(K) \).
To each quadratic alternative algebra we associate a Veronese variety.
Let $K$ be a field, $\text{char}(K) \neq 2$ (for simplicity).

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$x(xy) = (xx)y$ and $(xy)y = x(yy)$ for all $x, y$ in $A$. 

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Each elt. of $A$ is the root of quadratic equation over $K$.

∀$x \in A \setminus K$ there are (unique) $t_A(x)$, $n_A(x) \in K$ such that

$$x^2 - t_A(x) \cdot x + n_A(x) = 0$$
Let $K$ be a field, $\text{char}(K) \neq 2$ (for simplicity).

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Each elt. of $A$ is the root of quadratic equation over $K$.

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Denote the other root by $\mathbf{x}$ and put $\mathbf{x} = x$ for $x \in K$. 

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Let $K$ be a field, $\text{char}(K) \neq 2$ (for simplicity).
To each **quadratic** alternative algebra we associate a **Veronese variety**.

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∀$x \in A \setminus K$ there are (unique) $t_A(x), n_A(x) \in K$ such that

$$x^2 - t_A(x)x + n_A(x) = 0$$

Denote the other root by $x$ and put $x = x$ for $x \in K$.

Then $x \mapsto x$ is an **involution**, inducing a **norm function** on $A$:

$$n_A : A \to K : x \mapsto xx$$
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If $n_A(x) = 0$ implies $x = 0$  
$A$ is a **division algebra**
Let $K$ be a field, $\text{char}(K) \neq 2$ (for simplicity).

To each \textbf{quadratic} alternative algebra we associate a \textbf{Veronese variety}.

Each elt. of $A$ is the \textbf{root of quadratic equation over} $K$.

\[ \forall x \in A \setminus K \text{ there are (unique) } t_A(x), n_A(x) \in K \text{ such that } x^2 - t_A(x) x + n_A(x) = 0 \]

Denote the other root by $x$ and put $x = x$ for $x \in K$.

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\textbf{Property}: $x$ is \textbf{invertible} iff $n_A(x) \neq 0$;

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If $n_A(x) = 0$ implies $x = 0$ then $A$ is a \textbf{division algebra}.

If not $A$ is a \textbf{split algebra}.
Let $K$ be a field, $\text{char}(K) \neq 2$ (for simplicity).

To each quadratic alternative algebra we associate a Veronese variety.

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If $n_A(x) = 0$ implies $x = 0$ A is a division algebra

If not A is a split algebra

Example split

$2x2$ matrices over $K$

$n(M) = \det(M)$

$M = \text{Adj}(M)$
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$\dim(A/K) \in \{1,2,4,8\}$ and $A$ is one of the following:
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Let $K$ be a field, $\text{char}(K) \neq 2$ (for simplicity). To each quadratic alternative algebra we associate a Veronese variety.

**Remark:** If $K$ is a finite field, there is no division quaternion algebra nor an octonion algebra over it.

### Other Veronese Varieties

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Example 1. The Veronese variety $\mathcal{V}(K)$ - $K$ a field

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Example 2. The Veronese variety $\mathcal{V}(R)$ - $R = K[\sqrt{a}]$
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$\Rightarrow$ rewrite $\rho$, using that $xx = x^2 = n(x)$ for $x \in K$
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**ASSOCIATED VERONESE VARIETIES**

**Example 1.** The Veronese variety $\mathcal{V}(K) - K$ a field

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**Example 2.** The Veronese variety $\mathcal{V}(\mathbb{R}) - \mathbb{R} = K[\sqrt{a}]$

$\rho: PG(2,\mathbb{R}) \rightarrow PG(8,\mathbb{K}): (x,y,z) \mapsto (xx, yy, zz; yz, zx, xy)$
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Example 2. The Veronese variety $\mathcal{V}(R)$ - $R = K[\sqrt{a}]$

$\rho: \text{PG}(2,R) \rightarrow \text{PG}(8,K): (x,y,z) \mapsto (xx, yy, zz; yz, zx, xy)$

Warning: if $R = L'$ (split), then there is no projective plane over it.
ASSOCIATED VERONESE VARIETIES

Example 1. The Veronese variety $\mathcal{V}(K)$ - $K$ a field

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$$\rho: \text{PG}(2,R) \rightarrow \text{PG}(8,K): (x,y,z) \mapsto (xx, yy, zz; yz, zx, xy)$$

Warning: if $R = L'$ (split), then there is no projective plane over it.

$\Rightarrow$ take a ring geometry $G(2,R)$ as follows

- points: $\{(x,y,z)R^* \mid x, y, z \in R \land (x,y,z)r = 0 \text{ for } r \in R \text{ implies } r = 0\}$
- lines: $\{R^*[a,b,c] \mid a, b, c \in R \land r[a,b,c] = 0 \text{ for } r \in R \text{ implies } r = 0\}$
- incidence: $ax + by + cz = 0$

If $R = L$ (division algebra), then $G(2,L) = \text{PG}(2,L)$
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Example 2. The Veronese variety $\mathcal{V}(R)$ - $R = K[\sqrt{a}]$

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Warning: if $R = L'$ (split), then there is no projective plane over it.

$\Rightarrow$ take a ring geometry $G(2,R)$ as follows

- points: $\{(x,y,z)R^* \mid x, y, z \in R \& (x,y,z)r = 0 \text{ for } r \in R \text{ implies } r = 0\}$
- lines: $\{R^*[a,b,c] \mid a, b, c \in R \& r[a,b,c] = 0 \text{ for } r \in R \text{ implies } r = 0\}$
- incidence: $ax + by + cz = 0$

If $R = L$ (division algebra), then $G(2,L) = \text{PG}(2,L)$
ASSOCIATED VERONESE VARIETIES

Example 1. The Veronese variety $\mathcal{V}(K)$ - $K$ a field

| $\rho$: $\text{PG}(2,K) \rightarrow \text{PG}(5,K)$: $(x,y,z) \mapsto (xx, yy, zz; yz, zx, xy)$ |

$(0,y,z) \mapsto (0, yy, zz; yz, 0, 0)$ satisfies $X_1X_2 = X_3X_3 = n(X_3) = X_3^2$

Example 2. The Veronese variety $\mathcal{V}(R)$ - $R = K[\sqrt{a}]$

| $\rho$: $\text{G}(2,R) \rightarrow \text{PG}(8,K)$: $(x,y,z) \mapsto (xx, yy, zz; yz, zx, xy)$ |

$x_0 \ x_1 \ x_2 \ (x_3, x_4) \ (x_5, x_6) \ (x_7, x_8)$
ASSOCIATED VERONESE VARIETIES

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$(0,y,z) \mapsto (0, yy, zz; yz, 0, 0)$ satisfies $X_1X_2 = n(X_3, X_4) = X_3^2 - aX_4^2$
ASSOCIATED VERONESE VARIETIES

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- If $a \not\in K^2$ (i.e., $R$ division), we obtain an elliptic quadric in $\text{PG}(3,K)$
- If $a \in K^2$ (i.e., $R$ split), we obtain a hyperbolic quadric in $\text{PG}(3,K)$
ASSOCIATED VERONESE VARIETIES

Example 1. The Veronese variety $\mathcal{V}(K)$ - $K$ a field

$$\rho: PG(2,K) \rightarrow PG(5,K): (x,y,z) \mapsto (xx, yy, zz; yz, zx, xy)$$

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Axiomatic description of the Veronese variety $\mathcal{V}(R)$

$X$ = spanning point set of $PG(N,K)$

$\Xi$ = family of 3-spaces $\xi$ in $PG(N,K)$ s.th. $\xi \cap X$ is an ell./hyp. quadric in $\xi$

Each two points of $X$ belong to a [3] of $\Xi$

Two [3]s of $\Xi$ intersect in points of $X$

The tangent space of a point is contained in a [4]
ASSOCIATED VERONESE VARIETIES

What do we obtain for the other **quadratic alternative algebras** over K?

<table>
<thead>
<tr>
<th>dim</th>
<th>division</th>
<th>split</th>
<th>peculiarities</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>commutative field $K$</td>
<td></td>
<td>$x = x$, $n(x) = x^2$</td>
</tr>
<tr>
<td>2</td>
<td>L’ quadr. extension $K[\sqrt{a}]$ with $a \not\in K^2$</td>
<td>L’ quadr. extension $K[\sqrt{a}]$ with $a \in K^2$</td>
<td>commutative ring</td>
</tr>
<tr>
<td>4</td>
<td>H quaternion division algebra over $K$</td>
<td>H’ split quaternions over $K$ (2x2 matrices)</td>
<td>associative ring</td>
</tr>
<tr>
<td>8</td>
<td>O octonion division algebra over $K$</td>
<td>O’ split octonions over $K$</td>
<td>strictly alternative ring</td>
</tr>
</tbody>
</table>
ASSOCIATED VERONESE VARIETIES

What do we obtain for the other **quadratic alternative algebras** over $\mathbb{K}$? The **quadrics** are the following.

<table>
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<tr>
<th>dim</th>
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<tbody>
<tr>
<td>1</td>
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<td>5</td>
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<tr>
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<td>PG(2, H)</td>
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These Veronese varieties are **MM-sets** with quadrics in PG($d-1,K$) of *min/max* Witt index.
ASSOCIATED VERONESE VARIETIES

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These Veronese varieties are MM-sets with quadrics in $PG(d-1,K)$ of $\text{min/max}$ Witt index. Moreover, these are the only ones (up to a projectivity)
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What do we obtain for the other *quadratic alternative algebras* over $K$?

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These *Veronese varieties* are *MM-sets* with *quadrics* in $\text{PG}(d-1,K)$ of *min/max* Witt index.

Moreover, these are *the only ones* (up to a projectivity)

(unless if $N<3d-1$, then there also are 3 subvarieties when $A$ is split).
These Veronese varieties are MM-sets with quadrics in PG(d-1,K) of min/max Witt index. Moreover, these are the only ones (up to a projectivity) (unless if N<3d-1, then there also are 3 subvarieties when A is split).

(Krauss), Schillewaert, Van Maldeghem (2015)
GENERALISATIONS
OTHER SUCH SETS?

**Axiomatic description.**

\[ X = \text{spanning point set of } \text{PG}(N,K) \]
\[ \Xi = \text{family of } d\text{-spaces } \xi \text{ in } \text{PG}(N,K) \]
\[ \text{s.th. } \xi \cap X \text{ is a ???} \]

- **MM1**: each two points of \( X \) belong to a \([d]\) of \( \Xi \)
- **MM2**: two \([d]\)s of \( \Xi \) intersect in points of \( X \)
- **MM3**: the tangent space of a point of \( X \) is contained in a \([2(d-1)]\)
Axiomatic description.

\( X = \text{spanning point set of } \text{PG}(N,K) \)

\( \Xi = \text{family of d-spaces } \xi \text{ in } \text{PG}(N,K) \)

such that \( \xi \cap X \) is a 

Quadrics of a Witt index which is not minimal nor maximal?

Singular quadrics: cones over quadrics of min/max Witt index
Axiomatic description.

\( X = \) spanning point set of \( \text{PG}(N,K) \)
\( \Xi = \) family of \( d \)-spaces \( \xi \) in \( \text{PG}(N,K) \)
\( \text{s.th. } \xi \cap X \text{ is a ???} \)

**MM1**

each two *points* of \( X \)
belong to a \([d]\) of \( \Xi \)

**MM2**

two \([d]\)s of \( \Xi \)
intersect in *points* of \( X \)

**MM3**

the tangent space of a *point* of \( X \)
is contained in a \([2(d-1)]\)

Quadrics of a Witt index
which is not minimal nor maximal?

**Probably not!**
(conj.)

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Quadrics of a Witt index which is not minimal nor maximal?

**Probably not!**
(conj.)

Singular quadrics:
cones over quadrics of min/max Witt index

**Yes!**
SINGULAR QUADRRICS AS SYMPS

Axiomatic description.

\[ X = \text{spanning point set of } \text{PG}(N,K) \]
\[ \Xi = \text{family of } d'-\text{spaces } \xi \text{ in } \text{PG}(N,K) \]
\[ d' = d + v + 1 \]
\[ \text{s.th. } \xi \cap X \text{ is a } (d,v)-\text{tube} \]

\[ \text{vertex of dim } v \]

\[ \text{ovoid in } \text{PG}(d,K) \]
SINGULAR QUADRRICS AS SYMPS

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\[ Y = \text{set of vertices} \]

vertex of dim \( v \) (excluded!)

ovoid in \( \text{PG}(d,K) \)
**SINGULAR QUADRICS AS SYMPS**

**Axiomatic description.**

- \( X = \) spanning point set of \( \text{PG}(N,K) \)
- \( \Xi = \) family of \( d' \)-spaces \( \xi \) in \( \text{PG}(N,K) \)
- \( d' = d + v + 1 \) s.th. \( \xi \cap X \) is a \((d,v)\)-tube
- \( Y = \) set of vertices

- vertex of dim v (excluded!)
- ovoid in \( \text{PG}(d,K) \)

Each two points of \( X \) belong to a \([d']\) of \( \Xi \)

Two \([d']s\) of \( \Xi \) intersect in points of \( X \cup Y \)

The tangent space of a point of \( X \) is contained in a \([2(d'-1)]\)
X = spanning point set of PG(N,K)
Ξ = family of $d'$-spaces $\xi$ in PG(N,K)
\[ d' = d + v + 1 \] s.th. $\xi \cap X$ is a $(d,v)$-tube
Y = set of vertices
vertex of dim $v$
(excluded!)
ovoid in PG(d,K)
each two points of $X$
belong to a $[d']$ of $\Xi$
two $[d']$s of $\Xi$
intersect in points of $X \cup Y$
the tangent space of a point of $X$ is contained in a $[2(d' - 1)]$

Warning: You might want to avoid that two quads intersect in points of $Y$ only
SINGULAR QUADRRICS AS SYMPS

Axiomatic description.

\[ X = \text{spanning point set of } \text{PG}(N,K) \]
\[ \Xi = \text{family of } d'\text{-spaces } \xi \text{ in } \text{PG}(N,K) \]
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MM1

MM2'

MM3

Each two points of \( X \) belong to a \([d']\) of \( \Xi \)

Two \([d']\)s of \( \Xi \) intersect in points of \( X \cup Y \)

The tangent space of a point of \( X \) is contained in a \([2(d' - 1)]\)

Warning: You might want to avoid that two quads intersect in points of \( Y \) only
SINGULAR QUADRRICS AS SYMPS

Axiomatic description.

\( X = \) spanning point set of PG(\( N,K \))
\( \mathfrak{E} = \) family of \( d' \)-spaces \( \xi \) in PG(\( N,K \))
\( d' = d + v + 1 \)
\( \text{s.th. } \xi \cap X \text{ is a } (d,v)\text{-tube} \)
\( Y = \) set of vertices

The pair (\( X, \mathfrak{E} \)) together with MM1, MM2' and MM3
is called a singular MM-set.

- Each two points of \( X \) belong to a \([d']\) of \( \mathfrak{E} \)
- Two \([d']\)s of \( \mathfrak{E} \) intersect in points of \( X \cup Y \) but never in \( Y \) only
- The tangent space of a point of \( X \) is contained in a \([2(d'-1)]\)
CASE 1: ONE POINT AS A VERTEX

For any field K, let $(X, \Xi)$ be a singular MM-set with $(d,0)$-tubes.
CASE 1: ONE POINT AS A VERTEX

For any field $K$, let $(X, \Xi)$ be a singular MM-set with $(d,0)$-tubes.

$d=2$

Schillewaert, Van Maldeghem (2015)

(1) $(X, \Xi)$ is projectively equivalent to a point-cone over $\mathcal{Y}(K)$
CASE 1: ONE POINT AS A VERTEX

For any field $K$, let $(X, \Xi)$ be a singular MM-set with $(d,0)$-tubes.

Schillewaert, Van Maldeghem (2015)

1. $(X, \Xi)$ is projectively equivalent to a point-cone over $\mathcal{V}(K)$
2. $(X, \Xi)$ is projectively equivalent to a Hjelmslevian projective plane.
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For any field $K$, let $(X, \Xi)$ be a singular MM-set with $(d,0)$-tubes.

**d=2**

Schillewaert, Van Maldeghem (2015)

1. $(X, \Xi)$ is projectively equivalent to a point-cone over $\mathcal{V}(K)$
2. $(X, \Xi)$ is projectively equivalent to a Hjelmslevian projective plane.

**d>2**

ADS, Van Maldeghem (2017)

$(X, \Xi)$ is projectively equivalent to a cone over $\mathcal{V}(A)$, $A=K, L, H$ or $O$.

There are no non-trivial cases.
(X, ≡) is projectively equivalent to a Hjelmslevian projective plane.
\((X, \equiv)\) is projectively equivalent to a Hjelmslevian projective plane.

The vertices form a projective plane over \(K\).
(\(X, \equiv\)) is projectively equivalent to a Hjelmslevian projective plane.

The vertices form a projective plane over \(K\).

In a complementary subspace, the points of \(X\) form the Veronese variety \(\mathcal{V}(K)\).
HJELMSLEVIAN PROJECTIVE PLANES

\((X, \equiv)\) is projectively equivalent to a Hjelmslevian projective plane.

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In a complementary subspace, the points of $X$ form the Veronese variety $\mathcal{V}(K)$. 

HJELMSLEVIAN PROJECTIVE PLANES

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The mapping $\chi$ is a linear duality between $\mathcal{V}(K)$ and $\text{PG}(2,K)$. 
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(X, ≡) is projectively equivalent to a Hjelmslevian projective plane.

The vertices form a projective plane over K.

In a complementary subspace, the points of X form the Veronese variety \( \mathcal{V}(K) \).

The mapping \( \chi \) is a linear duality between \( \mathcal{V}(K) \) and \( PG(2,K) \).

The affine planes obtained by joining \( x \) and \( \chi(x) \), (x of \( \mathcal{V}(K) \)) give us X.
SIMILAR CONSTRUCTIONS

<table>
<thead>
<tr>
<th></th>
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<th>dim(im((r)))</th>
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\(\mathcal{V}(K)\)

\(\mathcal{V}(L)\)

\(\mathcal{V}(H)\)
### Similar Constructions

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**Quadrics:**
- **d-1:** Division of minimal Witt index in PG(d,K)
- **5:** Split of maximal Witt index in PG(d,K)
- **8:** Singular cones over quadrics of minimal Witt index
- **14:** Quadrics of maximal Witt index
- **26:** Quadrics of minimal Witt index

**Diagrams:**
- 8 (K)
- 14 (L)
- 26 (H)
The vertices form a projective 5-space over K.

In a complementary subspace, the points of X form the Veronese variety $\mathcal{V}(L)$.

The mapping $\chi$ is a linear duality between $\mathcal{V}(L)$ and the top structure.
SIMILAR CONSTRUCTIONS

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In a complementary subspace, the points of $X$ form the Veronese variety $\mathcal{V}(L)$.

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$\mathcal{V}(L) \cong \text{PG}(2,L) \Rightarrow$ represent PG(2,L) as a regular line spread in PG(5,K)
The \textbf{vertices} form a projective 5-space over $K$.

In a complementary subspace, the \textbf{points of $X$} form the Veronese variety $\mathcal{V}(L)$.

The mapping $\chi$ is a \textbf{linear duality} between $\mathcal{V}(L)$ and the \textbf{top structure}.

$\mathcal{V}(L) \approx \text{PG}(2,L) \implies$ represent $\text{PG}(2,L)$ as a \textbf{regular line spread} in $\text{PG}(5,K)$

points in $\text{PG}(2,K) = 1$-spaces in $V(3,K) \implies 2$-spaces in $V(6,K) = \text{lines in PG}(5,K)$
The **vertices** form a projective 5-space over K.

In a complementary subspace, the **points of X** form the Veronese variety $\mathcal{U}(L)$.

The mapping $\chi$ is a **linear duality** between $\mathcal{U}(L)$ and the **top structure**.

\[ \mathcal{U}(L) \approx \text{PG}(2,L) \]

**$\mathcal{U}(L)$** $\approx$ **PG(2,L)** $\rightarrow$ represent PG(2,L) as a **regular line spread** in PG(5,K)

points in PG(2,K) = 1-spaces in V(3,K) $\rightarrow$ 2-spaces in V(6,K) = lines in PG(5,K)
SIMILAR CONSTRUCTIONS

\( \mathcal{Y}(H) \approx PG(2,H) \)

The vertices form a projective 11-space over \( K \).

In a complementary subspace, the points of \( X \) form the Veronese variety \( \mathcal{V}(H) \).

The mapping \( \chi \) is a linear duality between \( \mathcal{V}(H) \) and the top structure.

\( \mathcal{V}(H) \approx PG(2,H) \Rightarrow \text{represent } PG(2,H) \text{ as a regular [3] spread in } PG(11,K) \)

points in \( PG(2,K) = 1\)-spaces in \( V(3,K) \Rightarrow 4\)-spaces in \( V(12,K) = 3\)-spaces in \( PG(11,K) \)
**SIMILAR CONSTRUCTIONS**

\[ v = d - 2 \]

\[ 50? \]

\[ \mathcal{V}(O) \approx PG(2, O) \]

- **PG(23, K)**
  - its vertex (a [7])
  - vertices of the conics through it (regular [7] spread in 15-space)

\[ \chi \]

- **\mathcal{V}(O)**
  - quadric
  - point

The **vertices** form a projective 23-space over K.

In a complementary subspace, the **points of X** form a the Veronese variety \( \mathcal{V}(O) \).

The mapping \( \chi \) is a linear duality between \( \mathcal{V}(O) \) and the top structure.

\( \mathcal{V}(O) \approx PG(2, O) \Rightarrow \) represent PG(2, O) as a regular [7] spread in PG(23, K)

points in PG(2, K) = 1-spaces in V(3, K) \( \Rightarrow \) 8-spaces in V(24, K) = 7-spaces in PG(23, K)
The vertices form a projective 23-space over $K$.

In a complementary subspace, the points of $X$ form a the Veronese variety $\mathcal{V}(O)$.

The mapping $\chi$ is a linear duality between $\mathcal{V}(O)$ and the top structure.

\[
\mathcal{V}(O) \cong PG(2,O) \Rightarrow \text{represent } PG(2,O) \text{ as a regular [7] spread in } PG(23,K)
\]

Warning: The top plane is Desarguesian, whereas $PG(2,O)$ is not. This does not work!
CASE 2: A HIGHER-DIMENSIONAL VERTEX

For any field $K$, let $(X, \exists)$ be a singular MM*-set with $(d, v)$-tubes, $v>0$. 
CASE 2: A HIGHER-DIMENSIONAL VERTEX

For any field $K$, let $(X, \Xi)$ be a singular MM*-set with $(d,v)$-tubes, $v>0$.

MM2' is replaced by MM2*: 

...
CASE 2: A HIGHER-DIMENSIONAL VERTEX

For any field K, let \((X, \Xi)\) be a \textit{singular MM\(^*\)-set} with \((d,v)\)-tubes, \(v>0\).

\textbf{MM2}\(^'\) is replaced by \textbf{MM2\(^*\)}:

- Two \([d']\)s of \(\Xi\) intersect in \textbf{points} of \(X\).
- And always contain a point of \(X\).
For any field $K$, let $(X, \Xi)$ be a singular MM*-set with $(d,v)$-tubes, $v>0$. 

\[ (1) \quad (X, \Xi) \text{ is projectively equivalent to one of the following} \]

**CASE 2: A HIGHER-DIMENSIONAL VERTEX**

**ADS, Van Maldeghem (2017)**
CASE 2: A HIGHER-DIMENSIONAL VERTEX

For any field $K$, let $(X, \Xi)$ be a singular $\text{MM}^*$-set with $(d,v)$-tubes, $v > 0$.

- $v > 0$
- $d > 1$

**ADS, Van Maldeghem (2017)**

1. $(X, \Xi)$ is projectively equivalent to one of the following
2. $(X, \Xi)$ is trivial (a cone over $\mathcal{V}(A)$ or over $\mathcal{HV}(K), \mathcal{HV}(L), \mathcal{HV}(H)$)
Thanks for your attention!