Projective planes **with** polar spaces

Joint work with:
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Projective planes with polar spaces
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Hyperbolic quadric

Symp = convex closure of two non-collinear points
INTRODUCTORY EXAMPLE

Projective planes with polar spaces

Hyperbolic quadric

Symp = convex closure of two non-collinear points
Projective planes with polar spaces

(Point, sym) is a projective plane.
Through each two points there is a unique sym.

(Point, symp) is a projective plane.

Hyperbolic quadric

Symp = convex closure of two non-collinear points
(Points, symps) is a projective plane.

Through each two points there is a unique symp.

Each two symps meet in at least one point.
Projective planes with polar spaces

Through each two points there is a unique symp.

Each two symps meet in at least one point.

There is a frame (four points, no three on a symp).
Through each two **points** there is a unique **symp**.

Each two **symps** meet in at least one **point**.

Not all **points** are on one **symp**.

**Hyperbolic quadric**

Symp = convex closure of two non-collinear points

**(Points, symps)** is a projective plane.
Projective planes with polar spaces

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Direct product of any projective plane $\pi$ and any projective line $L$ (Segre variety $S_{1,2}(K)$)

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INTRODUCTORY EXAMPLE

Projective planes with polar spaces

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Direct product of any projective n-space $\pi$ and any projective line $L$ (Segre variety $S_{1,n}(K)$)

$\pi$-space

$n$-space

Hyberbolic quadric

Symp = convex closure of two non-collinear points

(Point, symps) is a projective plane.
Projective planes with polar spaces

**Through each two points there is a unique symp.**

**Each two symps meet in at least one point.**

**Not all points are on one symp.**

This is not true if $n > 2$!

Direct product of any projective $n$-space $\pi$ and any projective line $L$ (Segre variety $S_{1,n}(K)$)

(Points, symps) is a projective plane.

Hyperbolic quadric

$\text{Symp} =$ convex closure of two non-collinear points
**INTRODUCTORY EXAMPLE**

**Projective planes with polar spaces**

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Direct product of any projective plane $\pi$ and any projective plane $\pi'$ (Segre variety $S_{2,2}(K)$)

$\textbf{Points, symps}$ is a projective plane.

$\textbf{Symp} = \text{convex closure of two non-collinear points}$

**Hyperbolic quadric**
THE QUESTION

Which point-symp-geometries are such that:

1. Through each two points there is a unique symp.
2. Each two symps meet in at least one point.
3. Not all points are on one symp.

Symps are convex in point-line geometry.
THE QUESTION

Which \textbf{point-symp}-geometries are such that:

- \textit{Symps} are convex in \textbf{point-line} geometry
- Through each two \textit{points} there is a unique \textit{symp}.
- Each two \textit{symps} meet in at least one \textit{point}.
- Not all \textit{points} are on one \textit{symp}.

\textbf{symps} can be of any kind, and need not be isomorphic or of same rank.
Which **point-symp**-geometries are such that:

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If there is a symp of rank 2:

- only the **direct products** $L \times \pi$ and $\pi \times \pi'$

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Both examples share that some lines are contained in a unique symp. This we want to avoid if there are no symps of rank 2.
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IF all symps have rank >2, there is a line contained in at least two symps.
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Only the line Grassmannian $A_{n,2}(L)$ of a projective n-space with $n \in \{4,5\}$ and the exceptional $E_{6,1}(K)$-variety.
EXAMPLE: THE LINE GRASSMANNIANS

Consider the line Grassmannian of a projective n-space. We verify the axioms:

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(new) **lines**: planar point pencils

**distance 1**
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### (new) **lines**:
- Planar point pencils
  - Distance 1

### **Symp**:
- Convex closure of $L_1, L_2$
  - All **lines** in $<L_1, L_2>$
  - Klein quadric $Q^+(5, K)$
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- $n=2$: projective plane (no symps)
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(n) lines: planar point pencils

- distance 1

- distance 2

$\text{n}=2$: projective plane (no symps)
$\text{n}=3$: Klein quadric (one symp)

Symp: convex closure of $L_1$, $L_2$

$\rightarrow$ all lines in $\langle L_1, L_2 \rangle$

$\rightarrow$ Klein quadric $Q^+(5,K)$
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**n=2**: projective plane (no **symps**)

**n=3**: Klein quadric (one **symp**)

**3 < n < 6**: each two **3-spaces** share a **line**

**一大批**: each two **symps** share a **point**

(new) **lines**: planar point pencils

- distance 1
- distance 2

**Symp**: convex closure of $L_1$, $L_2$

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EXAMPLE: THE LINE GRASSMANNIANS

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\( n > 2 \)
\( n < 6 \)
\( n > 3 \)

---

- **Distance 1**: (new) **lines**:
  - planar point pencils

- **Distance 2**: **Symp**: convex closure of \( L_1, L_2 \)
  - \( \Rightarrow \) all **lines** in \( <L_1, L_2> \)
  - \( \Rightarrow \) Klein quadric \( Q^+(5,K) \)

\( n=2 \): projective plane (no **symps**)
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\( 3 < n < 6 \): each two **3-spaces** share a **line**
  - \( \Rightarrow \) each two **symps** share a **point**
Consider a **variety** consisting of the following types of **objects**:

- **points**
- **lines**
- **planes**
- **4-spaces**
- **5-spaces**
- **symps $Q^+(9,K)$**
- 4'-spaces ($5 \cap Q$),
- 3-spaces ($4 \cap 4'$)
Consider a variety consisting of the following types of objects:

Then this is the exceptional $E_{6,1}$-variety over $K$ if:

4'-spaces ($5 \cap Q$),
3-spaces ($4 \cap 4'$)

and
THE EXCEPTIONAL E$_{6,1}$-VARIETY

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- **points**
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- **symps** $Q^+(9,K)$

4'-spaces ($5 \cap Q$),

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Then this is the **exceptional E$_{6,1}$-variety** over $K$ if:

- Two distinct points are on a **line** or their convex closure is a **symp**. *(both occur)*
- Two distinct **symps** meet in a **point** or a 4-dim space. *(both occur)*
THE EXCEPTIONAL $E_{6,1}$-VARIETY

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We verify the **axioms**:

**Symps** are convex in **point-line** geometry
Consider a variety consisting of the following types of objects:

- Points
- Lines
- Planes
- $4$-spaces
- Symps $Q^+(9,K)$
- $5$-spaces
- $4'$-spaces ($5 \cap Q$)
- $3$-spaces ($4 \cap 4'$)

Then this is the exceptional $E_{6,1}$-variety over $K$ if:

- Two distinct points are on a line or their convex closure is a symp. (both occur)
- Two distinct symps meet in a point or a $4$-dim space. (both occur)

We verify the axioms:

Symps are convex in point-line geometry
THE EXCEPTIONAL $E_{6,1}$-VARIETY

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- **points**
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- **4-spaces**
- **5-spaces**
- **symps** $Q^+(9, K)$
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Then this is the **exceptional $E_{6,1}$-variety** over $K$ if:

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We verify the **axioms**:

- **Symps** are convex in **point-line** geometry
  
  - Through each two **points** there is a unique **symp**.
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**Exercise**: show that each line is contained in a symp.
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THE EXCEPTIONAL $E_{6,1}$-VARIETY

Consider a **variety** consisting of the following types of objects:

![Diagram of objects: points, lines, planes, symps, Q⁺(9,K), 4'-spaces, 3-spaces]

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THE EXCEPTIONAL E\textsubscript{6,1}-VARIETY

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![Diagram](image.png)

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4'-spaces (5 \cap Q), 3-spaces (4 \cap 4')
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Which **point-symp**-geometries are such that:

- Through each two **points** there is a unique **symp**.
- Each two **symps** meet in at least one **point**.
- **IF** all **symps** have rank >2, there is a **line** contained in at least two **symps**.

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PARAPOLAR SPACES

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These geometries are **parapolar spaces**: connected **point-line** geometries such that:
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- For each non-incident point-line pair: 0, 1 or all (and 0 occurs)

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- For points at distance 2: either there is a unique path (i) or the convex closure is a symp (ii)

Or

(i) 1 path
(ii) $>1$ path
Which \textbf{point-symp}-geometries are such that:

- Through each two \textbf{points} there is a unique \textbf{symp}.
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- Each \textbf{line} is contained in a \textbf{symp}
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satisfying the following **additional requirements**:
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strong: always option (ii) and diameter 2

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Satisfying the following additional requirements:

- **Strong**: always option (ii) and **diameter 2**
- **IF** all symps have rank >2, there is a line contained in at least two symps.
The only **parapolar spaces** which satisfy the **additional properties**

**strong**: always option (ii) and **diameter 2**

Each two **symps** meet in at least one **point**.

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are

- the **direct products** of a projective **line/plane** with a projective **plane**
- the **line Grassmannian** of a projective 4- or 5-space over a skew field L
- the **exceptional E$_{6,1}$-variety** over a field K
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We can do better!
The only parapolar spaces which satisfy the additional properties

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are

1. the **direct products** of a projective **line/plane** with a projective **plane**
2. the **line Grassmannian** of a projective 4- or 5-space over a skew field \( \mathbb{L} \)
3. the **exceptional** **E\(_{6,1}\)-variety** over a field \( \mathbb{K} \)

**We can do better!**

**The Next Step**
The only **parapolar spaces** which satisfy the **additional properties**

- **IF** there is a **symp** of rank 2 **strong**: always option (ii) and diameter 2
- **IF** all **symps** have rank >2, there is a **line** contained in at least two **symps**.
- Each two **symps** meet in at least one **point**.

The only parapolar spaces which satisfy the additional properties are

- the **direct products** of a projective **line/plane** with a projective **plane**
- the **line Grassmannian** of a projective 4- or 5-space over a skew field L
- the **exceptional E_{6,1}-variety** over a field K

**THE NEXT STEP**

We can generalise this result!
The only parapolar spaces which satisfy the additional properties are

- **IF** there is a *symp* of rank 2
  - **strong**: always option (ii) and diameter 2
- **IF** all *symps* have rank >2, there is a line contained in at least two *symps.*
- No two *symps* share exactly a (-1)-dimensional subspace.

are

- the direct products of a projective line/plane with a projective plane
- the line Grassmannian of a projective 4- or 5-space over a skew field $L$
- the exceptional $E_{6,1}$-variety over a field $K$

We can generalise this result!
The only **parapolar spaces** which satisfy the **additional properties**

- **IF** there is a **symp** of rank 2
  - strong: always option (ii)
  - and diameter 2
- No two **symps** share exactly a **k-dimensional** subspace.
- **IF** all **symps** have rank >2,
  - there is a **line** contained in at least two **symps**.

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- the **direct products** of a projective **line/plane** with a projective **plane**
- the **line Grassmannian** of a projective 4- or 5-space over a skew field **L**
- the **exceptional** **E**\(_{6,1}\)-**variety** over a field **K**

We can generalise this result!

Take **any** integer \( k \geq 0 \)
The only **parapolar spaces** which satisfy the **additional properties**

- Each symp has rank at least \( k+3 \)
- No two symps share exactly a \( k \)-dimensional subspace.
- If all symps have rank >2, there is a line contained in at least two symps.

The only parapolar spaces which satisfy the additional properties

- The direct products of a projective line/plane with a projective plane
- The line Grassmannian of a projective 4- or 5-space over a skew field \( L \)
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---

**THE NEXT STEP**

We can generalise this result!

Take any integer \( k \geq 0 \)
The only **parapolar spaces** which satisfy the **additional properties**

- Each *symp* has rank at least \( k+3 \)
- No two *symps* share exactly a *k*-dimensional subspace.
- **locally connected**

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<table>
<thead>
<tr>
<th>$k=-1$</th>
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<th>3</th>
<th>4</th>
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<tbody>
<tr>
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The point-residue is a locally connected strong parapolar space, in which no two symps share a $(k-1)$-space.
The only **parapolar spaces** which satisfy

- Each symp has rank at least $k+3$
- No two symps share exactly a $k$-dimensional subspace.
- locally connected

are

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The **point-residue** is a locally connected strong parapolar space, in which no two symps share a $(k-1)$-space.

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**THE NEXT STEP**

Freudenthal-Tits magic square
Thank you for your attention!