Abstract

In this paper we study in detail an explicit example of a reversible system in which we can find a degenerate subharmonic bifurcation scenario (the so-called banana scenario) for which the stability behaviour of the solutions is different from that in a generic banana scenario. We explain the non-genericity as a consequence of the existence of a certain type of first integral. We also give some numerical evidence supporting the theory.

Keywords and Phrases: Reversible system, degenerate subharmonic bifurcation, banana scenario, stability, first integral.

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example of a system in which one can find one-dimensional branches of periodic orbits along which there is a change of stability behaviour without any accompanying further branching. Our system is 4-dimensional, reversible, and it has a first integral. The reversibility ensures that (symmetric) periodic orbits typically come in one-parameter families, while the existence of a first integral forms an important ingredient in explaining the stability behaviour that we will detect along certain of these branches of periodic orbits.

Before going into the details of the example let us first explain the key theoretical result that we will use. Suppose that we have a (smooth, finite-dimensional) system which has a one-parameter family \( \{ \Gamma_s \mid s \in \mathbb{R} \} \) of periodic orbits; such families appear for example generically in Hamiltonian systems and in reversible systems. Introducing a Poincaré map \( P : \Sigma \to \Sigma \) on a (flat) section \( \Sigma \) transversal to \( \Gamma_0 \) we obtain a one-dimensional curve \( \{ x(s) \mid s \in \mathbb{R} \} \subset \Sigma \) of fixed points of \( P \):

\[
P(x(s)) = x(s), \quad \forall s \in \mathbb{R}.
\]  

(1.1)

Differentiating at \( s = 0 \) shows that

\[
x'(0) \in \ker(DP(x_0) - \text{Id}_\Sigma), \quad x_0 = x(0) \in \Sigma \cap \Gamma_0.
\]  

(1.2)

Assuming that \( x'(0) \neq 0 \) this shows that 1 is an eigenvalue of the linearization \( DP(x_0) \) of the Poincaré map \( P \) at \( x_0 \), and hence that the periodic orbit \( \Gamma_0 \) has the multiplier 1 with multiplicity at least equal to 2. Typically this multiplicity will be equal to 2, and 1 will be a simple eigenvalue of \( DP(x_0) \), in the sense that

\[
\ker(DP(x_0) - \text{Id}_\Sigma) = \mathbb{R}x'(0) \quad \text{and} \quad \Sigma = \ker(DP(x_0) - \text{Id}_\Sigma) \oplus \text{im}(DP(x_0) - \text{Id}_\Sigma) \quad \text{(1.3)}
\]

Assume next that the system has also a first integral \( H \) which reaches a (local) maximum or minimum along the family \( \{ \Gamma_s \} \), say at \( s = 0 \). Denoting by \( H_\Sigma \) the restriction of \( H \) to \( \Sigma \) we have then that \( H_\Sigma(P(x)) = H_\Sigma(x) \) for all \( x \in \Sigma \), which by differentiation at \( x_0 \) shows that

\[
\text{im}(DP(x_0) - \text{Id}_\Sigma) \subset \nabla H_\Sigma(x_0)^\perp.
\]  

(1.4)

Since the map \( s \mapsto H_\Sigma(x(s)) \) has a critical point at \( s = 0 \) we have also that \( x'(0) \in \nabla H_\Sigma(x_0)^\perp \). Assuming that \( \nabla H_\Sigma(x_0) \neq 0 \) this implies that 1 can not be a simple eigenvalue of \( DP(x_0) \): indeed, if 1 would be a simple eigenvalue then, by a dimension argument, we need to have equality in (1.4), and hence \( x'(0) \in \text{im}(DP(x_0) - \text{Id}_\Sigma) \), which contradicts (1.3).
The conclusion is that when a system has a one-parameter family of periodic orbits along which a first integral of the system reaches a maximum or a minimum, then the critical periodic orbit has multiplier 1 with multiplicity at least equal to 3; in Hamiltonian or reversible systems other arguments then imply that this multiplicity must be at least 4. Typically such situation will entail a change of stability, from elliptic to hyperbolic or vice versa, as one moves along the given family of periodic orbits. However, contrary to what the folk theorem states, there may not be any other branches of periodic orbits bifurcating from the given branch. In the Hamiltonian case, and when the first integral which becomes critical is the Hamiltonian itself, one could argue that by plotting the (amplitude of) periodic orbits against the total energy one obtains a classical turning point with the corresponding change of stability, so there is a kind of bifurcation after all. But as the example below will show a similar behaviour can also appear without such additional conditions, and then such argument does not make much sense.

In the next section we introduce our system, which is 4-dimensional, reversible (but not Hamiltonian), and which has a first integral. We also describe how one could obtain branches of periodic orbits along which the first integral reaches a maximum. In the last section we prove then the existence of such branches, and give numerical evidence that the multipliers along these branches show indeed the expected behaviour.

2 The model system

Consider the system

\[
\begin{aligned}
\ddot{y} + y + y^2 + \gamma y^3 &= z, \\
\ddot{z} + \omega^2 z &= 0,
\end{aligned}
\]

for scalar variables \(y, z \in \mathbb{R}; \gamma \in \mathbb{R}\) and \(\omega > 0\) are parameters which we will adjust later on. This is a 4-dimensional system, with phase-space variable \(x\) which we choose for convenience as \(x = (y, \dot{y}, z, -\omega^{-1}\dot{z})\). The system is reversible, in the sense that if \((y(t), z(t))\) is a solution, then so is \((\tilde{y}(t), \tilde{z}(t)) = (y(-t), z(-t))\). There is also a first integral, namely \(H = H(z, \dot{z}) = z^2 + \omega^{-2} z^2\). The fact that the second equation of (2.1) is linear is not important, but was chosen to simplify the presentation; one can easily obtain similar results when this equation is replaced by \(\ddot{z} + g(z) = 0\), with an appropriate nonlinear \(g(z)\).

The two-dimensional \((y, \dot{y})\)-subspace (obtained by setting \(z = \dot{z} = 0\)) is in-
variant under (2.1), and the flow on this subspace is governed by the equation
\[ \ddot{y} + y + y^2 + \gamma y^3 = 0. \] (2.2)

A classical analysis of the potential function \( V(y) = \frac{1}{2} y^2 + \frac{1}{3} y^3 + \frac{1}{4} \gamma y^4 \) shows that for \( \gamma > 1/4 \) all orbits of (2.2) are periodic, surrounding the equilibrium at the origin. We denote these periodic orbits by \( \Gamma^0_a \), where \( (a, 0) \) (\( a > 0 \)) is the (unique) intersection point of the orbit with the half-axis \( y > 0 \), \( \dot{y} = 0 \). Observe that we cannot apply the argument of Section 1 to the family \( \{ \Gamma^0_a \mid a > 0 \} \) (considered as a family of periodic orbits of (2.1)) since \( H \) takes the constant value 0 along this family.

However, we will show that plenty of branches of subharmonic solutions bifurcate from the family \( \{ \Gamma^0_a \mid a > 0 \} \) (called the primary branch further on); along these subharmonics we have \( H > 0 \). Our strategy is then the following: if we can find positive numbers \( a \neq \tilde{a} \) and a continuous branch of subharmonics \( \{ \Gamma_b \mid \tilde{b} < b < \tilde{b} \} \) such that (in some appropriate sense) \( \Gamma_b \rightarrow \Gamma^0_{\tilde{a}} \) as \( b \downarrow \tilde{b} \) and \( \Gamma_b \rightarrow \Gamma^0_{\tilde{a}} \) as \( b \uparrow \tilde{b} \), then clearly \( H \) must reach a maximum along the branch \( \{ \Gamma_b \mid \tilde{b} < b < \tilde{b} \} \) and, as explained in Section 1, we should see a change of stability along this branch. We will work out this strategy in the remainder of this paper.

The model system (2.1) came around in our research on degenerate subharmonic bifurcations in reversible systems (we will report on this research in some forthcoming papers). It was our hope that this system would provide an illustration of our general theory, but we discovered from numerical calculations that the stability behaviour along the bifurcating branches of subharmonics did not match with the theoretical predictions. The reason for the discrepancy appeared to be the presence of the first integral, something which was not taken into account in the general theory.

To study bifurcations of periodic orbits from the primary branch we will use an appropriate Poincaré map, as follows. Denote the minimal period of \( \Gamma_a \) by \( T_0(a) \); later we will study this period map in more detail. Fix some \( a_0 > 0 \) and consider the section \( \Sigma = \{(a_0 + a, 0, z_1, z_2) \mid a, z_1, z_2 \in \mathbb{R}\} \) transversal to \( \Gamma_{a_0} \) at \( (a_0, 0, 0, 0) \). On \( \Sigma \) we use the coordinates \( (a, z_1, z_2) \in \mathbb{R}^3 \), or when it is convenient, the polar coordinates \( (a, \rho, \varphi) \in \mathbb{R} \times \mathbb{R}_+ \times S^1 \), with \( (z_1, z_2) = (\rho \cos \varphi, \rho \sin \varphi) \). Let \( P : \Sigma \rightarrow \Sigma \) denote the corresponding Poincaré map. To calculate \( P(a, \rho, \varphi) \) we denote by \( \phi(t; a, \rho, \varphi) \) the solution of the initial value problem
\[ \ddot{y} + y + y^2 + \gamma y^3 = \rho \cos(\omega t + \varphi), \quad y(0) = a_0 + a, \quad \dot{y}(0) = 0. \] (2.3)
The equation
\[ \frac{\partial \phi}{\partial t}(t; \alpha, \rho, \varphi) = 0 \] (2.4)
has for all small \((\alpha, \rho)\) a unique solution \(t = T(\alpha, \rho, \varphi)\) near \(T_0(a_0)\); actually, \(T(\alpha, 0, \varphi) = T_0(a_0 + \alpha)\). In polar coordinates the Poincaré map \(P\) then takes the form
\[ P(\alpha, \rho, \varphi) = (\phi(T(\alpha, \rho, \varphi); \alpha, \rho, \varphi) - a_0, \rho, \omega T(\alpha, \rho, \varphi) + \varphi). \] (2.5)

We have (in cartesian coordinates)
\[ P(\alpha, 0, 0) = (\alpha, 0, 0), \quad \forall \alpha \in \mathbb{R}, \] (2.6)
i.e. there is a line of fixed points corresponding to the primary branch. This implies that 1 is an eigenvalue of the linearization \(DP(\alpha, 0, 0)\), with eigenvector \((1, 0, 0)\). This linearization has the matrix form
\[ DP(\alpha, 0, 0) = \begin{pmatrix} 1 & \ast & \ast \\ 0 & \cos(\omega T_0(a_0 + \alpha)) & -\sin(\omega T_0(a_0 + \alpha)) \\ 0 & \sin(\omega T_0(a_0 + \alpha)) & \cos(\omega T_0(a_0 + \alpha)) \end{pmatrix}, \quad \forall \alpha \in \mathbb{R}, \] (2.7)
and therefore has the eigenvalues 1 and \(\exp(\pm \omega T_0(a_0 + \alpha))\).

It follows from the reversibility of our system that (using polar coordinates)
\[ \phi(t; \alpha, \rho, -\varphi) = \phi(-t; \alpha, \rho, \varphi). \] (2.8)

Introducing the reversor \(R : \Sigma \rightarrow \Sigma\) by
\[ R(\alpha, z_1, z_2) = (\alpha, z_1, -z_2) \iff R(\alpha, \rho, \varphi) = (\alpha, \rho, -\varphi), \] (2.9)
(2.8) implies that \(T(RP(\alpha, \rho, \varphi)) = T(\alpha, \rho, \varphi)\) and \(P(RP(\alpha, \rho, \varphi)) = (\alpha, \rho, -\varphi) = R(\alpha, \rho, \varphi)\), i.e.
\[ P \circ R \circ P = R \iff R \circ P \circ R = P^{-1}. \] (2.10)

This means that the Poincaré map \(P\) is a reversible diffeomorphism as studied for example in [1].

In the next section we briefly review the results of [1] and show how they apply to the bifurcation of \(q\)-periodic points of \(P\) from the line of fixed points \(\{(\alpha, 0, 0) \mid \alpha \in \mathbb{R}\}\). We will not consider the cases \(q = 1\) (branching of fixed points) and \(q = 2\) (period-doubling), but restrict ourselves to \(q \geq 3\) (further on even to \(q \geq 5\)). The bifurcation of \(q\)-periodic points for \(P\) then corresponds for the original system (2.1) to the branching of subharmonic solutions from the primary branch of periodic solutions.
3 Branching of subharmonic solutions

We fix some $q \geq 3$ and consider the problem to determine all $q$-periodic points of $P$ in a small neighborhood of the fixed point at the origin $(\alpha, z_1, z_2) = (0, 0, 0)$. A simple application of the implicit function theorem shows that besides the line of fixed points $\{ (\alpha, 0, 0) \}$ no such $q$-periodic points can be found arbitrarily close to the origin unless $DP(0, 0, 0)$ has a pair of complex conjugate eigenvalues which are $q$-th roots of unity. Comparing with (2.7) this means that we must impose the condition

$$\omega T(a_0) = \frac{2k\pi}{q}$$

with $k \geq 1$ an integer such that $\gcd(k, q) = 1$ (we want $q$ to be the minimal period).

The resonance condition (3.1) imposes a restriction on the possible choices of $a_0$.

Assuming (3.1) we will study the bifurcation of $q$-periodic points using a combination of Lyapunov-Schmidt reduction and normal form reduction. Since $S_0 = DP(0, 0, 0)$ has three different simple eigenvalues it is semi-simple; moreover $S_0^q = \text{Id}_\Sigma$ (from (3.1)) and $RS_0R = S_0^{-1}$ (from (2.10)). As explained in [1] it follows then that there is a smooth one-to-one relation between the small $q$-periodic points of the Poincaré map $P$ and the small $q$-periodic points of a reduced map $P_0 : \Sigma \to \Sigma$ which has the following properties:

(i) $P_0(\alpha, 0, 0) = (\alpha, 0, 0)$ for all $\alpha \in \mathbb{R}$;
(ii) $DP_0(0, 0, 0) = S_0$;
(iii) $P_0 \circ S_0 = S_0 \circ P_0$ ($\mathbb{Z}_q$-equivariance of $P_0$);
(iv) $R \circ P_0 \circ R = P^{-1}_0$ (reversibility);
(v) all small $q$-periodic points of $P_0$ are obtained by finding all small solutions $u = (\alpha, z_1, z_2) \in \Sigma$ of the bifurcation equation

$$P_0(u) = S_0u;$$

(vi) the bifurcation equation (3.2) is equivalent to the equation

$$B(u) = S_0^{-1}P_0(u) - S_0P_0^{-1}(u) = 0;$$

observe that $B(S_0u) = S_0B(u)$ and $B(Ru) = -RB(u)$.

Moreover, given any arbitrary large integer $N$, the Poincaré map $P$ can be transformed up to terms of order $N$ into the normal form

$$P_{\text{NF}}(u) = S_0\exp(X_{\text{NF}}(u)),$$
where “exp” means “time-one map” and where the normal form vectorfield $X_{NF}: \Sigma \to \Sigma$ has the following properties:

(vii) $X_{NF}(\alpha, 0, 0) = (0, 0, 0)$ and $DX_{NF}(0, 0, 0) = 0$;
(viii) $X_{NF}(S_0u) = S_0X_{NF}(u)$ for all $u \in \Sigma$;
(ix) $X_{NF}(Ru) = -RX_{NF}(u)$ for all $u \in \Sigma$.

The bifurcation equation (3.2) can then be approximated (again up to terms of order $N$) by

$$P_{NF}(u) = S_0u,$$  \hspace{1cm} (3.5)

which is equivalent to

$$X_{NF}(u) = 0.$$  \hspace{1cm} (3.6)

For each of the foregoing equations the solutions come in $\mathbb{Z}_q$-orbits, meaning that if $u \in \Sigma$ is a solution, then so are $S_0^ju$ for all $j \in \mathbb{Z}$; such $\mathbb{Z}_q$-orbit corresponds to the full orbit (under $P$) of the corresponding $q$-periodic point of $P$. By taking $N$ large enough one can (in most cases) guarantee that solutions of (3.6) (or equivalently (3.5)) persist when (3.5) is replaced by the full bifurcation equation (3.3) (or equivalently (3.2)). Also, the stability of these (approximate) solutions as equilibria of the vectorfield $X_{NF}$ coincides, up to terms of appropriate high order, with the stability of the corresponding $q$-periodic point of $P$. So further on we will concentrate on the equation (3.6), i.e. on the equilibria of $X_{NF}$ and their stability.

In order to apply this to our particular problem we set $\theta_0 = 2k\pi/q$ and choose a new basis in $\Sigma$ with respect to which $S_0$ is “diagonal”:

$$S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_0 & -\sin \theta_0 \\ 0 & \sin \theta_0 & \cos \theta_0 \end{pmatrix}.$$ \hspace{1cm} (3.7)

We still use $u = (\alpha, z_1, z_2)$ as coordinates, but also identify the $(z_1, z_2)$-plane with the complex plane $\mathbb{C}$, via $z = z_1 + iz_2$; this way $S_0$ and $R$ get the form $S_0(\alpha, z) = (\alpha, \exp(i\theta_0)z)$ and $R(\alpha, z) = (\alpha, \bar{z})$. The conditions (vii)-(viii)-(ix) above imply that the vectorfield $X_{NF}$ has the form

$$X_{NF}(\alpha, z) = (g(\alpha, z)\Im(z^q), ih_1(\alpha, z)z + ih_2(\alpha, z)\bar{z}^{q-1}),$$ \hspace{1cm} (3.8)

with $g, h_1, h_2 : \Sigma \to \mathbb{R}$ smooth real-valued functions which are invariant under the $D_q$-action generated on $\Sigma = \mathbb{R} \times \mathbb{C}$ by $S_0$ and $R$: $g(S_0u) = g(Ru) = g(u)$, and similarly for $h_1$ and $h_2$. Clearly $(\alpha, 0)$ is an equilibrium for all $\alpha$, confirming our
earlier requirements. Nontrivial equilibria \((\alpha, z) = (\alpha, \rho \exp(i\varphi)) (\rho \neq 0)\) of \(X_{\text{NF}}\) are given by the solutions of the equations
\[
\begin{align*}
g(\alpha, \rho \exp(i\varphi))\rho^n \sin(q\varphi) &= 0, \\
h_1(\alpha, \rho \exp(i\varphi)) + h_2(\alpha, \rho \exp(i\varphi))\rho^{n-2}\exp(-iq\varphi) &= 0.
\end{align*}
\] (3.9)

If either \(g(0, 0) \neq 0\) or \(h_2(0, 0) \neq 0\) then such non-trivial can only exist near the origin if \(\Im(z^q) = 0\), i.e. if \(\sin(q\varphi) = 0\); taking the \(\mathbb{Z}_q\)-equivariance into account this gives us two possibilities: either \(\varphi = 0 \pmod{\theta_0}\), or \(\varphi = \pi/q \pmod{\theta_0}\). Points \((\alpha, \rho \exp(i\varphi))\) with such values of \(\varphi\) have the property that they generate a \(\mathbb{Z}_q\)-orbit (by application of \(S_0\)) which is invariant under the reversor \(R\); hence, all solutions which we will find for these values of \(\varphi\) will correspond to \(q\)-subharmonics which are also symmetric. Even when both \(g(0, 0)\) and \(h_2(0, 0)\) are zero one can still try to find solutions with \(\varphi = 0 \pmod{\theta_0}\) or \(\varphi = \pi/q \pmod{\theta_0}\); however, in that degenerate case there might also exist other solutions, having other values for \(\varphi\), and hence corresponding to non-symmetric subharmonics.

For \(\varphi = 0 \pmod{\theta_0}\) the set of equations (3.9) reduces to one single (real) equation, namely
\[
h_1(\alpha, \rho) + \rho^{n-2}h_2(\alpha, \rho) = 0,
\] (3.10)
while for \(\varphi = \pi/q \pmod{\theta_0}\) we get in a similar way the single equation
\[
h_1(\alpha, \rho \exp(i\pi/q)) - \rho^{n-2}h_2(\alpha, \rho \exp(i\pi/q)) = 0.
\] (3.11)

Now we see from (3.8) that \(DX_{\text{NF}}(\alpha, 0)\) has the eigenvalues 0 and \(\pm ih_1(\alpha, 0)\), corresponding to the eigenvalues 1 and \(\exp(\pm ih_1(\alpha, 0))\) of \(DP(\alpha, 0)\). It follows that
\[
h_1(\alpha, 0) = \omega(T_0(a_0 + \alpha) - T_0(a_0)),
\] (3.12)
and hence \(\partial h_1/\partial \alpha(0, 0) \neq 0\) if we assume the transversality condition
\[
T'_0(a_0) \neq 0.
\] (3.13)

Under this condition we can solve (3.10) for \(\alpha = \alpha^*_+(\rho)\), and (3.11) for \(\alpha = \alpha^*_-(\rho)\), with \(\alpha^*_+(0) = \alpha^*_-(0) = 0\). This leads to two half-branches \(\{(\alpha^*_+(\rho), \rho) \mid \rho > 0\}\) and \(\{(\alpha^*_-(\rho), \rho \exp(i\pi/q)) \mid \rho > 0\}\) of equilibria of \(X_{\text{NF}}\), bifurcating from the trivial branch \(\{(\alpha, 0) \mid \alpha \in \mathbb{R}\}\) at the origin, and corresponding to two half-branches of symmetric \(q\)-subharmonics bifurcating from the primary branch \(\{\Gamma_a \mid a > 0\}\) at \(\Gamma_{a_0}\). Since the difference \(h_j(\alpha, \rho \exp(i\pi/q)) - h_j(\alpha, \rho) (j = 1, 2)\) is of the order \(\rho^n\) the two half-branches will, for \(q \geq 5\), be close to each other, behaving like the two
sides of an Arnol’d tongue of order \( q \); see for example Section 7 of [8] for more information on that.

The calculations to determine the stability of the equilibria of \( X_{NF} \) we have found are a bit lengthy but straightforward; some of the details are given in [1]. From these calculations it appears that generically along one of the bifurcating branches the solutions are stable, along the other one they are unstable; here “stable” means that the pair of nontrivial multipliers are on the unit circle (ellipticity), while “unstable” means that the nontrivial multipliers are real, one strictly inside the unit circle, the other one (the opposite of the first one) strictly outside the unit circle (hyperbolicity).

The conclusion of this section is that for each \( a_0 > 0 \) for which we can find integers \( k \geq 1 \) and \( q \geq 3 \) with \( \gcd(k, q) = 1 \) and such that both the resonance condition (3.1) and the transversality condition (3.13) are satisfied, there will be at \( \Gamma_{a_0} \) two half-branches of symmetric subharmonics bifurcating from the primary branch. This is however only part of what we wanted: our goal, as described in Section 2, was to find a branch of subharmonics which connects to the primary branch at two different points. To obtain those we will have to study a degenerate situation where the resonance condition is satisfied (otherwise there can be no bifurcation) but the transversality condition is not; we will also have to allow one of our parameters \( \omega \) or \( \gamma \) to change (up to now these parameters were supposed to be fixed). As far as the Lyapunov-Schmidt reduction and the normal form reduction are concerned the introduction of explicit parameters is absolutely no problem: both reductions can in a straightforward way be adapted to the parameter-dependent case. However, giving up the transversality condition (3.13) is not so obvious, since it requires that the period map \( T_0(a) \) for the system (2.2) has at least one critical point, something which is not so easily satisfied. To find out whether we can achieve this with our model system we will in the next section make a brief study of the period map.

4 The period map

There are a few obvious observations that can be made about the system (2.2). First, this is a classical oscillation equation, corresponding to a one-degree of freedom Hamiltonian system, with a potential function \( V(y) \) which is a quartic polynomial. Second, the total energy of the system is for \( \gamma > 1/4 \) (the range we have
considered up to now) a strictly monotonically increasing function of the parameter \(a > 0\) we have used to parametrize the primary branch and the corresponding minimal period \(T_0(a)\). It follows then from results of Chow and Sanders [2] and subsequent work of Gavrilov [4] that the mapping \(a \mapsto T_0(a)\) can have at most one critical point. In order to find out whether and when for our concrete system \(T_0(a)\) has actually such critical point we will analyse now the behaviour of \(T_0(a)\) near \(a = 0\) and for \(a \to \infty\).

We know that \(T_0(0) = 2\pi\), and it is not hard to prove that \(T_0'(0) = 0\) (see further); there are many (sometimes complicated) ways to determine \(T_0''(0)\), but here we will use a rather pedestrian approach, as follows. We set

\[
T_0(a) = \frac{2\pi}{\Omega(a)} = \frac{2\pi}{1 - \Omega_2 a^2 + \cdots} = 2\pi \left( 1 + \frac{10 - 9\gamma}{24} a^2 + \cdots \right).
\]
We conclude that $T_0(0) = 2\pi, T_0'(0) = 0$ and $T_0''(0) > 0$ if $\gamma < 10/9$. So from now on we will restrict $\gamma$ to the range $1/4 < \gamma < 10/9$.

To study the behaviour of $T_0(a)$ for $a \to \infty$ we set $a = 1/b$ and $y(t) = b^{-1}\zeta(b^{-1}t)$ in (4.1), with $b > 0$ and small. This gives the following initial value problem for $\zeta(\tau)$:

$$\ddot{\zeta} + \gamma\zeta^3 + b\zeta^2 + b^2\zeta = 0, \quad \zeta(0) = 1, \quad \dot{\zeta}(0) = 0. \quad (4.4)$$

This initial value problem has a unique solution which depends smoothly on the parameter $b$ and which is periodic with minimal period $\hat{T}(b)$. The classical expression for the period shows that

$$\hat{T}(0) = 4\sqrt{2} \frac{1}{\sqrt{\gamma}} \int_0^1 \frac{d\zeta}{\sqrt{1 - \zeta^4}} > 0. \quad (4.5)$$

It follows then that

$$T_0 \left( \frac{1}{b} \right) = b\hat{T}(b) = b\hat{T}(0) + O(b^2) \quad \text{as} \quad b \to 0, \quad (4.6)$$

and therefore $T_0(a) \to 0$ as $a \to \infty$.

Combining the foregoing elements we can conclude that for $1/4 < \gamma < 10/9$ the period map $T_0(a)$ will have exactly one critical point, namely a maximum at some $a = a_0$ which of course depends on $\gamma$. This is confirmed by numerical calculations, as shown in Figure 1 which depicts the period map for $\gamma = 0.3$.

## 5 Degenerate subharmonic bifurcation: banana’s

In this section we analyze how the subharmonic bifurcation discussed in section 3 is changed when the transversality condition (3.13) is not satisfied. We fix some $\gamma$ with $1/4 < \gamma < 10/9$ and set $a_0 > 0$ equal to the value where $T_0(a)$ reaches its maximum:

$$T'(a_0) = 0 \quad \text{and} \quad T''(a_0) < 0. \quad (5.1)$$

Next we fix some integer $q \geq 5$ (the cases $q = 3$ and $q = 4$ are special and will not be considered here); we also choose an integer $k \geq 1$ and some $\omega_0 > 0$ such that $\gcd(k, q) = 1$ and such that the resonance condition (3.1) is satisfied for $\omega = \omega_0$:

$$\omega_0 T(a_0) = \frac{2k\pi}{q}. \quad (5.2)$$
Observe that it follows from (5.1) and (5.2) (compare also with Figure 1) that for \( \omega > \omega_0 \) close to \( \omega_0 \) there are two values of \( a \), say \( a_0 < a_0 \) and \( \bar{a}_0 > a_0 \), where both the resonance and the transversality conditions are satisfied, and where, according to the results of Section 3, we should see two half-branches of bifurcating \( q \)-subharmonics. For \( \omega < \omega_0 \) there are no such branching points. By allowing \( \omega \) to vary near \( \omega_0 \) we will be able to obtain a more unified picture of the different bifurcations and transitions.

Under the foregoing conditions we can now study the bifurcation of \( q \)-subharmonics by exactly the same reduction techniques as explained in Section 3; the only difference is that here everything will depend explicitly on the parameter \( \omega \). For example, the Poincaré map \( P \), the normal form vectorfield \( X_{NF} \) and the scalar functions \( g, h_1 \) and \( h_2 \) will all depend on \( \omega \). In particular, (3.12) is replaced by

\[
    h_1(\alpha, 0, \omega) = \omega T_0(a_0 + \alpha) - \omega_0 T_0(a_0). \tag{5.3}
\]

The same arguments as in Section 3 lead again to two cases, corresponding to respectively \( \varphi = 0 \) (mod \( \theta_0 \)) and \( \varphi = \pi/q \) (mod \( \theta_0 \)), and each with a corresponding scalar bifurcation equation. However, as \( \partial h_1/\partial \alpha(0, 0, \omega_0) = \omega_0 T'(a_0) = 0 \) we can no longer solve these bifurcation equations for \( \alpha \) as a function of \( \rho \) and \( \omega \). Both equations have the same lowest order terms in each of the variables, namely both
have the form

\[(\omega - \omega_0)T_0(a_0) + \frac{\omega}{2} \gamma T''(a_0)\alpha^2 + \delta_0(\omega)\rho^2 + O((|\alpha| + |\rho|)^3) = 0, \tag{5.4}\]

with $\delta_0(\omega) \in \mathbb{R}$ a number for which we have not been able to find an analytical expression, but which for our example appears numerically to be strictly negative ($\delta_0(\omega) < 0$). The difference between the two cases is in the higher order terms.

The theory of Morse functions tells us that for each fixed $\omega$ the equation (5.4) can be transformed into

\[\frac{\omega}{2} T''(a_0)\alpha^2 - \delta_0(\omega)\rho^2 = (\omega - \omega_0)T_0(a_0). \tag{5.5}\]

Since the left hand side of (5.5) is positive definite (assuming that indeed $\delta_0(\omega) < 0$), and since we are only interested in solutions with $\rho > 0$, the bifurcation picture becomes clear: for $\omega > \omega_0$ we have in the $(\alpha, \rho)$-plane a half-ellipse which shrinks down to the origin as $\omega$ approaches $\omega_0$ and disappears for $\omega < \omega_0$. Transforming back and taking into account that we have two bifurcation equations (which coincide for $\rho = 0$) we obtain for $\omega > \omega_0$ two “solution arches”, both connecting the same two points along the trivial solution branch $\{(\alpha, 0) \mid \alpha \in \mathbb{R}\}$. These two arches are very close to each other (the larger $q$ is the closer the arches are); together they form what one could imagine as the two sides of a banana. As $\omega$ decreases towards $\omega_0$ the banana shrinks down to a point (namely $(a_0, 0)$) and then disappears for $\omega < \omega_0$. Figure 2 shows a numerically calculated banana for the system (2.1), obtained by setting $\gamma = 0.3, q = 5, k = 3$ and $\omega = 0.4003$.

We should observe here that if (contrary to what we have assumed and to what seems to come out of the numerics) we would have $\delta_0 > 0$, then a different branching scenario appears, namely the banana split scenario. For the details we refer to our forthcoming work; let us just mention that it appears to be hard to find concrete systems where such banana-split scenario takes place (we are still searching for one). The “banana terminology” has been taken over from the paper [7] by Peckham, Frousakis and Kevrikidis.

It should be apparent now that each of the two sides of a banana such as found above corresponds to a branch of subharmonics which connect two different periodic orbits along the primary branch. So, according to the arguments given in Section 2, we should see a stability change along each of these branches, with a transition from elliptic to hyperbolic or vice versa. These transitions are indeed confirmed by the numerics; for example, Figure 3, separately for the two
sides of the banana in Figure 2, both the logarithm of the modulus and the argument of the multipliers as one moves from one endpoint of the branch to the other. As one can clearly see there is in both cases a transition point where the nontrivial multipliers go from elliptic (zero logarithm of the modulus and argument different from zero) to hyperbolic (nonzero logarithm of the modulus and zero argument), or vice versa. The pictures show also that at the connection points with the primary branch the two bifurcating branches have opposite stability behaviour, which is in agreement with the generic subharmonic branching described in Section 3.

We have calculated some theoretical expressions from which the stability of the subharmonics along a banana can be determined. It appears from these expressions that when the function $h_2(\alpha, z, \omega)$ in (3.8) is such that $h_2(0, 0, \omega_0) \neq 0$, then at least for sufficiently small banana’ (that is, for $\omega > \omega_0$ sufficiently close to $\omega_0$) it should be such that all subharmonics along one side of the banana should be elliptic, and all subharmonics along the other side hyperbolic. This means no change of stability along either side of the banana, something which is clearly in contradiction with what we have found. We can only conclude that for our particular example the existence of a first integral which enforces the change of
stability also enforces the higher order coefficient \( h_2(0, 0, \omega_0) \) in the normal form of the system to be zero! This is kind of surprising, since \( h_2(0, 0, \omega_0) \) is a coefficient of order \( q - 1 \) in the normal form (see (3.8)), and our conclusion is valid independent of the choice of \( q \). This conclusion also implies that the tangencies between the two sides at the tips of the banana are higher than in the generic case when \( h_2(0, 0, \omega_0) \neq 0 \).

In order to further explore this issue we have modified our model system to

\[
\begin{align*}
\ddot{y} + y + y^2 + \gamma y^3 &= z, \\
\ddot{z} + (\omega^2 + \epsilon y) z &= 0.
\end{align*}
\] (5.6)

This system has an additional parameter \( \epsilon \), but remains reversible; it still has the same primary branch of periodic solution, with the same period map as before. However, for \( \epsilon \neq 0 \) there is no longer a first integral. Our numerical explorations of this modified system (on which we will report elsewhere, in combination with an adapted theory) show that even for small non-zero values of \( \epsilon \), like \( \epsilon = 0.04 \), all sufficiently small banana’s show a generic stability behaviour, i.e. there is no change of stability along the sides of the banana. By increasing \( \epsilon \) one can also see the transition from the non-generic conservative case to the generic non-conservative case.

We finish with a final note on what happens when one allows the banana to grow, by increasing \( \omega - \omega_0 \). It appears, at least for our example, that the banana not only grows, but also gets a very twisted shape, leading to many more stability transitions. In fact, in our first rather crude calculations we found at least five stability transitions, something which was so much in contradiction with the (generic) theory that we decided to explore the matter in detail, with this paper as a result.
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References


