A short tutorial on Hamiltonian systems and their reduction near a periodic orbit

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Preface

This tutorial grew out of my personal interest in subharmonic bifurcation in Hamiltonian (and other) systems. The classical approach to study the behaviour of an autonomous system near one of it's periodic orbits γ_0 consists in introducing a Poincaré (i.e. first return) map associated with a transversal section to the periodic orbit; this Poincaré map is a local diffeomorphism which has a fixed point, and the dynamics of this diffeomorphism near the fixed point completely determines the behaviour of the original system near γ_0. When the system has a continuous symmetry group and when γ_0 corresponds to a relative equilibrium, i.e. it is contained in a group orbit, then the dynamics near (the group orbit generated by) γ_0 can also be described by the flow of a vectorfield on a normal section to the group orbit. Now suppose that the original system is Hamiltonian; it is then a legitimate question to ask to what extend this Hamitonian structure will be reflected in either the Poincaré map or the normal vectorfield. The answer is that the Poincaré map, when restricted to level sets of the Hamiltonian, is symplectic; this is a kind of "folk theorem", in the sense that everyone believes it is true, but that almost nobody knows where to find a proof. Moreover, the one proof which I found (in [4]) was so solidly enwrapped in the formalism of differential geometry (wedge products, cotangent bundles, Cartan forms and the like) that one is easily scared away, certainly if one is only interested in understanding some of the dynamics. Something similar is true in the case of a relative equilibrium, where the normal vectorfield is again Hamiltonian (here the catchword is "momentum map" — see e.g. [1]). Amazingly enough, the proofs of these results are not as hard as one would expect, and since a number of people convinced me that this may be useful I have written down my own version in this tutorial. Also the Darboux Theorem plays an important role in the type of reduction we consider here; therefore I have included a proof of that result as well. Most of my inspiration came from the two books which I already mentionned, i.e. [1] and [4]. I have kept the differential geometry to a minimum, and for the symmetry reduction I have restricted myself to the case of an S^1-symmetry.

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1. The basic model: symplectic vectorspaces

Let V be an even-dimensional real vectorspace. For each $p \geq 1$ we denote by $\mathcal{L}_p(V;\mathbb{R})$ the vectorspace of p-linear forms $\alpha : V^p \to \mathbb{R}$; by $\mathcal{L}_p^{(a)}(V;\mathbb{R})$ we denote the subspace of alternating p-linear forms, i.e. the subspace of those $\alpha \in \mathcal{L}_p(V;\mathbb{R})$ such that

$$\alpha(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) = (\operatorname{sgn} \sigma)\, \alpha(x_1, \ldots, x_p)$$

for each permutation $\sigma : \{1, \ldots, p\} \to \{1, \ldots, p\}$. For convenience we set $\mathcal{L}_0^{(a)}(V;\mathbb{R}) = \mathcal{L}_0(V;\mathbb{R}) := \mathbb{R}$.

Definition. Suppose that $\omega_0 \in \mathcal{L}_2^{(a)}(V;\mathbb{R})$ is non-degenerate, i.e.

$$\omega_0(x, y) = 0, \quad \forall y \in V \quad \Longrightarrow \quad x = 0. \tag{1.1}$$

Then we call (V, ω_0) a *symplectic vectorspace*, and ω_0 the corresponding *symplectic form*. Observe that the non-degeneracy condition (1.1) forces V to be even-dimensional.

Given a symplectic vectorspace (V, ω_0), an open subset $\Omega_0 \subset V$ and a smooth function $H : \Omega_0 \to \mathbb{R}$ we can define a unique smooth vectorfield $X_H : \Omega_0 \to V$ by the relation

$$\omega_0(X_H(x), x_1) = DH(x) \cdot x_1, \quad \forall x \in \Omega_0,\ \forall x_1 \in V. \tag{1.2}$$

Definition. We call X_H the *Hamiltonian vectorfield* corresponding to the *Hamiltonian function* (or for short the *Hamiltonian*) H.

The Hamiltonian system

$$\dot{x} = X_H(x) \tag{1.3}$$

has H as a first integral; indeed, denoting by $\varphi_H^t(x)$ the flow of (1.3) we have

$$\frac{d}{dt} H(\varphi_H^t(x)) = DH(\varphi_H^t(x)) \cdot X_H(\varphi_H^t(x)) = \omega_0(X_H(\varphi_H^t(x)), X_H(\varphi_H^t(x))) = 0.$$

Before we proceed let us find out how Hamiltonian systems behave under coordinate transformations. Let $\Phi : \Omega \to \Omega_0$ be a smooth diffeomorphism; setting $x = \Phi(y)$ in (1.3) gives us the transformed equation

$$\dot{y} = (\Phi^* X_H)(y), \quad \text{with} \quad (\Phi^* X_H)(y) := D\Phi(y)^{-1} X_H(\Phi(y)),\ \forall y \in \Omega. \tag{1.4}$$

The system (1.4) has the first integral $\Phi^* H := H \circ \Phi$; moreover, if we associate to each $y \in \Omega$ a form $(\Phi^* \omega_0)(y) \in \mathcal{L}_2^{(a)}(V;\mathbb{R})$ given by

$$(\Phi^* \omega_0)(y)(y_1, y_2) := \omega_0(D\Phi(y) \cdot y_1, D\Phi(y) \cdot y_2), \quad \forall y_1, y_2 \in V,$$

then it follows from (1.2) that

$$\begin{aligned} D(\Phi^* H)(y) \cdot y_1 &= DH(\Phi(y)) \cdot D\Phi(y) \cdot y_1 \\ &= \omega_0(X_H(\Phi(y)), D\Phi(y) \cdot y_1) = (\Phi^* \omega_0)(y)((\Phi^* X_H(y), y_1). \end{aligned} \tag{1.5}$$

This relation between the transformed Hamiltonian $\Phi^* H$ and the transformed vectorfield $\Phi^* X_H$ is similar to the relation (1.2) between H and X_H, except that here we have a symplectic form $(\Phi^*\omega_0)(y)$ which depends on the "base point" $y \in \Omega$. For particular choices of Φ we may have that $(\Phi^*\omega_0)(y) = \omega_0$ for all $y \in \Omega$; then we say that Φ is a *symplectic diffeomorphism*; the foregoing shows that the transformed equation is then again Hamiltonian in the original symplectic vectorspace (V, ω_0), corresponding to the Hamiltonian $\Phi^* H$. However, if we want to allow transformations which are not symplectic (such as, for example, a chart of some submanifold of Ω), then (1.5) shows that we must extend our concept of a symplectic form. Before we elaborate on such extension in section 3 we first recall a few elements from differential geometry.

2. Some elements from differential geometry

Let Ω be an open subset of a real vectorspace V, and let $p \in \mathbb{N}$; we denote by $\Lambda_p(\Omega)$ the vectorspace of all smooth mappings $\alpha : \Omega \to \mathcal{L}_p^{(a)}(V;\mathbb{R})$; we call such α a *p-form* on Ω. We define an operator $d : \Lambda_p(\Omega) \to \Lambda_{p+1}(\Omega)$ by setting, for each $\alpha \in \Lambda_p(\Omega)$ and for each $x \in \Omega$,

$$d\alpha(x)\cdot(x_0, \ldots, x_p) := \sum_{i=0}^{p} (-1)^i \, (D\alpha(x)\cdot x_i)\cdot(x_0, \ldots, \hat{x}_i, \ldots, x_p), \quad \forall x_i \in V \ (0 \le i \le p). \tag{2.1}$$

In this formula $(x_0, \ldots, \hat{x}_i, \ldots, x_p)$ stands for $(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p)$; we hope that when further in the text we use similar notations they will be self-explaining.

Lemma 1. *$d(d\alpha) = 0$ for all $\alpha \in \Lambda_p(\Omega)$.*

Proof . We have

$$\begin{aligned} & d(d\alpha)(x) \cdot (x_0, x_1, \ldots, x_p, x_{p+1}) \\ &= \sum_{j=0}^{p+1} (-1)^j \, (D(d\alpha)(x) \cdot x_j) \cdot (x_0, \ldots, \hat{x}_j, \ldots, x_{p+1}) \\ &= \sum_{i<j} (-1)^{i+j} \, ((D^2\alpha)(x) \cdot (x_j, x_i)) \cdot (x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+1}) \\ &\quad + \sum_{i>j} (-1)^{i+j-1} \, ((D^2\alpha)(x) \cdot (x_j, x_i)) \cdot (x_0, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, x_{p+1}) \\ &= 0; \end{aligned}$$

the last two sums cancel because the second derivative is symmetric. ■

We call a p-form $\alpha \in \Lambda_p(\Omega)$ *closed* if $d\alpha = 0$. The following theorem (known as the Poincaré Lemma) gives a converse of lemma 1.

Theorem 2. *Let Ω be star-shaped around one of its points, and let $\alpha \in \Lambda_p(\Omega)$ $(p \ge 1)$ be closed, i.e. $d\alpha = 0$. Then there exists some $\beta \in \Lambda_{p-1}(\Omega)$ such that $\alpha = d\beta$.*

Proof . By a translation we can, without loss of generality, assume that $0 \in \Omega$ and that Ω is star-shaped around the origin; this means that $\{sx \mid 0 \leq s \leq 1\} \subset \Omega$ for all $x \in \Omega$.

Define $\beta \in \Lambda_{p-1}(\Omega)$ by

$$\beta(x) \cdot (x_1, \dots, x_{p-1}) := \int_0^1 s^{p-1} \alpha(sx) \cdot (x, x_1, \dots, x_{p-1})\, ds.$$

Then

$$\begin{aligned} & d\beta(x) \cdot (x_1, \dots, x_p) \\ &= \sum_{i=1}^{p} (-1)^{i-1} \int_0^1 s^p (D\alpha(sx) \cdot x_i) \cdot (x, x_1, \dots, \hat{x}_i, \dots, x_p)\, ds \\ &\qquad + \sum_{i=1}^{p} (-1)^{i-1} \int_0^1 s^{p-1} \alpha(sx) \cdot (x_i, x_1, \dots, \hat{x}_i, \dots, x_p)\, ds \\ &= \int_0^1 s^p (D\alpha(sx) \cdot x) \cdot (x_1, \dots, x_p)\, ds + p \int_0^1 s^{p-1} \alpha(sx) \cdot (x_1, \dots, x_p)\, ds \\ &= \int_0^1 \frac{d}{ds} \left[s^p \alpha(sx) \cdot (x_1, \dots, x_p) \right] ds = \alpha(x) \cdot (x_1, \dots, x_p). \end{aligned}$$

For the second equality we have used the fact that $d\alpha = 0$. ■

Definition. Let Ω (respectively $\widetilde{\Omega}$) be an open subset of the real vectorspace V (respectively $\widetilde{V}$), and let $\Phi : \Omega \to \widetilde{\Omega}$ be a smooth mapping. For $\alpha \in \Lambda_p(\widetilde{\Omega})$ we define $\Phi^*\alpha \in \Lambda_p(\Omega)$ by

$$(\Phi^*\alpha)(x) \cdot (x_1, \dots, x_p) := \alpha(\Phi(x)) \cdot (D\Phi(x) \cdot x_1, \dots, D\Phi(x) \cdot x_p), \ \forall x \in \Omega, \forall x_1, \dots, x_p \in V. \tag{2.2}$$

We call $\Phi^*\alpha$ the *pull-back* of α under Φ.

It is immediate to verify that if $\Phi : \Omega \to \widetilde{\Omega}$ and $\Psi : \widetilde{\Omega} \to \widehat{\Omega}$ are smooth mappings (with $\widehat{\Omega} \subset \widehat{V}$ open) then we have for each $\alpha \in \Lambda_p(\widehat{\Omega})$ that

$$(\Psi \circ \Phi)^*\alpha = \Phi^*(\Psi^*\alpha) \in \Lambda_p(\Omega). \tag{2.3}$$

Also, let $I \subset \mathbb{R}$ be an open interval, and let $\Phi : \Omega \to \widetilde{\Omega}$ and $\alpha : I \times \widetilde{\Omega} \to \mathcal{L}_p^{(a)}(\widetilde{V}, \mathbb{R})$ be smooth mappings; we have then that

$$\frac{\partial}{\partial t}(\Phi^*\alpha)(t, x) = \Phi^* \left(\frac{\partial \alpha}{\partial t} \right)(t, x), \qquad \forall (t, x) \in I \times \Omega. \tag{2.4}$$

(Here $\Phi^*\alpha : I \times \Omega \to \mathcal{L}_p^{(a)}(V; \mathbb{R})$ is defined in the obvious way: $(\Phi^*\alpha)(t, \cdot) := \Phi^*(\alpha(t, \cdot))$).

Lemma 3. *We have for each $\alpha \in \Lambda_p(\widetilde{\Omega})$ and for each $\Phi : \Omega \to \widetilde{\Omega}$ that*

$$d\,(\Phi^*\alpha) = \Phi^*(d\alpha). \tag{2.5}$$

Proof . It follows from the definitions that, for $x \in \Omega$ and $x_0, \ldots, x_p \in V$,

$$\begin{aligned} & d(\Phi^*\alpha)(x) \cdot (x_0, \ldots, x_p) \\ & = \sum_{i=0}^{p} (-1)^i \left(D\alpha(\Phi(x)) \cdot D\Phi(x) \cdot x_i\right) \cdot (D\Phi(x) \cdot x_0, \ldots, \widehat{D\Phi(x) \cdot x_i}, \ldots, D\Phi(x) \cdot x_p) \\ & \quad + \sum_{j<i} (-1)^i \alpha(\Phi(x)) \cdot (D\Phi(x) \cdot x_0, \ldots, D^2\Phi(x) \cdot (x_i, x_j), \ldots, \widehat{D\Phi(x) \cdot x_i}, \ldots, D\Phi(x) \cdot x_p) \\ & \quad + \sum_{i<j} (-1)^i \alpha(\Phi(x)) \cdot (D\Phi(x) \cdot x_0, \ldots, \widehat{D\Phi(x) \cdot x_i}, \ldots, D^2\Phi(x) \cdot (x_i, x_j), \ldots, D\Phi(x) \cdot x_p) \end{aligned}$$

These last two sums can be rewritten as

$$\begin{aligned} & \sum_{i>j} (-1)^{i+j} \alpha(\Phi(x)) \cdot (D^2\Phi(x) \cdot (x_i, x_j), D\Phi(x) \cdot x_0, \ldots, \hat{\jmath}, \ldots, \hat{\imath}, \ldots, D\Phi(x) \cdot x_p) \\ & \qquad + \sum_{i<j} (-1)^{i+j-1} \alpha(\Phi(x)) \cdot (D^2\Phi(x) \cdot (x_i, x_j), D\Phi(x) \cdot x_0, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, D\Phi(x) \cdot x_p) \\ & = 0. \end{aligned}$$

This proves (2.5). ■

One can consider each $\alpha_0 \in \mathcal{L}_p^{(a)}(V;\mathbb{R})$ as an element of $\Lambda_p(V)$, and clearly $d\alpha_0 = 0$. Combining this with lemma 3 gives the following.

Corollary 4. *Let $\alpha_0 \in \mathcal{L}_p^{(a)}(V;\mathbb{R})$, and let $\Phi : \Omega \to V$ be smooth. Then $\alpha := \Phi^*\alpha_0 \in \Lambda_p(\Omega)$ is closed, i.e. $d\alpha = 0$.*

Definition. Let $\Omega \subset V$ be open, $\alpha \in \Lambda_p(\Omega)$ $(p \geq 1)$, and let $X : \Omega \to V$ be a smooth vectorfield over Ω. Then we define $i(X)\alpha \in \Lambda_{p-1}(\Omega)$ by

$$(i(X)\alpha)(x) \cdot (x_1, \ldots, x_{p-1}) := \alpha(x) \cdot (X(x), x_1, \ldots, x_{p-1}), \quad \forall x \in \Omega, \forall x_1, \ldots, x_{p-1} \in V. \quad (2.6)$$

The $(p-1)$-form $i(X)\alpha$ is called the *inner product* of X and α.

Lemma 5. *Let $\Omega \subset V$ and $\widetilde{\Omega} \subset \widetilde{V}$ be open, let $\Phi : \Omega \to \widetilde{\Omega}$ be a smooth diffeomorphism of Ω onto $\Phi(\Omega) \subset \widetilde{\Omega}$, let $X : \widetilde{\Omega} \to \widetilde{V}$ be a smooth vectorfield over $\widetilde{\Omega}$, and let $\alpha \in \Lambda_p(\widetilde{\Omega})$ $(p \geq 1)$. Then*

$$\Phi^*(i(X)\alpha) = i(\Phi^*X)\Phi^*\alpha, \quad (2.7)$$

where the vectorfield $\Phi^*X : \Omega \to V$ is defined by

$$(\Phi^*X)(x) := D\Phi(x)^{-1} X(\Phi(x)), \qquad \forall x \in \Omega. \quad (2.8)$$

Proof . A straightforward calculation shows that

$$\begin{aligned} & \Phi^*(i(X)\alpha)(x) \cdot (x_1, \ldots, x_{p-1}) \\ & \quad = \alpha(\Phi(x)) \cdot (X(\Phi(x)), D\Phi(x) \cdot x_1, \ldots, D\Phi(x) \cdot x_{p-1}) \\ & \quad = \alpha(\Phi(x)) \cdot (D\Phi(x) \cdot \Phi^*X(x), D\Phi(x) \cdot x_1, \ldots, D\Phi(x) \cdot x_{p-1}) \\ & \quad = (i(\Phi^*X)\Phi^*\alpha)(x) \cdot (x_1, \ldots, x_{p-1}) \end{aligned}$$

for all $x \in \Omega$ and all $x_1, \ldots, x_{p-1} \in V$. ■

Lemma 6. *Let $I \subset \mathbb{R}$ be an open interval with $0 \in I$, let $\Omega \subset V$ and $\widetilde{\Omega} \subset V$ be open sets with $\Omega \subset \widetilde{\Omega}$, and let $\Phi : I \times \Omega \to \widetilde{\Omega}, (t, x) \mapsto \Phi(t, x) = \Phi_t(x)$ be a smooth mapping such that $\Phi_0(x) = x$ for all $x \in \Omega$. Define $X : \Omega \to V$ by*

$$X(x) := \frac{\partial \Phi}{\partial t}(0, x), \qquad \forall x \in \Omega.$$

Finally, let $\alpha \in \Lambda_p(\widetilde{\Omega})$ for some $p \geq 1$. Then

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_t^* \alpha = i(X) d\alpha + d(i(X)\alpha). \tag{2.9}$$

Proof . Let $x \in \Omega$ and $x_1, \ldots, x_p \in V$; then

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} (\Phi_t^* \alpha)(x) \cdot (x_1, \ldots, x_p) \\
&\quad = (D\alpha(x) \cdot X(x)) \cdot (x_1, \ldots, x_p) + \sum_{i=1}^{p} \alpha(x) \cdot (x_1, \ldots, DX(x) \cdot x_i \ldots, x_p) \\
&\quad = (i(X)d\alpha)(x) \cdot (x_1, \ldots, x_p) + \sum_{i=1}^{p} (-1)^{i-1} (D\alpha(x) \cdot x_i) \cdot (X(x), x_1, \ldots, \hat{x}_i, \ldots, x_p) \\
&\qquad + \sum_{i=1}^{p} (-1)^{i-1} \alpha(x) \cdot (DX(x) \cdot x_i, x_1, \ldots, \hat{x}_i, \ldots, x_p) \\
&\quad = (i(X)d\alpha)(x) \cdot (x_1, \ldots, x_p) + d(i(X)\alpha)(x) \cdot (x_1, \ldots, x_p).
\end{aligned}$$

This proves (2.9), in which the left hand side has to be interpreted pointwise, i.e. it is given at each $x \in \Omega$ by $\left.\frac{\partial}{\partial t}\right|_{t=0} (\Phi_t^* \alpha)(x)$. ■

As an application, let $J \subset \mathbb{R}$ be an interval containing 0, let $\widetilde{\Omega} \subset V$ be open, and let $X : J \times \widetilde{\Omega} \to V, (t, x) \mapsto X(t, x) = X_t(x)$ be a smooth non-autonomous vectorfield over $\widetilde{\Omega}$. Denote by $\Phi_t(x)$ the flow of the corresponding non-autonomous equation

$$\dot{x} = X_t(x), \tag{2.10}$$

i.e. we have

$$\frac{d}{dt} \Phi_t = X_t \circ \Phi_t \qquad \text{and} \qquad \Phi_0 = \mathrm{Id}.$$

Let $I \subset J$ be an open interval, and $\Omega \subset \widetilde{\Omega}$ an open subset such that $\Phi_t(x)$ is well defined for $(t, x) \in I \times \Omega$. Then we have for all $t \in I$ that

$$\begin{aligned}
\frac{d}{dt} \Phi_t^* \alpha &= \left.\frac{d}{ds}\right|_{s=0} \Phi_{t+s}^* \alpha \\
&= \left.\frac{d}{ds}\right|_{s=0} (\widetilde{\Phi}_{t,s} \circ \Phi_t)^* \alpha \qquad \text{with} \quad \widetilde{\Phi}_{t,s} := \Phi_{t+s} \circ \Phi_t^{-1} \\
&= \left.\frac{d}{ds}\right|_{s=0} \Phi_t^* (\widetilde{\Phi}_{t,s}^* \alpha) \\
&= \Phi_t^* \left(\left.\frac{d}{ds}\right|_{s=0} \widetilde{\Phi}_{t,s}^* \alpha \right), \qquad \text{(by (2.4))}.
\end{aligned}$$

Now
$$\frac{\partial}{\partial s}\Big|_{s=0} \widetilde{\Phi}_{t,s}(x) = \frac{\partial}{\partial s}\Big|_{s=0} \Phi_{t+s}(\Phi_t^{-1}(x)) = X_t(\Phi_t(\Phi_t^{-1}(x))) = X_t(x).$$
Using (2.9) we conclude that
$$\frac{d}{dt}\,\Phi_t^*\alpha = \Phi_t^*\left(i(X_t)d\alpha + d(i(X_t)\alpha)\right), \qquad \forall t \in I. \tag{2.11}$$
For $\alpha \in \Lambda_0(\widetilde{\Omega})$ this simplifies to
$$\frac{d}{dt}\,\Phi_t^*\alpha = \Phi_t^*\left(i(X_t)d\alpha\right). \tag{2.12}$$

3. Symplectic forms and Hamiltonian systems

In this section we return to our main topic, namely the formulation and proof of some basic results concerning Hamiltonian systems. We assume again that the vectorspace V is even-dimensional, and we start with the following definition.

Definition. Let $\Omega \subset V$ be open; we say that $\omega \in \Lambda_2(\Omega)$ is a *symplectic form* on Ω if

(i) $\omega(x) \in \mathcal{L}_2^{(a)}(V;\mathbb{R})$ is non-degenerate for all $x \in \Omega$, and
(ii) $d\omega = 0$.

We call (Ω, ω) a *symplectic structure.*

Example. Symplectic vectorspaces form trivial examples of symplectic structures: just take $\Omega = V$ and choose ω to be a constant non-degenerate 2-form, i.e. $\omega(x) = \omega_0 \in \mathcal{L}_2^{(a)}(V;\mathbb{R})$ for all $x \in V$. Starting from such symplectic vectorspace (V, ω_0) one can construct further symplectic structures by pull-backs, as follows. Let $\Phi : \Omega \to V$ be a smooth diffeomorphism of Ω onto $\Phi(\Omega)$. Then
$$\omega := \Phi^*\omega_0 \in \Lambda_2(\Omega) \tag{3.1}$$
is a symplectic form over Ω. Indeed, it follows from Corrolary 4 that $d\omega = 0$, while the non-degeneracy of $\omega(x)$ $(x \in \Omega)$ follows from the fact that ω_0 is non-degenerate and that $D\Phi(x)$ is invertible since Φ is a diffeomorphism.

The Darboux theorem shows that locally all symplectic forms have the form (3.1).

Theorem 7 (Darboux). *Let $\omega \in \Lambda_2(\widetilde{\Omega})$ be a symplectic form on an open $\widetilde{\Omega} \subset V$, and let $x_0 \in \widetilde{\Omega}$. Then there exist an open neighborhood $\Omega \subset \widetilde{\Omega}$ of x_0 and a smooth diffeomorphism $\Phi : \Omega \to \widetilde{\Omega}$ of Ω onto $\Phi(\Omega) \subset \widetilde{\Omega}$, with $\Phi(x_0) = x_0$ and such that*
$$\omega\big|_{\Omega} = \Phi^*\omega(x_0) \in \Lambda_2(\Omega). \tag{3.2}$$

Proof . Without loss of generality we can assume that $x_0 = 0$ and that $\widetilde{\Omega}$ is star-shaped around the origin (make a translation and shrink $\widetilde{\Omega}$ if necessary). For each $t \in \mathbb{R}$ we define $\omega_t \in \Lambda_2(\widetilde{\Omega})$ by
$$\omega_t(x) := (1-t)\,\omega(x) + t\,\omega(0) = \omega(x) - t\,\widetilde{\omega}(x), \qquad \forall x \in \widetilde{\Omega},$$

with $\widetilde{\omega} \in \Lambda_2(\widetilde{\Omega})$ given by

$$\widetilde{\omega}(x) := \omega(x) - \omega(0), \qquad \forall x \in \widetilde{\Omega}.$$

Clearly $d\widetilde{\omega} = 0$, and by Theorem 2 there exists a $\sigma \in \Lambda_1(\widetilde{\Omega})$ such that $\widetilde{\omega} = d\sigma$; we can without loss of generality assume that $\sigma(0) = 0$.

Now observe that $\omega_t(0) = \omega(0)$ is non-degenerate for all $t \in \mathbb{R}$; hence there exist an open neighborhood $\Omega_1 \subset \widetilde{\Omega}$ of the origin in V and an open interval $J \subset \mathbb{R}$ containing $[0,1]$ such that $\omega_t(x)$ is non-degenerate for all $(t,x) \in J \times \Omega_1$. It follows that there exists a uniquely defined smooth mapping $X : J \times \Omega_1 \to V$ such that

$$\sigma(x) \cdot (x_1) = \omega_t(x) \cdot (X_t(x), x_1), \qquad \forall (t,x) \in J \times \Omega_1, \forall x_1 \in V,$$

i.e. such that

$$\sigma = i(X_t)\omega_t \in \Lambda_2(\Omega_1), \qquad \forall t \in J. \tag{3.3}$$

Let $\Phi_t(x)$ denote the flow of the non-autonomous equation

$$\dot{x} = X_t(x);$$

it follows from $\sigma(0) = 0$ that $X_t(0) = 0$ for all $t \in J$, and hence $\Phi_t(0) = 0$ for all $t \in J$. This implies that there exists an open neighborhood $\Omega \subset \Omega_1$ of the origin in V and an open interval $I \subset V$ containing $[0,1]$ such that $I \times \Omega$ is contained in the domain of the flow $\Phi_t(x)$. Using (2.11) we find then that for all $t \in I$ we have

$$\begin{aligned} \frac{d}{dt}\Phi_t^*\omega_t &= \Phi_t^*\left(\frac{d}{dt}\omega_t + i(X_t)d\omega_t + d(i(X_t)\omega_t)\right) \\ &= \Phi_t^*(-\widetilde{\omega} + d\sigma) \;=\; 0. \end{aligned}$$

Therefore

$$\omega = (\Phi_t^*\omega_t)_{t=0} = (\Phi_t^*\omega_t)_{t=1} = \Phi_1^*\omega(0),$$

and (3.2) holds with $\Phi = \Phi_1$. ■

Remark. It is easily seen from the foregoing proof that the Darboux theorem also holds in the case of a smoothly parametrized family of symplectic forms, say $\omega_h \in \Lambda_2(\widetilde{\Omega})$ $(h \in \mathbb{R})$; by making the vectorfield X_t and hence also the flow Φ_t dependent on the parameter h one shows that there exists a parameter dependent local diffeomorphism Φ_h such that for sufficiently small h we have $\omega_h = \Phi_h^*\omega_0(0)$.

The Darboux Theorem as stated above says that by an appropriate coordinate transformation one can reduce any symplectic form to one which is locally constant, i.e. for local considerations one can always assume to be working in a symplectic vectorspace. In many texts one will find a version of the Darboux Theorem which states that locally one can always reduce to the so-called “standard symplectic form”. The proof of our next result shows how in any symplectic vectorspace one can find coordinates with respect to which the symplectic form becomes the standard one.

Theorem 8. *Let (V, ω_0) be a symplectic vectorspace. Then there exist a scalar product $\langle \cdot, \cdot \rangle$ on V and a linear operator $J \in \mathcal{L}(V)$ such that*

(i) $\omega_0(x_1, x_2) = \langle x_1, Jx_2 \rangle$ *for all* $x_1, x_2 \in V$*;*

(ii) J *is non-degenerate and anti-symmetric with respect to the scalar product* $\langle \cdot, \cdot \rangle$*, i.e.* $J^T = -J$*;*

(iii) $J^2 = -I_V$*.*

Such scalar product is called compatible *with the symplectic form ω_0 on V, and J is called the corresponding* symplectic operator *on V*

Proof . It is clear that (ii) follows from (i); hence we have to find $\langle \cdot, \cdot \rangle$ and J such that (i) and (iii) hold. Choose some non-zero $e_1 \in V$; since ω_0 is non-degenerate there exists some non-zero $\hat{f}_1 \in V$ such that $\omega_0(e_1, \hat{f}_1) \neq 0$. Let $f_1 := \omega_0(e_1, \hat{f}_1)^{-1}\hat{f}_1$; then $\omega_0(e_1, f_1) = 1$, and one verifies easily that the linear map $Q_1 : V \to V$ defined by $Q_1(x) := \omega_0(x, f_1)e_1 - \omega_0(x, e_1)f_1$ forms a projection in V. Hence $V = \mathrm{span}_{\mathbb{R}}\{e_1, f_1\} \oplus V_1$, where $V_1 := \ker Q_1$ is a symplectic subspace of V: the restriction of ω_0 to $V_1 \times V_1$ is non-degenerate. Repeating the argument on this subspace one finds after a finite number of steps a basis $\{e_i \mid 1 \leq i \leq n\} \cup \{f_i \mid 1 \leq i \leq n\}$ of V such that

$$\omega_0(e_i, f_j) = \delta_{i,j}, \quad \omega_0(e_i, e_j) = \omega_0(f_i, f_j) = 0, \qquad 1 \leq i, j \leq n. \tag{3.4}$$

Using this *standard basis* one then defines J and $\langle \cdot, \cdot \rangle$ by

$$Je_i := -f_i, \quad Jf_i := e_i, \qquad 1 \leq i \leq n, \tag{3.5}$$

and

$$\langle e_i, e_j \rangle = \langle f_i, f_j \rangle := \delta_{i,j}, \quad \langle e_i, f_j \rangle := 0, \qquad 1 \leq i, j \leq n. \tag{3.6}$$

The symplectic operator J takes with respect to the standard basis the classical form

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}. \tag{3.7}$$

The properties (i) and (iii) can then immediately be verified. ■

Next we introduce Hamiltonian vectorfields.

Definition. Let $\Omega \subset V$ be open, and $\omega \in \Lambda_2(\Omega)$ a symplectic form over Ω. Let $H : \Omega \to \mathbb{R}$ be a smooth function, i.e. $H \in \Lambda_0(\Omega)$. Then there is a uniquely defined smooth vectorfield $X_H : \Omega \to V$ such that

$$\omega(x) \cdot (X_H(x), x_1) = DH(x) \cdot x_1, \qquad \forall x \in \Omega, \forall x_1 \in V, \tag{3.8}$$

i.e. such that

$$dH = i(X_H)\omega. \tag{3.9}$$

We call $X_H = X_H^\omega$ the *Hamiltonian vectorfield* over Ω corresponding to the *Hamiltonian H* and to the symplectic form ω. In the case of a symplectic vectorspace (V, ω_0) such as in Theorem 8 one can rewrite (3.9) as

$$X_H(x) = J\,\nabla H(x), \tag{3.10}$$

where $\nabla H : \Omega \to V$ is the gradient of H with respect to the compatible scalar product $\langle \cdot, \cdot \rangle$, i.e. $DH(x) \cdot x_1 = \langle \nabla H(x), x_1 \rangle$ for all $x \in \Omega$ and $x_1 \in V$. We denote by $\varphi_H^t(x)$ the flow of the Hamiltonian system

$$\dot{x} = X_H(x). \tag{3.11}$$

The definition of a Hamiltonian vectorfield has a few immediate consequences which we mention here for later reference.

Lemma 9. *The Hamiltonian H is a first integral of the Hamiltonian system (3.11), that is we have for all (t, x) in the domain of the flow φ_H that*

$$H(\varphi_H^t(x)) = H(x).$$

Proof . A direct calculation shows that:

$$\frac{d}{dt} H(\varphi_H^t(x)) = DH(\varphi_H^t(x)) \cdot X_H(\varphi_H^t(x)) = \omega(\varphi_H^t(x)) \cdot (X_H(\varphi_H^t(x)), X_H(\varphi_H^t(x))) = 0.$$

Or more abstractly:

$$\frac{d}{dt} H \circ \varphi_H^t = \frac{d}{dt} (\varphi_H^t)^* H = (\varphi_H^t)^*(i(X_H)dH) = (\varphi_H^t)^*(i(X_H)(i(X_H)\omega)) = 0,$$

where we have used (2.12). ■

As a consequence of Lemma 9 we see that the flow φ_H^t leaves for each $h \in \mathbb{R}$ the level set

$$\mathcal{E}_h := \{x \in \Omega \mid H(x) = h\} \tag{3.12}$$

invariant; the next lemma gives a local description of these level sets.

Lemma 10. *Let $h \in \mathbb{R}$ and $x_0 \in \mathcal{E}_h$ be such that $X_H(x_0) \neq 0$. Then $\mathcal{E}_h$ is locally near x_0 a smooth codimension one submanifold, and*

$$T_{x_0}\mathcal{E}_h = \{y \in V \mid \omega(x_0)(X_H(x_0), y) = 0\} = \ker((i(X_H)\omega)(x_0)). \tag{3.13}$$

Proof . It follows from (3.8) and the non-degeneracy of $\omega(x_0)$ that $DH(x_0) \neq 0$ if and only if $X_H(x_0) \neq 0$; hence $\mathcal{E}_h$ is locally near x_0 a smooth codimension one submanifold. Moreover, $\ker DH(x_0) = \ker((i(X_H)\omega)(x_0))$, which proves (3.13). ■

Lemma 11. *Let $H \in \Lambda_0(\Omega)$ and $F \in \Lambda_0(\Omega)$ be such that $X_H(x) = X_F(x)$ for all $x \in \Omega$, with $\Omega \subset V$ connected. Then there exists some $c \in \mathbb{R}$ such that $H(x) = F(x) + c$ for all $x \in \Omega$.*

Proof . By (3.8) the condition $X_H = X_F$ means that $DH(x) = DF(x)$ for all $x \in \Omega$. The result then follows from classical analysis. ■

Definition. Let $\Omega_0 \subset \Omega$ be open, and let $\Phi : \Omega_0 \to \Omega$ be a diffeomorphism of Ω_0 onto $\Phi(\Omega_0) \subset \Omega$. Then we say that Φ is *symplectic* with respect to the symplectic form ω on Ω if

$$\Phi^*\omega = \omega\big|_{\Omega_0} \in \Lambda_2(\Omega_0). \tag{3.14}$$

Lemma 12. *Consider the flow φ_H of (3.11), and denote for each $t \in \mathbb{R}$ by $\Omega_t \subset \Omega$ the set of all $x \in \Omega$ such that (t, x) belongs to the domain of φ_H. Then $\varphi_H^t : \Omega_t \to \Omega$ is a symplectic diffeomorphism.*

Proof . We have

$$\frac{d}{dt}\,(\varphi_H^t)^*\omega = (\varphi_H^t)^* d(i(X_H)\omega) = (\varphi_H^t)^* d^2 H = 0,$$

by (2.11), $d\omega = 0$, (3.9) and Lemma 1. ■

Our next result gives a converse to Lemma 12.

Theorem 13. *Let (Ω, ω) be a symplectic structure, let $\Omega_0 \subset \Omega$ be open and star-shaped around one of it's points, let $I \subset \mathbb{R}$ be an open interval containing* 0*, and let $\Phi : I \times \Omega_0 \to \Omega$ be a smooth mapping such that*

(i) $\Phi(0, x) = x$, $\forall x \in \Omega_0$;

(ii) $\Phi_t = \Phi(t, \cdot) : \Omega_0 \to \Omega$ *is symplectic, for each* $t \in I$.

Then there exists a smooth function $H : \Omega_0 \to \mathbb{R}$ such that

$$\frac{\partial \Phi}{\partial t}(0, x) = X_H(x), \qquad \forall x \in \Omega_0. \tag{3.15}$$

Proof . Using (i), (ii), Lemma 6 and $d\omega = 0$ we see that

$$0 = \frac{d}{dt}\bigg|_{t=0} \Phi_t^*\omega = d(i(X)\omega), \qquad \text{with } X(x) := \frac{\partial \Phi}{\partial t}(0, x),\ \forall x \in \Omega_0.$$

By Theorem 2 there exists some $H \in \Lambda_0(\Omega_0)$ such that $i(X)\omega = dH$, which implies that $X = X_H$. ■

Corollary 14. *Let (Ω, ω) be a symplectic structure, with $\Omega \subset V$ open and star-shaped around one of it's points. Let Φ_t $(t \in \mathbb{R})$ be a one-parameter group of symplectic diffeomorphisms on Ω, i.e.*

(i) $\Phi : \mathbb{R} \times \Omega \to \Omega$ *is smooth;*

(ii) $\Phi_t : \Omega \to \Omega$ *is a symplectic diffeomorphism of Ω onto itself, for each $t \in \mathbb{R}$;*

(iii) $\Phi_0(x) = x$ *for each* $x \in \Omega$;

(iv) $\Phi_t \circ \Phi_s = \Phi_s \circ \Phi_t = \Phi_{t+s}$ *for all* $t, s \in \mathbb{R}$.

Then there exists some $H \in \Lambda_0(\Omega)$ such that $\Phi_t = \varphi_H^t$ for all $t \in \mathbb{R}$.

Proof . Differentiating the identity $\Phi_{t+s}(x) = \Phi_s(\Phi_t(x))$ in s at $s = 0$ and using Theorem 13 we see that there exists some $H \in \Lambda_0(\Omega)$ such that

$$\frac{\partial}{\partial t}\,\Phi_t(x) = X_H(\Phi_t(x)), \qquad \forall t \in \mathbb{R}.$$

Since $\Phi_0(x) = x$ the result follows, by uniqueness of solutions of ordinary differential equations. ∎

We also mention the following result for symplectic diffeomorphisms on a symplectic *vectorspace.*

Theorem 15. *Let (V, ω_0) be a symplectic vectorspace, and let $\Phi : \Omega_0 \to V$ be a symplectic diffeomorphism of an open $\Omega_0 \subset V$ onto $\Phi(\Omega_0) \subset V$. Then Φ is volume-preserving, i.e. we have for each (Lebesgue-measurable) $\Omega \subset \Omega_0$ that*

$$\int_{\Phi(\Omega)} dx = \int_{\Omega} dx. \tag{3.16}$$

Proof . If $x \in \Omega_0$ and $x_1, x_2 \in V$ then $\omega_0(D\Phi(x)\cdot x_1, D\Phi(x)\cdot x_2) = \omega_0(x_1, x_2)$; using Theorem 8 this can be rewritten as

$$D\Phi(x)^T J D\Phi(x) = J, \qquad \forall x \in \Omega_0. \tag{3.17}$$

Taking the determinant of both sides of (3.17) it follows that $(\det(D\Phi(x)))^2 = 1$ for all $x \in \Omega_0$, and hence

$$\int_{\Phi(\Omega)} dx = \int_{\Omega} |\det(D\Phi(x))|\, dx = \int_{\Omega} dx.$$

Observe also that (3.17) can be written as $J^{-1} D\Phi(x)^T J = D\Phi(x)^{-1}$, which implies that if $\mu \in \mathbb{C}$ is an eigenvalue of $D\Phi(x)$ then so is μ^{-1}, both with the same multiplicity. Further analysis (which we do not include here) shows that $\det(D\Phi(x)) = 1$ for all $x \in \Omega_0$. ∎

Next we consider transformations of Hamiltonian systems. Let $\omega \in \Lambda_2(\Omega)$ be a symplectic form over an open $\Omega \subset V$, and let $\Phi : \widetilde{\Omega} \to \Omega$ be a diffeomorphism from an open $\widetilde{\Omega} \subset V$ onto $\Phi(\widetilde{\Omega}) \subset \Omega$. Then $\widetilde{\omega} := \Phi^*\omega$ is a symplectic form over $\widetilde{\Omega}$: $d\widetilde{\omega} = d(\Phi^*\omega) = \Phi^* d\omega = 0$, and $\widetilde{\omega}(y) \in \mathcal{L}_2^{(a)}(V;\mathbb{R})$ is for each $y \in \widetilde{\Omega}$ non-degenerate since $\omega(\Phi(y))$ is non-degenerate and $D\Phi(y) \in \mathcal{L}(V)$ is invertible.

Lemma 16. *Let $\Psi : \Omega_0 \subset \Omega \to \Omega$ be a symplectic diffeomorphism on (Ω, ω). Then the pull-back of Ψ under Φ, namely*

$$\widetilde{\Psi} = \Phi^*\Psi := \Phi^{-1} \circ \Psi \circ \Phi : \widetilde{\Omega}_0 := \Phi^{-1}(\Omega_0 \cap \Psi^{-1}(\Phi(\widetilde{\Omega}))) \subset \widetilde{\Omega} \to \widetilde{\Omega},$$

is a symplectic diffeomorphism on $(\widetilde{\Omega}, \widetilde{\omega})$.

Proof . Using (2.3) we have $\widetilde{\Psi}^*\widetilde{\omega} = \Phi^*(\Psi^*((\Phi^{-1})^*(\Phi^*\omega))) = \Phi^*(\Psi^*\omega) = \Phi^*\omega = \widetilde{\omega}$. ∎

Now consider the Hamiltonian system (3.11) corresponding to the Hamiltonian $H \in \Lambda_0(\Omega)$. Setting $x = \Phi(y)$ in this equation gives the transformed equation

$$\dot{y} = (\Phi^* X_H)(y), \qquad \text{with} \quad (\Phi^* X_H)(y) := D\Phi(y)^{-1} X_H(\Phi(y)), \quad \forall y \in \widetilde{\Omega}. \tag{3.18}$$

Using Lemma 3 and Lemma 5 we obtain

$$i(\Phi^* X_H)\widetilde{\omega} = i(\Phi^* X_H)\Phi^*\omega = \Phi^*(i(X_H)\omega) = \Phi^*(dH) = d(\Phi^* H);$$

this shows that $\Phi^* X_H = \widetilde{X}_{\widetilde{H}}$ is the Hamiltonian vectorfield which in the symplectic structure $(\widetilde{\Omega}, \widetilde{\omega})$ corresponds to the Hamiltonian $\widetilde{H} := \Phi^* H$. The flow of the new Hamiltonian system (3.18) is given by

$$\widetilde{\varphi}^t_{\widetilde{H}} = \Phi^* \varphi^t_H = \Phi^{-1} \circ \varphi^t_H \circ \Phi. \tag{3.19}$$

In case $\widetilde{\Omega} \subset \Omega$ and Φ is a symplectic diffeomorphism we see that $\Phi^* X_H = X_{\widetilde{H}}$ is the Hamiltonian vectorfield corresponding to $\widetilde{H}$ in the original symplectic structure (Ω, ω). We summarize as follows.

Lemma 17. *Let $X_H = X^\omega_H$ be the Hamiltonian vectorfield corresponding to the Hamiltonian H in the symplectic structure (Ω, ω). Then the pull-back $\Phi^* X_H$ of X_H under a diffeomorphism $\Phi : \widetilde{\Omega} \to \Omega$ is the Hamiltonian vectorfield corresponding to the Hamiltonian $\Phi^* H$ in the symplectic structure $(\widetilde{\Omega}, \Phi^*\omega)$: $\Phi^* X^\omega_H = X^{\Phi^*\omega}_{\Phi^* H}$. If $\Phi : \widetilde{\Omega} \subset \Omega \to \Omega$ is a symplectic diffeomorphism on (Ω, ω), then $\Phi^* X_H$ is a Hamiltonian vectorfield on (Ω, ω), again corresponding to the Hamiltonian $\Phi^* H$.* ■

Next we consider submanifolds in a symplectic structure.

Definition. Let $\omega \in \Lambda_2(\Omega)$ be a symplectic form over an open $\Omega \subset V$, and let M be a smooth submanifold of Ω. We call M a *symplectic submanifold* of (Ω, ω) if for each $x \in M$ the restriction of $\omega(x) \in \mathcal{L}^{(a)}_2(V; \mathbb{R})$ to $T_x M \times T_x M$ is non-degenerate, i.e. if for all $x \in M$ and all $x_1 \in T_x M$ we have

$$\omega(x) \cdot (x_1, x_2) = 0, \;\; \forall x_2 \in T_x M \qquad \Longrightarrow \qquad x_1 = 0.$$

This condition necessarily implies that both the dimension and the codimension of M must be even. Given a chart $\psi : U \subset \mathbb{R}^{2m} \to M$ ($\dim M = 2m$) at a point $x_0 \in M$ of such symplectic submanifold we can represent the symplectic form ω on M by the symplectic form $\omega_\psi := \psi^*\omega \in \Lambda_2(U)$. It is straightforward to verify that ω_ψ is indeed a symplectic form over U.

We now describe a general way to construct symplectic submanifolds of codimension 2. Let (Ω, ω) be a symplectic structure, and let $H, G \in \Lambda_0(\Omega)$ be two smooth functions over Ω; we define the *Poisson bracket* of H and G as the function $\{H, G\} \in \Lambda_0(\Omega)$ given by

$$\{H, G\}(x) := \omega(x)(X_H(x), X_G(x)), \qquad \forall x \in \Omega. \tag{3.20}$$

Let $x_0 \in \Omega$ be such that $\{H, G\}(x_0) \neq 0$; let $h_0 := H(x_0)$ and $g_0 := G(x_0)$, and define $M \subset \Omega$ by

$$M := \{x \in \Omega \mid H(x) = h_0, \; G(x) = g_0\}. \tag{3.21}$$

Lemma 18. *Under the foregoing conditions there exists an open neighborhood $\Omega_0 \subset \Omega$ of x_0 such that $M \cap \Omega_0$ is a symplectic submanifold of (Ω, ω) of codimension 2.*

Proof . The condition $\{H,G\}(x_0) \neq 0$ implies that $X_H(x_0)$ and $X_G(x_0)$ are both non-zero and linearly independent; it follows then from Lemma 10 that $\{x \in \Omega \mid H(x) = h_0\}$ and $\{x \in \Omega \mid G(x) = g_0\}$ form near x_0 two transversal codimension one submanifolds, hence their intersection M is near x_0 a smooth codimension two submanifold. Moreover, (3.13) shows that

$$T_{x_0}M = \ker((i(X_H)\omega)(x_0)) \cap \ker((i(X_G)\omega)(x_0)), \tag{3.22}$$

which in combination with $\{H,G\}(x_0) \neq 0$ also implies

$$V = T_{x_0}M \oplus \mathrm{span}_{\mathbb{R}}\{X_H(x_0), X_G(x_0)\}. \tag{3.23}$$

Let $x_1 \in T_{x_0}M$ be such that $\omega(x_0) \cdot (x_1, x_2) = 0$ for all $x_2 \in T_{x_0}M$; since by (3.22) we also have $\omega(x_0) \cdot (x_1, X_H(x_0)) = \omega(x_0) \cdot (x_1, X_G(x_0)) = 0$ we conclude from (3.23) that $(\omega(x_0) \cdot (x_1, \tilde{x}_2) = 0$ for all $\tilde{x}_2 \in V$. Now $\omega(x_0)$ is non-degenerate, and therefore $x_1 = 0$, which proves that the restriction of $\omega(x_0)$ to $T_{x_0}M \times T_{x_0}M$ is non-degenerate. By continuity we have $\{H,G\}(x) \neq 0$ for all x in a neighborhood $\Omega_0 \subset \Omega$ of x_0, and repeating the foregoing arguments shows that $M \cap \Omega_0$ is a symplectic submanifold. ■

4. Poincaré maps in Hamiltonian systems are symplectic

In this Section we give a proof of the fact that Poincaré maps associated with periodic orbits of Hamiltonian systems and restricted to level sets of the Hamiltonian are symplectic. Let (Ω, ω) be a symplectic structure, $H \in \Lambda_0(\Omega)$, and let $\gamma_0 \subset \Omega$ be a non-trivial periodic orbit of the Hamiltonian system (3.11), with minimal period $T_0 > 0$. We can without loss of generality assume that $H(x) = 0$ for all $x \in \gamma_0$. Fix some $x_0 \in \gamma_0$, and let $G \in \Lambda_0(\Omega)$ be such that $G(x_0) = 0$ and $\{H,G\}(x_0) \neq 0$. Let

$$\Sigma := \{x \in \Omega \mid G(x) = 0\} \qquad \text{and} \qquad \Sigma_h := \Sigma \cap \mathcal{E}_h, \qquad (h \in \mathbb{R}). \tag{4.1}$$

Then the flow φ_H of (3.11) induces a Poincaré map (i.e. a first return map) $P : \Sigma \to \Sigma$, well defined in a neighborhood of x_0 on Σ and having x_0 as a fixed point. By Lemma 18 we know that there exists a neighborhood Ω_0 of x_0 in V such that $\Sigma_h \cap \Omega_0$ is a symplectic submanifold for each $h \in \mathbb{R}$, and since φ_H leaves $\mathcal{E}_h$ invariant (Lemma 9) it follows that P maps each Σ_h into itself. We denote the restriction of P to Σ_h by P_h, and we want to show that P_h is a symplectic diffeomorphism on Σ_h. We start with the following lemma.

Lemma 19. *There exist a neighborhood $\Omega_0 \subset \Omega$ of x_0, an open interval $I_0 \subset \mathbb{R}$ with $0 \in I_0$, and a uniquely defined smooth mapping $\tau : \Omega_0 \to I_0$ with $\tau(x_0) = 0$, such that for $(t, x) \in I_0 \times \Omega_0$ we have $\varphi_H^t(x) \in \Sigma$ if and only if $t = \tau(x)$.*

Proof . Let $F(t,x) := G(\varphi_H^t(x))$ for (t,x) in some neighborhood of $(0, x_0)$ in $\mathbb{R} \times \Omega$. Then F is smooth, $F(0, x_0) = 0$, and

$$\frac{\partial F}{\partial t}(0, x_0) = DG(x_0) \cdot X_H(x_0) = -\{H,G\}(x_0) \neq 0.$$

The result then follows from the implicit function theorem and the definition (4.1) of Σ. ■

Using Lemma 19 we define $\pi : \Omega_0 \to \Sigma$ by

$$\pi(x) := \varphi_H^{\tau(x)}(x), \qquad \forall x \in \Omega_0. \tag{4.2}$$

We also observe that from $\varphi_H^{T_0}(x_0) = x_0$, $\pi(x_0) = x_0$ and the continuity of the flow φ_H it follows that there exists a neighborhood $\Omega_1 \subset \Omega_0$ of x_0 such that $\varphi_H^{T_0}(x) \in \Omega_0$ and $\pi(\varphi_H^{T_0}(x)) \in \Omega_0$ for all $x \in \Omega_1$; the Poincaré map $P : \Sigma \cap \Omega_1 \to \Sigma \cap \Omega_0$ is then defined by

$$P(x) := \pi(\varphi_H^{T_0}(x)), \qquad \forall x \in \Sigma \cap \Omega_1. \tag{4.3}$$

For each (small) $h \in \mathbb{R}$ we define $P_h : \Sigma_h \cap \Omega_1 \to \Sigma_h \cap \Omega_0$ as the restriction of P to $\Sigma_h \cap \Omega_1$; we also choose Ω_0 sufficiently small such that $\Sigma_h \cap \Omega_0$ is a symplectic submanifold for all $h \in \mathbb{R}$ for which $\Sigma_h \cap \Omega_0 \neq \emptyset$. Under these conditions our main result is the following.

Theorem 20. *The mapping $P_h : \Sigma_h \cap \Omega_1 \to \Sigma_0 \cap \Omega_0$ is, for each small $h \in \mathbb{R}$, a symplectic diffeomorphism on the symplectic submanifold $\Sigma_h \cap \Omega_0$.*

Proof . It follows from (4.2) that for each $x \in \Omega_0$ we have

$$D\pi(x) \cdot \bar{x} = D\varphi_H^{\tau(x)}(x) \cdot \bar{x} + X_H(\pi(x))(D\tau(x) \cdot \bar{x}), \qquad \forall \bar{x} \in V.$$

For fixed t the mapping φ_H^t is symplectic (Lemma 12), and therefore we find for all $x \in \Omega_0$ and all $x_1, x_2 \in V$ that

$$\begin{aligned} &\omega(\pi(x)) \cdot (D\pi(x) \cdot x_1, D\pi(x) \cdot x_2) \\ &\quad = \omega(x) \cdot (x_1, x_2) + \omega(\pi(x)) \cdot (D\varphi_H^{\tau(x)}(x) \cdot x_1, X_H(\pi(x))(D\tau(x) \cdot x_2)) \\ &\qquad + \omega(\pi(x)) \cdot (X_H(\pi(x))(D\tau(x) \cdot x_1), D\varphi_H^{\tau(x)}(x) \cdot x_2) \\ &\quad = \omega(x) \cdot (x_1, x_2) + (D\tau(x) \cdot x_2)\, \omega(\pi(x)) \cdot (D\varphi_H^{\tau(x)}(x) \cdot x_1, X_H(\pi(x))) \\ &\qquad + (D\tau(x) \cdot x_1)\, \omega(\pi(x)) \cdot (X_H(\pi(x)), D\varphi_H^{\tau(x)}(x) \cdot x_2). \end{aligned} \tag{4.4}$$

Since φ_H^t maps $\mathcal{E}_h$ into itself it follows that for each $x \in \mathcal{E}_h \cap \Omega_0$ and for each $\bar{x} \in T_x\mathcal{E}_h$ we have that $D\varphi_H^{\tau(x)}(x) \cdot \bar{x} \in T_{\pi(x)}\mathcal{E}_h$; then Lemma 10 implies that the last two terms in (4.4) vanish if $x \in \mathcal{E}_h \cap \Omega_0$ and $x_1, x_2 \in T_x\mathcal{E}_h$, and hence

$$\omega(\pi(x)) \cdot (D\pi(x) \cdot x_1, D\pi(x) \cdot x_2) = \omega(x) \cdot (x_1, x_2), \qquad \forall x \in \mathcal{E}_h \cap \Omega_0, \forall x_1, x_2 \in T_x\mathcal{E}_h. \tag{4.5}$$

Now observe that if $x \in \Sigma_h \cap \Omega_1$, $\bar{x} \in T_x\Sigma_h$ and $y := \varphi_H^{T_0}(x)$, then $D\varphi_H^{T_0}(x) \cdot \bar{x} \in T_y\mathcal{E}_h$; combining (4.5) with the definition (4.3) of P and the fact that $\varphi_H^{T_0}$ is symplectic this gives us for all $x \in \Sigma_h \cap \Omega_1$ and for all $x_1, x_2 \in T_x\Sigma_h$ that

$$\begin{aligned} &\omega(P_h(x)) \cdot (DP_h(x) \cdot x_1, DP_h(x) \cdot x_2) \\ &\quad = \omega(\pi(\varphi_H^{T_0}(x)) \cdot (D\pi(\varphi_H^{T_0}(x)) \cdot D\varphi_H^{T_0}(x) \cdot x_1, D\pi(\varphi_H^{T_0}(x)) \cdot D\varphi_H^{T_0}(x) \cdot x_2) \\ &\quad = \omega(\varphi_H^{T_0}(x)) \cdot (D\varphi_H^{T_0}(x) \cdot x_1, D\varphi_H^{T_0}(x) \cdot x_2) \\ &\quad = \omega(x) \cdot (x_1, x_2). \end{aligned}$$

This proves that the mapping P_h is indeed symplectic. ■

We conclude this Section with the following remark. Using the parameter-dependent version of the Darboux Theorem and Lemma 16 we can reduce the one-parameter family P_h of symplectic maps on the symplectic submanifolds $\Sigma_h \cap \Omega_0$ to a one-parameter family Φ_h of (local) symplectic maps in a symplectic vectorspace $(\mathbb{R}^{2m}, \omega_0)$, where $2m = \dim V - 2 = \dim \Sigma_h$. It has been shown in [2] and [3] how this can be used to study subharmonics near a given periodic orbit of a Hamiltonian system.

5. Reduction of symmetric Hamiltonian systems

In the last Section of this tutorial we want to illustrate how symmetries in Hamiltonian systems allow to reduce the system to a lower-dimensional one which is still Hamiltonian. The usual approach to such reduction is based on symplectic Lie group actions and the associated momentum map (see e.g. [1]). Here we will limit ourselves to an illustration of the basic ideas in the case of an S^1-symmetry; also, our analysis will be local (near a "relative equilibrium"), and for simplicity we will assume that the original Hamiltonian system is defined on a symplectic vectorspace, i.e. the corresponding symplectic form is constant. However, we start with some generalities on symmetries in Hamiltonian systems.

Let $\Phi : \Omega_0 \subset \Omega \to \Omega$ be a symplectic diffeomorphism in the symplectic structure (Ω, ω), and let $H \in \Lambda_0(\Omega)$ be such that $\Phi^* H = H$ (on Ω_0). Then we have, by Lemma 17 and (3.19), that $\Phi^* X_H = X_{\Phi^* H} = X_H$ and $\Phi^* \varphi_H^t = \varphi_{\Phi^* H}^t = \varphi_H^t$, i.e. Φ is a *symplectic symmetry* of the Hamiltonian vectorfield X_H and the corresponding flow φ_H. The following lemma describes a condition under which there exists a one-parameter group of such symplectic symmetries.

Lemma 21. *Let $\Omega \subset V$ be open, $\omega \in \Lambda_2(\Omega)$ a symplectic form, and $H, F \in \Lambda_0(\Omega)$. Then the following statements are equivalent:*

(i) $\{H, F\} = 0$;

(ii) F is a first integral for the Hamiltonian vectorfield X_H;

(iii) H is a first integral for the Hamiltonian vectorfield X_F.

Moreover, if (i)-(iii) are satisfied, then also

(iv) $\varphi_F^s \circ \varphi_H^t = \varphi_H^t \circ \varphi_F^s$ *for all* $t, s \in \mathbb{R}$.

Proof . The equivalence of (i), (ii) and (iii) follows from the definition of the Poisson bracket:

$$\begin{aligned} \{H, F\}(x) &= \omega(x) \cdot (X_H(x), X_F(x)) = DH(x) \cdot X_F(x) \\ &= -\omega(x) \cdot (X_F(x), X_H(x)) = -DF(x) \cdot X_H(x). \end{aligned}$$

The statement (ii) implies that $F(\varphi_H^t(x)) = F(x)$ for all $t \in \mathbb{R}$ and for all x in the domain of φ_H^t, i.e. φ_H^t is for each $t \in \mathbb{R}$ a symplectic symmetry of the Hamiltonian vectorfield X_F. By the observation before this Lemma it follows that $(\varphi_H^t)^* \varphi_F^s = \varphi_F^s$ for all $t, s \in \mathbb{R}$, i.e. (iv) holds. ■

Remark. We can formulate (part of) the result of Lemma 21 by saying that $\{H, F\} = 0$ implies that $\{\varphi_F^s \mid s \in \mathbb{R}\}$ forms a one-parameter group of symplectic symmetries for the Hamiltonian vectorfield X_H. The converse is only true under some restricting conditions. Suppose that $\{\Phi_s \mid s \in \mathbb{R}\}$ forms a one-parameter group of symplectic symmetries of the Hamiltonian vectorfield X_H. Assuming the conditions of Corrolary 14 we have that $\Phi_s = \varphi_F^s$ for some $F \in \Lambda_0(\Omega)$, and our hypothesis means that the statement (iv) of Lemma 21 holds. Differentiating at $t = 0$ shows that $(\varphi_F^s)^* X_H = X_H$ or $X_{(\varphi_F^s)^* H} = X_H$ for all $s \in \mathbb{R}$. Lemma 11 then implies that $H(\varphi_F^s(x)) = H(x) + \eta(s)$ for all $x \in \Omega$ and all $s \in \mathbb{R}$, and with $\eta(s)$ a smooth real-valued function. Differentiating at $s = 0$ gives $\{H, F\}(x) = c$ for all $x \in \Omega$ and for some constant $c \in \mathbb{R}$. In case the group $\{\Phi_s = \varphi_F^s \mid s \in \mathbb{R}\}$ has some fixed point $x_0 \in \Omega$ then $\eta(s) = 0$ and $c = 0$, i.e. we have $\{H, F\} = 0$. The condition on the existence of a fixed point will in particular be satisfied if we are working in a symplectic vectorspace and if the group $\{\Phi_s\}$ consists of *linear* symplectic diffeomorphisms (see further on for an example).

The symplectic symmetries of a given Hamiltonian vectorfield X_H form a group; under certain conditions it is possible to use this group of symmetries to reduce the Hamiltonian system to a lower-dimensional one. Here we want to illustrate this reduction for the case where the symmetry group is isomorphic to the circle group $S^1 := \mathbb{Z}/(2\pi\mathbb{Z})$; we will also assume that we work in a symplectic vectorspace, and that the symmetries are linear. We start with a definition and some technical lemma's.

Definition. A *linear symplectic S^1-action* on a symplectic vectorspace (V, ω_0) is a smooth mapping $\Gamma : \mathbb{R} \to \mathcal{L}(V)$ with the following properties:

(a) $\Gamma(\theta + 2\pi) = \Gamma(\theta)$ for all $\theta \in \mathbb{R}$;
(b) $\Gamma(0) = I_V$;
(c) $\Gamma(\theta) \circ \Gamma(\psi) = \Gamma(\psi) \circ \Gamma(\theta) = \Gamma(\theta + \psi)$ for all $\theta, \psi \in \mathbb{R}$;
(d) $\Gamma(\theta)$ is symplectic for all $\theta \in \mathbb{R}$.

Differentiating the identities $\Gamma(\theta + \psi) = \Gamma(\psi) \circ \Gamma(\theta)$ and $\omega_0(\Gamma(\psi)x, \Gamma(\psi)y) = \omega_0(x, y)$ at $\psi = 0$ shows that

$$\frac{d}{d\theta}\Gamma(\theta) = J_0\Gamma(\theta), \qquad \text{with} \qquad J_0 := \left.\frac{d}{d\psi}\right|_{\psi=0} \Gamma(\psi) \in \mathcal{L}(V), \tag{5.1}$$

and

$$\omega_0(J_0 x, y) + \omega_0(x, J_0 y) = 0, \qquad \forall x, y \in V. \tag{5.2}$$

It follows from (5.1) that $\Gamma(\theta) = \exp(J_0\theta)$ for all $\theta \in \mathbb{R}$. The relation (5.2) means (by definition) that J_0 is *infinitesimally symplectic*. From (5.2) one can directly verify that J_0 is a linear Hamiltonian vectorfield corresponding to a quadratic Hamiltonian; more precisely

$$J_0 = X_S \qquad \text{with} \qquad S(x) := \frac{1}{2}\omega_0(J_0 x, x), \quad \forall x \in V, \tag{5.3}$$

and $\Gamma(\theta) = \varphi_S^\theta$ for all $\theta \in \mathbb{R}$. (These relations can also be obtained from Theorem 13, Corollary 14, the fact that γ is linear and the proof of Theorem 2). We call J_0 the *infinitesimal generator* of the S^1-action Γ.

Lemma 22. *Let* $\Gamma : \mathbb{R} \to \mathcal{L}(V)$ *be a linear symplectic* S^1*-action on a symplectic vectorspace* (V, ω_0)*, with infinitesimal generator* J_0*. Then there exists a* compatible *scalar product* $\langle \cdot, \cdot \rangle$ *on* V *such that*

$$\langle J_0 x, y\rangle + \langle x, J_0 y\rangle = 0 \qquad \text{and} \qquad \langle \Gamma(\theta)x, \Gamma(\theta)y\rangle = \langle x, y\rangle, \qquad \forall x, y \in V, \forall \theta \in \mathbb{R}, \quad (5.4)$$

i.e. J_0 *is anti-symmetric and the action* $\Gamma(\theta)$ *is orthogonal with respect to this scalar product.*

Proof . The condition $\Gamma(2\pi) = \exp(2\pi J_0) = I_V$ implies that J_0 is semisimple and can only have eigenvalues of the form $\pm ki$, with $k \in \mathbb{N}$. Therefore we can write V as

$$V = {\bigoplus_{k\in\mathbb{N}}}' V_k, \qquad V_k := \ker (J_0^2 + k^2 I_V), \tag{5.5}$$

where the prime indicates that only a finite number of the summands are nontrivial. We claim that each of the subspaces V_k is a symplectic subspace, and that the decomposition (5.5) is orthogonal with respect to ω_0. Indeed, we have for each fixed $k \in \mathbb{N}$ that

$$V = V_k \oplus W_k, \qquad \text{with} \qquad W_k := \mathrm{Im}\,(J_0^2 + k^2 I_V) = {\bigoplus_{\ell\neq k}}' V_\ell.$$

If $x \in V_k$ and $y \in W_k$ then, writing $y = (J_0^2 + k^2 I_V)z$ for some $z \in V$ and using (5.2) we find that

$$\omega_0(x, y) = \omega_0(x, (J_0^2 + k^2 I_V)z) = \omega_0((J_0^2 + k^2 I_V)x, z) = 0.$$

This proves our claim. Since J_0 and $\Gamma(\theta)$ leave each of the subspaces V_k invariant it is then sufficient to construct in each of these V_k a scalar product which satisfies the requirements; replacing V_k by V we can therefore assume that $J_0^2 = -k^2 I_V$ for some $k \in \mathbb{N}$.

In the case $k = 0$ we can take any compatible scalar product on V, as given by Theorem 8. In the case $k \geq 1$ consider the bilinear form $a_0 : V \times V \to \mathbb{R}$ defined by

$$a_0(x, y) := \omega_0(J_0 x, y), \qquad \forall x, y \in V. \tag{5.6}$$

Using (5.2) it is immediate to verify that this bilinear form is symmetric and invariant under the S^1-action:

$$a_0(x, y) = a_0(y, x) \qquad \text{and} \qquad a_0(\exp(J_0\theta)x, \exp(J_0\theta)y) = a_0(x, y), \qquad \forall x, y \in V,\ \forall \theta \in \mathbb{R}.$$

Also, a_0 is non-degenerate (since ω_0 is non-degenerate and J_0 is invertible), and the corresponding quadratic form $Q_0(x) := a_0(x, x)$ satisfies

$$Q_0(\exp(J_0\theta)x) = Q_0(x) \qquad \text{and} \qquad Q_0(J_0 x) = k^2 Q_0(x), \qquad \forall x \in V,\ \forall \theta \in \mathbb{R}. \tag{5.7}$$

Fix some $x_1 \in V$ such that $Q_0(x_1) \neq 0$ (such x_1 exists since otherwise a_0 would be zero); then the subspaces $V_1 := \mathrm{span}_{\mathbb{R}}\{x_1, J_0 x_1\}$ and $W_1 := \{y \in V \mid a_0(x, y) = 0,\ \forall x \in V_1\}$ are invariant under the S^1-action (since $J_0^2 = -k^2 I_V$), they are symplectic subspaces by the definition of a_0, and the quadratic form Q_0 is definite on V_1, by (5.7). Repeating the argument on W_1 we obtain a decomposition

$$V = \bigoplus_{1\leq i\leq p} V_i \tag{5.8}$$

of V into two-dimensional symplectic subspaces V_i which are S^1-invariant and such that the restriction of Q_0 to each of these subspaces is definite. Now define $J \in \mathcal{L}(V)$ by

$$J\left(\sum_{i=1}^{p} x_i\right) := k^{-1}\sum_{i=0}^{p} \epsilon_i J_0 x_i, \qquad \forall x_i \in V_i,\ 1 \le i \le p, \tag{5.9}$$

where $\epsilon_i := 1$ if Q_0 is positive definite on V_i, and $\epsilon_i := -1$ if Q_0 is negative definite on V_i. Then $J^2 = -I_V$, J is infinitessimally symplectic and commutes with the S^1-action, and the quadratic form

$$Q(x) := \omega_0(Jx, x) = k^{-1}\sum_{i=1}^{p} \epsilon_i Q_0(x_i), \qquad \forall x = \sum_{i=1}^{p} x_i \in V, \tag{5.10}$$

is positive definite. Hence

$$\langle x, y\rangle := \omega_0(Jx, y), \qquad \forall x, y \in V, \tag{5.11}$$

defines a scalar product on V for which the S^1-action is orthogonal (i.e. (5.4) holds), while we have for each $x, y \in V$ that

$$\omega_0(x, y) = -\omega_0(J^2 x, y) = \omega_0(Jx, Jy) = \langle x, Jy\rangle;$$

this shows that the scalar product $\langle\cdot,\cdot\rangle$ is compatible with the symplectic structure on V and finishes the proof of the lemma. ■

From now on we fix a scalar product in V which satisfies the conditions of Lemma 22. The next lemma describes a symmetry-adapted coordinate system in the neighborhood of a non-trivial S^1-orbit in V.

Lemma 23. *Let $x_0 \in V$ be such that $J_0 x_0 \neq 0$, and let $\gamma_0 := \{\exp(J_0\theta)x_0 \mid \theta \in \mathbb{R}\}$ be the orbit of x_0 under the S^1-action. Let $G_0 := \{\theta \in \mathbb{R} \mid \exp(J_0\theta)x_0 = x_0\}$ be the* isotropy subgroup *of x_0. Let $\Sigma := \{y \in V \mid \langle J_0 x_0, y\rangle = 0\}$. Then there exists some $\rho_0 > 0$ such that with $U_0 := \{y \in \Sigma \mid \langle y, y\rangle < \rho_0^2\}$ the following holds:*

- *(i) Σ and U_0 are invariant under the action of G_0;*
- *(ii) $\{\exp(J_0\theta)(x_0 + y) \mid \theta \in \mathbb{R}, y \in U_0\}$ forms an S^1-invariant open neighborhood of γ_0, called a* tubular neighborhood *of γ_0;*
- *(iii) for all $\theta_1, \theta_2 \in \mathbb{R}$ and all $y_1, y_2 \in U_0$ we have $\exp(J_0\theta_1)(x_0 + y_1) = \exp(J_0\theta_2)(x_0 + y_2)$ if and only if $\theta_2 - \theta_1 \in G_0$ and $y_1 = \exp(J_0(\theta_2 - \theta_1))y_2$.*

Remark. We have always that $2\pi\mathbb{Z} \subset G_0$, and $G_0/2\pi\mathbb{Z} \subset S^1$ is the actual isotropy subgroup of x_0; since $J_0 x_0 \neq 0$ this isotropy subgroup is a proper subgroup of S^1, and hence it must be isomorphic to the cyclic group $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ for some $p \ge 1$.

Proof . The property (i) is an direct consequence of the definitions and the fact that the S^1-action is orthogonal. Define $\Psi : \mathbb{R} \times \Sigma \to V$ by $\Psi(\theta, y) := \exp(J_0\theta)(x_0 + y)$; then $\Psi(0, 0) = x_0$ and $D\Psi(0, 0) \cdot (\theta, y) = \theta J_0 x_0 + y$. Since Σ is the orthogonal complement of

$J_0 x_0$ this implies that $D\Psi(0,0)$ is invertible, and that Ψ is a smooth diffeomorphism of an open neighborhood $I_0 \times U_0$ of $(0,0)$ in $\mathbb{R} \times \Sigma$ onto an open neighborhood of x_0 in V, where U_0 can be chosen as in the statement of the lemma, with $\rho_0 > 0$ sufficiently small. Then (ii) follows immediately, since

$$\{\exp(J_0\theta)(x_0 + y) \mid \theta \in \mathbb{R},\, y \in U_0\} = \bigcup_{\theta\in\mathbb{R}} \exp(J_0\theta)(\Psi(I_0 \times U_0)).$$

It is easily seen that the statement (iii) is equivalent to the following

(iv) We have for all $\theta \in \mathbb{R}$ and $y \in U_0$ that $\exp(J_0\theta)(x_0 + y) \in x_0 + U_0$ if and only if $\theta \in G_0$.

In one direction (iv) is obvious: if $\theta \in G_0$ and $y \in U_0$ then $\exp(J_0\theta)(x_0+y) = x_0+\exp(J_0\theta)y \in x_0 + U_0$. Now suppose the converse is not true, i.e. we cannot choose $\rho_0 > 0$ sufficiently small such that $\exp(J_0\theta)(x_0 + y) \in x_0 + U_0$ implies $\theta \in G_0$. Then there exist sequences $\{\theta_n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ and $\{y_n \mid n \in \mathbb{N}\} \subset \Sigma$ such that $y_n' := \exp(J_0\theta_n)(x_0 + y_n) - x_0$ belongs to Σ for each $n \in \mathbb{N}$, $y_n \to 0$ and $y_n' \to 0$ as $n \to \infty$, and $\theta_n \notin G_0$. Adding an appropriate multiple of 2π to θ_n we can without loss of generality assume that $\theta_n \in [0, 2\pi]$, and hence that θ_n converges to some $\bar{\theta}$. Then $0 = \lim_{n\to\infty} y_n' = \exp(J_0\bar{\theta})x_0 - x_0$, and so $\bar{\theta} \in G_0$. Replacing $\theta_n - \bar{\theta}$ by θ_n and $\exp(-J_0\bar{\theta})y_n'$ by y_n' we have then

$$\lim_{n\to\infty} y_n = 0, \quad \lim_{n\to\infty} \theta_n = 0, \quad x_0 + y_n' = \exp(J_0\theta_n)(x_0 + y_n) \quad \text{and} \quad \theta_n \notin G_0.$$

Since the mapping Ψ introduced in the first part of this proof is a diffeomorphism it follows that for sufficiently large n we must have $\theta_n = 0$ and $y_n' = y_n$; this however contradicts the fact that $\theta_n \notin G_0$. ■

Now suppose that we have a Hamiltonian vectorfield X_H corresponding to an S^1-invariant Hamitonian $H \in \Lambda_0(V)$:

$$H(\exp(J_0\theta)x) = H(x). \tag{5.12}$$

Then X_H is S^1-equivariant:

$$X_H(\exp(J_0\theta)x) = \exp(J_0\theta)X_H(x), \quad \forall x \in V, \forall\theta \in \mathbb{R}. \tag{5.13}$$

Let γ_0 be a non-trivial periodic orbit of X_H which is at the same time a group orbit under the S^1-action; this means that

$$\gamma_0 = \{\varphi_H^t(x_0) \mid t \in \mathbb{R}\} = \{\exp(J_0\theta)x_0 \mid \theta \in \mathbb{R}\} \tag{5.14}$$

for some $x_0 \in V$ with $J_0 x_0 \neq 0$. We want to describe the vectorfield X_H and it's flow φ_H in a tubular neighborhood of γ_0; we use the notations of Lemma 23. We clearly have that $V = \operatorname{span}_{\mathbb{R}} \{J_0 x_0\} \oplus \Sigma$, and hence, by continuity and by choosing ρ_0 sufficiently small:

$$V = \operatorname{span}_{\mathbb{R}} \{J_0(x_0 + y)\} \oplus \Sigma, \qquad \forall y \in U_0.$$

This allows us to write

$$X_H(x_0 + y) = \Omega(y)J_0(x_0 + y) + Y(y), \qquad \forall y \in U_0, \tag{5.15}$$

for some uniquely determined smooth mappings $\Omega : U_0 \to \mathbb{R}$ and $Y : U_0 \to \Sigma$. The uniqueness of this decomposition implies that Ω is G_0-invariant and Y is G_0-equivariant:

$$\Omega(\exp(J_0\theta)y) = \Omega(y) \quad \text{and} \quad Y(\exp(J_0\theta)y) = \exp(J_0\theta)Y(y), \quad \forall y \in U_0,\ \forall \theta \in G_0. \tag{5.16}$$

Also, (5.14) implies that $X_H(x_0) = \Omega_0 J_0 x_0$ for some $\Omega_0 \in \mathbb{R}$ ($\Omega_0 \neq 0$); hence $\Omega(0) = \Omega_0$ and $Y(0) = 0$. From (5.15) and the S^1-equivariance of X_H it follows then that

$$X_H\left(\exp(J_0\theta)(x_0 + y)\right) = \exp(J_0\theta)\left(\Omega(y)J_0(x_0 + y) + Y(y)\right), \qquad \forall y \in U_0,\ \forall \theta \in \mathbb{R}. \tag{5.17}$$

Fix some $y_0 \in U_0$, and consider the corresponding solution $\varphi_H^t(x_0+y_0)$; as far as this solution remains in our tubular neighborhood we can write it in the form

$$\varphi_H^t(x_0 + y_0) = \exp(J_0\theta(t))(x_0 + y(t)) \tag{5.18}$$

for some smooth functions $\theta : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to U_0$ satisfying $\theta(0) = 0$ and $y(0) = y_0$. Expressing that $\varphi_H^t(x_0 + y_0)$ forms a solution of the equation $\dot{x} = X_H(x)$ we find

$$\exp(J_0\theta(t))\left(\dot\theta(t)J_0(x_0 + y(t)) + \dot y(t)\right) = \exp(J_0\theta(t))\left(\Omega(y(t))(x_0 + y(t)) + Y(y(t))\right), \quad \forall t,$$

i.e. we have $\dot\theta(t) = \Omega(y(t))$ and $\dot y(t) = Y(y(t))$. This shows that

$$\varphi_H^t(x_0 + y) = \exp(J_0\tilde\theta(t, y))\left(x_0 + \tilde\varphi(t, y)\right), \qquad \forall y \in U_0,\ \forall t, \tag{5.19}$$

where $\tilde\varphi(t, y)$ denotes the flow of the G_0-equivariant equation

$$\dot y = Y(y), \tag{5.20}$$

and where

$$\tilde\theta(t, y) := \int_0^t \Omega(\tilde\varphi(\tau, y))\, d\tau. \tag{5.21}$$

We conclude that the flow φ_H of X_H is near γ_0 completely determined by the flow of (5.20): it consists of the flow $\tilde\varphi$ of (5.20) combined with a *drift along group orbits*. For example, equilibria of (5.20) correspond to periodic orbits of X_H which are at the same time group orbits: if $Y(y_0) = 0$ then $\varphi_H(x_0 + y_0) = \exp(J_0\Omega(y_0)t)(x_0 + y_0)$, which is periodic with period $2\pi/\Omega(y_0)$. In particular $\varphi_H^t(x_0) = \exp(J_0\Omega_0 t)x_0$. Such special periodic orbits are called *relative equilibria*. Closed orbits $\gamma \subset U_0$ of (5.20) corresponds to an invariant torus $\mathcal{T}$ of X_H, given by $\mathcal{T} := \{\exp(J_0\theta)(x_0 + y) \mid \theta \in \mathbb{R}, y \in \gamma\}$; if T_0 is the minimal period along γ and $T_1 := \tilde\theta(T_0, y_0)$ for any $y_0 \in \gamma$, then the flow on $\mathcal{T}$ is periodic or quasi-periodic, depending on whether T_1/T_0 is rational or irrational.

There remains the question whether the reduced equation (5.20) inherits any of the Hamiltonian structure of the starting equation $\dot x = X_H(x)$. To show that this is indeed the case we return to the function $S \in \Lambda_0(V)$ which generates the symplectic S^1-action on V (see (5.3)). It follows from (5.12) that $\{H, S\}(x) = DH(x) \cdot X_S(x) = DH(x) \cdot J_0 x = 0$; then Lemma 21 implies that the level sets $\mathcal{S}_\alpha := \{x \in V \mid S(x) = \alpha\}$ ($\alpha \in \mathbb{R}$) of S are invariant under the flow φ_H of X_H, and that $X_H(x) \in T_x\mathcal{S}_\alpha$ for each $x \in \mathcal{S}_\alpha$. For each α near $\alpha_0 := S(x_0)$ we set

$$\Sigma_\alpha := \{y \in U_0 \mid x_0 + y \in \mathcal{S}_\alpha\}, \tag{5.22}$$

and we define $Y_\alpha : \Sigma_\alpha \to \Sigma$ and $H_\alpha : \Sigma_\alpha \to \mathbb{R}$ by

$$Y_\alpha(y) := Y(y) \qquad \text{and} \qquad H_\alpha(y) := H(x_0 + y), \qquad \forall y \in \Sigma_\alpha. \tag{5.23}$$

Theorem 24. *Under the foregoing conditions and by choosing $\rho_0 > 0$ sufficiently small we have for all α near α_0 that:*

(i) Σ_α *is a symplectic submanifold;*

(ii) Y_α *defines a vectorfield on* Σ_α*, i.e.* $Y_\alpha(y) \in T_y\Sigma_\alpha$ *for all* $y \in \Sigma_\alpha$*;*

(iii) Y_α *is a Hamiltonian vectorfield, corresponding to the Hamiltonian* H_α*;*

(iv) $\{\exp(J_0\theta) \mid \theta \in G_0\}$ *defines a symplectic group action on* Σ_α*, and* Y_α *is equivariant with respect to this* G_0*-action.*

Proof . Define $F : V \to \mathbb{R}$ by $F(x) := \langle J_0x_0, x\rangle$; then

$$\{F, S\}(x_0) = DF(x_0) \cdot X_S(x_0) = \langle J_0x_0, X_S(x_0)\rangle = \langle J_0x_0, J_0x_0\rangle \neq 0,$$

and (i) follows from Lemma 18, since Σ_α is defined as the intersection of a level set of F with a level set of S. If $y \in \Sigma_\alpha$ then $x_0 + y \in \mathcal{S}_\alpha$, $X_H(x_0 + y) \in T_{x_0+y}\mathcal{S}_\alpha$, and since by (3.13) also $J_0(x_0 + y)$ belongs to $T_{x_0+y}\mathcal{S}_\alpha$ we conclude from (5.15) that $Y_\alpha(y) = Y(y) \in \Sigma \cap T_{x_0+y}\mathcal{S}_\alpha = T_y\Sigma_\alpha$. This proves (ii). With H_α defined by (5.23) and again using (3.13) we find for each $y \in \Sigma_\alpha$ and each $v \in T_y\Sigma_\alpha$ that

$$\begin{aligned} DH_\alpha(y) \cdot v &= DH(x_0 + y) \cdot v = \omega_0(X_H(x_0 + y), v) \\ &= \Omega(y)\omega_0(J_0(x_0 + y), v) + \omega_0(Y(y), v) \\ &= \omega_0(Y_\alpha(y), v). \end{aligned}$$

So we have (iii). Finally, (iv) is an obvious consequence of (5.16) and the fact that both Σ and $\mathcal{S}_\alpha$ are invariant under the G_0-action. ■

Combining Theorem 24 with the Darboux Theorem we conclude that the study of the system (5.20) amounts to the study of a one-parameter family

$$\dot{y} = X_{H_\alpha}(y) \qquad (\alpha \in \mathbb{R}) \tag{5.24}$$

of Hamiltonian systems on a symplectic vectorspace (W, ω_1), with $\dim W = \dim V - 2$. The equation (5.24) has for $\alpha = \alpha_0$ an equilibrium at $y = 0$. Also, (5.24) is equivariant with respect to a symplectic $G_0 \cong \mathbb{Z}_p$-action, for some $p \geq 1$; this group action is generated by a symplectic diffeomorphism Φ_α such that $\Phi_\alpha^p = I_W$. A priori, this generator Φ_α will be nonlinear and dependent on α. We believe that it should be possible to apply an appropriate equivariant version of the Darboux Theorem to show that in an appropriate coordinate system this generator will be linear and independent of α, but we have not worked out the details.

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