CONTINUATION OF PERIODIC SOLUTIONS IN CONSERVATIVE AND REVERSIBLE SYSTEMS

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In this note we show how very general continuation results can be used to describe (1) families of periodic orbits in conservative systems, and (2) families of symmetric and doubly-symmetric orbits in reversible conservative systems. We describe a general approach which repairs the lack of submersivity for such problems due to the presence of first integrals. Our results can in particular be applied in the Hamiltonian context; we very briefly describe how they can be used for the continuation of choreographies in the N-body problem.

Keywords: Continuation; Conservative Systems; Reversible Systems; Multi-Symmetric Solutions.

1. Introduction

Periodic orbits in systems having a first integral typically appear in one-parameter families, and the same is true for symmetric periodic orbits in reversible systems. When the number of (independent) first integrals increases, also the families of periodic orbits will become higher dimensional. This leads to the problem on how to parametrize the periodic orbits along these families, and, from a numerical point of view, how to calculate these families. A typical approach would be to fix the values of some appropriate first integrals and to work on level sets. However, theoretical results show that this is not always possible; moreover, it leads to implicit equations which one may want to avoid in numerical calculations.

In this note we describe an approach which avoids this type of problems. Instead of imposing internal constraints to reduce the problem we actually extend it by adding appropriate gradient terms to the equations; in the end these terms will appear to vanish along the solutions we calculate, such that we get indeed solutions of the original problem. But moreover, these additional terms regularize the problem, in such a way that standard techniques can be used for the augmented problem. Although most of the applications will be found in the Hamiltonian and reversible Hamiltonian context we do not specialize to that particular situation here. However, we very briefly indicate how our results can be used for the continuation of choreographies in the N-body problem, for example when one changes some of the masses. More details, both on the theory and the applications, can be found in a series of papers by Muñoz-Almaraz et al.[1–3]

2. Continuation of zeros of constrained mappings

Let $x_0 \in \mathbb{R}^m$ be a zero of some smooth mapping $f : \mathbb{R}^m \to \mathbb{R}^n$; if f is submersive at x_0, i.e. if $\mathrm{Im}Df(x_0) = \mathbb{R}^n$, then by the Implicit Function Theorem the zero set of f forms locally near x_0 a smooth $(m-n)$-dimensional submanifold of $\mathbb{R}^m$ (the submersivity condition requires $m \geq n$).

There are situations where the structure of the mapping f prevents the submersivity condition to be satisfied. Consider in particular the case where $f = g - h$ for some smooth $g, h : \mathbb{R}^m \to \mathbb{R}^n$, and such that the space

$$\mathcal{F} := \{F : \mathbb{R}^n \to \mathbb{R} \mid F \circ g = F \circ h\}$$

contains some non-constant functions. Differentiating the identity $F(g(x)) = F(h(x))$ at a zero x_0 of f (i.e. $g(x_0) = h(x_0) =: y_0$) we find that

$$DF(y_0) \cdot Dg(x_0) \cdot \tilde{x} = DF(y_0) \cdot Dh(x_0) \cdot \tilde{x}, \quad \forall \tilde{x} \in \mathbb{R}^m,$$

from which we deduce that

$$\mathrm{Im}Df(x_0) = \mathrm{Im}(Dg(x_0) - Dh(x_0)) \subset \cap_{F \in \mathcal{F}} \ker DF(y_0).$$

Or stated differently,

$$\mathrm{Im}Df(x_0) \subset W^{\perp}, \quad W := \{\nabla F(y_0) \mid F \in \mathcal{F}\}. \tag{1}$$

We call a mapping f such as described here a **constrained mapping**; we are interested in the continuation of a given zero x_0 of such constrained mapping. The structural lack of submersivity due to the inclusion (1) can be easily repaired, as follows.

Lemma 2.1. *Let $x_0 \in \mathbb{R}^m$ be a zero of a constrained mapping $f = g - h$. Then all solutions $(x, w) \in \mathbb{R}^m \times W$ of the equation*

$$g(x) = h(x) + w \tag{2}$$

sufficiently close to $(x_0, 0)$ are of the form $(x, 0)$, with $x \in \mathbb{R}^m$ a zero of f.

Proof. Let P be the orthogonal projection in $\mathbb{R}^n$ onto $W^\perp$, and let $F_i \in \mathcal{F}$ $(1 \le i \le k := \dim W)$ be such that $\{\nabla F_i(y_0) \mid 1 \le i \le k\}$ forms a basis of W. Then (2) and the definition of $\mathcal{F}$ imply

$$P(g(x)) = P(h(x)) \quad \text{and} \quad F_i(g(x)) = F_i(h(x)), \;\; (1 \le i \le k).$$

Since the mapping $y \mapsto (P(y), F_1(y), F_2(y), \ldots, F_k(y))$ is locally near y_0 a diffeomorphism from $\mathbb{R}^n$ onto $W^\perp \times \mathbb{R}^k$ it follows that $g(x) = h(x)$ and $w = 0$. □

It follows from the lemma that finding the zeros $x \in \mathbb{R}^m$ near x_0 of f is equivalent to finding the zeros $(x, w) \in \mathbb{R}^m \times W$ near $(x_0, 0)$ of the "augmented mapping" $\tilde{f} : \mathbb{R}^m \times W \to \mathbb{R}^n$ defined by

$$\tilde{f}(x, w) := g(x) - h(x) - w.$$

It is a trivial observation that $\tilde{f}$ is submersive at $(x_0, 0)$ if and only if the inclusion in (1) is actually an equality. This leads to the following definition.

Definition 2.1. We say that $x_0 \in \mathbb{R}^m$ is a **normal zero** of the constrained mapping $f = g - h : \mathbb{R}^m \to \mathbb{R}^n$ if $g(x_0) = h(x_0) =: y_0$ and if $\mathrm{Im}Df(x_0) = W^\perp$, where $W := \{\nabla F(y_0) \mid F \in \mathcal{F}\}$.

In combination with lemma 2.1 and the IFT this leads to our main abstract result.

Theorem 2.1. *Near each of its normal zeros the zero set of a constrained mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ forms a smooth $(m - n + k)$-dimensional manifold, with $k := \dim W$.*

3. Periodic orbits in conservative systems

A direct application of Theorem 2.1 is given by periodic orbits in conservative systems. Consider a smooth n-dimensional system

$$\dot{x} = X(x), \tag{3}$$

denote its flow by $\tilde{x}(t; x)$, and assume that the space

$$\mathcal{F} := \{F : \mathbb{R}^n \to \mathbb{R} \mid DF(x) \cdot X(x) \equiv 0\} \tag{4}$$

contains some non-constant functions. Typical examples of such systems are Hamiltonian systems having some (continuous) symmetries: in this case n is even (say, $n = 2N$), $X(x) = J\nabla H(x)$ for some smooth $H : \mathbb{R}^{2N} \to \mathbb{R}$ and with $J \in \mathcal{L}(\mathbb{R}^{2N})$ the standard symplectic matrix, and $\mathcal{F} = \{F : \mathbb{R}^{2N} \to \mathbb{R} \mid \{H, F\} \equiv 0\}$.

Periodic orbits of (3) with minimal period $T > 0$ are given by the zeros of the mapping

$$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n,\ (T, x) \mapsto f(T, x) := \tilde{x}(T; x) - x.$$

This mapping f is a constrained mapping, since $F(\tilde{x}(T; x)) = F(x)$ for all $F \in \mathcal{F}$. Given a zero (T_0, x_0) of f one can easily verify that

$$\mathrm{Im} Df(T_0, x_0) = \mathbb{R}X(x_0) + \mathrm{Im}(M - \mathrm{Id}),$$

where M is the monodromy matrix of the T_0-periodic solution $\tilde{x}(t; x_0)$. We say that $\tilde{x}(t; x_0)$ is a **normal periodic solution** if (T_0, x_0) is a normal zero of f, i.e. if

$$\mathbb{R}X(x_0) + \mathrm{Im}(M - \mathrm{Id}) = W^{\perp}, \quad W := \{\nabla F(x_0) \mid F \in \mathcal{F}\}.$$

By Theorem 2.1 such normal periodic orbits belong to a k-parameter family of periodic orbits of (3), where $k := \dim W$; indeed, $m = n + 1$, so we have a $(k + 1)$-dimensional manifold of zeros of f, but this manifold is of course foliated by 1-dimensional periodic orbits.

To calculate this family of periodic orbits we choose $F_j \in \mathcal{F}$ $(1 \leq j \leq k)$ such that $\{\nabla F_j(x_0) \mid 1 \leq j \leq k\}$ forms a basis of W, and we solve the "regularized equation"

$$\tilde{f}(T, x, \alpha) := \tilde{x}(T, x) - x - \sum_{j=1}^{k} \alpha_j \nabla F_j(x_0) = 0 \tag{5}$$

for $(T, x, \alpha) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$ near $(T_0, x_0, 0)$. All solutions will be of the form $(T, x, 0)$, with (T, x) a zero of f. There is however a different (and for numerical purposes better) way of regularizing the periodicity condition. It is based on the following lemma.

Lemma 3.1. *Let $F \in \mathcal{F}$ and let $\hat{x}(t)$ be a periodic solution of*

$$\dot{x} = X(x) + \nabla F(x).$$

Then $\nabla F(\hat{x}(t)) \equiv 0$, and $\hat{x}(t)$ is actually a periodic solution of (3).

Therefore, instead of solving (5) one can use the modified system

$$\dot{x} = X(x) + \sum_{j=1}^{k} \alpha_j \nabla F_j(x), \tag{6}$$

and obtain the desired family of periodic orbits of (1) by solving the new periodicity condition

$$\tilde{x}_{mod}(T; x, \alpha) = x; \tag{7}$$

here $\tilde{x}_{mod}(t; x, \alpha)$ denotes the flow of (6). In practice one needs to add to (7) some phase conditions[1] in order to prevent the calculation of some "trivial continuations" of the starting solution.

4. Reversible systems

In (time)-reversible systems there are different ways for obtaining some periodic orbits, but again, in the presence of first integrals similar regularity (submersivity) problems as explained in Section 3 do arise. The system (1) is **reversible** if there exists a compact group G of linear operators on $\mathbb{R}^n$ and a non-trivial character (group homomorphism) $\chi : G \to \{1, -1\}$ such that

$$X(g \cdot x) = \chi(g)\, g \cdot X(x), \quad \forall g \in G,\ \forall x \in \mathbb{R}^n.$$

Then the flow $\tilde{x}(t; x)$ will be such that $\tilde{x}(\chi(g)t; g \cdot x) = g \cdot \tilde{x}(t; x)$. An operator $R \in \Gamma$ such that $\chi(R) = -1$ is called a **reversor** of (1); then $\tilde{x}(-t, Rx) = R\tilde{x}(t, x)$. A (maximal) orbit γ of (1) is called **R-symmetric** if $R(\gamma) = \gamma$; it is easily shown that this will be the case if and only if $\gamma \cap \mathrm{Fix}(R) \neq \emptyset$, where $\mathrm{Fix}(R) := \{x \in \mathbb{R}^n \mid Rx = x\}$. Setting $t = 0$ at the intersection point the corresponding solution $x(t)$ will be such that $x(-t) = Rx(t)$ and $x(t) = R^2x(t)$, i.e. the orbit γ is contained in $\mathrm{Fix}(R^2)$. Therefore, when considering R-symmetric solutions one can w.l.o.g. assume that $R^2 = \mathrm{Id}$.

Roughly speaking, doubly-symmetric solutions are solutions (orbits) which are symmetric with respect to two reversors. More precisely, we use the following definition.

Definition 4.1. Let R_0 and R_1 be two reversors of the system (1) (the case $R_1 = R_0$ is allowed and, as explained before, we will assume that $R_0^2 = R_1^2 = \mathrm{Id}$). Then we say that a solution $x(t)$ of (1) is (R_0, R_1)-symmetric if there exist $t_0 < t_1$ such that $x(t_0) \in \mathrm{Fix}(R_0)$ and $x(t_1) \in \mathrm{Fix}(R_1)$; we call $[t_0, t_1]$ a basic domain of such (R_0, R_1)-symmetric solution. Usually we will set $t_0 = 0$ and $t_1 = T > 0$.

As shown in Muñoz Almaraz et al.[3] both the Figure Eight choreography for three equal bodies and Gerver's Supereight choreography for four equal bodies are examples of such doubly-symmetric solutions.

Assuming that the basic domain is $[0, T]$ the (R_0, R_1)-symmetric solution $x(t)$ will be such that $x(-t) = R_0x(t)$ and $x(T+t) = R_1x(T-t)$; from these it is easily shown that the solution $x(t)$ can be extended for all $t \in \mathbb{R}$. Moreover, we have for each $m \in \mathbb{Z}$ that

- $x(2mT+t) = (R_1R_0)^m x(t)$;
- $x(2mT) = (R_1R_0)^{2m-1}R_1x(2mT)$;
- $x((2m+1)T) = (R_1R_0)^m R_1 x((2m+1)T)$.

In case there exists some M such that $(R_1R_0)^M = \mathrm{Id}$ (something which appears in many applications) then $x(t)$ is automatically $2MT$-periodic, and also (R_0,R_0)-symmetric with basic domain $[0,MT]$, in other words, we have a R_0-symmetric periodic orbit. This holds in particular with $M=1$ in the case $R_1=R_0$ (due to our assumption that $R_0^2=\mathrm{Id}$).

Next we consider the continuation of such (R_0,R_1)-symmetric solutions. Since $R_0^2=R_1^2=\mathrm{Id}$ we can split the phase space $\mathbb{R}^n$ as

$$\mathbb{R}^n = \mathrm{Fix}(R_0)\oplus\mathrm{Fix}(-R_0) = \mathrm{Fix}(R_1)\oplus\mathrm{Fix}(-R_1);$$

we denote the corresponding projections as $\pi_0^\pm$ and $\pi_1^\pm$, respectively:

$$\pi_0^\pm := \frac{1}{2}(\mathrm{Id}\pm R_0),\quad \pi_1^\pm := \frac{1}{2}(\mathrm{Id}\pm R_1).$$

We obtain (R_0,R_1)-symmetric solutions with basic domain $[0,T]$ by looking for those $x\in\mathrm{Fix}(R_0)$ which are such that $\pi_1^-\tilde{x}(T;x)=0$. If the mapping $f_{0,1}:\mathbb{R}\times\mathrm{Fix}(R_0)\to\mathrm{Fix}(-R_1)$ given by $f_{0,1}(T,x):=\pi_1^-\tilde{x}(T;x)$ is submersive at some of its zeros (T_0,x_0) then this zero belongs to a d-dimensional manifold of zeros, where $d=1+\dim\mathrm{Fix}(R_0)-\dim\mathrm{Fix}(-R_1)$. In many applications the phase space is even-dimensional ($n=2N$) and $\dim\mathrm{Fix}(\pm R_0)=\dim\mathrm{Fix}(\pm R_1)=N$; in this case, and when the submersivity condition is satisfied, (R_0,R_1)-symmetric solutions belong to one-parameter families of such solutions. From now on we will assume these conditions on the fixed point subspaces of R_0 and R_1.

Similarly as for periodic solutions in conservative systems also here the existence of some first integrals for the reversible system may cause the submersivity condition to fail; however, this time not all first integrals come into play. More in particular, we will consider the subspace $\mathcal{F}_{0,1}$ of $\mathcal{F}$ given by

$$\mathcal{F}_{0,1} := \{F\in\mathcal{F}\mid F \text{ is constant on } \mathrm{Fix}(R_0)\cup\mathrm{Fix}(R_1)\}.$$

When $\mathcal{F}_{0,1}$ contains some non-constant functions then the mapping $f_{0,1}$ is a constrained mapping, as follows. We consider $f_{0,1}$ as a mapping from $\mathbb{R}\times\mathrm{Fix}(R_0)$ into the full phase space $\mathbb{R}^{2N}$ and write it in the form $f_{0,1}(T,x)=\tilde{x}(T;x)-\pi_1^+\tilde{x}(T;x)$. Then we have N linear constraints, given by $\pi_1^+\tilde{x}(T;x)=(\pi_1^+)^2\tilde{x}(T;x)$, and moreover, we have for each $F\in\mathcal{F}_{0,1}$ that

$$F(\tilde{x}(T;x)) = F(x) = F(\pi_1^+\tilde{x}(T;x));$$

for the last equality we use the facts that $x \in \mathrm{Fix}(R_0)$, $\pi_1^+ \tilde{x}(T;x) \in \mathrm{Fix}(R_1)$, and that F is constant on $\mathrm{Fix}(R_0) \cup \mathrm{Fix}(R_1)$. It follows that at each zero $(T_0, x_0) \in \mathbb{R} \times \mathrm{Fix}(R_0)$ of $f_{0,1}$ we have

$$\mathrm{Im} Df_{0,1}(T_0, x_0) \subset W_{0,1}^{\perp} \cap \mathrm{Fix}(-R_1),$$

where

$$W_{0,1} = \{\nabla F(x_1) \mid F \in \mathcal{F}_{0,1}\} \quad \text{and} \quad x_1 := \tilde{x}(T_0, x_0).$$

Observe that $x_1 \in \mathrm{Fix}(R_1)$ and that $\nabla F(x_1) \in \mathrm{Fix}(-R_1)$ for $F \in \mathcal{F}_{0,1}$.

Definition 4.2. We say that a zero $(T_0, x_0) \in \mathbb{R} \times \mathrm{Fix}(R_0)$ of $f_{0,1}$ generates a **normal** (R_0, R_1)**-symmetric solution** of the reversible conservative system (1) if

$$\mathrm{Im} Df_{0,1}(T_0, x_0) = W_{0,1}^{\perp} \cap \mathrm{Fix}(-R_1). \tag{8}$$

Theorem 4.1. *A normal (R_0, R_1)-symmetric solution belongs to a $(1 + k_{0,1})$-parameter family of such (R_0, R_1)-symmetric solutions, where $k_{0,1} := \dim W_{0,1}$.*

This $(1 + k_{0,1})$-parameter family can be obtained by considering the augmented system

$$\dot{x} = X(x) + \sum_{j=1}^{k_{0,1}} \alpha_j \nabla F_j(x), \tag{9}$$

where $F_j \in \mathcal{F}_{0,1}$ $(1 \le j \le k_{0,1})$ are chosen such that $\{\nabla F_j(x_1) \mid 1 \le j \le k_{0,1}\}$ forms a basis of $W_{0,1}$. Denoting by $\tilde{x}_{0,1}(t; x, \alpha)$ the flow of (9) one has to solve then the equation

$$\pi_1^- \tilde{x}_{0,1}(T; x, \alpha) = 0 \tag{10}$$

for $(T, x, \alpha) \in \mathbb{R} \times \mathrm{Fix}(R_0) \times \mathbb{R}^{k_{0,1}}$ near $(T_0, x_0, 0)$. This problem is now regular (i.e. the submersivity condition is satisfied) and all solutions will have the form $(T, x, 0)$, with (T, x) a zero of $f_{0,1}$. The proofs can be found in Muñoz Almaraz et al.[3]

5. Application to N-body problems

As we have already indicated above a large area of application of the foregoing ideas and results can be found in Hamiltonian systems, and in particular in the N-body problem from celestial mechanics. For example, they can be used to calculate continuations of the by now well known Figure Eight and Supereight choreographies for respectively three and four equal bodies. These choreographies can be considered either as periodic orbits, or as R-symmetric periodic orbits for an appropriate reversor R, or as (R_0, R_1)-symmetric solutions for several choices of reversors R_0 and R_1. The first integrals are the total energy and the components of the total linear momentum and the total angular momentum; also, since these choreographies are planar, one can work either in $\mathbb{R}^2$ or in $\mathbb{R}^3$.

There is a particular aspect of this application which we should mention here. The general N-body problem has in addition to the translational and rotational symmetries generated by the first integrals also a rescaling symmetry. When applying our continuation schemes for the continuation of the Figure Eight or the Supereight one finds families of choreographies which can be obtained from the starting choreography by symmetries and rescalings. So, this way we get nothing particularly new. The way out is to fix the period (this prevents rescalings) and to use some external parameter as one of the variables of the problem. For example, one can change one or more of the masses. Using this approach one in fact allows the system to change, which implies that one can only use those symmetries which are compatible with these changes. Again we refer to the papers of Muñoz Almaraz et al.[1–3] and to the paper by Doedel et al.[4] for the results of such continuation studies; on the website `http://www.maia.ub.es/~malmaraz/investigacion/Jaca/jaca.xml` one can find some numerical data and graphics related to this problem.

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