Quantified versions of the Ingham-Karamata Tauberian theorem

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Tauberian Theory

Tauberian theory: Extracting asymptotic information from integral transforms

$$\tau(t) \underset{\textbf{Tauberian theory}}{\overset{\textbf{Integral transforms}}{\rightleftarrows}} \int_0^\infty e^{-st}\tau(t)dt, \quad \int_{-\infty}^\infty \frac{\tau(t)}{t+z}dt, \quad \dots$$

The Ingham-Karamata theorem

Theorem (Ingham, Karamata, 1934)

Let $\tau : \mathbb{R}_+ \to \mathbb{R}$ *be such that* $\tau(x) + Ax$ *is non-decreasing for certain* $A > 0$*. Suppose that*

$$\mathcal{L}\{\tau; s\} = \int_0^\infty e^{-su}\tau(u)\mathrm{d}u$$

converges for $\operatorname{Re} s > 0$ *and admits an analytic continuation beyond* $\operatorname{Re} s = 0$*, then*

$$\tau(x) = o(1), \quad x \to \infty.$$

An application: short proof of the PNT

Ingredients:

- $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ admit an meromorphic extension beyond $\operatorname{Re} s = 1$ with a unique simple pole at $s = 1$ with residue 1.
- $\zeta(1 + it) \neq 0$

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Let

$$\psi_1(x) := \sum_{n \leq x} \frac{\Lambda(n)}{n}$$

We aim to show that

$$\psi_1(x) = \log x - \gamma + o(1),$$

where γ is the Euler-Mascheroni constant.

Proof PNT (continued)

We set

$$\tau(x) := \sum_{n \leq e^x} \frac{\Lambda(n)}{n} - x + \gamma.$$

Its Laplace transform is

$$-\frac{\zeta'(s+1)}{s\zeta(s+1)} - \frac{1}{s^2} + \frac{\gamma}{s}.$$

From the ingredients, it follows that τ satisfies all the hypotheses for Ingham-Karamata, thus

$$\tau(x) = o(1).$$

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- Consider flexible one-sided Tauberian conditions/Treat more general singularities on the Laplace transform.
- Establish optimality of the quantified rate.
- Consider different types of boundary behavior of the Laplace transform.

Sketch of proof of unquantified Ingham-Karamata theorem:
Laplace transform behavior implies via Riemann-Lebesgue lemma

$$\left\langle \hat{\tau}(t), e^{iht}\hat{\phi}(t) \right\rangle = o_{\phi}(1), \quad h \to \infty,$$

for all $\phi \in \mathcal{F}(\mathcal{D}(\mathbb{R}))$.

Treatment of one-sided Tauberian conditions

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for all $\phi \in \mathcal{F}(\mathcal{D}(\mathbb{R}))$.
This translates to

$$\int_{-\infty}^{\infty} \tau(x+h)\phi(x)\mathrm{d}x = o_{\phi}(1), \quad h \to \infty.$$

Treatment of one-sided Tauberian conditions (continued)

The Tauberian condition implies

$$\tau(x) = \tau(x) + Ax - Ax \leq \tau(x+y) + Ay, \quad y \geq 0$$

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$$\tau(x) = \tau(x) + Ax - Ax \geq \tau(x+y) + Ay, \quad y \leq 0.$$

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So, if $\int_{-\infty}^{\infty} \phi = 1$, $x\phi(x) \ge 0$ and $\int_{-\infty}^{\infty} x\phi(x)\mathrm{d}x = C < \infty$,

$$\begin{aligned} \tau(y) &= \int_{-\infty}^{\infty} \tau(y)\phi(\lambda x)\lambda \mathrm{d}x \\ &\le \int_{-\infty}^{\infty} \tau(x+y)\phi(\lambda x)\lambda \mathrm{d}x + A\int^{\infty} \lambda y \phi(\lambda y)\mathrm{d}y \\ &\le o_{\lambda,\phi}(1) + \frac{AC}{\lambda}. \end{aligned}$$

One-sided Tauberian conditions: Remarks

- Selection of test functions crucial! One can show that admissible test functions exist.
- This proof of the Tauberian theorem in combination with the above deduction of the PNT is one of the quickest proofs of the prime number theorem available.
- Technique also leads to simpler proofs for other one-sided Tauberian theorems, such as the Berry-Esseen inequality.

Stahn's quantified theorem

Theorem (Stahn, 2018)

Let $\tau : [0,\infty) \to \mathbb{C}$ *be a Lipschitz continuous function. Let* $M, K : \mathbb{R}_+ \to (0,\infty)$ *be two continuous non-decreasing functions for which there exists* $\varepsilon \in (0,1)$ *such that*

$$K(t) \ll \exp\left(\exp\left((tM(t))^{1-\varepsilon}\right)\right), \quad t \to \infty.$$

If $\mathcal{L}\{\tau; s\}$ *admits an analytic extension to*

$$\Omega_M := \{s := |\operatorname{Re} s| \leq 1/M(|\operatorname{Im} s|)\},$$

where $|\mathcal{L}\{\tau; s\}| \ll K(|s|)/|s|$ *as* $|s| \to \infty$, *then*

$$\tau(x) \ll M_{K,\log}^{-1}(x)^{-1}, \quad x \to \infty,$$

where $M_{K,\log}^{-1}$ *is the inverse function of* $M_{K,\log}(t) = M(t)(\log t + \log\log t + \log K(t))$.

Stahn's quantified theorem (continued)

Theorem

Furthermore, if K is of positive increase, that is, there exists $a, t_0 > 0$ such that $t^{-a}K(t) \ll R^{-a}K(R)$ for all $t_0 \leq t \leq R$ as $R \to \infty$, then

$$\tau(x) \ll M_K^{-1}(x)^{-1}, \quad x \to \infty,$$

with M_K^{-1} the inverse function of $M_K(t) = M(t)(\log t + \log K(t))$.

Our main quantified Tauberian theorem (simplified)

Theorem (D., 2024)

Let $\tau : [0, \infty) \to \mathbb{R}$ *be such that* $\tau(x) + Ax$ *is non-decreasing. Let* M, K *be continuous non-decreasing functions on* $\mathbb{R}_+$ *such that* $\mathcal{L}\{\tau; s\}$ *admits an analytic extension to* Ω_M *where it satisfies the bound* $K(|s|)/|s|$ *as* $|s| \to \infty$*. Then, for any* $c < 1$*,*

$$\tau(x) \ll M_{K,\log}^{-1}(cx)^{-1}, \quad x \to \infty.$$

The above estimate holds with $c = 1$ *if, additionally,* $M_{K,\log}$ *is of* $675M(0)^{-1}$*-regular growth, that is, there exists* $C, t_0 > 0$ *such that*

$$\frac{M_{K,\log}(Ct)}{M_{K,\log}(t)} \geq 1 + \frac{675}{M(0)t}, \quad t \geq t_0.$$

If, additionally, $K(t)$ *is of positive increase or* $M(t)/\log^{\beta} t$ *is eventually non-decreasing for some* $\beta > 0$*, then*

$$\tau(x) \ll M_K^{-1}(x)^{-1}, \quad x \to \infty.$$

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- We also have (sharp) quantified theorems under the flexible one-sided Tauberian condition

$$\tau(x) + F(x) \text{ is non-decreasing,}$$

where $F : (0, \infty) \to \mathbb{R}$ is some functions satisfying

$$|F(x+y) - F(x)| \ll f(x)|y| \exp(|x|^{\alpha}), \quad x, y \in \mathbb{R},$$

for some $0 < \alpha < 1$ and a function $f : (0, \infty) \to \mathbb{R}$.

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for some $0 < \alpha < 1$ and a function $f : (0, \infty) \to \mathbb{R}$.
- Different boundary assumptions are also treated.

Optimality: a quantified model theorem

Theorem

Let $N \in \mathbb{N}$, $M > -1$ and $\tau : \mathbb{R}_+ \to \mathbb{R}$ be such that $\tau(x) + Ax$ is non-decreasing for certain $A > 0$. Suppose that

$$\mathcal{L}\{\tau; s\} = \int_0^\infty e^{-su}\tau(u)\mathrm{d}u$$

converges for $\operatorname{Re} s > 0$ and admits an N times differentiable extension $g(t) := \mathcal{L}\{\tau; it\}$ to $\operatorname{Re} s = 0$, satisfying

$$\left|g^{(N)}(t)\right| \ll (1 + |t|)^M, \quad t \in \mathbb{R},$$

then

$$\tau(x) \ll x^{-N/(M+2)}, \quad x \to \infty.$$

Question: Is the decay rate optimal?

Structure of the optimality theorem

Theorem (D., 2018)

Suppose that all functions τ *satisfying the hypotheses of the model theorem admit the decay rate*

$$\tau(x) \ll \frac{1}{V(x)}, \quad x \to \infty,$$

then

$$V(x) \ll x^{N/(M+2)}, \quad x \to \infty.$$

The key proof idea

Collect the functions τ who satisfy the (more restrictive) hypotheses of the model theorem into a Banach space X_1, topologized via

$$\|\tau\|_1 = \sup_{x \geq 0} |\tau'(x)| + \sup_{t \in \mathbb{R}} \frac{|g^{(N)}(t)|}{(1+|t|)^M}.$$

Collect the functions τ which additionally satisfy the decay rate $1/V(x)$ in another Banach space X_2, topologized via

$$\|\tau\|_2 = \|\tau\|_1 + \sup_{x \geq 0} |\tau(x)V(x)|.$$

The key proof idea: open mapping theorem

Consider the canonical inclusion mapping $\iota : X_2 \to X_1$, which is clearly continuous.
If $V(x)$ is an acceptable decay rate for the model theorem, then ι is *surjective* and therefore by the **open mapping theorem** an open mapping, that is, ι^{-1} is also continuous:

$$\sup_{x\geq 0} |\tau(x)V(x)| \ll \sup_{x\geq 0} |\tau'(x)| + \sup_{t\in\mathbb{R}} \frac{|g^{(N)}(t)|}{(1+|t|)^M}.$$

The rest of the proof consists in considering the families $\tau_{y,\lambda}(x) := \kappa(\lambda(x-y))$ for a well-chosen function κ and optimizing the parameters y and λ.

The unquantified Ingham-Karamata theorem

Theorem (D.-Vindas, 2018)

Let $-1 < \alpha < 0$*. Suppose every function who satisfies the hypotheses of the unquantified Ingham-Karamata theorem with even an analytic extension to the half-plane* $\operatorname{Re} s > -\alpha$ *satisfies* $\tau(x) \ll V(x)$*, then*

$$V(x) \ll 1.$$

For a constructive proof (Broucke-D.-Vindas, 2021)

An optimality theorem for analytic extensions

Theorem (D., 2024)

Let $M, K : \mathbb{R}_+ \to \mathbb{R}$ *be non-decreasing positive functions. Let*

$$K(x) \ll \exp(\exp(CxM(x)))$$

for some $C > 0$*. Suppose that for all functions* τ *for which* $\tau(x) + Ax$ *is non-decreasing and whose Laplace transforms admit analytic extensions to*

$$\Omega_M = \left\{ \sigma + it : \sigma > -\frac{1}{M(|t|)} \right\}.$$

where they satisfy the bound $K(|t|)/(1 + |t|)$*, satisfy the decay estimate*

$$\tau(x) \ll 1/V(x).$$

Then

$$V(x) \ll M_K^{-1}(x).$$

Analytic extensions: conclusions

- The proof is based on the open mapping theorem, and again the choice of test functions is crucial. Responsible for the superexponential restriction.

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  1. What if the superexponential hypothesis fails?
  2. What if the bounds are so strong (M and K relatively close to being constant) that the M_K^{-1}-estimate is not reached in the Tauberian theorem?
- We also obtained optimality results under more general flexible Tauberian conditions and other boundary behavior for the Laplace transform.