Zero-density theorems for some L-functions

Gregory Debruyne

Ghent University

August 2024
ELAZ 2024
Rostock

The Riemann zeta function

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} \frac{1}{1 - p^{-s}}, \quad \operatorname{Re} s > 1,$$

admits a meromorphic extension to $\mathbb{C}$.

Zeros of $\zeta(s)$ encode information from the distribution of the primes.

Consequences of zero-free regions

Zero-free regions lead to error terms in the prime number theorem

$$\psi(x) := \sum_{n \le x} \Lambda(n) = x + O(E(x)).$$

- de la Vallée-Poussin (1899),

$$\sigma \ge 1 - c/\log t \implies E(x) = x \exp(-c' \sqrt{\log x}),$$

- Littlewood (1922),

$$\sigma \ge 1 - c \log_2 t/\log t \implies E(x) = x \exp(-c' \sqrt{\log x \log_2 x}),$$

- Vinogradov-Korobov (1958),

$$\sigma \ge 1 - c(\log_2 t)^{-1/3}(\log t)^{-2/3}$$
$$\implies E(x) = x \exp(-c' \log^{3/5} x \log_2^{-1/5} x),$$

- Riemann hypothesis (??), for each $\varepsilon > 0$,

$$\sigma > 1/2 \implies E(x) = O_\varepsilon(x^{1/2+\varepsilon}).$$

Zeros in half-planes

Let $N(\sigma, T)$ be the number of zeros of $\zeta(s)$ in the region $\{s : \operatorname{Re} s \geq \sigma, |\operatorname{Im} s| \leq T\}$.

For suitable $a > 0$, we have for each $\varepsilon > 0$,

$$N(\sigma, T) \ll_\varepsilon T^{a(1-\sigma)+\varepsilon}, \quad \sigma > 1 - 1/a.$$

The current record is $a = 30/13 \approx 2.307$ (Guth, Maynard, 2024).

Application: Prime number theorem in short intervals

The estimate $N(\sigma, T) \ll_\varepsilon T^{a(1-\sigma)+\varepsilon}$ in combination with the validity of the Littlewood zero-free region (asymptotically for all $c > 0$) yields the prime number theorem in short intervals:

$$\psi(x+h) - \psi(x) \sim h, \quad h > x^{1-1/a+\varepsilon}, \quad x \to \infty.$$

Guth and Maynard's zero-density theorem delivers the PNT in short intervals for $h > x^{17/30+\varepsilon}$ where $17/30 \approx 0.566$.

Density hypothesis

The density hypothesis is the (unproven) bound

$$N(\sigma, T) \ll_\varepsilon T^{2(1-\sigma)+\varepsilon}.$$

Its validity would entail $\psi(x+h) - \psi(x) \sim h$ for all $h > x^{1/2+\varepsilon}$ which cannot be improved by assuming RH.

The current record for the halfplane $\operatorname{Re} s \geq \alpha$ where the density hypothesis is valid is $\alpha \geq 25/32 \approx 0.781$ (Bourgain, 2000).

Other L-functions

Our goal is to investigate the range of validity of the density hypothesis for other Dirichlet series.

Let $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$ be a normalized holomorphic cusp Hecke eigenform of even integral weight κ and let $L(s, f) = \sum_{n=1}^{\infty} a_f(n) n^{-(\kappa-1)/2-s}$ with $N_f(\sigma, T)$ its number of zeros in $\{\operatorname{Re} s > \sigma, |\operatorname{Im} s| \le T\}$.

Theorem (Chen, D., Vindas, 2024)

We have

$$N_f(\sigma, T) \ll_\varepsilon T^{2(1-\sigma)+\varepsilon},$$

for $\sigma \ge 1407/1601 \approx 0.8788$.

This improves Ivić who obtained $\sigma \ge 53/60 \approx 0.8833$ in 1992.

Necessary ingredients of $L(s, f)$

- A "Ramanujan" bound: $\lambda_f(n) := a_f(n) n^{-(\kappa-1)/2} \ll_\varepsilon n^\varepsilon$,
- A good second moment estimate:

$$\int_0^T |L(1/2 + it, f)|^2 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon},$$

- Polynomial bounds for $L(s, f)$ on $\mathrm{Re}\, s \geq 1/2$.

Reasoning: zero-detection

Consider a truncated approximate inverse of $L(s, f)$. Let $\mu_f(n)$ be the Dirichlet inverse of $\lambda_f(n)$. Set

$$M_X(s) = \sum_{n \leq X} \mu_f(n) n^{-s}$$

Introducing weight $e^{-n/Y}$, one finds after calculations

$$\begin{aligned} L(s, f) M_X(s) = 1 &+ \text{Dirichlet polynomial} \\ &+ \text{integral related to } L(1/2 + it, f) \\ &+ \text{negligible error.} \end{aligned}$$

If s is a zero of $L(s, f)$ then either the Dirichlet polynomial is "large"(class-I zero) or the integral is "large"(class-II zero).

Class-II zeros

The number of class-II zeros are estimated via Gallagher's lemma and the second moment estimate.

Let R be a set of well-spaced class-I zeros. After a few reductions one obtains

$$|R| \ll N^{-\sigma+\varepsilon} \sum_{\rho \in R} \left| \sum_{N<n\leq 2N} b(n) n^{-it} \right|,$$

where $|b(n)| \leq 1$ and one may assume that $T^{1/2} \leq N \leq T^{1+\varepsilon}$.

The Halász-Montgomery inequality

Lemma (Halász, Montgomery)

Let $\xi, \phi_1, \ldots \phi_R$ be elements of an inner product space over $\mathbb{C}$. Then

$$\sum_{r=1}^{R} |\langle \xi, \phi_r \rangle| \leq \|\xi\| \left(\sum_{r,s} |\langle \phi_r, \phi_s \rangle| \right)^{1/2}$$

Applying this together with a few more calculations gives (apart a factor T^{ε})

$$|R| \ll N^{2-2\sigma} + N^{3/4-\sigma} \left[\sum_{\ell \in \mathbb{Z}} \Delta(\ell) \int_{-2\log^2 T}^{2\log^2 T} \left| \zeta\left(\frac{1}{2} + iv + i\ell\right) \right| \mathrm{d}v \right]^{\frac{1}{2}},$$

where

$$\Delta(\ell) = \#\{(\beta + it, \beta' + it') \in R \times R : |t - t' - \ell| < 1\}.$$

Mixed moment bounds for $\zeta(s)$

To estimate the integral of ζ Ivić then used the mixed moment bounds

$$\int_0^T |\zeta(1/2+it)|^6 \chi_{\{t:|\zeta(1/2+it)|\le T^{2/13}\}} \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon},$$

$$\int_0^T |\zeta(1/2+it)|^{19} \chi_{\{t:|\zeta(1/2+it)|> T^{2/13}\}} \mathrm{d}t \ll_\varepsilon T^{3+\varepsilon}.$$

Combining this with Huxley's subdivision argument Ivić arrived at the range $\sigma \ge 53/60$.

Our improvements

- We select better mixed moments bounds based on the exponent pair $(13/84+\varepsilon, 55/84+\varepsilon)$ Bourgain obtained in 2016.
- We use a dichotomy technique developed by Bourgain in 2000 to invoke Heath-Brown's estimate which leads to an improvement in some ranges.

Lemma (Heath-Brown, 1979)

Let R be a finite set of well-spaced points such that $|t| \leq T$ for each $t \in R$. Then

$$\sum_{t,t' \in R} \left| \sum_{N<n\leq 2N} n^{i(t-t')} \right|^2 \ll_\varepsilon T^\varepsilon (|R|N^2 + |R|^2 N + N|R|^{5/4} T^{1/2}).$$