A proof of the linearity conjecture for \( k \)-blocking sets in \( \text{PG}(n, p^3) \), \( p \) prime

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Abstract

In this paper, we show that a small minimal \( k \)-blocking set in \( \text{PG}(n, q^h) \), \( q = p^h \), \( h \geq 1 \), \( p \) prime, \( p \geq 7 \), intersecting every \((n-k)\)-space in \( 1 \) (mod \( q \)) points, is linear. As a corollary, this result shows that all small minimal \( k \)-blocking sets in \( \text{PG}(n, p^3) \), \( p \) prime, \( p \geq 7 \), are \( \mathbb{F}_p \)-linear, proving the linearity conjecture (see [7]) in the case \( \text{PG}(n, p^3) \), \( p \) prime, \( p \geq 7 \).

1 Introduction and preliminaries

Throughout this paper \( q = p^h \), \( p \) prime, \( h \geq 1 \) and \( \text{PG}(n, q) \) denotes the \( n \)-dimensional projective space over the finite field \( \mathbb{F}_q \) of order \( q \). A \( k \)-blocking set \( B \) in \( \text{PG}(n, q) \) is a set of points such that any \((n-k)\)-dimensional subspace intersects \( B \). A \( k \)-blocking set \( B \) is called trivial when a \((n-k)\)-dimensional subspace is contained in \( B \). If an \((n-k)\)-dimensional space contains exactly one point of a \( k \)-blocking set \( B \) in \( \text{PG}(n, q) \), it is called a tangent \((n-k)\)-space to \( B \). A \( k \)-blocking set \( B \) is called minimal when no proper subset of \( B \) is a \( k \)-blocking set. A \( k \)-blocking set \( B \) is called small when \( |B| < 3(q^k + 1)/2 \).

Linear blocking sets were first introduced by Lunardon [3] and can be defined in several equivalent ways.

In this paper, we follow the approach described in [1]. In order to define a linear \( k \)-blocking set in this way, we introduce the notion of a Desarguesian spread. Suppose \( q = q_0^t \), with \( t \geq 1 \). By "field reduction", the points of \( \text{PG}(n, q) \) correspond to \((t-1)\)-dimensional subspaces of \( \text{PG}((n+1)t-1, q_0) \), since a point of \( \text{PG}(n, q) \) is a 1-dimensional vector space over \( \mathbb{F}_q \), and so a \( t \)-dimensional vector space over \( \mathbb{F}_{q_0} \). In this way, we obtain a partition \( \mathcal{D} \) of the pointset of \( \text{PG}((n+1)t-1, q_0) \) by \((t-1)\)-dimensional subspaces. In general, a partition of the point set of a projective space by subspaces of a given dimension \( d \) is called a spread, or a \( d \)-spread if we want to specify the dimension. The spread obtained by field reduction is called a Desarguesian spread. Note that the Desarguesian spread satisfies the property that each subspace spanned by spread elements is partitioned by spread elements.

Let \( \mathcal{D} \) be the Desarguesian \((t-1)\)-spread of \( \text{PG}((n+1)t-1, q_0) \). If \( U \) is a subset of \( \text{PG}((n+1)t-1, q_0) \), then we define \( B(U) := \{ R \in \mathcal{D} | U \cap R \neq \emptyset \} \), and we identify the elements of \( B(U) \) with the corresponding points of \( \text{PG}(n, q_0^t) \). If \( U \) is subspace of \( \text{PG}((n+1)t-1, q_0) \), then we call \( B(U) \) a linear set or an \( \mathbb{F}_{q_0^t} \)-linear

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set if we want to specify the underlying field. Note that through every point in $\mathcal{B}(U)$, there is a subspace $U'$ such that $\mathcal{B}(U') = \mathcal{B}(U)$ since the elementwise stabiliser of the Desarguesian spread $\mathcal{D}$ acts transitively on the points of a spread element of $\mathcal{D}$. If $U$ intersects the elements of $\mathcal{D}$ in at most a point, i.e. $|\mathcal{B}(U)|$ is maximal, then we say that $U$ is scattered with respect to $\mathcal{D}$; in this case $\mathcal{B}(U)$ is called a scattered linear set. We denote the element of $\mathcal{D}$ corresponding to a point $P$ of $\text{PG}(n, q_0^h)$ by $\mathcal{S}(P)$. If $U$ is a subset of $\text{PG}(n, q)$, then we define $\mathcal{S}(U) := \{\mathcal{S}(P) | P \in U\}$. Analogously to the correspondence between the lines of $\text{PG}(n, q)$ and the $(2t-1)$-dimensional subspaces of $\text{PG}((n+1)t-1, q_0)$ spanned by two elements of $\mathcal{D}$, and in general, we obtain the correspondence between the $(n-k)$-spaces of $\text{PG}(n, q)$ and the $((n-k+1)t-1)$-dimensional subspaces of $\text{PG}((n+1)t-1, q_0)$ spanned by $n-k+1$ elements of $\mathcal{D}$. With this in mind, it is clear that any $tk$-dimensional subspace $U$ of $\text{PG}(t(n+1)-1, q_0)$ defines a $k$-blocking set $\mathcal{B}(U)$ in $\text{PG}(n, q)$. A $(k)$-blocking set constructed in this way is called a linear $(k)$-blocking set, or an $\mathbb{F}_{q_0}$-linear $(k)$-blocking set if we want to specify the underlying field.

By far the most challenging problem concerning blocking sets is the so-called linearity conjecture. Since 1998 it has been conjectured by many mathematicians working in the field. The conjecture was explicitly stated in the literature by Sziklai in [7].

(LC) All small minimal $k$-blocking sets in $\text{PG}(n, q)$ are linear.

Various instances of the conjecture have been proved; for an overview we refer to [7]. In this paper we prove the linearity conjecture for small minimal $k$-blocking sets in $\text{PG}(n, p^h)$, $p \geq 7$, as a corollary of the following main theorem:

**Theorem 1.** A small minimal $k$-blocking set in $\text{PG}(n, q^3)$, $q = p^h$, $p$ prime, $h \geq 1$, $p \geq 7$, intersecting every $(n-k)$-space in $1 \mod q$ points is linear.

### 1.1 Known characterisation results

In this section we mention a few results, that we will rely on in the sequel of this paper. First of all, observe that a subspace intersects a linear set of $\text{PG}((n+1)t-1, q_0)$ in $1 \mod p$ or zero points. The following result of Szőnyi and Weiner shows that this property holds for all small minimal blocking sets.

**Result 2.** [8, Theorem 2.7] If $B$ is a small minimal $k$-blocking set of $\text{PG}(n, q)$, $p > 2$, then every subspace intersects $B$ in $1 \mod p$ or zero points.

Result 2 answers the linearity conjecture in the affirmative for $\text{PG}(n, p^2)$. For $\text{PG}(n, p^3)$, the linearity conjecture was proved by Weiner (see [9]). For $1$-blocking sets in $\text{PG}(n, q^3)$, we have the following theorem of Polverino $(n = 2)$ and Storme and Weiner $(n \geq 3)$.

**Result 3.** [5/6] A minimal $1$-blocking set in $\text{PG}(n, q^3)$, $q = p^h$, $h \geq 1$, $p$ prime, $p \geq 7$, $n \geq 2$, of size at most $q^3 + q^2 + q + 1$, is linear.

In Theorem 8 we show that this implies the linearity conjecture for small minimal $1$-blocking sets $\text{PG}(n, q^3)$, $p \geq 7$, that intersect every hyperplane in $1 \mod q$ points.

The following Result by Szőnyi and Weiner gives a sufficient condition for a blocking set to be minimal.

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Result 4. [8, Lemma 3.1] Let $B$ be a $k$-blocking set of $\text{PG}(n, q)$, and suppose that $|B| \leq 2q^k$. If each $(n - k)$-dimensional subspace of $\text{PG}(n, q)$ intersects $B$ in $1 \pmod{p}$ points, then $B$ is minimal.

1.2 The intersection of a subline and an $\mathbb{F}_q$-linear set

The possibilities for an $\mathbb{F}_q$-linear set of $\text{PG}(1, q^3)$, other than the empty set, a point, and the set $\text{PG}(1, q^3)$ itself are the following: a subline $\text{PG}(1, q)$ of $\text{PG}(1, q^3)$, corresponding to the $a$ line of $\text{PG}(5, q)$ not contained in an element of $\mathcal{D}$; a set of $q^2 + 1$ points of $\text{PG}(1, q^3)$, corresponding to a plane of $\text{PG}(5, q)$ that intersects an element of $\mathcal{D}$ in a line; a set of $q^2 + q + 1$ points of $\text{PG}(1, q^3)$, corresponding to a plane of $\text{PG}(5, q)$ that is scattered w.r.t. $\mathcal{D}$.

The following results describe the possibilities for the intersection of a subline with an $\mathbb{F}_q$-linear set in $\text{PG}(1, q^3)$, and will play an important role in this paper.

Result 5. [2] A subline $\cong \text{PG}(1, q)$ intersects an $\mathbb{F}_q$-linear set of $\text{PG}(1, q^3)$ in $0, 1, 2, 3$, or $q + 1$ points.

Result 6. [4, Lemma 4.4, 4.5, 4.6] Let $q$ be a square. A subline $\text{PG}(1, q)$ and a Baer subline $\text{PG}(1, q\sqrt{q})$ of $\text{PG}(1, q^3)$ share at most a subline $\text{PG}(1, \sqrt{q})$. A Baer subline $\text{PG}(1, q\sqrt{q})$ and an $\mathbb{F}_q$-linear set of $q^2 + 1$ or $q^2 + q + 1$ points in $\text{PG}(1, q^3)$ share at most $q + \sqrt{q} + 1$ points.

2 Some bounds and the case $k = 1$

The Gaussian coefficient $\binom{n}{k}_q$ denotes the number of $(k - 1)$-subspaces in $\text{PG}(n - 1, q)$, i.e.,

$$\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

Lemma 7. If $B$ is a subset of $\text{PG}(n, q^3)$, $q \geq 7$, intersecting every $(n - k)$-space, $k \geq 1$, in $1 \pmod{q}$ points, and $\pi$ is an $(n - k + s)$-space, $s \leq k$, then either

$$|B \cap \pi| < q^{3s} + q^{3s-1} + q^{3s-2} + 3q^{3s-3}$$

or

$$|B \cap \pi| > q^{3s+1} - q^{3s-1} - q^{3s-2} - 3q^{3s-3}.$$

Proof. Let $\pi$ be an $(n - k + s)$-space of $\text{PG}(n, q^3)$, and put $B_\pi := B \cap \pi$. Let $x_i$ denote the number of $(n - k)$-spaces of $\pi$ intersecting $B_\pi$ in $i$ points. Counting the number of $(n - k)$-spaces, the number of incident pairs $(P, \pi)$ with $P \in B_\pi$, $P \in \sigma, \pi$ an $(n - k)$-space, and the number of triples $(P_1, P_2, \sigma)$, with $P_1, P_2 \in B_\pi$, $P_1 \neq P_2$, $P_1, P_2 \in \sigma, \pi$ an $(n - k)$-space yields:

$$\sum_i x_i = \binom{n - k + s + 1}{n - k + 1}_q, \quad (1)$$

$$\sum_i ix_i = |B_\pi| \binom{n - k + s}{n - k}_q, \quad (2)$$

$$\sum i(i - 1)x_i = |B_\pi|(|B_\pi| - 1) \binom{n - k + s - 1}{n - k - 1}_q. \quad (3)$$
Since we assume that every \((n - k)\)-space intersects \(B\) in \(1 \pmod{q}\), it follows that every \((n - k)\)-space of \(\pi\) intersect \(B_{\pi}\) in \(1 \pmod{q}\) points, and hence 
\[
\sum_i (i - 1)(i - 1 - q)x_i \geq 0.
\]
Using Equations (1), (2), and (3), this yields that 
\[
|B_{\pi}|((B_{\pi} - 1)(q^{3(n - 3k)} - 1)(q^{3n - 3k + 3} - 1) - (q + 1))B_{\pi}((q^{3n - 3k + 3} - 1)(q^{3n - 3k + 3} - 1) + (q + 1)(q^{3n - 3k + 3} - 1)(q^{3n - 3k + 3} - 1) - (q + 1))B_{\pi}((q^{3n - 3k + 3} - 1)(q^{3n - 3k + 3} - 1) + (q + 1)(q^{3n - 3k + 3} - 1)(q^{3n - 3k + 3} - 1) \geq 0.
\]
Putting \(|B_{\pi}| = q^{3s} + q^{3s-1} + q^{3s-2} + 3q^{3s-3}\) or \(|B_{\pi}| = q^{3s+1} + q^{3s-2} - 3q^{3s-3}\) in this inequality, with \(q \geq 7\), gives a contradiction. Hence the statement follows.

**Theorem 8.** A small minimal 1-blocking set in \(PG(n, q^3)\), \(p \geq 7\), intersecting every hyperplane in \(1 \pmod{q}\) points, is linear.

**Proof.** Lemma 7 implies that a small minimal 1-blocking set \(B\) in \(PG(n, q^3)\), intersecting every hyperplane in \(1 \pmod{q}\) points, has at most \(q^3 + q^2 + q + 3\) points. Since every hyperplane intersects \(B\) in \(1 \pmod{q}\) points, it is easy to see that \(|B| \equiv 1 \pmod{q}\). This implies that \(|B| \leq q^3 + q^2 + q + 1\). Result 3 shows that \(B\) is linear.

**Corollary 9.** A small minimal 1-blocking set in \(PG(n, p^3)\), \(p\) prime, \(p \geq 7\), is \(F_p\)-linear.

**Proof.** This follows from Result 2 and Theorem 8.

For the remaining of this section, we use the following assumption:

(B) \(B\) is small minimal \(k\)-blocking set in \(PG(n, q^3)\), \(p \geq 7\), intersecting every \((n - k)\)-space in \(1 \pmod{q}\) points.

For convenience let us introduce the following terminology. A full line of \(B\) is a line which is contained in \(B\). An \((n - k + s)\)-space \(S\), \(s < k\), is called large if \(S\) contains more than \(q^{3s+1} - q^{3s-1} - q^{3s-2} - 3q^{3s-3}\) points of \(B\), and \(S\) is called small if it contains less than \(q^{3s} + q^{3s-1} + q^{3s-2} + 3q^{3s-3}\) points of \(B\).

**Lemma 10.** Let \(L\) be a line such that \(1 < |B \cap L| < q^3 + 1\).

1. For all \(i \in \{1, \ldots, n - k\}\) there exists an \(i\)-space \(\pi_i\) on \(L\) such that \(B \cap \pi_i = B \cap L\).

2. Let \(N\) be a line, skew to \(L\). For all \(j \in \{1, \ldots, k - 2\}\), there exists a small \((n - k + j)\)-space \(\pi_j\) on \(L\), skew to \(N\).

**Proof.** (1) It follows from Result 2 that every subspace on \(L\) intersects \(B \setminus L\) in zero or at least \(p\) points. We proceed by induction on the dimension \(i\). The statement obviously holds for \(i = 1\). Suppose there exists an \(i\)-space \(\pi_i\) on \(L\) such that \(\pi_i \cap B = L \cap B\), with \(i \leq n - k - 1\). If there is no \((i + 1)\)-space intersecting \(B\) only on \(L\), then the number of points of \(B\) is at least 
\[
|B \cap L| + p(q^{3(n-i)-3} + q^{3(n-i)-6} + \ldots + q^3 + 1),
\]
but by Lemma 7 \(|B| \leq q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}\). If \(i < n - k - 1\) this is a contradiction. If \(i = n - k - 1\) then in the above count we may replace the factor \(p\) by a factor \(q\), using the hypothesis (B), and hence also in this case we get a contradiction. We may conclude that there exists an \(i\)-space \(\pi_i\) on \(L\) such that \(B \cap L = B \cap \pi_i\), \(\forall i \in \{1, \ldots, n - k\}\).
Part (1) shows that there is an \((n - k - 1)\)-space \(\pi_{n-k-1}\) on \(L\), skew to \(N\), such that \(B \cap L = B \cap \pi_{n-k-1}\). If an \((n - k)\)-space through \(\pi_{n-k-1}\) contains an extra element of \(B\), it contains at least \(q^2\) extra elements of \(B\), since a line containing 2 points of \(B\) contains at least \(q + 1\) points of \(B\). This implies that there is an \((n - k)\)-space \(\pi_{n-k}\) through \(\pi_{n-k-1}\) with no extra points of \(B\), and skew to \(N\).

We proceed by induction on the dimension \(i\). Lemma 12(1) shows that there are at least \((q^{3k} - 1)/(q^3 - 1 - q^{3k-5} - 5q^{3k-6} + 1 > q^3 + 1\) small \((n - k + 1)\)-spaces through \(\pi_{n-k}\) which proves the statement for \(i = 1\).

Suppose that there are \((n - k + t)\)-space \(\pi_{n-k+t}\) on \(L\), skew to \(N\), such that \(B \cap \pi_{n-k+t}\) is a small minimal \(t\)-blocking set of \(\pi_{n-k+t}\). An \((n - k + t + 1)\)-space through \(\pi_{n-k+t}\) contains at most \((q^{3t+4} - 1)(q - 1)\) or more than \(q^{3t+4} - q^{3t+2} - q^{3t+1} - 3q^{3t}\) points of \(B\) (see Lemmas 7 and 13).

Suppose all \((q^{3k+3} - 1)(q^3 - 1 - q^3 - 1)\) \((n - k + t)\)-spaces through \(\pi_{n-k+t-1}\), skew to \(N\), contain more than \(q^{3t+4} - q^{3t+2} - q^{3t+1} - 3q^{3t}\) points of \(B\). Then the number of points in \(B\) is larger than \(q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}\) if \(t \leq k - 3\), a contradiction.

We may conclude that there exists an \((n - k + j)\)-space \(\pi_j\) on \(L\) such that \(B \cap \pi_j\) is a small minimal \(t\)-blocking set, skew to \(N\), \(\forall j \in \{1, \ldots, k - 2\}\).

**Theorem 11.** A line \(L\) intersects \(B\) in a linear set.

**Proof.** Note that it is enough to show that \(L\) is contained in a subspace of \(\text{PG}(n, q^3)\) intersecting \(B\) in a linear set. If \(k = 1\), then \(B\) is linear by Theorem 8, and the statement follows. Let \(k > 1\), let \(L\) be a line, not contained in \(B\), intersecting \(B\) in at least two points. It follows from Lemma 10 that there exists an \((n - k)\)-space \(\pi_j\) such that \(B \cap L = B \cap \pi_j\). Then each of the \((q^{3k} - 1)/(q^3 - 1)\) \((n - k + 1)\)-spaces through \(\pi_j\) is large, then the number of points in \(B\) is at least

\[
\frac{q^{3k} - 1}{q^3 - 1} (q^4 - q^2 - q - 3 - q^3) + q^3 > q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3},
\]

a contradiction. Hence, there is a small \((n - k + 1)\)-space \(\pi\) through \(L\), so \(B \cap \pi\) is a small 1-blocking set which is linear by Theorem 8. This concludes the proof.

**Lemma 12.** Let \(\pi\) be an \((n - k)\)-space of \(\text{PG}(n, q^3), k > 1\).

1. If \(B \cap \pi\) is a point, then there are at most \(q^{3k-5} + 4q^{3k-6} - 1\) large \((n - k + 1)\)-spaces through \(\pi\).

2. If \(\pi\) intersects \(B\) in \((q\sqrt{q} + 1), q^2 + 1\) or \(q^2 + q + 1\) collinear points, then there are at most \(q^{3k-5} + 5q^{3k-6} - 1\) large \((n - k + 1)\)-spaces through \(\pi\).

3. If \(\pi\) intersects \(B\) in \(q + 1\) collinear points, then there are at most \(3q^{3k-6} - q^{3k-7} - 1\) large \((n - k + 1)\)-spaces through \(\pi\).

**Proof.** Suppose there are \(y\) large \((n - k + 1)\)-spaces through \(\pi\). Then the number of points in \(B\) is at least

\[
y(q^4 - q^2 - q - 3 - |B \cap \pi|) + ((q^{3k} - 1)/(q^3 - 1) - y)x + |B \cap \pi|, \quad (*)
\]

where \(x\) depends on the intersection \(B \cap \pi\).
(1) In this case, \( x = q^3 \) and \( |B \cap \pi| = 1 \). If \( y = q^{3k-5} + 4q^{3k-6} \), then (*) is larger than \( q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3} \), a contradiction.

(2) In this case \( x = q^3 \) and \( |B \cap \pi| \leq q^2 + q + 1 \). If \( y = q^{3k-5} + 5q^{3k-6} \), then (*) is larger than \( q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3} \), a contradiction.

(3) By Result 3 we know that an \((n-k+1)\)-space \( \pi' \) through \( \pi \) intersects \( B \) in at least \( q^3 + q^2 + 1 \) points, since a \((q+1)\)-secant in \( \pi' \) implies that the intersection of \( \pi' \) with \( B \) is non-trivial and not a Baer subplane, hence \( x = q^3 + q^2 - q \), and \( |B \cap \pi| = q + 1 \). If \( 3q^{3k-6} - q^{3k-7} \), then (*) is larger than \( q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3} \), a contradiction. □

3 The proof of Theorem 1

In the proof of the main theorem, we distinguish two cases. In both cases we need the following two lemmas.

We continue with the following assumption

(B) \( B \) is small minimal \( k \)-blocking set in \( \text{PG}(n, q^3) \), \( p \geq 7 \), intersecting every \((n-k)\)-space in \( 1 \) \((\bmod\ q)\) points;

and we consider the following properties:

\((H_1)\) \( \forall s < k \): every small minimal \( s \)-blocking set, intersecting every \((n-s)\)-space in \( 1 \) \((\bmod\ q)\) points, not containing a \((q\sqrt{q} + 1)\)-secant, is \( \mathbb{F}_q \)-linear;

\((H_2)\) \( \forall s < k \): every small minimal \( s \)-blocking set, intersecting every \((n-s)\)-space in \( 1 \) \((\bmod\ q)\) points, containing a \((q\sqrt{q} + 1)\)-secant, is \( \mathbb{F}_q\sqrt{q} \)-linear.

Lemma 13. If \((H_1)\) or \((H_2)\), and \( S \) is a small \((n-k+s)\)-space, \( 0 < s < k \), then \( B \cap S \) is a small minimal linear \( s \)-blocking set in \( S \), and hence \( |B \cap S| \leq (q^{3s+1} - 1)/(q - 1) \).

Proof. Clearly \( B \cap S \) is an \( s \)-blocking set in \( S \). Result 2 implies that \( B \cap S \) intersects every \((n-k+s-s)\)-space of \( S \) in \( 1 \) \((\bmod\ q)\) points, and it follows from Result 4 that \( B \cap S \) is minimal. Now apply \((H_1)\) or \((H_2)\).

Lemma 14. Suppose \((H_1)\) or \((H_2)\). Let \( k > 2 \) and let \( \pi_{n-2} \) be an \((n-2)\)-space such that \( B \cap \pi_{n-2} \) is a non-trivial small linear \((k-2)\)-blocking set, then there are at least \( q^3 - q + 6 \) small hyperplanes through \( \pi_{n-2} \).

Proof. Applying Lemma 13 with \( s = k - 2 \), it follows that \( B \cap \pi_{n-2} \) contains at most \( (q^{3k-3} - 1)/(q - 1) \) points. On the other hand, from Lemmas 7 and 13 with \( s = k - 1 \), we know that a hyperplane intersects \( B \) in at most \( q^{3k-2} - q^{3k-4} - q^{3k-5} - 3q^{3k-6} \) points. In the first case, a hyperplane \( H \) intersects \( B \) in at least \( q^{3k-3} + 1 + (q^{3k-3} + q)/(q + 1) \) points, using a result of Szönyi and Weiner [8, Corollary 3.7] for the \((k-1)\)-blocking set \( H \cap B \). If there are at least \( q - 4 \) large hyperplanes, then the number of points in \( B \) is at least

\[
(q - 4)(q^{3k-2} - q^{3k-4} - q^{3k-5} - 3q^{3k-6} - \frac{q^{3k-5} - 1}{q - 1}) +
\]

\[
(q^3 - q + 5)(q^{3k-3} + 1 + \frac{q^{3k-3} + q}{q + 1} - \frac{q^{3k-5} - 1}{q - 1}) + \frac{q^{3k-5} - 1}{q - 1},
\]

which is larger than \( q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3} \) if \( q \geq 7 \), a contradiction. Hence, there are at most \( q - 5 \) large hyperplanes through \( \pi_{n-2} \). □
3.1 Case 1: there are no $q\sqrt{q}+1$-secants

In this subsection, we will use induction on $k$ to prove that small minimal $k$-blocking sets in $\text{PG}(n, q^3)$, intersecting every $(n-k)$-space in $1 \pmod{q}$ points and not containing a $(q\sqrt{q}+1)$-secant, are $F_q$-linear. The induction basis is Theorem 8. We continue with assumptions $(H_1)$ and

$$(B_1) \text{ } B \text{ is small minimal } k\text{-blocking set in } \text{PG}(n, q^3), \text{ p } \geq 7, \text{ intersecting every (n-k)-space in 1 } \pmod{q} \text{ points, not containing a } (q\sqrt{q}+1)\text{-secant.}$$

Lemma 15. If $B$ is non-trivial, there exist a point $P \in B$, a tangent $(n-k)$-space $\pi$ at the point $P$ and small $(n-k+1)$-spaces $H_i$, through $\pi$, such that there is a $(q+1)$-secant through $P$ in $H_i$, $i = 1, \ldots, q^{3k-3} - 2q^{3k-4}$.

Proof. Since $B$ is non-trivial, there is at least one line $N$ with $1 < |N \cap B| < q^3 + 1$. Lemma 10 shows that there is an $(n-k)$-space $\pi_N$ through $N$ such that $B \cap N = B \cap \pi_N$. It follows from Theorem 11 and Lemma 12 that there is at least one $(n-k+1)$-space $H$ through $\pi_N$ such that $H \cap B$ is a small minimal linear 1-blocking set of $H$. In this non-trivial small minimal linear 1-blocking set, there are $(q+1)$-secants (see Result 3). Let $M$ be one of those $(q+1)$-secants of $B$. Again using Lemma 10, we find an $(n-k)$-space $\pi_M$ through $M$ such that $B \cap M = B \cap \pi_M$.

Lemma 12(3) shows that through $\pi_M$, there are at least $\frac{q^3k}{q^3-1} - 3q^{3k-6} + q^{3k-7} + 1$ small $(n-k+1)$-spaces. Let $P$ be a point of $M$. Since in each of these intersections, $P$ lies on at least $q^2 - 1$ other $(q+1)$-secants, a point $P$ of $M$ lies in total on at least $(q^2 - 1)(\frac{q^3k}{q^3-1} - 3q^{3k-6} + q^{3k-7} + 1)$ other $(q+1)$-secants. Since each of the $\frac{q^3k}{q^3-1} - 3q^{3k-6} + q^{3k-7} + 1$ small $(n-k+1)$-spaces contains at least $q^3 + q^2 - q$ points of $B$ not on $M$, and $|B| < q^{3k} + q^{3k-1} + q^{3k-2} + 3q^{3k-3}$ (see Lemma 7), there are less than $2q^{3k-2} + 6q^{3k-3}$ points of $B$ left in the large $(n-k+1)$-spaces. Hence, $P$ lies on less than $2q^{3k-5} + 6q^{3k-6}$ full lines.

Since $B$ is minimal, $P$ lies on a tangent $(n-k)$-space $\pi$. There are at most $q^{3k-5} + 4q^{3k-6} - 1$ large $(n-k+1)$-spaces through $\pi$ (Lemma 12(1)). Moreover, since at least $\frac{q^3k}{q^3-1} - (q^{3k-5} + 4q^{3k-6} - 1) - (2q^{3k-5} + 6q^{3k-6}) (n-k+1)$-spaces through $\pi$ contain at least $q^3 + q^2$ points of $B$, and at most $2q^{3k-5} + 6q^{3k-6}$ of the small $(n-k+1)$-spaces through $\pi$ contain exactly $q^3 + q^2$ points of $B$, there are at most $2q^{3k-2} + 23q^{3k-3}$ points of $B$ left. Hence, $P$ lies on at most $2q^{3k-3} + 23q^{3k-4}(q+1)$-secants of the large $(n-k+1)$-spaces through $\pi$. This implies that there are at least $(q^2 - 1)(\frac{q^{3k}}{q^3-1} - 3q^{3k-6} + q^{3k-7} + 1) - (2q^{3k-3} + 23q^{3k-4}) (q+1)$-secants through $P$ left in small $(n-k+1)$-spaces through $\pi$. Since in a small $(n-k+1)$-space through $\pi$, there can lie at most $q^2 + q + 1$ $(q+1)$-secants through $P$, this implies that there are at least $q^{3k-3} - 2q^{3k-4} (n-k+1)$-spaces $H_i$ through $\pi$ such that $P$ lies on a $(q+1)$-secant in $H_i$. \hfill \Box

Lemma 16. Let $\pi$ be an $(n-k)$-dimensional tangent space of $B$ at the point $P$. Let $H_1$ and $H_2$ be two $(n-k+1)$-spaces through $\pi$ for which $B \cap H_i = B(\pi_i)$, for some 3-space $\pi_i$ through $x \in S(P), B(x) \cap \pi_i = \{x\}$ $(i = 1, 2)$ and $B(\pi_i)$ not contained in a line of $\text{PG}(n, q^3)$. Then $B(\langle \pi_1, \pi_2 \rangle) \subseteq B$.

Proof. Since $(B(\pi_i))$ is not contained in a line of $\text{PG}(n, q^3)$, there is at most one element $Q$ of $B(\pi_i)$ such that $(S(P), Q)$ intersects $\pi_i$ in a plane. If there is such a plane, then we denote its pointset by $\mu_i$, otherwise we put $\mu_i = \emptyset$.
Let $M$ be a line through $x$ in $\pi_1 \setminus \mu_1$, let $s \neq x$ be a point of $\pi_2 \setminus \mu_2$, and note that $B(s) \cap \pi_2 = \{s\}$.

We claim that there is a line $T$ through $s$ in $\pi_2$ and an $(n-2)$-space $\pi_M$ through $(B(M))$ such that there are at least 4 points $t_i \in T$, $t_i \notin \mu_2$, such that $(\pi_M, B(t_i))$ is small and hence has a linear intersection with $B$, with $B \cap \pi_M = M$ if $k = 2$ and $B \cap \pi_M$ is a small minimal $(k-2)$-blocking set if $k > 2$.

If $k = 2$, the existence of $\pi_M$ follows from Lemma 10(1), and we know from Lemma 12(1) that there are at most $q + 3$ large hyperplanes through $\pi_M$. Denote the set of points of $B(\pi_2)$, contained in one of those hyperplanes by $F$. Hence, if $Q$ is a point of $B(\pi_2) \setminus F$, $(Q, \pi_M)$ is a small hyperplane.

Let $T_1$ be a line through $s$ in $\pi_2 \setminus \mu_2$ and not through $x$, and suppose that $B(T_1)$ contains at least $q - 3$ points of $F$.

Let $T_2$ be a line in $\pi_2 \setminus \mu_2$, through $s$, not in $(x, T_1)$, and through $x$. There are at most $q + 3 - (q - 3)$ reguli through $x$ of $S(F)$, not in $(x, T_1)$, and if $\mu \neq \emptyset$ one element of $B(\mu_2)$ is contained $B(T_2)$. Since it is possible that $B(s)$ is an element of $F$, this gives in total at most 8 points of $B(T_2)$ that are contained in $F$. This implies, if $q > 11$, that at least 5 of the hyperplanes $(\pi_M, B(t))|t \in T_2$ are small.

If $q = 11$, it is possible that $B(T_2)$ contains at least 8 points of $F$. If $T_3$ is a line in $\pi_2 \setminus \mu_2$, through $s$, $(x, T_1)$, $(x, T_2)$ and not through $x$, then there are at least 5 points $t \in T_3$ such that $(\pi_M, B(t))$ is a small hyperplane.

If $q = 7$ and if $B(s) \in B(F)$, it is possible that $B(T_2), B(T_3), B(T_4)$, with $T_i$ a line through $s$ in $\pi_2 \setminus \mu_2$, not in $(x, T_i)$, $j < i$, not through $x$, and contain 4 points of $F$. A fifth line $T_5$ through $s$ in $\pi_2 \setminus \mu_2$, not in $(x, T_i)$, $j < i$, not through $x$, contains at least 5 points such that $(\pi_M, B(t))$ is a small hyperplane.

If $k > 2$, let $T$ be a line through $s$ in $\pi_2 \setminus \mu_2$, not through $x$. It follows from Lemma 10(2) that there is an $(n-2)$-space $\pi_M$ through $(B(M))$ such that $B \cap \pi_M$ is a small minimal $(k-2)$-blocking set of $\text{PG}(n, q^3)$, skew to $B(T)$. Lemma 14 shows that at most $q - 5$ of the hyperplanes through $\pi_M$ are large. This implies that at least 5 of the hyperplanes $(\pi_M, B(t))|t \in B(T)$ are small. This proves our claim.

Since $B \cap (B(t_i), \pi_M)$ is linear, also the intersection of $(B(t_i), B(M))$ with $B$ is linear, i.e., there exist subspaces $\tau_i, \tau_i \cap S(P) = \{x\}$, such that $B(\tau_i) = (B(t_i), B(M)) \cap B$. Since $\tau_i \cap (B(M))$ and $M$ are both transversals through $x$ to the same regulus $B(M)$, they coincide, hence $M \subseteq \tau_i$. The same holds for $\tau_i \cap (B(t_i), S(P))$, implying $t_i \in \tau_i$. We conclude that $B(M, t_i) \subseteq B(\tau_i) \subseteq B$.

We show that $B((M, T)) \subseteq B$. Let $L'$ be a line of $(M, T)$, not intersecting $M$. The line $L'$ intersects the planes $(M, t_i)$ in points $p_i$ such that $B(p_i) \in B$. Since $B(L')$ is a subline intersecting $B$ in at least 4 points, Result 5 shows that $B(L') \subseteq B$. Since every point of the space $(M, T)$ lies on such a line $L'$, $B((M, T)) \subseteq B$.

Hence, $B((M, s)) \subseteq B$ for all lines $M$ through $x$, $x$ in $\pi_1 \setminus \mu_1$, and all points $s \neq x \in \pi_2 \setminus \mu_2$, so $B((\pi_1, \pi_2) \setminus (\mu_1, \pi_2) \cup (\mu_2, \pi_1)) \subseteq B$. Since every point of $(\mu_1, \pi_2) \cup (\mu_2, \pi_1)$ lies on a line $N$ with $q - 1$ points of $(\pi_1, \pi_2) \setminus (\mu_1, \pi_2) \cup (\mu_2, \pi_1)$, Result 5 shows that $B(N) \subseteq B$. We conclude that $B((\pi_1, \pi_2)) \subseteq B$.

Theorem 17. The set $B$ is $F_q$-linear.

Proof. If $B$ is a $k$-space, then $B$ is $F_q$-linear. If $B$ is non-trivial small minimal $k$-blocking set, Lemma 15 shows that there exists a point $P$ of $B$, a tangent $(n-k)$-space $\pi$ at the point $P$ and at least $q^{k-3} - 2q^{k-4} (n-k+1)$-spaces $H_i$ through
\( \pi \) for which \( B \cap H_i \) is small and linear, where \( P \) lies on at least one \((q+1)\)-secant of \( B \cap H_i \). \( i = 1, \ldots, s \), \( s \geq q^{3k-3} - 2q^{3k-4} \). Let \( B \cap H_i = B(\pi_i) \), \( i = 1, \ldots, s \), with \( \pi_i \) a 3-dimensional space.

Lemma 16 shows that \( B((\pi_i, \pi_j)) \subseteq B \), \( 0 \leq i \neq j \leq s \).

If \( k = 2 \), the set \( B((\pi_1, \pi_2)) \) corresponds to a linear 2-blocking set \( B' \) in \( \text{PG}(n, q^3) \). Since \( B \) is minimal, \( B = B' \), and the Theorem is proven.

Let \( k > 2 \). Denote the \((n-k+1)\)-spaces through \( \pi \), different from \( H_i \), by \( K_{j}, j = 1, \ldots, z \). It follows from Lemma 15 that \( z \leq 2q^{3k-4} + (q^{3k-3} - 1)/(q^3 - 1) \).

There are at most \((q^{3k-3} - 2q^{3k-4} - 1)/q^3 \) different \((n-k+2)\)-spaces \( (H_1, H_2) \), \( 1 < j \leq s \). If all \((n-k+2)\)-spaces \( (H_1, H_2) \), contain at least \( 5q^2 - 49 \) of the spaces \( K_i \), then \( z \geq (5q^2 - 49)(q^{3k-3} - 2q^{3k-4} - 1)/q^3 \), a contradiction if \( q \geq 7 \). Let \( (H_1, \ldots, H_{k+1}) \) be an \((n-k+i+1)\)-space containing less than \( 5q^{3i-1} - 49q^{3i-6} \) of the spaces \( K_i \). Suppose by induction that for any \( 1 < i < k \), there is \((n-k+i)\)-space \( (H_1, H_2, \ldots, H_{i}) \) containing at most \( 5q^{3i-4} - 49q^{3i-6} \) of the spaces \( K_i \), such that \( B(\langle \pi_1, \ldots, \pi_{i} \rangle) \subseteq B \).

There are at least \( q^{3k-3} - 2q^{3k-4} - (q^{3i-1} - 1)/(q^3 - 1) \) different \((n-k+i+1)\)-spaces \( (H_1, H_2, \ldots, H_{i+1}, H) \), \( H \not\subseteq \langle H_1, H_2, \ldots, H_i \rangle \). If all of these contain at least \( 5q^{3i-1} - 49q^{3i-6} \) of the spaces \( K_i \), then

\[
\begin{align*}
z \geq & \quad (5q^{3i-1} - 49q^{3i-3} - 5q^{3i-4} + 49q^{3i-6})^{\frac{q^{3k-3} - 2q^{3k-4} - (q^{3i-1} - 1)/(q^3 - 1)}{q^3}} \\
& + 5q^{3i-4} - 49q^{3i-6},
\end{align*}
\]

a contradiction if \( q \geq 7 \). Let \( (H_1, \ldots, H_{k+1}) \) be an \((n-k+i+1)\)-space containing less than \( 5q^{3i-1} - 49q^{3i-3} \) spaces \( K_i \). We still need to prove that \( B(\langle \pi_1, \ldots, \pi_{i+1} \rangle) \subseteq B \). Since \( B(\langle \pi_{i+1}, \pi \rangle) \subseteq B \), with \( \pi \) a 3-space in \( \langle \pi_1, \ldots, \pi_{i} \rangle \) for which \( B(\pi) \) is not contained in one of the spaces \( K_i \), there are at most \( 5q^{3i-1} - 49q^{3i-6} \) 6-dimensional spaces \( \langle \pi_1, \mu \rangle \) for which \( B(\langle \pi_{i+1}, \mu \rangle) \) is not necessarily contained in \( B \), giving rise to at most \( (5q^{3i-4} - 49q^{3i-6})(q^6 + q^5 + q^4) \) points \( t \) for which \( B(t) \) is not necessarily contained in \( B \). Let \( u \) be a point of such a space \( \langle \pi_{i+1}, \mu \rangle \). Suppose that each of the \((q^{3i+3} - 1)/(q - 1) \) lines through \( u \) in \( \langle \pi_1, \ldots, \pi_{i+1} \rangle \) contains at least \( q - 2 \) of the points \( t \) for which \( B(t) \) is not in \( B \). Then there are at least \( (q - 3)(q^{3i+3} - 1)/(q - 1) + 1 > (5q^{3i-4} - 49q^{3i-6})(q^6 + q^5 + q^4) \) such points \( t \), if \( q \geq 7 \), a contradiction. Hence, there is a line \( N \) through \( t \) for which for at least 4 points \( v \in N \), \( B(v) \in B \).

Result 5 yields that \( B(t) \in B \). This implies that \( B(\langle \pi_1, \ldots, \pi_{i+1} \rangle) \subseteq B \).

Hence, the space \( \langle H_1, H_2, \ldots, H_{k} \rangle \), which spans the space \( \text{PG}(n, q^3) \), is such that \( B(\langle \pi_1, \ldots, \pi_k \rangle) \subseteq B \). But \( B(\langle \pi_1, \ldots, \pi_k \rangle) \) corresponds to a linear \( k \)-blocking set \( B' \) in \( \text{PG}(n, q^3) \). Since \( B \) is minimal, \( B = B' \).

\[ \square \]

**Corollary 18.** A small minimal \( k \)-blocking set in \( \text{PG}(n, q^3) \), \( p \) prime, \( p \geq 7 \), is \( F_p \)-linear.

\[ \square \]

**Proof.** This follows from Results 2 and Theorem 17.

\[ \square \]

### 3.2 Case 2: there are \((q \sqrt{q} + 1)\)-secants to \( B \)

In this subsection, we will use induction on \( k \) to prove that small minimal \( k \)-blocking sets in \( \text{PG}(n, q^3) \), intersecting every \((n-k)\)-space in 1 (mod \( q \)) points and containing a \( q \sqrt{q} + 1 \)-secant, are \( F_{q \sqrt{q}} \)-linear. The induction basis is Theorem 8. We continue with assumptions \((H_2)\) and
(B₂) B is small minimal k-blocking set in PG(n, q³) intersecting every (n − k)-space in 1 (mod q) points, containing a (q√q + 1)-secant.

In this case, S maps PG(n, q³) onto PG(2n + 1, q√q) and the Desarguesian spread consists of lines.

**Lemma 19.** If B is non-trivial, there exist a point P ∈ B, a tangent (n − k)-space π at P and small (n − k + 1)-spaces Hᵢ through π, such that there is a (q√q + 1)-secant through P in Hᵢ, i = 1, ..., q³k−3 − q³k−4 − 2√q³k−5.

**Proof.** There is a (q√q + 1)-secant M. Lemma 10(1) shows that there is an (n − k)-space πᵢ through M such that B ∩ M = P ∩ πᵢ.

Lemma 12(3) shows that there are at least \( \frac{q³k−1}{q−1} - q³k−5 - 5q³k−6 + 1 \) small (n − k + 1)-spaces through πᵢ. Moreover, the intersections of these small (n − k + 1)-spaces with B are Baer subplanes PG(2, q√q), since there is a (q√q + 1)-secant M. Let P be a point of M ∩ B.

Since in any of these intersections, P lies on q√q other (q√q + 1)-secants, a point P of M ∩ B lies in total on at least \( q³k−1 - q³k−5 - 5q³k−6 + 1 \) other (q√q + 1)-secants. Since any of the \( \frac{q³k−1}{q−1} - q³k−5 - 5q³k−6 + 1 \) small (n − k + 1)-spaces through πᵢ contains q³ points of B not in πᵢ, and |B| < q³k + q³k−1 + q³k−2 + 3q³k−3 (see Lemma 7), there are less than q³k−4 + 4q³k−2 points of B left in the other (n − k + 1)-spaces through πᵢ. Hence, P lies on less than q³k−4 + 4q³k−5 full lines.

Since B is minimal, there is a tangent (n − k)-space π through P. There are at most q³k−5 + 4q³k−6 − 1 large (n − k + 1)-spaces through π (Lemma 12(1)). Moreover, since at least \( \frac{q³k−1}{q−1} - q³k−5 - 4q³k−6 + 1 \) small (n − k + 1)-spaces through π contain q³ + q√q + 1 points of B, and at most q³k−4 + 4q³k−5 of the small (n − k + 1)-spaces through π contain exactly q³ + 1 points of B, there are at most \( q³k−1 - q³k−2 - q³k−5 + 4q³k−2 \) points of B left. Hence, P lies on at most \( q³k−1 - q³k−2 - q³k−5 + 4q³k−2 \) different (q√q + 1)-secants of the large (n − k + 1)-spaces through π. This implies that there are at least \( q³k−1 - q³k−5 - 5q³k−6 + 1 \) different (q√q + 1)-secants through P in small (n − k + 1)-spaces through π. Since in a small (n − k + 1)-space through π, there lie q√q + 1 different (q√q + 1)-secants, we have at least q³k−3 − q³k−4 − 2q³k−5 small (n − k + 1)-spaces Hᵢ through π such that P lies on a (q√q + 1)-secant in Hᵢ.

**Lemma 20.** Let π be an (n − k)-dimensional tangent space of B at the point P. Let H₁ and H₂ be two (n − k + 1)-spaces through π for which B ∩ Hᵢ = B(πᵢ), for some plane πᵢ through x ∈ S(P), B(x) ∩ πᵢ = \{x\} (i = 1, 2) and B(πᵢ) not contained in a line of PG(n, q³). Then B(⟨π₁, π₂⟩) ⊆ B.

**Proof.** Let M be a line through x in π₁, let s ≠ x be a point of π₂.

We claim that there is a line T through s, not through x, in π₂ and an (n − 2)-space πᵢ through (B(M)) such that there are at least q√q − q − 2 points tᵢ ∈ T, such that ⟨πᵢ, B(tᵢ)⟩ is small and hence has a linear intersection with B, with B ∩ πᵢ = M if k = 2 and B ∩ πᵢ is a small minimal (k − 2)-blocking set if k > 2. From Lemma 12(1), we know that there are at most q + 3 large hyperplanes through πᵢ if k = 2, and at most q − 5 if k > 2 (see Lemma 14).
Let $T$ be a line through $s$ in $\pi_2$, not through $x$. The existence of $\pi_M$ follows from Lemma 10(1) if $k = 2$, and Lemma 10(2) if $k > 2$. Since $B(T)$ contains $q\sqrt{q} + 1$ spread elements, there are at least $q\sqrt{q} - q - 2$ points $t_i \in T$ such that $(\pi_M, B(t_i))$ is small. This proves our claim.

Since $B \cap (B(t_i), \pi_M)$ is linear, also the intersection of $(B(t_i), B(M))$ with $B$ is linear, i.e., there exist subspaces $\tau_i$, $\tau_i \cap S(P) = \{x\}$, such that $B(\tau_i) = (B(t_i), B(M)) \cap B$. Since $\tau_i \cap (B(M))$ and $M$ are both transversals through $x$ to the same regulus $B(M)$, they coincide, hence $M \subseteq \tau_i$. The same holds for $\tau_i \cap (B(t_i), S(P))$, implying $t_i \in \tau_i$. We conclude that $B(\langle M, t_i \rangle) \subseteq B(\tau_i) \subseteq B$.

We show that $B(\langle M, T \rangle) \subseteq B$. Let $L'$ be a line of $\langle M, T \rangle$, not intersecting $M$. The line $L'$ intersects the planes $\langle M, t_i \rangle$ in points $p_i$ such that $B(p_i) \subseteq B$. Since $B(L')$ is a subline intersecting $B$ in at least $q\sqrt{q} - q - 2$ points, Result 6 shows that $B(L') \subseteq B$. Since every point of the space $\langle M, T \rangle$ lies on such a line $L'$, $B(\langle M, T \rangle) \subseteq B$.

Hence, $B(\langle M, s \rangle) \subseteq B$ for all lines $M$ through $x$ in $\pi_2$, and all points $s \neq x \in \pi_2$. We conclude that $B(\langle \pi_1, \pi_2 \rangle) \subseteq B$.

**Theorem 21.** The set $B$ is $\mathbb{F}_q\sqrt{q}$-linear.

**Proof.** Lemma 19 shows that there exists a point $P$ of $B$, a tangent $(n - k)$-space $\pi$ at the point $P$ and at least $q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5} - (n - k + 1)$-spaces $H_i$ through $\pi$ for which $B \cap H_i$ is a Baer subplane, $i = 1, \ldots, s$, $s \geq q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5}$. Let $B \cap H_i = B(\pi_i), i = 1, \ldots, s$, with $\pi_i$ a plane.

Lemma 20 shows that $B(\langle \pi_i, \pi_j \rangle) \subseteq B$, $0 \leq i \neq j \leq s$. If $k = 2$, the set $B(\langle \pi_1, \pi_2 \rangle)$ corresponds to a linear 2-blocking set $B'$ in $\text{PG}(n, q^3)$. Since $B$ is minimal, $B = B'$, and the Theorem is proven.

Let $k > 2$. Denote the $(n - k + 1)$-spaces trough $\pi$ different from $H_i$ by $K_j, j = 1, \ldots, z$. There are at least $(q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5} - 1)/q^3$ different $(n - k + 2)$-spaces $\langle H_1, H_j \rangle, 1 < j \leq s$. If all $(n - k + 2)$-spaces $\langle H_1, H_j \rangle$, contain at least $2q^2$ of the spaces $K_i$, then $z \geq 2q^2(q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5} - 1)/q^3$, a contradiction if $q \geq 49$. Let $\langle H_1, H_2 \rangle$ be an $(n - k + 2)$-spaces containing less than $2q^2$ spaces $K_i$.

Suppose, by induction, that for any $1 < i < k$, there is an $(n - k + i)$-space $\langle H_1, H_2, \ldots, H_i \rangle$ containing at most $2q^{3i-4}$ of the spaces $K_i$, such that $B(\langle \pi_1, \pi_i \rangle) \subseteq B$.

There are at least $q^{3i-3} - q^{3i-4} - 2\sqrt{q}q^{3i-5} - (q^{3i-1} - 1)/q^3$ different $(n - k + i + 1)$-spaces $\langle H_1, H_2, \ldots, H_i, H \rangle, H \not\subseteq \langle H_1, H_2, \ldots, H_i \rangle$.

If all of these contain at least $2q^{3i-1}$ of the spaces $K_i$, then $z \geq 2q^{3i-1} - 2q^{3i-4}q^{3k-3} - q^{3k-4} - 2\sqrt{q}q^{3k-5} - (q^{3i-1} - 1)/q^3 + 2q^{3i-4}$, a contradiction if $q \geq 49$. Let $\langle H_1, \ldots, H_{i+1} \rangle$ be an $(n - k + i + 1)$-space containing less than $2q^{3i-1}$ spaces $K_i$. We still need to prove that $B(\langle \pi_1, \ldots, \pi_{i+1} \rangle) \subseteq B$.

Since $B(\langle \pi_{i+1}, \pi \rangle) \subseteq B$, with $\pi$ a plane in $\langle \pi_1, \ldots, \pi_i \rangle$ for which $B(\pi)$ is not contained in one of the spaces $K_i$, there are at most $2q^{3i-4}$ 4-dimensional spaces $\langle \pi_{i+1}, \mu \rangle$ for which $B(\pi_{i+1}, \mu)$ is not necessarily contained in $B$, giving rise to at most $2q^{3i-4}(q^6 + q^4\sqrt{q})$ points $Q_i$, for which $B(Q_i)$ is not necessarily in $B$.

Let $Q$ be a point of such a space $\langle \pi_{i+1}, \mu \rangle$.

There are $(q\sqrt{q})^{2i+2} - 1)/(q\sqrt{q} - 1)$ lines through $Q$ in $\langle \pi_1, \ldots, \pi_{i+1} \rangle \cong \text{PG}(2i + 2, q\sqrt{q})$, and there are at most $2q^{3i-4}(q^6 + q^4\sqrt{q})$ points $Q_i$ for which
$B(Q_i)$ is not necessarily in $B$. Suppose all lines through $Q$ in $(\pi_1, \ldots, \pi_{i+1}) \cong \text{PG}(2i + 2, q\sqrt{q})$ contain at least $q\sqrt{q} - q - \sqrt{q}$ points $Q_i$ for which $B(Q_i)$ is not necessarily in $B$, then there are at least $(q\sqrt{q} - q - \sqrt{q} - 1)((q\sqrt{q})^{2i+2} - 1)/(q\sqrt{q} - 1) + 1 > 2q^{3i-4}(q^6 + q^4\sqrt{q})$ points $Q_i$ for which $B(Q_i)$ is not necessarily in $B$, a contradiction.

Hence, there is a line $N$ through $Q$ in $(\pi_1, \ldots, \pi_{i+1})$ with at most $q\sqrt{q} - q - \sqrt{q} - 1$ points $Q_i$ for which $B(Q_i)$ is not necessarily contained in $B$, hence, for at least $q + \sqrt{q} + 2$ points $R \in N$, $B(R) \subseteq B$. Result 6 yields that $B(Q) \subseteq B$.

This implies that $B((\pi_1, \ldots, \pi_{i+1})) \subseteq B$.

Hence, the space $B((H_1, H_2, \ldots, H_k))$ is such that $B((\pi_1, \ldots, \pi_k)) \subseteq B$. But $B((\pi_1, \ldots, \pi_k))$ corresponds to a linear $k$-blocking set $B'$ in $\text{PG}(n, q^3)$. Since $B$ is minimal, $B = B'$.

References