18th Balkan Mathematics Olympiad

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(communicated by Dusan Djukic)

1. Let \( n \) be a natural number. Show that, if \( a, b \) are natural numbers greater than 1 such that \( 2^n - 1 = ab \), then \( ab - (a - b) - 1 \) is a number of the form \( k2^{2m} \), where \( k \) is odd and \( m \) natural.

2. In a pentagon all interior angles are congruent and all its sides have rational lengths. Prove that this pentagon is regular.

3. Let \( a, b, c \) be positive real numbers such that \( a + b + c \geq abc \). Prove that \( a^2 + b^2 + c^2 \geq \sqrt{3}abc \).

4. A cube of dimension \( 3 \times 3 \times 3 \) is divided into 27 unit cube cells. One of the cells is empty, and all others are filled with unit cubes which are, on an arbitrary way, denoted with 1, 2, ..., 26. A legal move consists of a move of a unit cube to its neighbouring empty cell. Does there exist a finite sequence of legal moves after which the unit cubes denoted with \( k \) and \( 27 - k \) will exchange their positions for all \( k = 1, 2, ..., 13 \) (two cells are neighbouring if they have a common face)