20th Balkan Mathematical Olympiad
May 4, 2003

1. Can one find 4004 positive integers such that the sum of any 2003 of them is not divisible by 2003?

2. Let ΔABC be a triangle with $AB \neq AC$, and let D be the point where the tangent from A to the circumcircle of ΔABC meets BC. Consider the points E, F which lie on the perpendiculars raised from B and C on BC respectively, and on the perpendicular bisectors of $[AB]$, respectively $[AC]$. Prove that D, E and F are collinear.

3. Find all functions $f : \mathbf{Q} \to \mathbf{R}$ which fulfill the following conditions:

   a) $f(1) + 1 > 0$;

   b) $f(x+y) - xf(y) - yf(x) = f(x)f(y) - x - y + xy$, for all $x, y \in \mathbf{Q}$;

   c) $f(x) = 2f(x+1) + x + 2$, for every $x \in \mathbf{Q}$.

4. Let $ABCD$ be a rectangle of lengths m, n, made up of $m \times n$ unit squares, where m, n are two odd and coprime positive integers. The main diagonal AC intersects the unit squares in the points $A_1, A_2, \ldots, A_k$, where k is a positive integer, $k \geq 2$, and $A_1 = A$, and $A_k = C$. Prove that

$$A_1A_2 - A_2A_3 + A_3A_4 - \cdots + (-1)^k A_{k-1}A_k = \frac{\sqrt{m^2+n^2}}{mn}.$$

Work time: 4 hours.