54th Mathematical Olympiad in Poland
Second Round, February 21–22, 2003

First Day

1. Prove that there exists a positive integer $n > 2003$ such that the sequence $a_k = \binom{n}{k}$, $k = 0, 1, \ldots, 2003$ has the following property: a_k divides a_m for all $m > k$ and $k = 0, 1, \ldots, 2002$.

2. Let o be the circumcircle of a quadrilateral $ABCD$. The bisectors of the angles DAB and ABC intersect in the point P and the bisectors of the angles BCD and CDA intersect in the point Q. The point M is the midpoint of the arc BC of o which does not contain the points D and A. The point N is the midpoint of the arc DA of o which does not contain the points B and C. Prove that the points P and Q lie on a line perpendicular to MN.

3. The polynomial $W(x) = x^4 - 3x^3 + 5x^2 - 9x$ is given. Determine all pairs of different integers a, b satisfying the equation

$$W(a) = W(b)\,.$$

Second Day

4. Show that for each prime $p > 3$ there exist integers x, y, k satisfying the conditions: $0 < 2k < p$ and $kp + 3 = x^2 + y^2$.

5. Point A lies inside a circle o with the center O. Through A two tangent lines to o are drawn and the tangent points of these lines with o are called B and C, respectively. Some tangent line to o intersects the segments AB and AC in the points E and F, respectively. The lines OE and OF intersect the segment BC in the points P and Q, respectively. Prove that the segments BP, PQ and QC form a triangle, which is similar to the triangle AEF.

6. A real function f defined on all pairs of nonnegative integers is given. This function satisfies the following conditions:
$f(0,0) = 0, \quad f(2x, 2y) = f(2x+1, 2y+1) = f(x, y),$
$f(2x+1, 2y) = f(2x, 2y+1) = f(x, y) + 1$
for all nonnegative integers x, y. Let n be a nonnegative integer and a, b be nonnegative integers such that $f(a, b) = n$. Find out how many nonnegative integers x satisfy the equation

$$f(a, x) + f(b, x) = n\,.$$