50th Mathematical Olympiad in Poland

Problems of the first round, September – December 1998

1. Prove that among the numbers of the form $50^n + (50n+1)^{50}$, where n is a natural number, there exist infinitely many composite numbers.

2. Show that for all real numbers a, b, c, d, the following inequality holds:
$$(a+b+c+d)^2 \le 3(a^2+b^2+c^2+d^2) + 6ab\,.$$

3. In the isosceles triangle ABC the angle BAC is a right angle. Point D lies on the side BC and satisfies $BD = 2 \cdot CD$. Point E is the foot of the perpendicular of the point B on the line AD. Find the angle CED.

4. Let x, y be real numbers such that the numbers $x+y$, x^2+y^2, x^3+y^3 and x^4+y^4 are integers. Prove that for all positive integers n, the number x^n+y^n is an integer.

5. Find all pairs of positive integers x, y satisfying the equation $\;y^x = x^{50}$.

6. Diagonals AC and BD of the convex quadrilateral $ABCD$ meet at the point P. The point M is the midpoint of the side AB. The line MP meets the side CD at the point Q. Prove that the ratio of the areas of the triangles BCP and ADP equals the ratio of the lengths of CQ and DQ.

7. Let $n \ge 2$ be a positive integer. Find all polynomials $P(x) = a_0 + a_1x + \ldots + a_nx^n$ having exactly n roots not greater than -1 and satisfying
$$a_0^2 + a_1a_n = a_n^2 + a_0a_{n-1}\,.$$

8. Let $n \ge 2$ be a positive integer and S be a set containing n-elements. Find the smallest positive integer k for which there exist subsets $A_1, A_2, \ldots, A_k$ of the set S with the following property:
for any two different elements $a, b \in S$, there exists a number $j \in \{1, 2, \ldots, k\}$, such that $A_j \cap \{a, b\}$ has exactly one element.

9. Points D, E, F lie on the sides BC, CA, AB of the triangle ABC respectively. The incircles of the triangles AEF, BFD, CDE are tangent to the incircle of the triangle DEF. Prove that the lines AD, BE, CF are concurrent.

10. Let $x_1 > 0$ be a given real number. The sequence (x_n) is defined by the formula:
$$x_{n+1} = x_n + \frac{1}{x_n^2}\,, \qquad \text{for} \;\; n = 1, 2, 3, \ldots\,.$$
Prove that the limit $\displaystyle\lim_{n\to\infty} \frac{x_n}{\sqrt[3]{n}}$ exists and find it.

11. There are two balls in an urn: one white and one black. We have Moreover we have 50 white and 50 black balls. We repeat 50 times the following procedure: we take out one ball from the urn at random and next we put it into the urn again together with another ball of the same colour. Finally we have 52 balls in the urn. What is the most probable number of white balls in the urn?

12. All vertices of the cube with edge length a lie on the surface of the regular tetrahedron with edge length 1. Find all possible values of a.