First Day

1. Let $f: \langle 0, 1 \rangle \to \mathbb{R}$ be a given function, such that $f\left(\frac{1}{n}\right) = (-1)^n$ for $n = 1, 2, \ldots$. Prove that there do not exist increasing functions $g: \langle 0, 1 \rangle \to \mathbb{R}$, $h: \langle 0, 1 \rangle \to \mathbb{R}$, such that $f = g - h$.

2. The cube $S$ with edge length 2 consists of 8 unit cubes. We will call the cube $S$ with one unit cube removed a piece. The cube $T$ with edge length $2^n$ consists of $(2^n)^3$ unit cubes. Prove that if one unit cube is removed from $T$, then the remaining solid can be built with pieces.

3. The convex quadrilateral $ABCD$ is inscribed in a circle. The points $E$ and $F$ lie on the sides $AB$ and $CD$ respectively and satisfy the condition $AE:EB = CF:FD$. The point $P$ lies on the straight segment $EF$ and satisfies $EP:PF = AB:CD$. Prove that the ratio of the areas of the triangles $APD$ and $BPC$ does not depend on the choice of the points $E$ and $F$.

Second Day

4. Point $P$ lies inside the triangle $ABC$ and satisfies $\angle PAB = \angle PCA$ and $\angle PAC = \angle PBA$. The point $O$ is the center of the circumcircle of the triangle $ABC$. Prove that $\angle APO$ is a right angle if $O \neq P$.

5. Let $S = \{1, 2, 3, 4, 5\}$. Find out how many functions $f: S \to S$ exist with the following property: $f^{50}(x) = x$ for all $x \in S$.

   Note: $f^{50}(x) = f \circ f \circ \ldots \circ f(x)$.

6. Let the positive integer $k \geq 2$ and the integers $a_1, a_2, \ldots, a_n$ be given and have the following properties:

   $a_1 + 2^ia_2 + 3^ia_3 + \ldots + n^ia_n = 0$, \hspace{1cm} \text{for } i = 1, 2, \ldots, k-1.

   Prove that the number $a_1 + 2^ka_2 + 3^ka_3 + \ldots + n^ka_n$ is divisible by $k!$. 