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σ-HOMOTOPY GROUPS OF COXETERCOMPLEXES

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communicated by J.A. Thas

The notion of a σ-homotopy group of an arbitrary chamber system has been introduced by J.Tits in his work on local characterisations of buildings [2]. A method is given to calculate some σ-homotopy groups for arbitrary Coxetercomplexes.

INTRODUCTION

Consider an arbitrary chamber system Σ of rank n (for definition see Tits [2] and Ronan [1]). If $\Delta=\{1,2,\ldots,n\}$ and $\sigma\subseteq 2^{\Delta}$, two galleries γ and γ' are called elementary σ-homotopic if and only if γ and γ' can be written as the juxtaposition of 3 galleries $\gamma=\alpha\delta\beta$ and $\gamma'=\alpha\delta'\beta$ respectively, where $\alpha\delta$ and $\alpha\delta'$ are both i-adjacent and $\delta\beta$ and $\delta'\beta$ are both j-adjacent ($i,j\in\Delta;\alpha,\beta$ possibly empty) and where δ and δ' are both J-galleries with $J\in\sigma$, and have both the same end chambers. We call two galleries γ and γ' σ-homotopic if they can be connected by a sequence of elementary σ-homotopies and we define

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$[\gamma]_\sigma$ to be the σ-homotopy class of galleries containing γ. We define the σ-homotopy group of $\Sigma : \pi^\sigma(\Sigma)$ as the group where elements are all σ-homotopy classes of galleries of Σ (who is supposed to be connected) with an arbitrary chamber c of Σ as end chambers, and with binary operation $[\gamma].[\delta]=[\gamma\delta]$ where $\gamma\delta$ is the juxtaposition of γ and δ. It has been proved by J.Tits that if Σ is a building and if $\sigma=\Delta \cup \binom{\Delta}{2}$ that $\pi^\sigma(\Sigma)$ is the trivial group of one element.

1. NUMBER OF FLAGS OF A GIVEN TYPE OF AN ARBITRARY COXETERCOMPLEX

We denote always the cardinality of a set A by $|A|$. If Σ_n is an arbitrary Coxetercomplex (reducible or not) of rank n and $\Delta_n=\{1,2,\ldots,n\}$, $J\subseteq\Delta_n$ and $i\in\Delta_n$, then we denote by Σ_n^J, Σ_n^i and $\Sigma_n^{(i)}$ respective the set of flags of type J of Σ_n, the set of varieties of type i of Σ_n and the set of flags of type J with $|J|=i$ of Σ_n. Hence $|\Sigma_n^{(n)}|=|\Sigma_n^{\Delta_n}|$ is the number of chambers of Σ_n and since the Weylgroup $W(\Sigma_n)$ of Σ_n acts sharply 1-transitive on the set of chambers, this number is also the order of the Weylgroup. If $J=\{i_1,i_2,\ldots,i_k\}\subseteq\Delta_n$, then the flags of type Δ_n-J are in fact the left cosets of the parabolic subgroup W_J generated by the fundamental reflections $\{w_{i_1},w_{i_2},\ldots,w_{i_k}\}$, and so $|\Sigma_n^J|$ is the index of the parabolic subgroup W_{Δ_n-J} in $W(\Sigma_n)$. Also

W_J is the Weylgroup of the Coxetercomplex with diagram J. Also if Σ_n is a reducible Coxetercomplex and if $\Sigma_{n_1}, \Sigma_{n_2}, \ldots, \Sigma_{n_k}$ (with $\sum_{i=1}^{k} n_i = n$) are its irreducible components, then $W(\Sigma_n) = W(\Sigma_{n_1}) \oplus W(\Sigma_{n_2}) \oplus \ldots \oplus W(\Sigma_{n_k})$ and so $|\Sigma_n^{(n)}| = \prod_{i=1}^{k} |\Sigma_{n_i}^{(n_i)}|$.

Hence if Σ_n is an arbitrary Coxetercomplex and $J=\{i_1, i_2, \ldots, i_k\}$ and if we denote the connected components of the diagram $\Delta_n - J$ by $B_1, B_2, \ldots, B_l$, then

$$|W_{\Delta_n - J}| = \prod_{t=1}^{l} |W_{B_t}|$$

and so the number of flags of type J in Σ_n is

$$\frac{|W_{\Sigma_n}|}{\prod_{t=1}^{l} |W_{B_t}|} .$$

For $J=\{i\}$ we have then the number of varieties of type i. Hence the number of varieties or of flags of given type of a given Coxetercomplex can be derived directly from the well known orders of the Weylgroups of the irreducible Coxetercomplexes.

REMARK. Since a Coxetercomplex Σ_n of rank n defines always a triangulation of the hypersphere $S^{n-1} = \{(x_1, \ldots, x_n) \in R^n \| \sum_{i=1}^{n} x_i^2 = 1\}$ we have by the Euler-Poincaré formula the following linear equation between the number of flags of a given type of Σ_n :

$$\sum_{k=0}^{n} (-1)^{n-k} |\Sigma_n^{(k)}| = 1.$$

2. CHAMBER GRAPHS

The chamber graph $Ch(\Sigma)$ of a chamber system Σ of rank n is the linear graf $\Gamma\equiv(X,E)$ where X is the set of chambers of Σ and where $\{c,c'\}\in E$ if and only if c and c' are adjacent chambers. If we denote the set of edges $\{\{c,c'\}\| c$ and c' are i-adjacent$\}$ by E_i, then $E=E_1\sqcup E_2\sqcup\ldots\sqcup E_n$ is a natural partition of the set of edges E of $Ch(\Sigma)$. If $\Delta=\{1,2,\ldots,n\}$ and $J\subseteq\Delta$, then we denote $E_J=\bigcup_{t\in J} E_t$.

If Σ_n is a Coxetercomplex, then since there exist a 1-1 correspondence between $\Sigma_n^{(n)}$ and $W(\Sigma_n)$ we can take $X=W(\Sigma_n)$ and two elements w and w' of the Weylgroup are then i-adjacent (or $\{w,w'\}\in E_i$) if and only if $w^{-1}w'$ is the fundamental reflection w_i (Hence $Ch(\Sigma_n)$ is in fact the Cayley graph of the Weylgroup $W(\Sigma_n)$ with the fundamental reflections $\{w_1,w_2,\ldots,w_n\}$ as set of generators).If $J=\{i,j\}\in\binom{\Delta_n}{2}$, then each connected component of (X,E_J) is the chamber graph of a Coxetercomplex of rank 2. If this Coxetercomplex is reducible or of type $\underset{i}{\circ}\ \underset{j}{\circ}$, then a connected component of (X,E_j) is the Cayley graph of the four group of Klein or a square

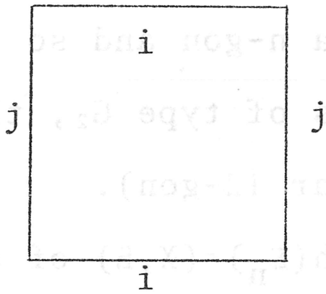


Fig.[1]

If $J=\{i,j\}$ is of type A_2 or $\underset{i}{\circ}\!-\!\underset{j}{\circ}$ then a connected component of (X,E_J) is the Cayley graph of Sym(3) or an hexagon

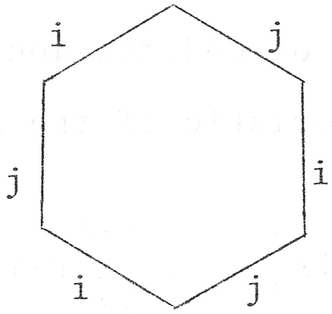


Fig.[2]

We can conceive this as the first barycentric subdivision of a triangle (the thin projective plane).

If $J=\{i,j\}$ is of type C_2 or $\underset{i}{\circ}\!=\!\underset{j}{\circ}$ then a connected component of (X,E_J) is the Cayley graph of the dihedral group D_8 of symmetries of the square or an octagon

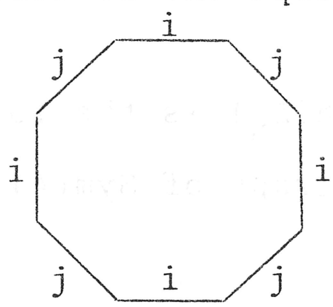


Fig.[3]

Again, we can conceive this as the first barycentric subdivision of a square (a "thin generalized quadrangle").

In general if $J=\{i,j\}$ is of type I_m, then each connected component of (X,E_J) is the Cayley graph of the dihedral

group D_{2m} of symmetries of a regular n-gon or the first barycentric subdivision of a n-gon and so a 2m-gon. (Hence in particular if $J=\{i,j\}$ is of type G_2, then each component of (X,E_J) is a regular 12-gon).
Hence each Chamber graph $Ch(\Sigma_n)=(X,E)$ of a Coxetercomplex Σ_n contains quadrangles, hexagons, octagons, 10-gons and so on, as "elementary $\{i,j\}$ - subgraphs or $\{i,j\}$-cells.

REMARK 1. Since $|X|=|\Sigma_n^{(n)}|$ and $|E|=|\Sigma_n^{(n-1)}|$ is the cardinality of the set of flags of codimension 1 we have for the Euler-Poincaré characteristic of the linear graph $Ch(\Sigma_n)$:

$$\chi(Ch(\Sigma_n))=|\Sigma_n^{(n-1)}|-|\Sigma_n^{(n)}|+1=\sum_{k=0}^{n-2}(-1)^{n-k}|\Sigma_n^{(k)}|$$

REMARK 2. Of course each elementary $\{i,j\}$-subgraph is the chambergraph of the residu $R(F)$ of some flag of type $\Delta_n-\{i,j\}$. Hence the number of $\{i,j\}$-cells is in fact the number of flags of type $\Delta_n-\{i,j\}$ or $|\Sigma_n^{\Delta_n-\{i,j\}}|$.

EXAMPLE. If $\Sigma_n\equiv A_3$, then $Ch(\Sigma_n)$ is the following linear graph which is the Cayley graph of Sym(4)).

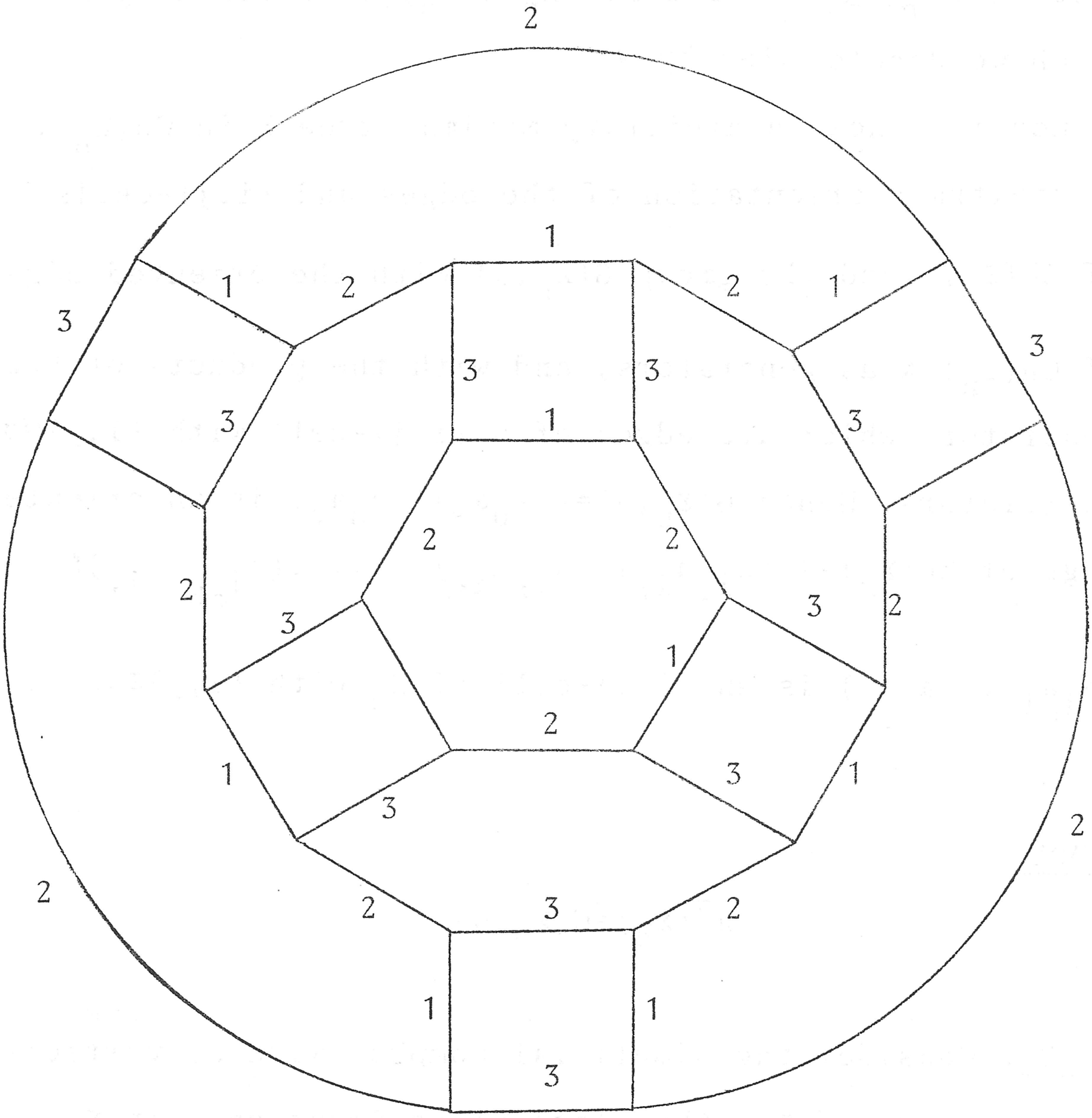


Fig.[4] ($\Delta=\{1,2,3\}$)

3. σ-HOMOTOPY GROUPS

With each gallery of Σ_n corresponds a path in $Ch(\Sigma_n)$. Moreover with each J-gallery ($J\subseteq\Delta_n$) corresponds a path in (X,E_J). Two galleries of Σ_n are elementary $\{i,j\}$ - homotopic equivalent if and only if the corresponding paths in $Ch(\Sigma_n)$ differ only in an $\{i,j\}$-cell. Also we know that each $\{i,j\}$-cell is a regular 2h-gon and if $n\geq 3$, then $h\in\{2,3,4,5\}$. We shall consider here only the case that $\Delta_n \subseteq \sigma \subseteq \Delta_n\binom{n}{2}$ and so $(\Delta_n, \sigma-\Delta_n)$ is a linear graph which we denote also by a

Consider now an arbitrary maximal tree K in $Ch(\Sigma_n)$, an arbitrary orientation of the edges and {i,j}-cells of $Ch(\Sigma_n)$; and the group $G(\Sigma_n,\sigma)$ with the oriented edges of $Ch(\Sigma_n)$-K as generators, and with the products of the generators which are edges of a {i,j}-cell with $\{i,j\}\in\sigma$ as relators. Hence $G(\Sigma_n,\sigma)=\langle(a_u a_v) \| (a_u a_v)$ is an oriented edge of $Ch(\Sigma_n)-K \mid (a_{i_1} a_{i_2}).(a_{i_2} a_{i_3}). \ldots .(a_{i_{2h}} a_{i_1}) \|$

$(a_{i_1} a_{i_2} \ldots a_{i_{2h}})$ is an {i,j}-cell of Σ_n with $\{i,j\}\in\sigma\rangle$.

LEMMA

$$\Pi^{\sigma}(\Sigma_n) \cong G(\Sigma_n,\sigma)$$

PROOF. Consider the simplicial complex with as vertices the chambers of Σ_n, the flags of codimension 1 in Σ_n, and the flags of cotype {i,j} with $\{i,j\}\in\sigma$; and with as 2-dimensional simplexes the 3-subsets of flags$\{\alpha,\beta,\gamma\}$ where cotyp $\alpha=\phi$, cotyp $\beta=\{i\}$for some $i\in\Delta_n$, cotyp$\gamma=\{i,j\}$, for some $\{i,j\}\in\sigma$ and $\gamma\subset\beta\subset\alpha$. This is the Ronan-complex $\Gamma_\sigma(\Sigma_n)$ and it has been proved by Ronan [1] that $\Pi^{\sigma}(\Sigma_n)\cong\Pi_1[\Gamma_\sigma(\Sigma_n),p]$ for an arbitrary vertex p (Σ_n is connected).

Hence in $\Gamma_\sigma(\Sigma_n)$ in fact each {i,j}-cell which is a regular 2h-gon is triangulated in 4h triangles , and the 3 vertices of each triangle are : a vertex of $Ch(\Sigma_n)$ (or a chamber of Σ_n), a midpoint of an edge of $Ch(\Sigma_n)$

(or a flag of codimension 1) and the center of an {i,j}-cell with {i,j}∈σ as a regular 2h-gon (or a flag of cotype {i,j})[see figure [5]].

An other way to triangulate these {i,j}-cells is by taking an arbitrary point of it, and by joining this vertex to the other vertices (see figure [6]). We denote such simplicial complex $\Gamma^*_\sigma(\Sigma_n)$

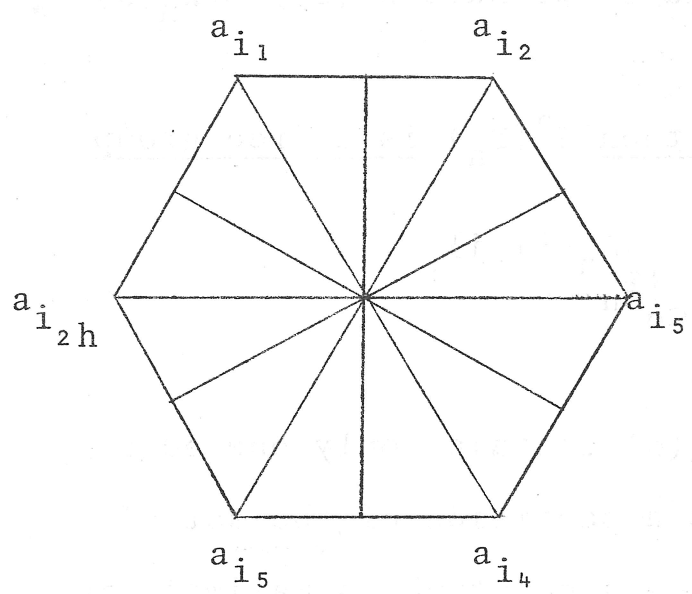


Fig.[5]

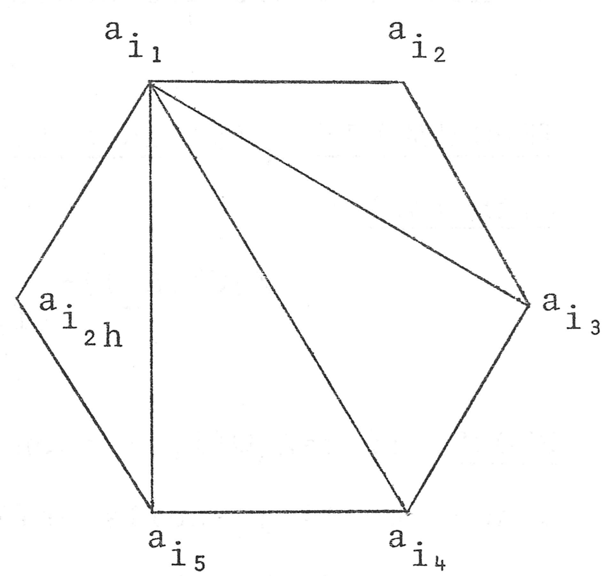


Fig.[6]

Since each such simplicial complex is a triangulation of the same polyheder of which the Ronancomplex is also a triangulation they have all $\Pi^\sigma(\Sigma_n)$ as fundamental group. But in the last triangulation no new vertices are added, and so if K is a maximal tree of $Ch(\Sigma_n)$, then K is also a maximal tree of $\Gamma^*_\sigma(\Sigma_n)$. Hence we have as a set of generators of $\Pi^\sigma(\Sigma_n)$ the set of generators of $G(\Sigma_n,\sigma)$ together with for each {i,j}-cell where {i,j}∈σ the set of oriented edges with the exceptional point as one endpoint and the vertices of the $\{i,j\}$-cell not adjacent to it in $Ch(\Sigma_n)$ as the other endpoints.

If we give now an arbitrary orientation to each such "interior" simplex of each $\{i,j\}$-cell with $\{i,j\}\in\sigma$, then it is clear that each new generator can be expressed as a product of generators of $G(\Sigma_n,\sigma)$ and that each $\{i,j\}$-cell with $\{i,j\}\in\sigma:(a_{i_1}a_{i_2}\ldots a_{i_{2h}})$ gives rise to only one relator $(a_{i_1}a_{i_2})(a_{i_2}a_{i_3})\ldots(a_{i_{2h-1}}a_{i_{2h}})(a_{i_{2h}}a_{i_1})$.
No new relators are added and so we have $\Pi^{\sigma}(\Sigma_n)\cong G(\Sigma_n,\sigma)$. ∎

<u>THEOREM 1.</u> <u>If</u> $\Delta_n(\sigma)$ <u>is a tree then</u> $\Pi^{\sigma}(\Sigma_n)$ <u>is a free group with rank</u>

$$\chi(Ch(\Sigma_n)) - \sum_{\{i,j\}\in\sigma} |\Sigma_n^{\Delta_n-\{i,j\}}|$$

<u>PROOF</u>. If $\sigma=\Delta_n\cup\{i,j\}$ then $\Delta_n(\sigma)$ contains only one edge . Since the $\{i,j\}$-cells define a partition of the set of vertices of $Ch(\Sigma_n)$ we can always construct a maximal tree K of $Ch(\Sigma_n)$ by taking all but one edge of each $\{i,j\}$-cell and by joining these $\{i,j\}$-cells in a suitable way (one has first to construct a maximal tree of the linear graph where vertices are the $\{i,j\}$-cells and where two vertices are adjacent if and only if there exists an edge in $Ch(\Sigma_n)$ joining them; this graph is connected since the $\{i,j\}$-cells define a partition of the set of vertices of $Ch(\Sigma_n)$). Of course we can take in particulary as remaining edge in an $\{i,j\}$-cell, an edge that joins two j-adjacent chambers (or briefly : a j-edge). Let us denote such a

maximal tree by $K(i,j)$. If $\sigma'=\Delta_n$ then of course $\Pi^{\sigma'}(\Sigma_n)$ is the free group with $\chi(Ch(\Sigma_n))$ generators. But with this tree $K(i,j)$ and by the lemma , exactly one edge of each $\{i,j\}$-cell will be a new relator in $\Pi^{\sigma}(\Sigma_n)$ and no other relators are added. Hence $\Pi^{\Delta_n\cup\{i,j\}}(\Sigma_n)$ is the free group with $\chi(Ch(\Sigma_n))-|\Sigma^{\Delta_n-\{i,j\}}|$ generators.

We shall now prove inductively on $p=|\sigma-\Delta_n|$. Suppose p : $|\sigma-\Delta_n|>1$ (in fact the case $p=1$ that we have just proved will follow directly by induction from the case $p=0$, where the statement is obvious). If $\Delta_n(\sigma)$ is a tree, then it has some endpoint j_0.

Suppose $\{i_0,j_0\}\in\sigma$, then $j_0\notin\cup((\sigma-\Delta_n)-\{i_0,j_0\})$. Therefore the set of generators of the free group $\Pi^{\sigma-\{i\ ,j\}}(\Sigma_n)$ contains all oriented j_0-edges that are not in the maximal tree. If we take now as maximal tree a $K(i_0,j_0)$. It is clear that $\Pi^{\sigma}(\Sigma_n)$ has the same generators as $\Pi^{\sigma-\{i_0,j_0\}}(\Sigma_n)$ and by the lemma, each remaining oriented j_0-edge becomes a relator and no other relators are added. If m is the rank of $\Pi^{\sigma-\{i_0,j_0\}}(\Sigma_n)$ and since there are $|\Sigma_n^{\{i_0,j_0\}}|$ remaining j_0-edges, $\Pi^{\sigma}(\Sigma_n)$ is a free group of rank

$$m-|\Sigma_n^{\Delta_n-\{i_0,j_0\}}| = \chi(Ch(\Sigma_n))-\sum_{\{i,j\}\in\sigma}|\Sigma_n^{\Delta_n-\{i,j\}}|$$

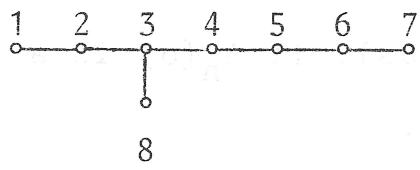
Hence we have proved the theorem. ■

<u>REMARK 1</u>.If σ is not a tree, then in general $\Pi^{\sigma}(\Sigma_n)$ is not a free group. If for instance $n=4$,

$\sigma=\Delta_4 \cup \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$ (Thus σ is the linear graph : the square) and $\Sigma_4=A_1\oplus A_1\oplus A_1\oplus A_1$, the Coxetercomplex with diagram ∘ ∘ ∘ ∘

then $\pi^{\sigma}(\Sigma_4) \cong Z\oplus Z$ (see example 2)

<u>EXAMPLE 1</u>. Consider the Coxeter complex E_8 and take for σ the Dynkindiagram of E_8 which is itself a tree

$\sigma =$ 

Then $\Pi^{\sigma}(E_8)$ is the free group with rank m=

$$\chi(Ch(E_8)) - \sum_{\{i,j\}\in\sigma} |E_8^{\Delta_8-\{i,j\}}| = 1.045.094.401-406.425.600$$

$= 638.668.801$ (using the results of paragraph 1)

<u>THEOREM 2</u>. <u>If</u> $J\subseteq\Delta_n$, $\sigma=\Delta_n\cup\binom{J}{2}$, <u>then</u> $\Pi^{\sigma}(\Sigma_n)$ <u>is the free group with rank</u> :

$$\chi(Ch(\Sigma_n)) - \sum_{\substack{L\in(2^J-J)\\ L\neq\phi}} (-1)^{|L|}\,|\Sigma_n^{\Delta_n-L}|$$

<u>PROOF</u> : We examine first the case $J=\Delta_n$. Since $\Pi^{\sigma}(\Sigma_n)\cong\Pi_1(\Gamma_\sigma(\Sigma_n))$ and $\Gamma_\sigma(\Sigma_n)$ is a triangulation of the hypersphere S^{n-1} in R^n, clearly $\Pi^{\sigma}(\Sigma_n)\cong\{1\}$, in con-

formity with the results of J. Tits [2] : the universal 2-cover of a building is isomorphic to the building itself. In this case $\chi(Ch(\Sigma_n)) = \sum_{k=0}^{n-2} (-1)^{n-k} |\Sigma_n^{(k)}|$

$$= \sum_{\substack{L \subseteq \Delta_n \\ |L| \geq 2}} (-1)^{|L|} |\Sigma_n^{\Delta n - L}|$$ and $\{1\}$ is the free group with rank 0.

If $J \subset \Delta_n$, consider the set of connected components of (X, E_J) which is a partition of the set of vertices of (X, E). The number of partition classes is clearly $|\Sigma_n^{\Delta n - J}|$ and each connected component is a copy of $Ch(\Sigma_{|J|})$ where $\Sigma_{|J|}$ is the Coxeter complex with diagram $J \subset \Delta_n$. Consider in each connected component of (X, E_J) a maximal tree. Joining this components in a suitable way in $Ch(\Sigma_n)$, we obtain a maximal tree in $Ch(\Sigma_n)$ that we denote by $K(J)$. By the first part of the proof, the result of the relators in a connected component of (X, E_j) is $\{1\}$. Since the sets of edges of these connected components are disjoint, the result of the relators in (X, E_j) is $\{1\}$. Hence $\Pi^\sigma(\Sigma_n)$ is the free group with rank

$m = |\bigsqcup_{i \in \Delta_n - J} E_i - K(J)|$. By joining the connected components of (X, E_j) in $Ch(\Sigma_n)$ to obtain $K(J)$, we needed exactly $|\Sigma_n^{\Delta n - J}| - 1$ edges.

Hence,

$$m=\sum_{i\in\Delta_n-J}|\Sigma_n{}^{n-\{i\}}|-|\Sigma_n{}^{n-J}|+1$$

$$=\chi(Ch(\Sigma_n))+|\Sigma_n^{(n)}|-\sum_{i\in J}|\Sigma_n^{\Delta n-\{i\}}|-|\Sigma_n^{\Delta n-J}|$$

$$=\chi(Ch(\Sigma_n))-|\Sigma_n^{\Delta n-J}|[\sum_{i\in J}|\Sigma_{|J|}^{J-\{i\}}|-|\Sigma_{|J|}^{|J|}|+1]$$

$$=\chi(Ch(\Sigma_n))-\chi(Ch(\Sigma_{|J|}))|\Sigma_n^{\Delta n-J}|$$

$$=\chi(Ch(\Sigma_n))-|\Sigma_n^{\Delta n-J}|(\sum_{\substack{L\subset J\\|\overline{L}|\geqslant 2}}(-1)^{|L|}\Sigma_{|J|}^{J-L})$$

Since $|\Sigma_{|J|}^{J-L}|\ |\Sigma_n^{\Delta n-J}|=|\Sigma_n^{\Delta n-L}|$ with $L\subseteq J\subseteq\Delta_n$, the result follows. ■

REMARK 2. from this, it follows that if $J\subseteq\Delta_n$, $\binom{J}{2}\subseteq\sigma$, $\sigma'=\sigma\cup 2^J$ then $\Pi^{\sigma}(\Sigma_n)=\Pi^{\sigma'}(\Sigma_n)$.

DEFINITIONS 1. Let $\Gamma\equiv(V_n,E)$ be the chambergraph of a Coxetercomplex Σ_n of rank n (V_n is the set of vertices or the chambers of Σ_n and E is the set of edges or the pairs of adjacent chambers of Σ). Let the points of V_n be in general position in R^n, with the additional property that the points of the chambergraph of each element of $\Sigma_n{}^{\Delta_n-\Delta_m}$, $\forall\ \Delta_m\subseteq\Delta_n$ lies in some $R^m\subseteq R^n$, where $m=|\Delta_m|$, and forms a convex set.

If $\{x_1,x_2\}\in E$, we write $x_1\sim x_2$, and then $\{x\in R^n \| x=x_1+t(x_2-x_1),\ x_1\sim x_2,\ t\in I=[0,1]\}$ is the set of points of the Euclidean model $EM(\Sigma_n)$ of $Ch(\Sigma_n)$ in R^n.

Consider $\Delta_n \subseteq \sigma \subseteq \Delta_n \cup \binom{\Delta_n}{2}$. We define

$$\Omega = \{\sigma \,\|\, \Delta_n \subseteq \sigma \subseteq \Delta_n \cup \binom{\Delta_n}{2}, n \in N\}$$

$$\Omega^* = \{\sigma \,\|\, \Delta_n \subseteq \sigma \subseteq 2^{\Delta_n}, n \in N\}$$

$* : \Omega \to \Omega^* : \sigma \to \sigma^*$

where σ^* has the property that if $\Delta_m \subseteq \Delta_n$, and $\binom{\Delta_m}{2} \subseteq \sigma$, then $2^{\Delta_m} \subseteq \sigma^*$ And we define $EM_1(\Sigma_n, \sigma)$ by $EM(\Sigma_n) \subseteq EM_1(\Sigma_n, \sigma) \subseteq R^n$ as the set $\{x \in R^n \,\|\, x \in C(V_J), \forall V_J \subseteq V$ and V_J is the set of points that corresponds with $\Lambda \in \Sigma^{\Delta_n - J}$, $\forall \Lambda$, $\forall J \in \sigma$, $|J| \geqslant 2\}$ where $C(W)$ is the convex hull of W in R^n or the intersection of all convex sets $W' \subseteq R^n$, with $W \subseteq W'$.

Finely we define $EM(\Sigma_n, \sigma) = EM_1(\Sigma_n, \sigma^*)$.

Since the homotopy type of s^{m-1}, the $(m-1)$-sphere in R^m is the same of the homotopytype of E^m, the m-ball in R^m, $\forall m \leqslant n, m > 2$, we see that $EM(\Sigma_n, \sigma)$ en $EM_1(\Sigma_n, \sigma)$ have the same homotopy-type. From the definitions, it follows also that $\Gamma_\sigma(\Sigma_n)$ is a triangulation of $EM_1(\Sigma_n, \sigma)$. So we have

$$\pi^\sigma(\Sigma_n) \cong \pi_1(EM(\Sigma_n, \sigma))$$

If $Ch(\Sigma_{n_1})$ and $Ch(\Sigma_{n_2})$ are the chambergraphs of respective Σ_{n_1} and Σ_{n_2}, we define the graph $G \equiv (A, F)$ with set of vertices $A = \{(t, u) \mid t$ is vertex of $Ch(\Sigma_{n_1})$, u is vertex of $Ch(\Sigma_{n_2})\}$ and set of edges F.F is defined by :

$$(t_1, u_1) \sim (t_2, u_2) \iff \begin{cases} (t_1 = t_2 \text{ and } u_1 \sim u_2) \\ \text{or} \\ (t_1 \sim t_2 \text{ and } u_1 = u_2) \end{cases} \qquad (1)$$

If T and U are the maximal flags in respective Σ_{n_1} and Σ_{n_2} that correspond with respective t and u, then every maximal flag of $\Sigma_{n_1}+\Sigma_{n_2}$ can be written as (T,U) and we can identify the flag (T,U) with the vertex (t,u) by the bijection :

$$\begin{aligned} b:\Sigma_{n_1}\oplus\Sigma_{n_2} &\to G \\ (T,U) &\mapsto (t,u) \end{aligned}$$

which transforms adjacent flags into adjacent vertices. If we define, for $\Delta_n=\Delta_{n_1}\sqcup\Delta_{n_2}$, $\Delta_{n_i}\subseteq\sigma_i\subseteq\Delta_{n_i}\cup\binom{\Delta_{n_i}}{2}$, i=1,2, $\sigma_1\oslash\sigma_2=\sigma_1\cup\sigma_2\cup\{\{i,j\}\parallel i\in\Delta_{n_1},\ j\in\Delta_{n_2}\}$, then since $C(W_1 x W_2)=C(W_1)xC(W_2)$ as product topology in $\mathbb{R}^n$, $W_i\subseteq\mathbb{R}^{n_i}$, $n_i=|\Delta_{n_i}|$, $n_1+n_2=n$, and using (1), one sees that

$$EM(\Sigma_{n_1}\oplus\Sigma_{n_2},\sigma_1\oslash\sigma_2)=EM(\Sigma_{n_1},\sigma_1)xEM(\Sigma_{n_2},\sigma_2)$$

<u>THEOREM 3</u>. <u>If</u> Σ_{n_i}, i=1,2 <u>is the Coxetercomplex with diagram</u> Δ_{n_i} <u>and if</u> $\Delta_{n_i}\subseteq\sigma_i\subseteq\Delta_{n_i}\cup\binom{\Delta_{n_i}}{2}$, i=1,2, <u>then we have</u>

$$\pi^{\sigma_1\oslash\sigma_2}(\Sigma_{n_1}\oplus\Sigma_{n_2})=\pi^{\sigma_1}(\Sigma_{n_1})\oplus\pi^{\sigma_2}(\Sigma_{n_2}).$$

<u>PROOF</u>.

$$\begin{aligned} \pi^{\sigma_1\oslash\sigma_2}(\Sigma_{n_1}\oplus\Sigma_{n_2}) &= \pi_1(EM(\Sigma_{n_1}\oplus\Sigma_{n_2},\sigma_1\oslash\sigma_2)) \\ &=\pi_1(EM(\Sigma_{n_1},\sigma_1)xEM(\Sigma_{n_2},\sigma_2)) \\ &=\pi_1(EM(\Sigma_{n_1},\sigma_1))\oplus\pi_1(EM(\Sigma_{n_2},\sigma_2)) \\ &=\pi^{\sigma_1}(\Sigma_{n_1})\oplus\pi^{\sigma_2}(\Sigma_{n_2}). \quad \blacksquare \end{aligned}$$

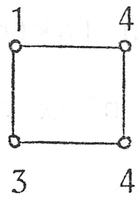
EXAMPLE 2

Consider the Coxetercomplex $A_1 \oplus A_2 = \Sigma_2$ with diagram $\{1,2\}: \overset{1}{\circ}\ \overset{2}{\circ}$ and a copy of Σ_2 with diagram $\{3,4\}: \overset{3}{\circ}\ \overset{4}{\circ}$

Then $\Sigma_2 \oplus \Sigma_2 = A_1 \oplus A_1 \oplus A_1 \oplus A_1$ and $EM(\Sigma_2 \oplus \Sigma_2)$ is the hypercube in R^4. Take $\sigma_1 = \{\{1\},\{2\}\}$, $\sigma_2 = \{\{3\},\{4\}\}$. Then it is trivial that

$$\pi^{\sigma_1}(\Sigma_1) = \pi^{\sigma_2}(\Sigma_2) = Z$$

and

$\sigma = \sigma_1 \oslash \sigma_2 =$ 

So we have

$$\pi^{\sigma}(\Sigma_1 \oplus \Sigma_2) \cong Z \oplus Z$$

and $EM(\Sigma_2 \oplus \Sigma_2, \sigma)$ has the homotopytype of a torus.

EXAMPLE 3

Suppose $\Sigma_{2n} = (A_1 \oplus A_1) \oplus (A_1 \oplus A_1) \oplus \ldots \oplus (A_1 \oplus A_1)$ (n terms) and $\sigma = \sigma_1 \oslash \sigma_2 \oslash \sigma_3 \oslash \ldots \oslash \sigma_n \oslash \sigma_i = \{\{2i-1\},\{2i\}\}$, $1 \leqslant i \leqslant n$ then $\sigma = 2^{\Delta_{2n}} - \{\{1,2\},\{3,4\},\ldots,\{2n-1,n\}$ with $\Delta_{2n} = \{1,2,\ldots,2n\}$ and using theorem 3 n-1 times we obtain :

$$\pi^{\sigma}(\Sigma_{2n}) \cong Z \oplus Z \oplus \ldots \oplus Z \qquad \text{(n terms)}$$

EXAMPLE 4

If Σ_n is a Coxetercomplex with diagram $\Delta_n = \{1,2,\ldots,n\}$ and $\Delta_n \subseteq \sigma \subseteq \Delta_n \cup \binom{\Delta_n}{2}$, $\pi^{\sigma}(\Sigma_n) \cong G$, then it follows with theorem 3 that for $\Sigma_{n+1} = A_1 \oplus \Sigma_n$ and $\sigma' = \{n+1\} \oslash \sigma$ that

$$\pi^{\sigma'}(\Sigma_{n+1}) \cong \pi^{\sigma}(\Sigma_n) \cong G.$$

DEFINITIONS 2

1. Let G and H be groups with respective presentation

$$G=\langle a_1,\dots,a_n \| R_1,\dots,R_m\rangle$$

$$H=\langle b_1,\dots,b_p \| S_1,\dots,S_q\rangle$$

Then we define the free product $G*H$:

$$G*H=\langle a_1,\dots,a_n,b_1,\dots,b_p \| R_1,\dots,R_m,S_1,\dots,S_q\rangle$$

One can easily see that $(G*H)*K = G*(H*K)$, so we denote $G^n=G*G*\dots*G$ (n faktors).

2. Let Σ_n be a Coxetercomplex of rank n, with diagram $\Delta_n=\{1,2,\dots,n\}$. Suppose

$$J_0\subseteq J_1\subseteq\dots\subseteq J_l\subseteq\Delta_n=J_{l+1}.$$

Let us denote the set of connected components of (X,E_J) by S_J, then we prove that there exists a maximal tree K in $Ch(\Sigma_n)\equiv(X,E)$ so that the property (a) holds for $i=0,1,\dots,l$.

(a) $\forall s_j\in S_J$, $s_J\cap K$ is a maximal tree in s_{J_i}.

We construct first in each element of S_{J_0} a maximal tree. We denote this set of edges by K_0, so $K_0\subseteq E_J$ and (a) holds for $i=0$ if we replace K by K_0. We complete the proof by induction. Suppose we have a set of edges K_j, $K_j\subseteq E_{J_j}$ and (a) holds for $i=0,1,\dots,j$ if we replace K by K_j. Then in each $s_{J_{j+1}}\in S_{J_{j+1}}$, there exists a subset $P\subseteq S_{J_j}$ so that the union of vertices of all elements of P is exactly the set of vertices of $s_{J_{j+1}}$. By joining these subgraphs (the elements of P) for each $s_{J_{j+1}}$ apartly, in a suitable way, we obtain a maximal tree in each $s_{J_{j+1}} \in S_{J_{j+1}}$. The union of these trees, say K_{j+1}, satisfies (a) for $i=0,1,\ldots,j,j+1$ if we replace K by K_{j+1}. If we denote $K_{l+1}=K$, we obtain (a) for $i=1,\ldots,l$. We denote such a tree K by $K(J_0;J_1;\ldots;J_l)$

<u>THEOREM 4</u>

Suppose F_1 and F_2 are flags of the Coxetercomplex Σ_n, with typ $F_1 \cap \mathrm{typ} F_2 = \phi$.

Suppose

$$\mathrm{cotyp} F_i = \Delta_{n_i} \quad |\Delta_{n_i}| = n_i \qquad i=1,2.$$

$$J = \Delta_{n_1} \cap \Delta_{n_2}$$

$$R(F_i) = \Sigma_{n_i} \quad i=1,2.$$

$$R(F_1 \cup F_2) = \Sigma_{|J|}$$

$$\Delta_{n_i} \cup \binom{J}{2} \subseteq \sigma_i \subseteq \Delta_{n_i} \cup \binom{\Delta_{n_i}}{2} \qquad i=1,2.$$

If $\pi^{\sigma_1}(\Sigma_{n_1}) \cong G$

$$\pi^{\sigma_2}(\Sigma_{n_2}) \cong H$$

$$\sigma = \sigma_1 \cup \sigma_2$$

then $\pi^{\sigma}(\Sigma_n) \cong G^{|\Sigma_n^{\Delta_n - \Delta_{n_1}}|} * H^{|\Sigma_n^{\Delta_n - \Delta_{n_2}}|} * F_k$

with

$$k = \sum_{i \in \Delta_n - J} |\Sigma_n^{\Delta_n - \{i\}}| - |\Sigma_n^{\Delta_n - \Delta_{n_1}}| - |\Sigma_n^{\Delta_n - \Delta_{n_2}}| + 2$$

$$-X(\Sigma_n) + X(\Sigma_{|J|}) \cdot |\Sigma_n^{\Delta_n - J}|$$

where $X(\Sigma)$ denotes the Euler-Poincaré characteristic of the Coxetercomplex Σ.

PROOF : We consider first the case $J=\phi$, $\sigma_2=\Delta_{n_2}$. Then we have $\Delta_n=\Delta_{n_1}\sqcup\Delta_{n_2}$. Consider in $Ch(\Sigma_n)$ a tree $K(\Delta_{n_1})$. Then we have in each connected component of $(X,E\Delta_{n_1})$ the representation of the group G. There are no other relators , but there are still

$k'=\chi(\Sigma_n)-|\Sigma_n^{\Delta_n-\Delta_{n_1}}|.\chi(\Sigma_{n_1})$ generators left.

So $\pi^{\sigma}(\Sigma_n)=G'^{|\Sigma_n^{\Delta_n-\Delta_{n_1}}|}*F_{k'}$.

In this case

$$H\cong\pi^{\sigma_2}(\Sigma_{n_2})\cong F_{\chi(\Sigma_{n_2})}$$

and we write

$$F_{k'}=F_{\chi(\Sigma_{n_2})}^{|\Sigma_n^{\Delta_n-\Delta_{n_2}}|}*F_k$$

with :

$$k=k'-|\Sigma_n^{\Delta_n-\Delta_{n_2}}|.\chi(\Sigma_{n_2})$$

$$=\chi(\Sigma_n)-[\,|\Sigma_n^{\Delta_n-\Delta_{n_1}}\,.\chi(\Sigma_{n_1})+|\Sigma_n^{\Delta_n-\Delta_{n_2}}|\,.\chi(\Sigma_{n_2})$$

Since $\chi(\Sigma_{n_j})=1+\sum_{i_j\in\Delta_{n_j}}|\Sigma_{n_j}^{\Delta_{n_j}-\{i_j\}}|-|\Sigma_{n_j}^{\Delta_{n_j}}|$ $\quad j=1,2,$

we have

$$k=\chi(\Sigma_n)-|\Sigma_n^{\Delta_n-\Delta_{n_1}}|-\sum_{i_1\in\Delta_{n_1}}|\Sigma_n^{\Delta_n-\Delta_{n_1}}|.|\Sigma_{n_1}^{\Delta_{n_1}-\{i_1\}}|$$

$$+|\Sigma_n^{\Delta_n-\Delta_{n_1}}|\,.|\Sigma_{n_1}^{\Delta_{n_1}}|-|\Sigma_n^{\Delta_n-\Delta_{n_2}}|-\sum_{i_2\in\Delta_{n_2}}|\Sigma_n^{\Delta_n-\Delta_{n_2}}|.|\Sigma_n^{\Delta_{n_2}-\{i_2\}}|$$

$$+|\Sigma_n^{\Delta_n-\Delta_{n_2}}| \cdot |\Sigma_{n_2}^{\Delta_{n_2}}| .$$

Since $|\Sigma_{|\Delta_n|}^{\Delta_n-\Delta_m}| \cdot |\Sigma_{|\Delta_m|}^{\Delta_m-\Delta_p}| = \Sigma_{|\Delta_n|}^{\Delta_n-\Delta_p}|$ for $\Delta_p \subseteq \Delta_m \subseteq \Delta_n$

and using $\Delta_n = \Delta_{n_1} \sqcup \Delta_{n_2}$, we have :

$$k = \chi(\Sigma_n) - |\Sigma_n^{\Delta_n-\Delta_{n_1}}| - |\Sigma_n^{\Delta_n-\Delta_{n_2}}| - \sum_{i\in\Delta_n} |\Sigma_n^{\Delta_n-\{i\}}| + 2|\Sigma_n^{(n)}|$$

$$k = \sum_{i\in\Delta_n} |\Sigma_n^{\Delta_n-\{i\}}| - |\Sigma_n^{\Delta_n-\Delta_{n_1}}| - |\Sigma_n^{\Delta_n-\Delta_{n_2}}| + 2$$

$$-[2 \sum_{i\in\Delta_n} |\Sigma_n^{\Delta_n-\{i\}}| - 2|\Sigma_n^{(n)}| + 2 - \chi(\Sigma_n)]$$

the sum between brackets is $[2\chi(\Sigma_n)-\chi(\Sigma_n)] = \chi(\Sigma_n)$, so the theorem follows in this particular case.

We consider now the general case. In $Ch(\Sigma_n)$, we take a tree $K(J;\Delta_{n_1}-J)$. Since $\binom{J}{2} \subseteq \sigma$, the calculation in each connected component of (X,E_j) will lead to the trivial group by theorem 2. So we have $u=\phi$ (the empty word) for each generator u that corresponds with a j-edge, $j\in J$. Let us denote the set of generators that correspond with i_k-edges ($i_k \in \Delta_{n_k}-J, k=1,2$) by R_k and the set of relators that correspond with $\{i_{k_1}, i_{k_2}\}$-subgraphs by Ω_k $\forall \{i_{k_1}, i_{k_2}\} \in \sigma_k$, $k=1,2$. No other relators appear since $\sigma=\sigma_1\cup\sigma_2$. Since $J=\Delta_{n_1}\cap\Delta_{n_2}$, we have $R_1\cap R_2=\phi$ and we can write formally

$\Omega_1 \equiv \Omega_1(R_1)$; $\Omega_2 \equiv \Omega_2(R_2)$ and we have

$$\pi^\sigma(\Sigma_n) = \langle R_1, R_2 | \Omega_1, \Omega_2 \rangle .$$

We compute $<R_1,R_2 \| \Omega_1>$. Since Ω_2 corresponds with $\sigma_2-(_2)$, we have $<R_1,R_2\|\Omega_1>=\Pi^{\sigma_1\cup\Delta_{n_2}}(\Sigma_n)$ and by the first part of the proof :

$$<R_1,R_2\|\Omega_1>\cong G^{|\Sigma_n^{\Delta_n-\Delta_{n_1}}|} * F_{k_1}$$

By the choice of the maximal tree, $<R_1\|\Omega_1>\cong G^{|\Sigma_n^{\Delta_n-\Delta_{n_1}}|}$ and thus $k_1=|R_2|$ (notice that a tree $K(J;\Delta_{n_1}-J)$ is also a tree $K(\Delta_{n_1})$). So we have

$$k_1=|R_1|=\sum_{i\in\Delta_n-\Delta_{n_1}} |\Sigma_n^{\Delta_n-\{i\}}|-|\Sigma_n^{\Delta_n-\Delta_{n_1}}|+1.$$

Similar

$$<R_1,R_2\|\Omega_2>=H^{|\Sigma_n^{\Delta_n-\Delta_{n_2}}|} * F_{k_2}$$

with

$$k_2=\sum_{i\in\Delta_n-\Delta_{n_2}} |\Sigma_n^{\Delta_n-\{i\}}|-|\Sigma_n^{\Delta_n-\Delta_{n_2}}|+1$$

Among the k_2 generators of F_{k_2} are all generators that correspond with i_1-edges, $i_1\in\Delta_{n_1}$. Let us call them briefly n_1-generators. We have

$$|R_1|=k_2'=[\chi(\Sigma_{n_1})-|\Sigma_{n_1}^{\Delta_{n_1}-J}|.\chi(\Sigma_{|J|}]\,.\,|\Sigma_n^{\Delta_n-\Delta_{n_1}}|$$

If we add now Ω_1 to $<R_1,R_2\|\Omega_2>$, these k_2' n_1-generators from together $G^{|\Sigma_n^{\Delta_n-\Delta_{n_1}}|}$ and k_2-k_2' generators remain, which form the free group $F_{k_2-k_2'}=F_k$ with $k=k_2-k_2'$. Hence

$$\pi^{\sigma}(\Sigma_n) \cong G^{|\Sigma_n^{\Delta_n - \Delta_{n_1}}|} * H^{|\Sigma_n^{\Delta_n - \Delta_{n_2}}|} * F_k.$$

with

$$k = k_2 - k_2' = \sum_{i \in \Delta_n - \Delta_{n_2}} |\Sigma_n^{\Delta_n - \{i\}}| - |\Sigma_n^{\Delta_n - \Delta_{n_2}}| + 1 - \chi(\Sigma_n).|\Sigma_n^{\Delta_n - \Delta_{n_1}}|$$

$$+ |\Sigma_{n_1}^{\Delta_n - J}|.|\Sigma_n^{\Delta_n - \Delta_{n_1}}|.\chi(\Sigma_{|J|})$$

$$= \sum_{i \in \Delta_n - \Delta_{n_2}} |\Sigma_n^{\Delta_n - \{i\}} - \Sigma_n^{\Delta_n - \Delta_{n_2}}| + 1 - [\sum_{i \in \Delta_{n_1}} |\Sigma_{n_1}^{\Delta_{n_1} - \{i\}}| - |\Sigma_{n_1}^{(n_1)}| + 1]$$

$$.|\Sigma_n^{\Delta_n - \Delta_{n_1}}| + \chi(\Sigma_{|J|}).|\Sigma_n^{\Delta n - J}|$$

$$= \sum_{i \in \Delta_n - \Delta_{n_2}} |\Sigma_n^{\Delta_n - \{i\}}| - |\Sigma_n^{\Delta_n - \Delta_{n_2}}| + 1 - \sum_{i \in \Delta_{n_1}} |\Sigma_n^{\Delta_n - \{i\}}| + \Sigma_n^{(n)}|$$

$$- |\Sigma_n^{\Delta_n - \Delta_{n_1}}| + \chi(\Sigma_{|J|}).|\Sigma_n^{\Delta_n - J}|$$

Since

$$\sum_{i \in \Delta_n - \Delta_{n_1}} |x_i - \sum_{i \in \Delta_{n_2}} x_i = \sum_{i \in \Delta_n} x_i - \sum_{i \in \Delta_{n_1} \cup \Delta_{n_2}} |x_i - \sum_{i \in \Delta_{n_1} \cap \Delta_{n_2}} x_i$$

$$= - \sum_{i \in J} x_i$$

$$= \sum_{i \in \Delta_n - J} x_i - \sum_{i \in \Delta_n} x_i$$

we have

$$k = \sum_{i \in \Delta_n - J} |\Sigma_n^{\Delta_n - \{i\}}| - |\Sigma_n^{\Delta_n - \Delta_{n_1}}| - |\Sigma_n^{\Delta_n - \Delta_{n_2}}| + 2$$

$$+ [\Sigma_n^{(n)} - 1 - \sum_{i \in \Delta_n} |\Sigma_n^{\Delta_n - \{i\}}|] + \chi(\Sigma_{|J|}).\ |\Sigma_n^{\Delta_n - J}|$$

Since $\Sigma_n^{(n)} - 1 - \sum_{i\in\Delta_n} |\Sigma_n^{\Delta_n - \{i\}}| = -\chi(\Sigma_n)$, the result follows. ∎

REMARK 3

It is an easy exercise to prove that, if we can obtain σ by means of theorem 1,2 and 4, then $\pi^\sigma(\Sigma_n)$ is the free group with rank

$$\chi(\Sigma_n) - \sum_{\substack{J\in\sigma^* \\ |J|\geq 2}} (-1)^{|J|} |\Sigma_n^{\Delta_n - J}|$$

EXAMPLE 5

Let Σ_n be a Coxetercomplex with diagram $\{1,2,\ldots,n\} = \Delta_n$. Take for $\sigma = 2^{\Delta_n} - 2^J$, $J \subseteq \Delta_n$. Then we have

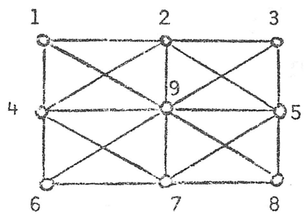
$$\sigma = \bigcup_{i\in J} A_i \text{ with } A_i = 2^{(\Delta_n - J)\ \{i\}}$$

and $(\bigcup_{i\in L} A_i) \cap A_j = 2^{\Delta_n - J}$ with $j \notin L \subseteq J$ so we can use theorem 4 $(|J|-1)$-times and with theorem 2, we conclude that $\pi^\sigma(\Sigma_n)$ is a free group with rank as in remark 3.

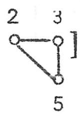
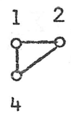
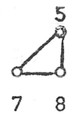
EXAMPLE 6

Let $\Sigma_9 = A_1 \oplus A_1 \oplus \ldots \oplus A_1$ (9 times)

and $\sigma =$



then we can represent σ as follows

$\sigma = \overset{9}{\circ} \otimes [[(\overset{2}{\circ}\ \overset{7}{\circ}) \otimes (\overset{5}{\circ}\ \overset{4}{\circ})] \cup [$  $] \cup [$  $] \cup [$  $] \cup [$  $]]$

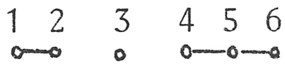
and by using theorem 4 and 3, we have

$$\pi^{\sigma}(\Sigma_9)=(Z\oplus Z)^{16}*F_{113}.$$

The homotopy type of $EM(\Sigma_9;\sigma)$ is a tree of 16 disjoint tori and one linear graph G with $\chi(G)=113$.

EXAMPLE 7

Let $\Sigma_6=\Sigma_3\oplus\Sigma_3^!$ with $\Sigma_3=A_2\oplus A_1$ and $\Sigma_3^!=A_3$ and Δ_6 the diagram of Σ_6

$\Delta_6=\{1,2,3,4,5,6\}$, 

Suppose

$\sigma_1=$ 

$\sigma_2=$ 

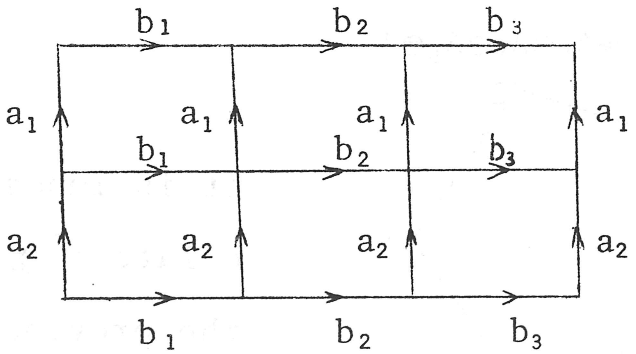
then by use of theorem 1 we know $\Pi^{\sigma_1}(\Sigma_3)\cong F_2$

and $\Pi^{\sigma_2}(\Sigma_3^!)\cong F_3$

and if $\sigma=\sigma_1\otimes\sigma_2=\binom{\Delta_6}{2}-\{\{1,3\},\{5,6\}\}$ we have

$$\Pi^{\sigma}(\Sigma_6)\cong F_2\oplus F_3=\langle a_1,a_2,b_1,b_2,b_3 \| a_i b_j a_i^{-1} b_j^{-1}\rangle,\ i=1,2;j=1,2,3\rangle$$

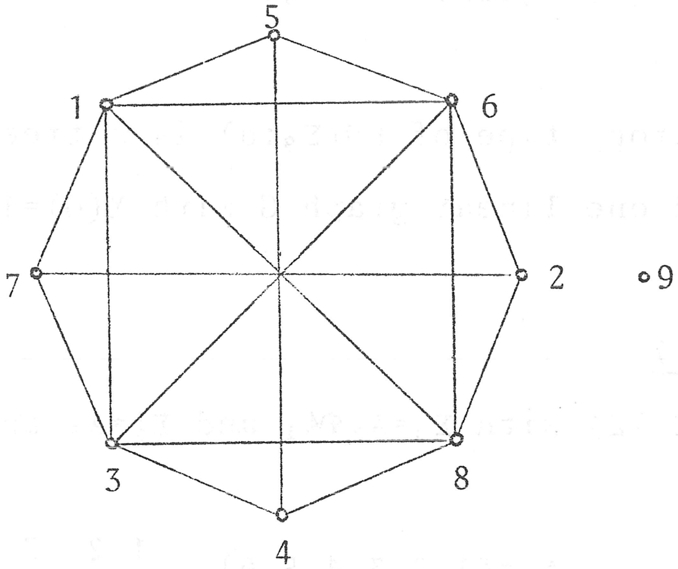
$EM(\Sigma_6,\sigma)$ has the homotopy type of the following surface :



EXAMPLE 8

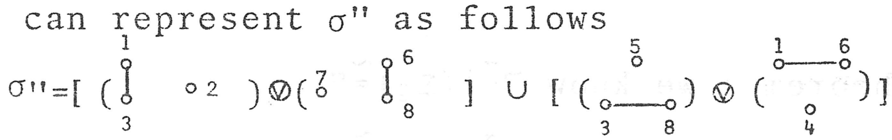
If we take back the σ from example 6 and call in general for $\Delta_n \subseteq \sigma_1 \subseteq \Delta_n \cup \binom{\Delta_n}{2}$, $co\sigma_1 = \Delta_n \cup \left[\binom{\Delta_n}{2} - \sigma_1\right]$, then

$\sigma' = co\sigma =$



Now we take $\sigma'' = \sigma' - \{\{9\}\}$ and consider $\Sigma_8 = A_8$ with diagram $\Delta_8 = \{1,2,3,4,5,6,7,8\}$

We can represent σ'' as follows



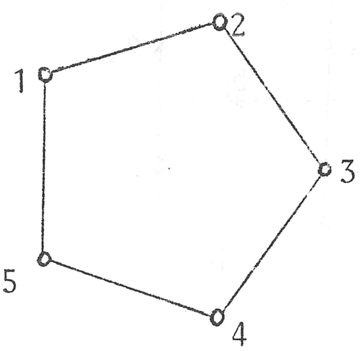
and by using ,four times theorem 1, two times theorem 3 and one time theorem 4, we find

$$\pi^{\sigma''}(A_8) = (F_7 \oplus F_7)^{630} * (F_7 \oplus F_4)^{1260} * F_{20790}$$

EXAMPLE 9

Consider $\Sigma_5 = A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$

and $\sigma =$



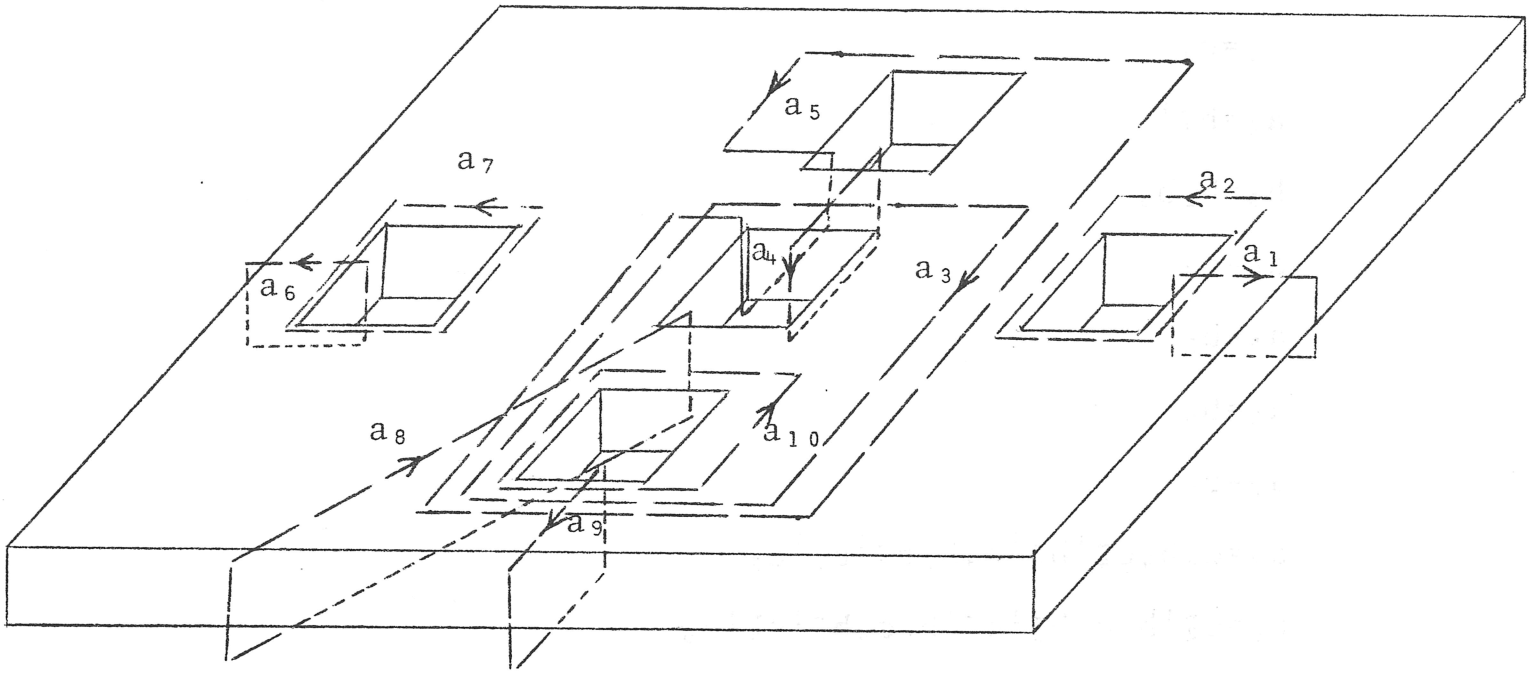
It is impossible to calculate $\pi^{\sigma}(\Sigma_5)$ only by using the previous theorems.

$EM(\Sigma_5,\sigma)$ is a hypercube in R^5 where some faces are missing
After a suitable triangulation, one finds

$$\pi^{\sigma}(\Sigma_5)\cong\langle a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10} \parallel$$

$$a_1a_2a_1^{-1}a_3a_4a_3^{-1}a_5a_4^{-1}a_6a_7a_6^{-1}a_7^{-1}a_8a_5^{-1}a_9a_{10}a_9^{-1}a_8^{-1}a_{10}^{-1}a_2^{-1}\rangle$$

which is the homotopy type of the surface below :



This is a surface with 5 handles. The fundamental group can be written as :

$$\langle b_1,c_1,b_2,c_2,\ldots,b_5,c_5 \parallel b_1c_1b_1^{-1}c_1^{-1}b_2c_2b_2^{-1}c_2^{-1}\ldots b_5c_5b_5^{-1}c_5^{-1}\rangle$$

by the groupisomorphism :

$b_1=a_2^{-1}$

$c_1=a_1$

$b_2=a_5$

$c_2=a_5^{-1}a_3a_4a_3^{-1}a_5a_4^{-1}a_6a_7a_6^{-1}a_7^{-1}a_8$

$b_3=a_5^{-1}a_3$

$c_3=a_4$

$b_4=a_6$

$c_4=a_7$

$b_5=a_8a_9$

$c_5=a_{10}$

and the inverse formula's are

$a_1=c_1$

$a_2=b_1^{-1}$

$b_3=b_2b_3$

$a_4=c_3$

$a_5=b_2$

$a_6=b_4$

$a_7=c_4$

$a_8=c_4b_4c_4^{-1}b_4^{-1}c_3b_3c_3^{-1}b_3^{-1}c_2$

$a_9=c_2^{-1}b_3c_3b_3^{-1}c_3^{-1}b_4c_4b_4^{-1}c_4^{-1}b_5$

$a_{10}=c_5$

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