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ALGEBRAIC PROPERTIES OF QUADRATIC QUATERNARY RINGS

ABSTRACT. A quadratic quaternary ring (abbreviated QQR) was defined for generalized quadrangles [1] as the analogue of a planar ternary ring (abbreviated PTR) for projective planes. To be useful, this algebraic structure should have nice properties whenever the generalized quadrangle has a large automorphism group. This interaction is described in this paper.

INTRODUCTION

One of the most powerful tools in the theory of projective geometry is the coordinatization of projective spaces. An analogous theory for generalized quadrangles has been introduced by the authors in [1] and the coordinatizing algebraic structure was called a 'quadratic quaternary ring' (QQR), a wild generalization of the notion of a planar ternary ring (PTR) in the theory of projective planes. One of the properties of PTRs is that, the more automorphisms the corresponding projective plane has, the nicer the PTR looks (there is more symmetry around). This is explained in detail in [4, Ch. VI]. One of the reasons to look at this is that it seems to be easier to construct PTRs with much symmetry than to construct PTRs with very little or no symmetry! So lots of projective planes arise by constructing PTRs with some symmetry. In the present paper, we establish the analogue of this for the generalized quadrangles and the QQRs. We cannot do this without going into technical details. But this is inevitable and clears the way for some interesting applications. We mention three of them.

1. Geometers are still interested in an elementary proof of Tits' Moufang theorem for generalized quadrangles which states that every Moufang generalized quadrangle has to be classical. Using the techniques of this paper, one can go very far in that proof and the authors hope to be able to complete it.

2. In his thesis, the second author uses the results of this paper heavily in order to construct new non-classical examples of affine buildings of type $\tilde{C}_2$. Along the way, new (infinite) generalized quadrangles arise.

3. The authors still hope to find new classes of finite generalized quadrangles using, e.g., Section 4.4 of this paper. It should be a challenge for the interested reader to try to find QQRs giving rise to new finite generalized quadrangles. We really hope someone succeeds!

*Both authors are supported by the National Fund of Scientific Research Belgium (NFWO).

Geometriae Dedicata **30** (1989) 43–67.

1. Coordinatization, elations

1.1. DEFINITIONS. A (*thick*) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ with point set $\mathcal{P}$ and line set $\mathcal{B}$, satisfying the following axioms:

(i) each point is incident with at least three lines;
(ii) each line is incident with at least three points;
(iii) if P is a point and l is a line not incident with P, then there is a unique pair $(Q, M) \in \mathcal{P} \times \mathcal{B}$ for which $P \mathrm{I} M \mathrm{I} Q \mathrm{I} L$.

It is a nice exercise to show that two distinct points are incident with at most one line, and dually, two distinct lines are incident with at most one point. Moreover, the number $s+1$ of points on a line is independent of the chosen line, and the number $t+1$ of lines on a point is independent of the point. We say that $\mathcal{S}$ has *order* (s, t), where $s, t \in \mathbb{N} \cup \{\infty\}$. In view of the point-line duality for GQ, we assume that the dual of a given definition or theorem has also been given implicitly.

Given two points P and Q of $\mathcal{S}$, we write $P \perp Q$ and say that P and Q are *collinear* provided that there is some line L incident with both. If this is not the case, we write $P \not\perp Q$.

For $P \in \mathcal{P}$, put $P^{\perp} = \{Q \in \mathcal{P} \mid Q \perp P\}$ and note that $P \in P^{\perp}$. If $\mathcal{A} \subset \mathcal{P}$, we write $\mathcal{A}^{\perp} = \cap \{P^{\perp} \mid P \in \mathcal{A}\}$. For distinct points P and Q, $\{P, Q\}^{\perp}$ is called the *trace* of P and Q and $\{P, Q)^{\perp\perp}$ the *span*.

1.2. *Introduction of Coordinates for GQ*

We recall how coordinates for a generalized quadrangle $\mathcal{S}$ of order (s, t), $s, t > 1$, are introduced. We start by choosing a point (∞) and a line $[\infty]$ incident with it, and consider sets R_1 and R_2 of respective cardinalities s and t, containing the distinct elements 0 and 1 but not ∞. Coordinates of lines, denoted by square brackets, are defined dually to those of points, which are denoted by parentheses.

Complete now the elements (∞), $[\infty]$ to a non-degenerate quadrangle (∞), $[\infty]$, (0), $[0, 0]$, $(0, 0, 0)$, $[0, 0, 0]$, $(0, 0)$, $[0]$. Choose bijectively a coordinate (a), $a \in R_1$, for the points of $[\infty]$ distinct from (∞), such that 0 corresponds to (0). We do the same for points of $[0, 0]$ distinct from (0), which are given coordinates $(0, 0, a)$, $a \in R_1$.

A point P collinear with (∞) but not lying on $[\infty]$ has coordinate $(k, a) \in R_2 \times R_1$ if and only if P lies on $[k]$ and is collinear with $(0, 0, a)$. Finally, a point not collinear with (∞) is assigned the coordinate $(a, l, a') \in R_1 \times R_2 \times R_1$ if and only if it lies on $[a, l]$ and is collinear with $(0, a')$.

We summarize these incidences in Figure 1.

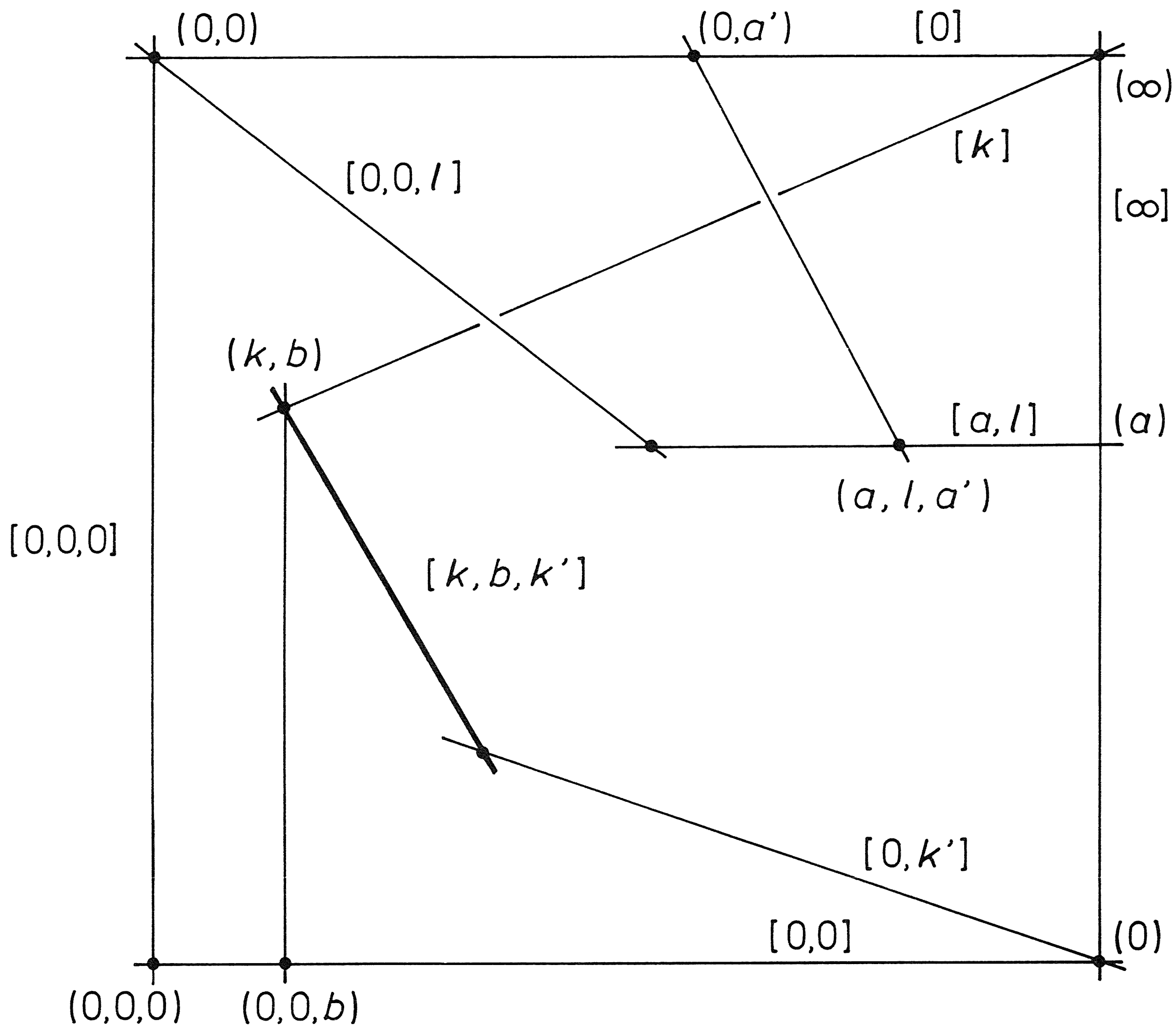

Fig. 1.

We define two (dual) quaternary operations Q_1 and Q_2 as follows: if $a, a', b \in R_1$ and $k, k', l \in R_2$, then

$$Q_1(k, a, l, a') = b$$
$$Q_2(a, k, b, k') = l$$

if and only if (a, l, a') is on $[k, b, k']$. For details we refer to [1]. We only mention here:

THEOREM. *Let S be a GQ coordinatized by* (R_1, R_2, Q_1, Q_2), *where* $Q_1\colon R_1 \times R_2 \times R_1 \times R_2 \to R_1$ *and* $Q_2\colon R_2 \times R_1 \times R_2 \times R_1 \to R_2$ *are quaternary operations defined by*

$$Q_1(k, a, l, a') = b$$
$$Q_2(a, k, b, k') = l$$

where (a, l, a') *is on* $[k, b, k']$. *Then the following properties hold:*

(O) $Q_1(k, 0, 0, a) = a$

$Q_1(0, a, k, a') = a'$.

($\bar{\mathrm{O}}$) $Q_2(a, 0, 0, k) = k$

$Q_2(0, k, a, k') = k'$.

(A) *If* $a, b \in R_1$ *and* $k, l \in R_2$, *then there is a unique* $x \in R_1$ *such that*

$Q_1(k, a, l, x) = b$.

($\bar{\mathrm{A}}$) *If* $a, b \in R_1$ *and* $k, l \in R_2$, *then there is a unique* $p \in R_2$ *such that*

$Q_2(a, k, b, p) = l$.

(B) *If* $a, b \in R_1$ *and* $k, l, k' \in R_2$ *with* $k \neq l$, *then there is a unique pair* $(x, y) \in R_1^2$ *such that*

$Q_1(k, x, Q_2(x, k, a, k'), y) = a$
$Q_1(l, x, Q_2(x, k, a, k'), y) = b$.

($\bar{\mathrm{B}}$) *If* $a, b, a' \in R_1$ *and* $k, l \in R_2$ *with* $a \neq b$, *then there is a unique pair* $(p, q) \in R_2^2$ *such that*

$Q_2(a, p, Q_1(p, a, k, a'), q) = k$

$Q_2(b, p, Q_1(p, a, k, a'), q) = l$.

(C) *If* $a, a', b \in R_1$ *and* $k, k', l \in R_2$, *then*

$Q_1(k, x, Q_2(x, k, b, k'), x') = b$

$Q_1(p, x, Q_2(x, k, b, k'), x') = Q_1(p, a, l, a')$

$Q_2(a, p, Q_1(p, a, l, a'), p') = l$

$Q_2(x, p, Q_1(p, a, l, a'), p') = Q_2(x, k, b, k')$

has a unique solution $(x, p) \in R_1 \times R_2$ *if* $Q_1(k, a, l, a') \neq b$ *and* $Q_2(a, k, b, k') \neq l$ *and none if exactly one of the equalities holds.*

1.3. *-*Normalization of a QQR*

In defining an addition and multiplication, we shall adopt in this paper a slightly different point of view from the one taken in [1]. (For the classical generalized quadrangles these additions are the same.) We shall come back to this matter in Section 3.8.

Let $\mathscr{S}$ be a GQ coordinatized by a QQR (R_1, R_2, Q_1, Q_2). Define quaternary operations Q_1^* on $R_1 \times R_2 \times R_1 \times R_2$ and Q_2^* on $R_2 \times R_1 \times R_2 \times R_1$:

$$Q_1^*(a, k, b, k') = a' \Leftrightarrow Q_1(k, a, Q_2(a, k, b, k'), a') = b,$$

$$Q_2^*(k, a, l, a') = k' \Leftrightarrow Q_2(a, k, Q_1(k, a, l, a'), k') = l.$$

This is possible in view of (A) and ($\bar{\text{A}}$). As a consequence, the following statements are equivalent:

(i) $(a, l, a')\ \mathrm{I}\ [k, b, k']$;

(ii) $Q_1(k, a, l, a') = b$,

$Q_2^*(k, a, l, a') = k'$;

(iii) $Q_2(a, k, b, k') = l$,

$Q_1^*(a, k, b, k') = a'$.

We can choose the bijections from R_1 to $[\infty] - \{(\infty)\}$ and to $[0, 0] - \{(0)\}$ such that (a) and $(0, a)$ are collinear with the same point on $[1, 0, 0]$. We obtain for the QQR:

$$\text{(N*)} \qquad Q_1^*(a, 1, 0, 0) = a.$$

Dually we have

$$(\bar{\text{N}}^*) \qquad Q_2^*(k, 1, 0, 0) = k.$$

If (R_1, R_2, Q_1, Q_2) is a QQR not satisfying (N*) and ($\bar{\text{N}}$*), then define permutations α of R_1 and β of R_2 by:

$$Q_1^*(a, 1, 0, 0) = a^\alpha;$$
$$Q_2^*(k, 1, 0, 0) = k^\beta.$$

Then $(R_1, R_2, \tilde{Q}_1, \tilde{Q}_2)$ with

$$\tilde{Q}_1(k, a, l, a') = [Q_1(k, a, l^\beta, a'^\alpha)]^{\alpha^{-1}},$$
$$\tilde{Q}_2(a, k, b, k') = [Q_2(a, k, b^\alpha, k'^\beta)]^{\beta^{-1}},$$

is a QQR satisfying (N*) and ($\bar{\text{N}}$*). We shall call such a QQR **-normalized,* and the procedure just described, **-normalization.*

Let (R_1, R_2, Q_1, Q_2) be a *-normalized QQR. We define an addition in R_1 and in R_2 by:

$$a + b = Q_1^*(a, 1, b, 0) \in R_1,$$
$$k + 1 = Q_2^*(k, 1, l, 0) \in R_2$$

for all $a, b \in R_1$ and $k, l \in R_2$, and also a 'scalar' multiplication for $a \in R_1$ and $k \in R_2$:

$$k \cdot a = Q_1^*(a, k, 0, 0) \in R_1,$$
$$a \cdot k = Q_2^*(k, a, 0, 0) \in R_2.$$

Finally, we introduce the notion of an 'exterior' multiplication (which we shall read as 'cross'):

$$k \times l = Q_1^*(0, k, 0, l) \in R_1,$$
$$a \times b = Q_2^*(0, a, 0, b) \in R_2.$$

1.4. *Elations*

Let θ be a collineation of the generalized quadrangle. The point P is called a *center* of θ if θ fixes each line incident with P. Dually, L is called an *axis* of θ if θ fixes L pointwise.

If P and Q are distinct points on the line L, we call a collineation θ a (P, L, Q)-*elation* if P and Q are centers and L an axis of θ. Clearly the set of all (P, L, Q)-elations for fixed P, L and Q form a group under composition. If this group acts transitively on the points distinct from P, of a line M on P, then $\mathcal{S}$ is said to be (P, L, Q)-*transitive*. If for fixed P and L, $\mathcal{S}$ is (P, L, Q)-transitive for all points Q on L distinct from P, then $\mathcal{S}$ is (P, L)-*transitive*.

Elations have some nice properties that can easily be proved.

1.4.1. PROPOSITION. *A non-trivial (P, L, Q)-elation fixes no point which is not incident with L. More generally, a non-trivial collineation with center P and axis $L, P \mathrm{I} L$, fixes no point not collinear with P.*

1.4.2. PROPOSITION. *For distinct points P, Q and distinct lines L, M, with $M \mathrm{I} P \mathrm{I} L \mathrm{I} Q$, let θ be a (P, L, Q)-and ψ a (L, P, M)-elation. Then θ and ψ commute.*

Proof. Choose a point $A \neq P$ on M, and a line $X \neq L$ through Q. Call B the unique point of X collinear with A, then $B^\theta \perp A^\theta$ and $B^\theta \in X$, so $B^{\beta\varphi} \perp A^\theta$ and $B^{\theta\varphi} \in X^\varphi$. On the other hand, $B^\varphi \perp A$ and $B^\varphi \in X^\varphi$, so $B^{\varphi\theta} \perp A^\theta$ and $B^{\varphi\theta} \in X^\varphi$. By uniqueness, $B^{\varphi\theta} = B^{\theta\varphi}$, and hence by Proposition 1.4.1 it follows that φ and θ commute.

1.5. *Examples*

EXAMPLE 1.5.1. Moufang quadrangles $Q(L, L', K, K')$ of characteristic 2. Let L' be a subfield of the field L (L has characteristic 2) containing L^2. Let K, K' be subspaces of L, L' respectively, the latter considered as a vector space over L', L^2 respectively. Put $R_1 = K$ and $R_2 = K'$. Define

$$Q_1^*(a, k, b, k') = k \cdot a + b,$$
$$Q_2^*(k, a, l, a') = a^2 \cdot k + l,$$

with the addition and multiplication of the field L. One can check that the corresponding generalized quadrangle represents the Moufang quadrangle of characteristic 2 described by Tits in [5].

EXAMPLE 1.5.2. The Hermitian variety $H(4,q^2)$. Let $R_1 = GF(q^2)$ and $R_2 = GF(q^2) \times K$, where K is the set of solutions of the equation $t^q + t = 0$ ($|K| = q$). Denote by θ a fixed non-zero element of $GF(q^2)$ satisfying $(1+\theta)^{q+1} = 1$. Then $H(4,q^2)$ is a generalized quadrangle described by (use [2]):

$$Q_1^*(a,k,b,k') = a\left(k_0^{q+1} + \frac{k_1}{\theta}\right) + b + \theta^q k_0^q k_0',$$

$$Q_2^*(k,a,l,a') = (ak_0 + l_0; a^{q+1}k_1 + l_1 - \theta a^q a' + \theta^q a a'^q - \theta a^q k_0^q l_0 + \theta^q a k_0 l_0^q),$$

where $k = (k_0; k_1)$, etc.

2. Algebraic properties of quadratic quaternary rings

The following identities will be useful.

2.1. PROPOSITION. *For any QQR* (R_1, R_2, Q_1, Q_2) *and any* $a, b, a' \in R_1, k, l, k' \in R_2$ *hold*:

(i) $Q_1(k,0,0,a') = a', \quad Q_1(0,a,k,a') = a'$;

(ii) $Q_2(a,0,0,k') = k', \quad Q_2(0,k,b,k') = k'$;

(iii) $Q_1^*(0,k,b,0) = b, \quad Q_1^*(a,0,b,k') = b$;

(iv) $Q_2^*(0,a,l,0) = l, \quad Q_2^*(k,0,l,a') = l$.

The proof is obvious.

2.2. PROPOSITION. *If* (R_1, R_2, Q_1, Q_2) *is a *-normalized QQR, then the following properties hold (denote* $R_i - \{0\}$ *by* $R_i^*, i = 1,2$):

(i) $a + 0 = a = 0 + a$ *for all* $a \in R_1$,

(ii) $k + 0 = k = 0 + k$ *for all* $k \in R_2$,

(iii) $x + a = b$ *has a unique solution for any* $a, b \in R_1$,

(iv) $p + k = l$ *has a unique solution for any* $k, l \in R_2$,

(v) $1 \cdot a = a$ *for all* $a \in R_1$,

(vi) $1 \cdot k = k$ *for all* $k \in R_2$,

(vii) $k \cdot x = a$ *has a unique solution for any* $a \in R_1, k \in R_2^*$,

(viii) $a \cdot p = k$ *has a unique solution for any* $a \in R_1^*, k \in R_2$.

Proof. By duality we have only to show the odd numbered parts. We have by

(N*) that

$$a + 0 = Q_1^*(a, 1, 0, 0) = a,$$

and by 2.1(iii) that

$$0 + a = Q_1^*(0, 1, a, 0) = a.$$

If $x + a = b$, then by definition,

$$Q_1(1, x, Q_2(x, 1, a, 0), b) = a,$$
$$Q_1(0, x, Q_2(x, 1, a, 0), b) = b.$$

Because $0 \neq 1$, we can apply (B) proving (iii). Also (v) follows from (N*). Finally, to prove (vii) it suffices to put $a = 0$, $k' = 1 = 0$, and to substitute b for a in (B).

We shall denote the element x described in (iii) for $b = 0$ by $-a$, and write $b - a$ for $b + (-a)$.

3. Additive properties of a quadratic quaternary ring

In this section we relate the additive structure of (R_1, R_2, Q_1, Q_2) to the existence of elations with center (∞) and axis $[\infty]$.

3.1. THEOREM. *The generalized quadrangle $\mathcal{S}$ is* $([\infty], (\infty), [0])$*-transitive if and only if*

$$Q_1(k, a, l + L, a') = Q_1(k, 0, L, Q_1(k, a, l, a')), \tag{3.1.1}$$

$$Q_2^*(k, a, l + L, a') = Q_2^*(k, a, l, a') + L. \tag{3.1.2}$$

for all $a, a' \in R_1$, $k, l, L \in R_2$. *In this case,* $(R_2, +)$ *is a group, and* Q_2^* *is linear in the third argument.*

Proof. (1) Suppose θ is a $([\infty], (\infty), [0])$-elation, mapping the line $[0, 0]$ on $[0, L]$. Then $[0, 0, 0]^\theta = [0, 0, L]$, so $[a, 0]^\theta = [a, L]$ and $(a, 0, a')^\theta = (a, L, a')$. In particular, the unit point $(1, 0, 0)$ is mapped on $(1, L, 0)$. Therefore,

$$[k, Q_1(k, 1, 0, 0), k]^\theta = [k, Q_1(k, 1, L, 0), Q_2^*(k, 1, L, 0)].$$

It follows that

$$[k, Q_1(k, 1, 0,0), k]^\theta \perp [0, k]^\theta = [0, Q_2^*(k, 1, L, 0] = [0, k + L]$$

and

$$[0, 0, k]^\theta = [0, 0, k + L].$$

So, $[0, 0, k]^\theta \perp [a, k]^\theta = [a, k + L]$, and $(a, l, a')^\theta = (a, l + L, a')$ because $(0, a')$

is fixed. In particular, $(0,0,a)^\theta = (0, L, a)$, and $[k]^\theta\ I\ (k,a)^\theta \perp (0,0,a)^\theta$ implies

$$(k,a)^\theta = (k, Q_1(k, 0, L, a))$$

for $(0, L, a)\ I\ [k, Q_1(k, 0, L, a), Q_2^*(k, 0, L, a)]$. Hence,

$$[k, b, k']^\theta = [k, Q_1(k, 0, L, b), k' + L].$$

Next we express that θ is a collineation; i.e.

$$(a, l, a')\ \mathrm{I}\ [k, b, k']$$

$$\Leftrightarrow\ (a, l + L, a')\ \mathrm{I}\ [k, Q_1(k, 0, L, b], k' + L],$$

or

$$Q_1(k, a, l, a') = b$$
$$Q_2^*(k, a, l, a') = k'$$
$$\Leftrightarrow\ Q_1(k, a, l + L, a') = Q_1(k, 0, L, b)$$
$$Q_2^*(k, a, l + L, a') = k' + L$$

or

$$Q_1(k, a, l + L, a') = Q_1(k, 0, L, Q_1(k, a, l, a'))$$
$$Q_2^*(k, a, l + L, a') = Q_2^*(k, a, l, a') + L.$$

These equations are identities for all a, $a' \in R_1$, and all k, l, $L \in R_2$.

(2) Conversely, suppose that (3.1.1) and (3.1.2) hold. Then the following map θ is actually a collineation:

$$(\infty)^\theta = (\infty); \qquad [\infty]^\theta = [\infty];$$
$$(a)^\theta = (a); \qquad [k]^\theta = [k];$$
$$(k, b)^\theta = (k, Q_1(k, 0, L, b)); \quad [a, l]^\theta = [a, l + L];$$
$$(a, l, a')^\theta = (a, l + L, a'); \qquad [k, b, k']^\theta = [k, Q_1(k, 0, L, b), k' + L].$$

Moreover, it fixes $[\infty]$ and $[0]$ pointwise, and (∞) linewise, while it maps $[0, 0]$ on $[0, L]$.

(3) Take $l = 0$ in (3.1.2); then

$$Q_2^*(k, a, L, a') = Q_2^*(k, a, 0, a') + L,$$

which means that Q_2^* is linear in the third argument. Take $a = 1$ and $a' = 0$ in (3.1.2) then

$$k + (l + L) = (k + l) + L,$$

which is exactly the associativity of the addition in R_2.

3.2. COROLLARY. *If the GQ $\mathcal{S}$ is ([∞], (∞), [0])-transitive then we have also the following identities for all $a, b \in R_1$ and all $k, k', L \in R_2$:*

(3.2.1) $Q_1^*(a, k, Q_1(k, 0, L, b), k' + L) = Q_1^*(a, k, b, k')$;

(3.2.2) $Q_2(a, k, Q_1(k, 0, L, b), k' + L) = Q_2(a, k, b, k') + L$;

(3.2.3) $Q_2(a, 0, b, 0) = -a \times b$;

(3.2.4) $Q_2(a, 0, b, k') = -a \times b + k'$.

Proof. Expressing the incidence of $(a, l, a')^\theta$ and $[k, b, k']^\theta$ by means of Q_1^* and Q_2 we obtain (3.2.1) and (3.2.2).

By definition,

$$Q_2(a, 0, Q_1(0, a, 0, b), a \times b) = 0,$$

so $Q_2(a, 0, b, 0) + a \times b = 0$ by (3.2.2), and hence $Q_2(a, 0, b, 0) = -a \times b$ by definition. Using (3.2.2) again for $k = k' = 0$, we get

$$\begin{aligned} Q_2(a, 0, b, L) &= Q_2(a, 0, b, 0) + L \\ &= -a \times b + L. \end{aligned}$$

3.3. THEOREM. *The generalized quadrangle $\mathcal{S}$ is ((∞), [∞], (0))-transitive if and only if*

(3.3.1) $Q_1^*(a, k, b + B, k') = Q_1^*(a, k, b, k') + B$

(3.3.2) $Q_2(a, k, b + B, k') = Q_2(a, 0, B, Q_2(a, k, b, k'))$

for all $a, b, B \in R_1$, $k, k' \in R_2$. In this case, $(R_1, +)$ is a group, and Q_1^* is linear in the third argument.

3.4. COROLLARY. *If the generalized quadrangle $\mathcal{S}$ is ((∞), [∞], (0))-transitive then we have also the following identities for all $a, a', B \in R_1$ and $k, l \in R_2$:*

(3.4.1) $Q_2^*(k, a, Q_2(a, 0, B, l), a' + B) = Q_2^*(k, a, l, a')$;

(3.4.2) $Q_1(k, a, Q_2(a, 0, B, l), a' + B) = Q_1(k, a, l, a') + B$;

(3.4.3) $Q_1(k, 0, l, 0) = -k \times l$;

(3.4.4) $Q_1(k, 0, l, a') = -k \times l + a'$.

3.5. THEOREM. *The generalized quadrangle $\mathcal{S}$ is ([∞], (∞), [0])- and ((∞), [∞], (0))-transitive if and only if both additions are associative and*

(3.5.1) $Q_1^*(a, k, b, k') = k \cdot a + k \times k' + b$, (3.5.2) $Q_2^*(k, a, l, a') = a \cdot k + a \times a' + l,$

(3.5.3) $a \times (b + c) = a \times b + a \times c,$

(3.5.4) $k \times (l + m) = k \times l + k \times m,$

for all $a, a', b, c \in R_1$ *and* $k, k', l, m \in R_2$. *In this case, we have also:*

(3.5.5) $Q_1(k, a, l, a') = -k \times l - k \times (a \times a') - k \times (a \cdot k) - k \cdot a + a',$

(3.5.6) $Q_2(a, k, b, k') = -a \times b - a \times (k \times k') - a \times (k \cdot a) - a \cdot k + k',$

for all $a, a', b \in R_1$ *and* $k, k', l \in R_2$.

Proof. Suppose $\mathscr{S}$ is a $([\infty], (\infty), [0])$- and $((\infty), [\infty], (0))$-transitive GQ. Then it follows from 3.1 and 3.2 that both additions are associative. By (3.1.1), for $l = 0$, we have

(1) $$Q_1(k, a, L, a') = Q_1(k, 0, L, Q_1(k, a, 0, a'))$$
$$= -k \times L + Q_1(k, a, 0, a') \quad \text{(by 3.4.4)}.$$

By (3.2.1), together with (3.3.1), we obtain

$$Q_1^*(a, k, 0, k' + L) + Q_1(k, 0, L, b) = Q_1^*(a, k, b, k')$$
$$\Rightarrow Q_1^*(a, k, b, k') = Q_1^*(a, k, 0, k' + L) - k \times L + b \quad \text{(by 3.4.4)}.$$

Take $L = -k'$, then

(2) $$Q_1^*(a, k, b, k') = Q_1^*(a, k, 0, 0) - k \times (-k') + b$$
$$= k \cdot a - k \times (-k') + b.$$

Now, by (3.1.1) for $a = a' = 0$, we get:

$$Q_1(k, 0, l + L, 0) = Q_1(k, 0, L, Q_1(k, 0, l, 0))$$

or

$$-k \times (l + L) = -k \times L + Q_1(k, 0, l, 0)$$
$$= -k \times L - k \times l \quad \text{(by 3.4.4)}$$

or

$$k \times (l + L) = k \times l + k \times l,$$

proving (3.5.4). It follows that $k \times (-l = -k \times l)$, so (2) becomes:

$$Q_1^*(a, k, b, k') = k \cdot a + k \times k' + b,$$

proving (3.5.1). By duality, also (3.5.2) and (3.5.3) hold. To prove (3.5.5) we start from (3.4.2) and apply (1) on the left-hand side:

$$-k \times Q_2(a, 0, B, l) + Q_1(k, a, 0, a' + B) = Q_1(k, a, l, a') + B.$$

Take $B = -a'$, and also use (3.2.4):

$$-k \times (a \times a' + 1) + Q_1(k, a, 0, 0) = Q_1(k, a, l, a') + a',$$

or

$$Q_1(k, a, l, a') = -k \times l - k \times (a \times a') + Q_1(k, a, 0, 0) + a'.$$

From the fact that

$$(a, 0, 0) \mathrm{I} [k, Q_1(k, a, 0, 0), Q_2^*(k, a, 0, 0)] = [k, Q_1(k, a, 0, 0), a \cdot k]$$

it easily follows that

$$\begin{aligned} & Q_1^*(a, k, Q_1(k, a, 0, 0), a \cdot k) = 0 \\ \Rightarrow\ & k \cdot a + k \times (a \cdot k) + Q_1(k, a, 0, 0) = 0 \\ \Rightarrow\ & Q_1(k, a, 0, 0) = -k \times (a \cdot k) - k \cdot a. \end{aligned}$$

Hence,

$$Q_1(k, a, l, a') = -k \times l - k \times (a \times a') - k \times (a \cdot k) - k \cdot a + a'.$$

Clearly, (3.5.6) is the dual of (3.5.5). The converse is an easy verification.

3.6. *Explicit Form of Collineations*

The collineation θ with center (∞) and axis $[\infty]$ which maps $(0, 0, 0)$ on $(0, L, B)$ is, in the case of a $([\infty], (\infty), [0])$- and $((\infty), [\infty], (0))$-transitive GQ, given by:

$$\begin{aligned} & [a, l]^\theta = [a, -a \times B + l + L], \\ & (k, b)^\theta = (k, -k \times L + b + B), \\ & (a, l, a')^\theta = (a, -a \times B + l + L, a' + B), \\ & [k, b, k']^\theta = [k, -k \times L + b + B, k' + L], \end{aligned}$$

where all remaining elements are fixed.

3.7. REMARK. It follows from Theorem 3.5 that if $\mathcal{S}$ is both $([\infty], (\infty), [0])$- and $((\infty), [\infty], (0))$-transitive, the knowledge of Q_1^* and Q_2^* suffices to determine the incidence.

3.8 *Connection with the Operations Defined in* [1]

3.8.1. DEFINITIONS. In 1.3 we made a particular choice for the bijections from R_1 to $[\infty] - \{(\infty)\}$ and to $[0,0] - \{(0)\}$. In [1] this was done in a different way: there $(0,0,a)$ was collinear with $(1,a)$. For the QQR we obtained:

$$\text{(N)} \qquad Q_1(1,a,0,0) = a \quad \text{for all } a \in R_1.$$

Dually,

$$(\bar{\text{N}}) \qquad Q_2(1,k,0,0) = k, \quad \text{for all } k \in R_2.$$

If this is not the case for a given QQR, we can normalize by defining permutations α of R_1 and β of R_2:

$$a^\alpha = Q_1(1,a,0,0),$$
$$k^\beta = Q_2(1,k,0,0),$$

and proceed in the same way as in 1.3. If (R_1, R_2, Q_1, Q_2) is a normalized QQR, then we can introduce additions, 'scalar' and 'exterior' multiplications as follows: for any $a, b \in R_1$ and $k, l \in R_2$, we have:

$$a \oplus b = Q_1(1,a,0,b) \in R_1,$$
$$k \oplus l = Q_2(1,k,0,l) \in R_2,$$
$$k \odot a = Q_1(k,a,0,0) \in R_1,$$
$$a \odot k = Q_2(a,k,0,0) \in R_2,$$
$$a \otimes b = Q_2(a,0,b,0) \in R_2,$$
$$k \otimes l = Q_1(k,0,l,0) \in R_1.$$

3.8.2. THEOREM. *Suppose that* $\mathscr{S}$ *is a* $([\infty],(\infty),[0]$- *and* $((\infty),[\infty],(0))$-*transitive GQ such that*

$$a \times (k \times l) = 0,$$
$$k \times (a \times b) = 0,$$

for all $a, b \in R_1$ *and* $k, l \in R_2$. *Then both additions defined in* 3.8.1 *are associative and*

$$\tilde{Q}_1(k,a,l,a') = k \otimes l \oplus k \odot a \oplus a',$$
$$\tilde{Q}_2(a,k,b,k') = a \otimes b \oplus a \odot k \oplus k',$$
$$a \otimes (b \oplus c) = a \otimes c \oplus a \otimes b,$$
$$k \otimes (l \oplus m) = k \otimes m \oplus k \otimes l,$$

where $(R_1, R_2, \tilde{Q}_1, \tilde{Q}_2)$ *is a normalized QQR of* $\mathscr{S}$.

Proof. Under the stated assumptions, we have

$$Q_1(k,a,l,a') = -k \times l - k \times (a \cdot k) - k \cdot a + a',$$
$$Q_2(a,k,b,k') = -a \times b - a \times (k \cdot a) - a \cdot k + k'.$$

If we define bijections α and β by

$$a^\alpha = Q_1(1,a,0,0) = -1 \times (a \cdot 1) - a,$$
$$k^\beta = Q_2(1,k,0,0) = -1 \times (k \cdot 1) - k,$$

then $\tilde{Q}_1(k,a,l,a') = [Q_1(k,a,l^\beta,a'^\alpha)]^{\alpha^{-1}}$,

$$\tilde{Q}_2(a,k,b,k') = [Q_2(a,k,b^\alpha,k'^\beta)]^{\beta^{-1}}$$

are normalized.

To prove that $\oplus$ is associative, we must show that

$$\tilde{Q}_1(1,a \oplus b,0,c) = \tilde{Q}_1(1,a,0,b \oplus c).$$

This is equivalent to

$$Q_1(1,Q_1(1,a,0,b^\alpha)^{\alpha^{-1}},0,c^\alpha) = Q_1(1,a,0,Q_1(1,b,0,c^\alpha)).$$

Putting $Q_1(1,a,0,b^\alpha)^{\alpha^{-1}} = x$, it follows by definition that

$$-1 \times (x \cdot 1) - x = -1 \times (a \cdot 1) - a - 1 \times (b \cdot 1) - b.$$

So we now get:

$$\begin{aligned} &-1 \times (x \cdot 1) - x - 1 \times (c \cdot 1) - c = \\ &\quad -1 \times (a \cdot 1) - a - 1 \times (b \cdot 1) - b - 1 \times (c \cdot 1) - c, \end{aligned}$$

proving the claim.

Dually, also the addition in R_2 is associative. Next we prove that

$$\tilde{Q}_1(k,a,l,a') = \tilde{Q}_1(k,a,l,0) \oplus a'.$$

Indeed, we can write this as follows:

$$\begin{aligned} &\tilde{Q}_1(k,a,l,a') = \tilde{Q}_1(1,\tilde{Q}_1(k,a,l,0),0,a') \\ \Leftrightarrow\ & Q_1(k,a,1^\beta,a'^\alpha) = Q_1(1,Q_1(k,a,l^\beta,0)^{\alpha^{-1}},0,a'^\alpha) \\ \Leftrightarrow\ & -k \times l^\beta - k \times (a \cdot k) - k \cdot a + a'^\alpha = -1 \times (x \cdot 1) - x + a' \end{aligned}$$

with $x = Q_1(k,a,l^\beta,0)^{\alpha^{-1}}$. Hence,

$$-1 \times (x \cdot 1) - x = k \times l^\beta - k \times (a \cdot k) - k \cdot a,$$

proving the claim. Also, we have

$$
\begin{aligned}
& \tilde{Q}_1(k,a,l,0) = k \otimes l \oplus \tilde{Q}_1(k,a,0,0) \\
\Leftrightarrow\ & \tilde{Q}_1(k,a,l,0) = \tilde{Q}_1(1, k \otimes l, 0, \tilde{Q}_1(k,a,0,0)) \\
\Leftrightarrow\ & Q_1(k,a,l^\beta,0) = Q_1(1, k \otimes l, 0, Q_1(k,a,0,0)) \\
\Leftrightarrow\ & -k \times l^\beta - k \times (a \cdot k) - k \cdot a = -1 \times ((k \otimes l) \cdot 1) \\
& \qquad\qquad - k \otimes l - k \times (a \cdot k) - k \cdot a \\
\Leftrightarrow\ & -k \times l^\beta = -1 \times (k \otimes l) \cdot 1) - k \otimes l \\
\Leftrightarrow\ & Q_1(k,0,l^\beta,0) = (k \otimes l)^\beta \\
\Leftrightarrow\ & \tilde{Q}_1(k,0,l,0) = k \otimes l
\end{aligned}
$$

which is exactly the definition. Putting these results together, we get

$$\tilde{Q}_1(k,a,l,a') = k \otimes l \oplus k \odot a \oplus a'.$$

Finally, we show $k \otimes (l \oplus m) = k \otimes m \oplus k \otimes l$. Indeed,

$$
\begin{aligned}
& Q_1(k,0,(l \oplus m)^\beta,0) = Q_1(1, k \otimes m, 0, (k \otimes l)^\alpha) \\
\Leftrightarrow\ & -k \times [Q_2(1,l,0,m^\beta)] = (k \otimes m)^\alpha + (k \otimes l)^\alpha \\
\Leftrightarrow\ & -k \times [l^\beta + m^\beta] = (k \otimes m)^\alpha + (k \otimes l)^\alpha
\end{aligned}
$$

and it is easily seen that

$$
\begin{aligned}
k \times l^\beta &= Q_1(k,0,l^\beta,0) \\
&= \tilde{Q}_1(k,0,l,0)^\alpha \\
&= (k \otimes l)^\alpha.
\end{aligned}
$$

3.9. *Algebraic Defining Properties for a* $([\infty],(\infty),[0])$- *and* $((\infty),[\infty],(0))$-*Transitive GQ*

Let $(R_1, +)$ and $(R_2, +)$ be two groups with identity 0 and another element called 1. Suppose the following four operations are given:

$$
\begin{aligned}
&\cdot : R_1 \times R_2 \to R_2: \quad (a,k) \mapsto a \cdot k \\
&\cdot : R_2 \times R_1 \to R_1: \quad (k,a) \mapsto k \cdot a \\
&\times : R_1 \times R_1 \to R_2: \quad (a,b) \mapsto a \times b \\
&\times : R_2 \times R_2 \to R_1: \quad (k,l) \mapsto k \times l
\end{aligned}
$$

such that

(1) $0 \cdot k = a \cdot 0 = 0$ and $k \cdot 0 = 0 \cdot a = 0$ for all $a \in R_1, k \in R_2$;

(2) $1 \cdot a = a$ and $1 \cdot k = k$ for all $a \in R_1, k \in R_2$;

(3) $a \times (b + c) = a \times b + a \times c$ for all $a, b, c \in R_1$;
$k \times (l + m) = k \times l + k \times m$ for all $k, l, m \in R_2$;

(4) if $a \in R_1, k, l \in R_2$ with $k \neq l$, then there is a unique $x \in R_1$ such that

$$- l \times (x \cdot l - x \cdot k) - l \cdot x + k \cdot x + a = 0;$$

(5) if $k \in R_2, a, b \in R_1$ with $a \neq b$, then there is a unique $p \in R_2$ such that

$$- b \times (p \cdot b - p \cdot a) - b \cdot p + a \cdot p + k = 0.$$

(6) if $a, b \in R_1, k, l \in R_2$ with

$$\left.\begin{array}{l} k \cdot a + k \times (a \cdot k) + k \times l + b \neq 0 \\ a \cdot k + a \times (k \cdot a) + a \times b + l \neq 0 \end{array}\right\} (*)$$

then there is a unique pair $(x, p) \in R_1 \times R_2$ such that

$$- p \times (x \cdot p - x \cdot k) - p \cdot x + k \cdot x + b = - p \times (a \cdot p + l) - p \cdot a$$
$$- x \times (p \cdot x - p \cdot a) - x \cdot p + a \cdot p + l = - x \times (k \cdot x + b) - x \cdot k$$

and none if, in (*), exactly one of the equalities holds.

Then (R_1, R_2, Q_1, Q_2) defined by (3.5.5) and (3.5.6) is a QQR corresponding to a $([\infty], (\infty), [0])$- and $((\infty), [\infty], (0))$-transitive GQ. Conversely, every such GQ can be obtained in this way.

Proof. We consider first the converse, and apply therefore the theorem in Section 1.2. Clearly, (O), ($\bar{\text{O}}$), (A), ($\bar{\text{A}}$) are satisfied if we use (3.5.5) and (3.5.6). In view of 3.6, we can suppose $b = 0$ and $k' = 0$ in (B), so we get:

$$Q_1^*(x, k, a, 0) = y$$

or

$$k \cdot x + a = y$$

and

$$- l \times Q_2^*(l, x, Q_2(x, k, a, 0), y) - l \cdot x + y = 0$$

or

$$- l \times (x \cdot l + x \times y - x \times a - x \times k \cdot x - x \cdot k) - l \cdot x + y = 0$$

or

$$- l \times (x \cdot l - x \cdot k) - l \cdot x + k \cdot x + a = 0,$$

and this equation must have a unique solution in x.

Of course ($\bar{\text{B}}$) is just the dual, so we are left with (C). Again, by 3.6, we may suppose that $a' = 0$ and $k' = 0$. Hence,

$$Q_1(k, x, Q_2(x, k, b, 0), x') = b,$$

or

$$Q_1^*(x, k, b, 0) = x' \quad \text{or} \quad k \cdot x + b = x',$$

and dually $a \cdot p + l = p'$.

The other expression is:

$$\begin{aligned} &Q_1(p, x, Q_2(x, k, b, 0), x') = Q_1(p, a, l, 0) \\ \Rightarrow &- p \times Q_1^*(p, x, Q_2(x, k, b, 0), x') - p \cdot x + x' = \\ &\quad - p \times Q_1^*(p, a, l, 0) - p \cdot a \\ \Rightarrow &- p \times (x \cdot p + x \times x' - x \times b - x \times (k \cdot x) - x \cdot k) \\ &\quad - p \cdot x + x' = - p \times (a \cdot p + l) - p \cdot a \\ \Rightarrow &- p \times (x \cdot p - x \cdot k) - p \cdot x + k \cdot x + b = \\ &\quad - p \times (a \cdot p + l) - p \cdot a \end{aligned}$$

This equation, together with

$$- x \times (p \cdot x - p \cdot a) - x \cdot p + a \cdot p + l = - x \times (k \cdot x + b) - x \cdot k$$

must have a unique solution (x, p) if $Q_1(k, a, l, 0) \neq b$ and $Q_2(a, k, b, 0) \neq l$ and none otherwise. The direct statement of the theorem should now also be clear.

3.10. THEOREM. *Let $\mathcal{S}$ be an $([\infty], (\infty), [0])$- and $((\infty), [\infty], (0))$-transitive GQ. Then $\mathcal{S}$ is $((\infty), [\infty])$-transitive if and only if $k + a \times b = a \times b + k$ for all $a, b \in R_1$, $k \in R_2$, and $\mathcal{S}$ is $([\infty], (\infty))$-transitive if and only if $a + k \times l = k \times l + a$ for all $a \in R_1, k, l \in R_2$.*

Proof. It is not so difficult to give a proof analogous to the one of Theorem 3.1. We shall show the converse which can also be used to check the computations of the first part of this theorem.

We define an $((\infty), [\infty], (A))$-elation θ as follows:

$$\begin{aligned} &(\infty)^\theta = (\infty) \quad \text{and} \quad [\infty]^\theta = [\infty], \\ &(a)^\theta = (a) \quad \text{and} \quad [k]^\theta = [k], \\ &(k, b)^\theta = (k, k \times (- A \times B) + b + B), \\ &[a, l]^\theta = [a, a \times B + l - A \times B], \\ &(a, l, a')^\theta = (a, a \times B + l - A \times B, a' + B), \\ &[k, b, k']^\theta = [k, k \times (- A \times B) + b + B, k' - A \times B]. \end{aligned}$$

To prove that θ is indeed a morphism, it suffices to check that

$$\begin{aligned}
&Q_1^*(a, k, k \times (-A \times B) + b + B, k' - A \times B)\\
&= k \cdot a - k \times (k' - A \times B) + k \times (-A \times B) + b + B\\
&= k \cdot a - k \times k' + b + B\\
&= a' + B,
\end{aligned}$$

and also

$$\begin{aligned}
&Q_2^*(k, a, a \times B + l - A \times B, a' + B)\\
&= a \cdot k - a \times (a' + B) + a \times B + l - A \times B\\
&= a \cdot k - a \times a' + l - A \times B\\
&= k' - A \times B.
\end{aligned}$$

Moreover, it is easy to see that θ^{-1} is given by replacing B by $-B$, and that (∞) is a center and $[\infty]$ an axis. It follows also that (A) is a center if and only if for any $l \in R_2$ and $B \in R_1$:

$$l + A \times B = A \times B + l.$$

3.11. EXAMPLES. Consider Example 1.5.1. The additions are the addition of L, both crosses are zero and the multiplications reduce to $k \cdot a$ and $a^2 \cdot k$. Clearly, the quadrangle satisfies the assumptions of Theorems 3.5 and 3.10.

Consider now Example 1.5.2. The addition in R_1 is the same as in the field GF(q^2), but let us write down the other operations:

$$(k_0; k_1) + (l_0; l_1) = (k_0 + l_0; k_1 + l_1 - \theta k_0^q l_0 + \theta^q k_0 l_0^q)$$

$$(k_0; k_1) \cdot a = \left(k_0^{q+1} + \frac{k_1}{\theta}\right) a$$

$$a \cdot (k_0; k_1) = (a k_0; a^{q+1} k_1)$$

$$a \times a' = (0; \theta^q a a'^q - \theta a^q a')$$

$$(k_0; k_1) \times (k_0'; k_1') = \theta^q k_0^q k_0'.$$

We can see that the addition is not commutative in R_2. The center of R_2 is the subgroup consisting of all elements of the form $(0; k_1)$, in agreement with the dual of Theorem 3.10. This group is connected to the subquadrangle $H(3, q^2)$ of $H(4, q^2)$.

4. Distributive properties of a quadratic quaternary ring

In this section we investigate the connection between distributive properties of (R_1, R_2, Q_1, Q_2) and the existence of elations with center (∞) or axis $[\infty]$.

4.1. THEOREM. *Suppose that $\mathcal{S}$ is $([\infty], (\infty), [0])$- and $((\infty), [\infty], (0))$-transitive. Then $\mathcal{S}$ is $([\infty], (0), [0,0])$-transitive if and only if*

(4.1.1) $\quad a \cdot (k + l) = a \cdot k + a \cdot l,$

(4.1.2) $\quad (k + l) \cdot a = l \cdot a + k \cdot a - l \times (a \cdot k),$

(4.1.3) $\quad a + k \times l = k \times l + a,$

(4.1.4) $\quad a \times (k \times (a \times b)) = 0,$

(4.1.5) $\quad (k + l) \times m = k \times m + l \times m,$

(4.1.6) $\quad a \times b + a \cdot k + k \times (k \cdot a) = a \cdot k + k \times (k \cdot a) + a \times b$

for all $a, b \in R_1, k, l \in R_2$.

Proof. Let $\mathcal{S}$ be a GQ satisfying the assumptions of this theorem, and which is $([\infty], (0), [0,0])$-transitive. To prove that $\mathcal{S}$ is also $([\infty], (\infty))$-transitive and hence to prove (4.1.3) it suffices to construct for any $K \in R_2$, and any line $[0, L]$ on (0), an $([\infty], (\infty), [K])$-elation mapping $[0,0]$ on $[0, L]$. Call φ the $([\infty], (\infty), [0])$-elation such that $[0,0]^\varphi = [0, L]$, and θ the $([\infty], (0), [0,0])$-elation, such that $[0]^\theta = [K]$. We claim that $\bar{\varphi} = \theta^{-1}\varphi\theta$ is the desired elation. Clearly $\bar{\varphi}$ has axis $[\infty]$. Also, $[k]^\varphi = ([k]^{\theta^{-1}})^{\varphi\theta} = [k]^{\theta^{-1}\theta} = [k]$, because (∞) is a center for φ, and hence also for $\bar{\varphi}$.

Now, $(K, b)^{\bar{\varphi}} = ((K, b)^{\theta^{-1}})^{\varphi\theta} = (0, b)^{\varphi\theta} = (0, b)^\theta = (K, b)$, hence $[K]$ is an axis of $\bar{\varphi}$. Finally, we check that $[0,0]^{\bar{\varphi}} = ([0,0]^{\theta^{-1}})^{\varphi\theta} = ([0,0]^\varphi)^\theta = [0, L]^\theta = [0, L]$. By Theorem 3.7 we have already (4.1.3).

Let θ be an $([\infty], (0), [0,0])$-elation mapping $[0]$ on $[L]$. Then we can compute successively the following images:

$$[0,0,0]^\theta = [L, 0, 0],$$

$$(1,0,0)^\theta = (1, Q_2(1, L, 0, 0), Q_1^*(1, L, 0, 0))$$

$$= (1, -1 \times (L \cdot 1) - L, L \cdot 1),$$

$$[0, k]^\theta = [0, \mathrm{k}],$$

$$[k]^\theta = [p] = [k + L],$$

for

$$\begin{aligned} k &= Q_2^*(p, 1, -1 \times (L\cdot 1) - L, L\cdot 1) \\ &= p + 1 \times (L\cdot 1) - 1 \times (L\cdot 1) - L \\ &= p - L, \\ [k, b, k']^\theta &= (k + L, b, k'], \\ [0, 0, l]^\theta &= [L, 0, l], \\ [a, l]^\theta &= [a, Q_2(a, L, 0, l)] \\ &= [a, -a \times (L \times l) - a \times (L\cdot a) - a\cdot L + l], \\ (0, a')^\theta &= (L, a'), \end{aligned}$$

To find the image of (a, l, a') we proceed in the following way. We know that

$$(a, l, a')^\theta = (a, p, x)$$

and

$$(a, l, a')^\theta \, \mathrm{I} \, [O, a', a \times a' + l]^\theta = [L, a', a \times a' + l].$$

Hence

$$\begin{aligned} p &= Q_2(a, L, a', a \times a' + l) \\ &= -a \times a' - a \times (L \times (a \times a' + l)) - a \times (L\cdot a) - a\cdot L + a \times a' + l, \\ x &= Q_1^*(a, L, a', a \times a' + l) \\ &= L\cdot a + L \times (a \times a' + l) + a'. \end{aligned}$$

Next we express the incidence of the images, given that

$$\begin{aligned} a' &= Q_1^*(a, k, b, k') = k\cdot a + k \times k' + b, \\ k' &= Q_2^*(k, a, l, a') = a\cdot k + a \times a' + l. \end{aligned}$$

Firstly,

$$\begin{aligned} L\cdot a + L \times (a \times a' + l) + a' &= Q_1^*(a, k + L, b, k') \\ &= (k + L)\cdot a + (k + L) \times k' + b. \end{aligned}$$

Taking $a = 0$, we get:

$$L \times l + a' = (k + L) \times k' + b,$$

or

$$L \times l + k \times k' = (k + L) \times k',$$

or

$$L\times k' + k\times k' = (k+L)\times k'.$$

In the general case we can use this:

$$L\cdot a + L\times(-a\cdot k + k') + k\cdot a + k\times k' + b =$$
$$(k+L)\cdot a + (k+L)\cdot a + (k+L)\times k' + b,$$

or

$$L\cdot a + L\times(-a\cdot k) + k\cdot a = (k+L)\cdot a.$$

This is (4.1.2) in view of (4.1.3).

Secondly, we have also

$$\begin{aligned} p &= Q_2(a, k+L, b, k') \\ &= -a\times Q_1^*(a, k+L, b, k') - a\cdot(k+L) + k' \\ &= -a\times x - a\cdot(k+L) + k' \\ &= -a\times(L\times(-a\cdot k + k') + L\cdot a + a') - a\cdot(k+L) + k' \\ &= -a\times a' - a\times(L\cdot a) - a\times(L\times k') + a\times(L\times(a\cdot k)) \\ &\quad - a\cdot(k+L) + k', \end{aligned}$$

and this should equal

$$\begin{aligned} p = &-a\times a' - a\times(L\times k') + a\times(L\times(a\cdot k)) \\ &- a\times(L\cdot a) - a\cdot L - a\cdot k + k'. \end{aligned}$$

This is so, in view of (4.1.3), if and only if

$$-a\cdot(k+L) = -a\cdot L - a\cdot k \quad \text{or} \quad a\cdot(k+L) = a\cdot k + a\cdot L.$$

Hence we obtain (4.1.1) and it remains to show (4.1.5). It suffices to remark that $[a, l]^\theta \; \mathrm{I} \; (a, l, a')^\theta$ to have:

$$\begin{aligned} &-a\times(L\times l) - a\times(L\cdot a) - a\cdot L + l = \\ &\quad -a\times a' - a\times(L\times(a\times a')) - a\times(L\times l) - a\times(L\cdot a) \\ &\quad - a\cdot L + a\times a' + l. \end{aligned}$$

In view of (4.1.3) this is equivalent to

$$\begin{aligned} a\times(L\times(a\times a')) = &-a\times(L\cdot a) - a\cdot L + a\times a' + a\cdot L + \\ &a\times(L\cdot a) - a\times a', \end{aligned}$$

which is (4.1.5).

Conversely, suppose (4.1.1)–(4.1.6) are satisfied. By Theorem 3.7, $\mathscr{S}$ is $([\infty],(\infty))$-transitive.

Consider the following map:

$$(\infty)^\theta = (\infty);\quad [\infty]^\theta = [\infty];$$
$$(a)^\theta = (a);\quad [k]^\theta = [k + L];$$
$$(k,b)^\theta = (k + L, b);$$
$$[a,l]^\theta = [a, -a \times (L\times l) - a \times (L\cdot a) - a\cdot L + l];$$
$$(a,l,a')^\theta = (a, -a \times (L\times l) - a \times (L\cdot a) - a\cdot L + l, L\cdot a + L \times (a \times a' + l) + a');$$
$$[k,b,k']^\theta = [k + L, b, k'].$$

To show that θ is a collineation, we check:

$$\begin{aligned} k' &= Q_2^*(k + L, a, -a \times (L \times l) - a \times (L\cdot a) \\ &\quad - a\cdot L + l, L\cdot a + L \times (a \times a' + l) + a') \\ &= a\cdot(k + L) + a \times (L\cdot a + L\times(a \times a' + l) + a') \\ &\quad - a \times (L\times l) - a \times (L\cdot a) - a\cdot L + l \\ &= a\cdot k + a\cdot L + a \times (L\cdot a) + a \times (L\times l) \\ &\quad + a \times a' - a \times (L\times l) - a \times (L\cdot a) - a\cdot L + l \\ &= a\cdot k + a \times a' + l; \end{aligned}$$

$$\begin{aligned} L\cdot a + L\times(a \times a' + l) + a' &= Q_1^*(a, k + L, b, k') \\ &= (k + L)\cdot a + (k + L) \times k' + b \\ &= L\cdot a + k\cdot a - L\times(a\cdot k) + L\times k' \\ &\quad + k \times k' + b \\ &= L\cdot a + L\times(a \times a' + l) + a'. \end{aligned}$$

4.2. THEOREM. *Suppose that $\mathscr{S}$ is $([\infty],(\infty),[0])$- and $((\infty),[\infty],(0))$-transitive. Then $\mathscr{S}$ is $((\infty),[0],(0,0))$-transitive if and only if*

(4.2.1) $k\cdot(a + b) = k\cdot a + k\cdot b,$

(4.2.2) $(a + b)\cdot k = b\cdot k + a\cdot k - b \times (k\cdot a),$

(4.2.3) $k + a \times b = a \times b + k,$

(4.2.4) $(a + b) \times c = a \times c + b \times c,$

(4.2.5) $k \times (a \times (k \times l)) = - k \times (a\cdot k) - k\cdot a + k \times l + k\cdot a + k \times (a\cdot k) - k \times l.$

4.3. COROLLARY. *Suppose $\mathcal{S}$ satisfies the assumptions of Theorems* 4.1 *and* 4.2, *then clearly conditions* (4.1.5) *and* (4.2.5) *reduce to*

$$a \times (k \times (a \times b)) = 0$$

$$k \times (a \times (k \times 1)) = 0$$

for all $a, b \in R_1$ and $k, l \in R_2$.

4.4. *Algebraic Defining Properties for a GQ Satisfying the Assumptions of* 4.1 *and* 4.2

Let $(R_1, +)$ and $(R_2, +)$ be two groups with identity 0 and another element called 1. Suppose there are given operations:

$$\cdot : R_1 \times R_2 \to R_2: \quad (a, k) \mapsto a \cdot k$$

$$\cdot : R_2 \times R_1 \to R_1: \quad (k, a) \mapsto k \cdot a$$

$$\times : R_1 \times R_1 \to R_2: \quad (a, b) \mapsto a \times b$$

$$\times : R_2 \times R_2 \to R_1: \quad (k, l) \mapsto k \times l$$

such that

(1) $0 \cdot k = a \cdot 0 = 0$ and $k \cdot 0 = 0 \cdot a = 0$ for all $a \in R_1, k \in R_2$;
(2) $1 \cdot a = a$ and$1 \cdot k = k$ for all $a \in R_1, k \in R_2$;
(3) both operations $\times$ are bi-additive;
(4) both operations $\cdot$ are additive in the second argument;
(5) $(a + b) \cdot k = b \cdot k + a \cdot k - b \times (k \cdot a)$,
$(k + l) \cdot a = l \cdot a + k \cdot a - l \times (a \cdot k)$ for all $a, b \in R_1, k, l \in R_2$;
(6) $k + a \times b = a \times b + k$,
$a + k \times l = k \times l + a$ for all $a, b \in R_1, k, l \in R_2$;
(7) $a \times (k \times (a \times b)) = 0$,
$k \times (a \times (k \times l)) = 0$ for all $a, b \in R_1, k, l \in R_2$;
(8) if $a \in R_1, k \in R_2^*$, then there is a unique $x \in R_1$ such that $k \cdot x = a$;
(9) if $a \in R_1^*, k \in R_2$, then there is a unique $p \in R_2$ such that $a \cdot p = k$;
(10) if $b \in R_1^*, k \in R_1^*$, then there is a unique pair $(x, p) \in R_1 \times R_2$ such that

$$p \cdot x = b \qquad x \cdot p = l$$

and none if $b = 0$ and $l \neq 0$, or $b \neq 0$ and $l = 0$.

Then (R_1, R_2, Q_1, Q_2) defined by (3.5.5) and (3.5.6) is a QQR corresponding to a([∞], (∞), [0])-, ((∞), [∞], (0))-, ([∞], (0), [0, 0])- and ((∞), [0], (0, 0))- transitive GQ. Conversely, every such GQ can be obtained in this way.

Proof. Using Theorem 3.9 and the explicit expressions of the elations mentioned in Theorem 4.1 and dually 4.2, it is easy to see that we can assume that in (4) $l = 0$, in (5) $b = 0$, and in (6) $a = 0$ and $k = 0$. The rest of the proof is straightforward.

4.5. EXAMPLES. As both crosses vanish for the Moufang quadrangle $Q(L, L'; K, K')$, it is easy to see that the corresponding QQR (see Example 1.5.1) satisfies the properties of 4.4.

It is really worth seeing how the magic works for $H(4, q^2)$ when checking the distributive properties. But it also shows how complicated this quadrangle is and how difficult it will be to discover a non-classical generalized quadrangle with the same parameters.

As a further example, we now write down the coordinatizing ring of the so-called 'bad eggs' discovered by W. M. Kantor. A description of those quadrangles can be found in [4].

Let $R_2 = GF(q)$, q odd and suppose $-\theta$ is a non-square in $GF(q)$. Let χ be an arbitrary automorphism of the field $GF(q)$. Put $R_1 = R_2 \times R_2$. Define

$$Q_1^*(a, k, b, k') = (a_1 k + b_1; a_2 k^\chi + b_2)$$

$$Q_2^*(k, a, l, a') = a_1^2 k + \theta a_2^2 k^\chi + 1 - 2a_1 a_1' - 2\theta a_2 a_2'$$

where $a = (a_1; a_2)$, etc. Also this quadrangle satisfies the assumptions of Theorems 4.1 and 4.2.

4.6. APPLICATION. *Suppose* $\mathcal{S}$ *is* $((\infty), [\infty])$- *and* $([\infty], (\infty))$-*transitive*, (∞) *is a regular point and* $[\infty]$ *is a regular line* (see [4] for definitions). *Then all elations are involutions.*

Proof. From the assumptions and the above theorems, it follows that both crosses are zero and that $(2a)\cdot k = a\cdot(2k)$ and $k\cdot(2a) = (2k)\cdot a$. But by 4.4(10), we must have either $2a = 0$ or $2k = 0$. Provided $k \neq 0$ and $a \neq 0$, this implies $2a = 0$ and $2k = 0$. In view of the explicit form of the elations, we see that these are involutions now.

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(Received, February 24, 1988; revised version, October 4, 1988)