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BN-pairs with projective or affine lines

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Abstract. Let G be a simple group with an irreducible BN-pair of (Tits-) rank 2. If the groups induced by G on the panels are permutation equivalent to groups of type PSL_2 or AGL_1 over arbitrary fields with at least 4 elements, then the associated polygon satisfies the Moufang condition and G contains its little projective group. This result is applied to groups of finite Morley rank.

1. Introduction

The first author's interest in the question considered here is motivated by the Cherlin-Zil'ber Conjecture, which states that an infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field. In this paper, we prove a group theoretic result on BN-pairs of (Tits-) rank 2 and show how it can be applied to groups of finite Morely rank.

One main obstacle in the classification of the infinite simple groups of finite Morley rank is a missing classification of all BN-pairs belonging to such groups. The conjecture has been proved to hold for several major classes of groups, such as groups with Moufang BN-pairs of Tits rank at least 2 [KTVM] or groups of k-rational points of isotropic linear algebraic groups [KRT]. Since BN-pairs of Tits rank at least 3 are automatically Moufang, it remains to classify BN-pairs of Tits rank 2 (corresponding exactly to generalized polygons) and of finite Morley rank, not assuming the Moufang condition, and BN-pairs of Tits rank 1, i.e. the 2-transitive permutation groups. These problems seem to be rather hopeless for the moment, so one needs additional assumptions.

In the Tits rank 1 case, Hrushovski [Hr] classified 2-transitive permutation groups acting on strongly minimal sets, proving that they are isomorphic to either $\mathrm{PSL}_2(K)$ or $\mathrm{AGL}_1(K)$ in their natural action, for K an algebraically closed field. Hence we may apply Theorem 1.1 to extend the classification to all BN-pairs of finite Morley rank which have *panels* of Morley rank 1. A *panel* of the building associated to a BN-pair can be identified with the coset space P/B for a minimal proper parabolic subgroup P. We will also call this

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a panel of the BN-pair. Since algebraic groups over algebraically closed fields always contain a BN-pair with projective lines as panels, the Cherlin-Zil'ber conjecture implies in particular that any infinite simple group of finite Morley rank has a BN-pair whose panels in the pure group structure have Morley rank 1. Thus we obtain a natural characterization of all known simple groups of finite Morley rank.

One approach towards a proof of the Cherlin-Zil'ber conjecture is modeled on the revision of the classification of the finite simple groups. Hence one tries to disprove the existence of a counterexample of minimal Morley rank. In this setting (the so-called K^*-case), all proper definable simple sections of a given group of finite Morley rank and of Tits rank 2 are algebraic groups over an algebraically closed field. One consequence of the K^*-assumption should be that the groups induced on the panels are isomorphic to either $\mathrm{PSL}_2(K)$ or $\mathrm{AGL}_1(K)$ for an algebraically closed field K (with possibly different K's for the two types of panels). Hence our result will then also apply in the K^*-case.

Taking a geometric approach we prove the following result:

1.1. Theorem. *Let G be a simple group with an irreducible* BN-*pair of* (*Tits-*) *rank* 2. *If the groups induced by G on the panels are permutation equivalent to groups of type* PSL_2 *or* AGL_1 *over arbitrary fields with at least* 4 *elements, then the associated polygon satisfies the Moufang condition and G contains its little projective group.*

Using [Hr] we then apply this result to groups of finite Morley rank:

1.2. Corollary. *Let G be a simple group of finite Morley rank with a definable irreducible* BN-*pair of* (*Tits-*) *rank* 2. *If all panels have Morley rank* 1, *then G is definably isomorphic to* $\mathrm{PSL}_3(K)$, $\mathrm{PSp}_4(K)$, *or* $\mathrm{G}_2(K)$ *for some algebraically closed field K.*

Note that we do not have to assume that the BN-pair is spherical. If G is not assumed to be simple or even connected then the corresponding conclusions hold for G/R where R is the kernel of the action of G on the associated polygon (see Section 2). The case where the associated building is reducible is rather less interesting and easy to classify, see the appendix.

Note also, that in all known examples of simple groups of finite Morley rank, the panels do indeed have Morley rank 1 (considered in the pure group structure). In particular, all Moufang BN-pairs of finite Morley rank automatically have Morley rank 1 panels as shown in [KTVM]. However, the construction in [Te2] yields 'wild' groups having spherical BN-pairs of Tits rank 2 with panels of Morley rank 1. So the assumption of G having finite Morley rank is crucial in Corollary 1.2.

We will show in Section 2 that Corollary 1.2 translates into the language of polygons as the following theorem which we will prove below (and which is slightly more general):

1.3. Theorem. (i) *If* $\mathfrak{P}$ *is a generalized n-gon with strongly minimal point rows and line pencils,* $n \geqq 3$, *and* $G \leqq \mathrm{Aut}(\mathfrak{P})$ *is a group of finite Morley rank which acts transitively and definably on the set of ordered ordinary n-gons contained in* $\mathfrak{P}$, *then one of the following holds*:

($n = 3$) *G is definably isomorphic to* $\mathrm{PSL}_3(K)$ *for some algebraically closed field K, and* $\mathfrak{P}$ *is the projective plane over K.*

$(n = 4)$ *G is definably isomorphic to* $\mathrm{PSp}_4(K)$ *and* $\mathfrak{P}$ *is the symplectic quadrangle over K.*

$(n = 6)$ *G is definably isomorphic to* $\mathrm{G}_2(K)$ *and* $\mathfrak{P}$ *is the split Cayley hexagon over K.*

(ii) *Let* $\mathfrak{P}$ *be as in* (i) *and let G be a group of automorphisms of the incidence graph of* $\mathfrak{P}$ (*so G possibly contains dualities which interchange points and lines*). *If G acts transitively and definably on the set of ordered ordinary n-gons contained in* $\mathfrak{P}$, *then G is as in* (i) *or is definably isomorphic to* $\mathrm{PSL}_3(K) \rtimes 2$ *for some algebraically closed field K.*

In contrast, for any n there are many generalized n-gons with strongly minimal point rows and line pencils (not interpreting any infinite groups) the automorphism groups of which act transitively on ordered ordinary $n+1$-gons. Thus the situation is completely different if we do not assume that the group G in Theorem 1.3 has finite Morley rank. Note also that all known examples of generalized n-gons of finite Morley rank have strongly minimal point rows and line pencils and are indeed almost strongly minimal.

While by the results in [KTVM] it would suffice to prove that any generalized polygon satisfying the assumptions of Theorem 1.3 is a Moufang polygon, we here give a full independent proof. We would like to stress the fact that our approach here does not need the full classification of Moufang polygons (in particular the yet unpublished parts about quadrangles can be avoided).

As a corollary we obtain:

1.4. Corollary. *If G is an infinite group of finite Morley rank with an irreducible definable* BN-*pair whose panels have Morley rank* 1, *then G satisfies the Cherlin-Zil'ber Conjecture.*

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2. Background on polygons

Buildings were introduced by Tits in the early 60s to give a geometric interpretation of simple groups of (exceptional) Lie-type. Group-theoretically, the groups of Lie-type were characterized by Tits as groups with a so-called spherical BN-pair (*spherical* means that the associated Weyl group is finite). For more information about BN-pairs the reader might look at [Hu], Section 29.

The buildings belonging to irreducible spherical BN-pairs of (Tits-) rank 2 (which means that there are exactly two proper parabolic subgroups containing B) are exactly the generalized polygons, introduced by Tits already in 1959 (see [Ti1]). We briefly recall some of the definitions we will use. A detailed account can be found in [VM1] or for the model theoretic point of view in [KTVM].

2.1. Incidence structures. An *incidence structure* is a triple $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ consisting of a set $\mathcal{P}$ of *points*, a set $\mathcal{L}$ of *lines*, and a set $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L}$ of *flags*. We always assume that $\mathcal{P}$ and $\mathcal{L}$ are disjoint nonempty sets. If (a, ℓ) is a flag, then we say that the point a and the line ℓ are *incident*.

A *k-chain* is a sequence $(x_0, x_1, \dots, x_k)$ of elements $x_i \in \mathscr{P} \cup \mathscr{L}$ with the property that x_i is incident with x_{i-1} for $i = 1, \dots, k$. In this case, we say that the *distance* of x_0 and x_k is $d(x_0, x_k) \leqq k$, and we say that $d(x_0, x_k) = k$ if there is no j-chain joining x_0 and x_k for $j < k$. Note that $d(x_0, x_k)$ is necessarily even if x_0 and x_k are of the same sort, i.e. if x_0 and x_k are both points or both lines.

2.2. Polygons. Let $n \geqq 2$ be an integer. An incidence structure $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$ is called a *generalized n-gon*, if it satisfies the following three axioms:

(i) For all elements $x, y \in \mathscr{P} \cup \mathscr{L}$ we have $d(x, y) \leqq n$.

(ii) If $d(x, y) = k < n$, then there is a unique k-chain $(x_0 = x, x_1, \dots, x_k = y)$ joining x and y (and we denote $x_1 = \mathrm{proj}_x y$).

(iii) $\mathfrak{P}$ is *thick*, i.e. every element $x \in \mathscr{P} \cup \mathscr{L}$ is incident with at least three other elements.

An *ordered ordinary n-gon* is a $2n$-chain $(x_0, \dots, x_{2n} = x_0)$ of distinct elements, i.e. instead of (iii) above we require that every element is incident with exactly two elements.

The case $n = 2$ (the so-called *generalized digons*), gives rise to a trivial geometry $(\mathscr{P}, \mathscr{L}, \mathscr{P} \times \mathscr{L})$ where every point is incident with every line. As we will need them in our appendix, we include them in most of our definitions or properties. Note also that for triangles (3-gons) the axioms translate precisely into the axioms of projective planes. By exchanging the sets of lines and points of a generalized n-gon, we obtain the *dual n-gon*.

If $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$ is a generalized n-gon, then $\mathrm{Aut}(\mathfrak{P})$ denotes the group of automorphisms of the first order structure $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$. A map which preserves the incidence but exchanges points and lines is called a *duality*. A duality of order 2 is a *polarity*. (Caution: in [Ti2], $\mathrm{Aut}(\mathfrak{P})$ denotes the graph automorphisms, the type preserving automorphisms are denoted by $\mathrm{Spe}(\mathfrak{P})$.)

For $x \in \mathscr{P} \cup \mathscr{L}$ we put $D_k(x) = \{y \in \mathscr{P} \cup \mathscr{L} \mid d(x, y) = k\}$. If a is a point, then $D_1(a)$ is called a *line pencil*; if ℓ is a line, then $D_1(\ell)$ is called a *point row*.

2.3. Levi-factors. If $G \leqq \mathrm{Aut}(\mathfrak{P})$ acts n-gon transitively, the stabilizer G_x of x for $x \in \mathscr{P} \cup \mathscr{L}$ induces on $D_1(x)$ a 2-transitive permutation group, which is called the Levi-factor of G. Note that since G acts transitively on the set of points (and on the set of lines, resp.), the isomorphism type depends only on whether x is a point or a line.

2.4. Projectivities. If $d(x, y) = n$, then there is a bijection $[y; x]\colon D_1(x) \to D_1(y)$, mapping each $w \in D_1(x)$ to the unique $z \in D_1(y)$ at distance $n - 2$ from w; a concatenation of such maps is called a *projectivity*, and we put

$$[x_3; x_2] \circ [x_2; x_1] = [x_3; x_2; x_1]\colon D_1(x_1) \to D_1(x_3)$$

etc. For $n > 2$, the set $\Pi(x)$ of all projectivities from x to x is a 2-transitive permutation group on $D_1(x)$, see [Kn], Lemma 1.2.

2.5. Coordinatization. Generalized n-gons, $n \geqq 2$, can be coordinatized analogously to projective planes ([VM1], see also [KTVM]). This coordinatization says in particular that the whole polygon is in the *definable closure* of the set

$$D_1(x_0) \cup D_1(x_1) \cup \{x_0, \ldots, x_{2n-1}\}$$

for any ordinary n-gon $(x_0, \ldots, x_{2n-1}, x_{2n})$. Note also that $x_{n+1}, \ldots, x_{2n-2}$ are determined by $(x_{2n-1}, x_0, \ldots, x_n)$, so any automorphism which fixes each element of

$$(x_0, \ldots, x_n) \cup D_1(x_0) \cup D_1(x_1)$$

must be the identity.

Example. Let K be any field. The symplectic quadrangle over K can be described with coordinates as follows. Put, with the above notation, x_0 equal to (∞) and x_1 equal to $[\infty]$ (it is convenient to denote the labels (= coordinates) of points with parentheses and those of lines with square brackets). We put $D_1(x_0) = \{[\infty]\} \cup \{[k] \mid k \in K\}$ and $D_1(x_1) = \{(\infty)\} \cup \{(a) \mid a \in K\}$. The lines incident with (a), $a \in K$, distinct from $[\infty]$ have coordinates $[a, l]$, $l \in K$. Dually, the points on $[k]$, $k \in K$, different from (∞) are (k, b), $b \in K$. The lines through (k, b), $k, b \in K$, distinct from $[k]$ are the lines $[k, b, k']$, $k' \in K$, and, dually, the points on $[a, l]$, $a, l \in K$, different from (a) are (a, l, a'), $a' \in K$. Finally, a point (a, l, a') is incident with the line $[k, b, k']$ if and only if $ak + b = a'$ and $a^2k + k' + 2ab = l'$. This is a very explicit description of the symplectic quadrangle over K. A similar description exists for the split Cayley hexagon over K, see [VM1], 3.5.1.

2.6. Connection between groups and polygons. Let G be a group with a spherical irreducible BN-pair of rank 2, and suppose that the associated Weyl group has order $2n$, for $n \geqq 3$. Let G_a and G_ℓ be the proper parabolic subgroups of G containing B. We define an incidence structure—which is a generalized n-gon—on the coset spaces $\mathscr{P} = G/G_a$ and $\mathscr{L} = G/G_\ell$ by defining $p = gG_a$ to be incident with $l = g'G_\ell$ if and only if $gG_a \cap g'G_\ell \neq \emptyset$. Hence, the panels of the BN-pair (as defined in the introduction) exactly correspond to the point rows and line pencils of the polygon, which are the (geometric) *panels* of the associated building.

For the converse direction, i.e. for getting from a polygon to the group, let $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$ be a generalized n-gon and let (a, ℓ) be a flag. Suppose that a group $G \subseteqq \mathrm{Aut}(\mathfrak{P})$ acts transitively on the set of ordered ordinary n-gons. Then a BN-pair of G (which is not necessarily unique) can be seen as follows: Let (a, ℓ) be a flag, and Γ an ordinary n-gon containing (a, ℓ). Then B is the stabilizer of (a, ℓ), N is the setwise stabilizer of Γ, and $\mathfrak{P}$ is isomorphic to the coset geometry $(G/G_a, G/G_\ell, \{(gG_a, gG_\ell) \mid g \in G\})$, where G_a and G_ℓ denote the stabilizer in G of the elements a and ℓ, respectively.

2.7. Definable BN-pairs. By a definable BN-pair of G we mean that there are definable subgroups B and N satifying the axioms of a BN-pair (see e.g. [Hu], 29.1).

Note that the BN-pair property is not a first order property unless the Weyl group (and the (Tits-)rank) is finite. However, for stable groups, the Weyl group and the (Tits-)rank are necessarily finite:

2.8. Proposition. *Let G be a stable group with a definable* BN-*pair. Then the Weyl-group $W = N/(N \cap B)$, and thus the (Tits-) rank are finite.*

Proof. Clearly, W is interpretable. Suppose towards a contradiction that W is infinite. We will obtain a formula $\phi(x,y)$ and a sequence a_i, $i < \omega$ of elements in W such that $\phi(a_i, a_j)$ holds if and only if $i \leqq j$, contradicting stability.

Let S be the set of distinguished involutions generating W. By the axioms of a BN-pair, we always have

(T1) $sBw \subseteqq BswB \cup BwB$ for $s \in S$, $w \in W$.

Furthermore by [Hu], 29.3A, for $s \in S$, $w \in W$,

$(*)$ $\ell(sw) > \ell(w)$ if and only if $sBw \subseteqq BswB$.

So it follows from (T1) that $\ell(sw) < \ell(w)$ if and only if $sBw \cap BwB$ is non-empty. We use this to show the following

Claim. Let $u, w \in W$ with $\ell(uw) = \ell(u) + \ell(w)$. Then

$$uBw \subseteqq BuwB \quad \text{and} \quad u^{-1}Buw \nsubseteqq BwB.$$

Proof of Claim. First assume that $\ell(uw) = \ell(u) + \ell(w)$. To see that $uBw \subseteqq BuwB$, or equivalently, $BuBwB = BuwB$ we use $(*)$ applied to a reduced presentation for uw which allows us to move B stepwise across u.

Now suppose that $u^{-1}Buw \subseteqq BwB$ and write $u = s_1 \dots s_k$ reduced. Applying $(*)$ and (T1) k times, it follows that $u^{-1}Buw$ contains an element of $B\tilde{u}uwB$ for some subexpression $\tilde{u}$ of u with $\ell(\tilde{u}) < \ell(u)$. But $B\tilde{u}uwB$ and BwB are disjoint, yielding a contradiction.

Let a_i, $i < \omega$ be elements in W such that $a_{i+1} = s_{k_i}a_i$ for some $s_{k_i} \in S$ is reduced. (Such elements always exist.)

Thus $\ell(a_i) \leqq \ell(a_j)$ if and only if $i \leqq j$. Using the claim, we obtain a formula $\phi(x,y)$ given by $yx^{-1}Bx \subseteqq ByB$. Then $\phi(a_i, a_j)$ holds if and only if $i \leqq j$, contradicting stability. □

Under the additional assumptions that the group G has finite Morley rank and finite (Tits-) rank a different proof can be found in [BN], 12.39. (However, the proofs of the theorems claimed in loc. cit. 12.39 and 12.40 are not correct, a correct proof and an explanation of the problems arising there can be found in [KTVM].)

2.9. Reduction of Corollary 1.2 to 1.3. If G is a group of finite Morley rank with a definable irreducible BN-pair of (Tits-) rank 2 (in fact it suffices that B is definable), then the BN-pair is automatically spherical by Proposition 2.8 and the parabolic subgroups are definable (see [KTVM], proof of Thm. 5.3) Clearly, the generalized n-gon $\mathfrak{P}$ on the coset geometry is definable in the structure $(G, G_a, G_\ell, 1, \cdot)$ and thus has finite Morley rank. If $\mathfrak{P}$ is a polygon defined in this way from a group of finite Morley rank, then by [Te1] the point rows and line pencils of $\mathfrak{P}$ have Morley degree 1 and thus are strongly minimal by assumption on G. (Alternatively, by 4.2 this follows from the fact that the Levi-factors are interpretable groups which act 2-transitively on the point rows and line pencils.)

Note that G is contained in a natural way in $\mathrm{Aut}(\mathfrak{P})$ (thus preserves lines and points) and acts transitively on the set of ordered n-gons contained in $\mathfrak{P}$. So the first part of Theorem 1.3 is sufficient to prove Corollary 1.2.

2.10. Examples. In particular, starting with some standard BN-pair of the simple algebraic groups $\mathrm{PSL}_3(K)$, $\mathrm{PSp}_4(K) \cong \mathrm{PSO}_5(K)$, $\mathrm{G}_2(K)$ over some algebraically closed field K, the associated *Pappian polygons*, namely the projective plane, the symplectic quadrangle, and the split Cayley hexagon over K, have finite Morley rank and all point rows and line pencils are strongly minimal and the Levi-factors are all permutation equivalent to $\mathrm{PSL}_2(K)$ acting naturally on the projective line. (They are called Pappian because all *derived geometries* in these polygons are Pappian projective planes, see e.g. [VM1], 3.5.2.) It was shown in [KTVM] that these are exactly the Moufang polygons of finite Morley rank.

2.11. Root groups and the little projective group. Let $\mathfrak{P}$ be a generalized n-gon, $n \geqq 2$. Let $G = \mathrm{Aut}(\mathfrak{P})$ if $n > 2$, and $G \leqq \mathrm{Aut}(\mathfrak{P})$ if $n = 2$. A *root* in $\mathfrak{P}$ is an n-chain $\gamma = (x_0, \ldots, x_n)$ with $x_{i-1} \neq x_{i+1}$, $i = 1, 2, \ldots, n-1$. The *root group* U_γ of γ (with respect to G if $n = 2$) is the subgroup of G which fixes the set $\bigcup_{i=1}^{n-1} D_1(x_i)$ elementwise. Note that, for $n > 2$, the coordinatization implies that the root group acts freely on $D_1(x_n)$. The root γ is a *Moufang* root (with respect to G if $n = 2$) if its root group acts transitively (and thus regularly for $n > 2$) on the set $D_1(x_n) \setminus \{x_{n-1}\}$. The generalized n-gon satisfies the *Moufang condition* if all roots are Moufang.

The *little projective group* of $\mathfrak{P}$ (with respect to G if $n = 2$) is the subgroup of G generated by the root groups. An element of the root group of γ is called a *root elation for* γ. If x_0 is a line (respectively a point) then a corresponding root elation for γ will also be called a *point elation* (respectively *line elation*). If n is odd, then point elations are line elations, but if n is even, then the set of elations is the disjoint union of the set of point elations and the set of line elations.

Note that if $\mathfrak{P}$ is a Moufang polygon which is defined as in 2.6 from a group of finite Morley rank, then the root groups are connected since the line pencils and point rows are, and hence the little projective group is definable by Zil'ber's Indecomposability Theorem.

Moufang n-gons for $n > 2$ have been completely classified by Tits and Weiss [TW], they exist only for $n = 3, 4, 6$ and 8.

For more background on groups of finite Morley rank the reader is referred to [BN]. However, Chapter 12 of loc. cit., which deals with geometries and BN-pairs, contains a number of inaccuracies and mistakes. Background for this area can be found in [KTVM].

3. Proof of Theorem 1.1 and Corollary 1.2

Let G be a simple group with an irreducible BN-pair of (Tits-) rank 2 and let $\mathfrak{P}$ be the associated polygon. For any $x \in \mathfrak{P}$, the group G induces a 2-transitive group H_x on the panels $D_1(x)$, the *Levi factor* of the parabolic subgroup G_x. Assume that these Levi factors are isomorphic to groups of type PSL_2 or AGL_1 over arbitrary fields with at least 4 elements. We now aim at showing that $\mathfrak{P}$ is a Moufang polygon and that G contains the little projective group of $\mathfrak{P}$.

3.1. Proposition. *The polygon $\mathfrak{P}$ is a Moufang polygon and G contains all root elations.*

Proof. There are three cases to consider depending on whether the Levi-factors are isomorphic to groups of type PSL_2 or AGL_1:

Case 1: All the Levi-factors H_x are of type AGL_1.

Let $\gamma = (x_0, \ldots, x_n)$ be a root. Since G acts transitively on ordered ordinary n-gons, the subgroup G_γ of G fixing γ elementwise acts transitively on $D_1(x_n)\backslash\{x_{n-1}\}$ and must fix all of $D_1(x_i)$, $i = 1, \ldots, n-1$ elementwise showing that $\mathfrak{P}$ is Moufang with G containing all root elations.

Case 2: All the Levi-factors H_x are permutation equivalent to some $\mathrm{PSL}_2(K)$, $\mathrm{PSL}_2(K')$ for fields K and K' with at least 4 elements.

In this case Corollary 2.4 in [Te3] implies that $\mathfrak{P}$ is Moufang and that G contains all root elations.

Case 3: It is left to consider the case where $H_x \cong \mathrm{PSL}_2(K)$ for $x \in \mathscr{L}$ and $H_x \cong K' \rtimes K'^*$ for $x \in \mathscr{P}$.

Clearly, this cannot happen for odd n since in that case any elation is at the same time a point elation and a line elation, i.e.; the root groups are the same.

So we may assume that n is even.

Claim. G contains all root elations on point rows.

Let $\gamma = (x_0, \ldots, x_n)$ be a root with $x_0, x_n \in \mathscr{L}$. Then since H_x for $x \in \mathscr{P}$ is sharply 2-transitive, the subgroup G_γ of G fixing γ pointwise fixes $D_1(x_1) \cup D_1(x_3) \cup \cdots \cup D_1(x_{n-1})$ pointwise. It now follows exactly as in [Te3] that the commutator subgroup $[G_\gamma, G_\gamma]$ is a group of root elations for γ inducing the additive group of K on $D_1(x_0)\backslash\{x_1\}$.

We want to show that G also contains all root elations on line pencils.

For quadrangles, this follows from

3.2. Proposition (cf. [BTVM]). *Suppose $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$ is a generalized quadrangle which is half-Moufang, i.e. all point rows have transitive root groups. If for all $(p, l) \in \mathscr{F}$, and* *roots* $\gamma = (x_0, x_1, x_2, p, l)$ *the group induced by* U_γ *on* $D_1(l) \setminus \{p\}$ *does not depend on the root* γ, *then* $\mathfrak{P}$ *is Moufang.*

Proof. Let $\gamma = (p_0, l_0, p_1, l_1, p_2)$ be a root with $p_0, p_2 \in \mathscr{P}$ and complete γ in two distinct ways $(p_2, l_2, p_3, l_3, p_0)$ and $(p_2, l_2', p_3', l_3', p_0)$ to an ordinary quadrangle. We have to show that there is a line elation for γ mapping l_2 to l_2'.

Let p be a point on l_2 distinct from p_3, p_2, let (p, l, q, l_3') be the (well-defined) path from p to l_3' and let (p_1, l', q', l) be the path from p_1 to l. We will now compose three point elations: one mapping the point p_3 to p, a second from there to q and the last from there to p_3'.

Let α_1 be the point elation fixing $(l_2, p_2, l_1, p_1, l_0)$ and mapping p_3 to p, let α_2 be the point elation fixing (l_0, p_1, l', q', l) and mapping p to q, and let α_3 be the point elation fixing $(l_3', p_0, l_0, p_1, l_1)$ and mapping q to p_3'. Since by assumption all these elations do not depend on the different roots and the composition $\alpha = \alpha_1\alpha_2\alpha_3$ fixes p_0, p_1 and p_2, it follows that l_0 and l_1 stay pointwise fixed and so does the line pencil in p_1. Thus α is the required line elation. □

$n = 4$. We know that all root groups induce the additive group of the field K on the point rows. Hence we can apply Proposition 3.2 to see that $\mathfrak{P}$ and G are as claimed.

$n \geqq 6$. Put $n = 2m$. We fix two lines ℓ and ℓ' at distance n. Then the group $G_{\ell,\ell'}$ fixing $\{\ell, \ell'\}$ induces a 2-transitively subgroup of $\mathrm{PSL}_2(K)$ on ℓ. Since $\mathrm{PSL}_2(K)$ does not contain 2-transitive proper subgroups (see e.g. [KTVM], 4.9) and since the groups H_x are sharply 2-transitive for $x \in \mathscr{P}$, it follows from the coordinatization that $G_{\ell,\ell'}$ acts effectively on ℓ. So $G_{\ell,\ell'}$ is definably isomorphic to $\mathrm{PSL}_2(K)$.

We now consider an element $g \in G_{\ell,\ell'}$ having exactly two fixpoints x, y on ℓ. Then g fixes an ordinary n-gon Γ through ℓ, ℓ', x, y. Moreover, since g fixes two lines through every point of Γ, it fixes every line through any point of Γ. By (the dual of) 4.4.2(iii) of [VM1], the set of fixed elements of g forms a weak non-thick ideal subpolygon $\mathfrak{P}'$ (i.e., a structure satisfying conditions (i) and (ii) of Definition 2.2, but instead of (iii), we have exactly two points on each line, and at least three lines through each point; *ideal* means that all lines in $\mathfrak{P}$ incident with a point of $\mathfrak{P}'$ belong to $\mathfrak{P}'$). By Theorem 1.6.2 of [VM1], $\mathfrak{P}'$ arises in a canonical (and definable) way from a (thick) generalized m-gon $\mathfrak{P}^*$. Since $\mathfrak{P}'$ is ideal in $\mathfrak{P}$, it also arises from a BN-pair, and since $\mathfrak{P}'$ has a group transitive on the points, it is easy to see that $\mathfrak{P}^*$ is isomorphic to its dual. Moreover, the BN-pair associated with $\mathfrak{P}^*$ satisfies the assumptions of Case 1 of the present proof. Hence $\mathfrak{P}^*$ is a Moufang m-gon and we have $m = 3, 4, 6, 8$ (see e.g. [Ti4], [W]). For Moufang polygons Knarr [Kn] proved that the group of even projectivities coincides with the group induced by the little projective group. Hence the groups of even projectivities are sharply 2-transitive. This already rules out the case $m = 3$ as in projective planes the projectivity groups are always 3-transitive. The case $m = 4$ is likewise eliminated by [VM2]. If $m = 6$, then we know from [Ro] that, up to duality, $\mathfrak{P}^*$ contains an ideal split Cayley subhexagon, and [VM1] (Table 8.1) implies that the group of projectivities of a line pencil is 3-transitive (namely, containing $\mathrm{PGL}_2(k)$, for some field k), a contradiction. Finally, if $m = 8$, then the Moufang polygon can never be self-dual, as shown by Weiss in [W].

Hence Case 3 cannot occur for $n \geqq 6$.

Thus, $\mathfrak{P}$ is Moufang and G contains the little projective group. This finishes the proof of Proposition 3.1 and of Theorem 1.1. □

Now the first part of Theorem 1.3 and Corollary 1.2 follows from [KTVM], 3.14. and the following result where as usual we denote the Morley rank of the group G by $\mathrm{RM}(G)$:

3.3. Theorem ([Hr]). *If G is a group of finite Morley rank which acts definably, effectively and transitively on a strongly minimal set X, then one of the following occurs*:

(i) $\mathrm{RM}(G) = 1$ *and the connected component* G^0 *of G acts regularly on X*;

(ii) $\mathrm{RM}(G) = 2$ *and the action of G on X is definably permutation equivalent to the affine group* $K^+ \rtimes K^*$ *acting on the affine line of K for some algebraically closed field K*; *or*

(iii) $\mathrm{RM}(G) = 3$ *and the action of G on X is definably permutation equivalent to the simple algebraic group* $\mathrm{PSL}_2(K)$ *acting on the projective line of K for some algebraically closed field K.*

Now let $\mathfrak{P}$ be a generalized n-gon whose point rows and line pencils are strongly minimal, $n \geqq 3$, and let G be a group of automorphisms of the incidence graph of $\mathfrak{P}$ of finite Morley rank acting transitively and definably on the (definable) set of ordered ordinary n-gons contained in $\mathfrak{P}$.

In order to prove Theorem 1.3, first we assume that $G \leqq \mathrm{Aut}(\mathfrak{P})$, i.e., G preserves lines and points. By 3.3, for any $x \in \mathfrak{P}$ the Levi factor H_x (which clearly is interpretable) is definably isomorphic to either $\mathrm{PSL}_2(K)$ or $K^+ \rtimes K^*$ for some algebraically closed field K.

Theorem 1.1 together with 3.3 immediately implies the first part of Corollary 1.2 and the rest now follows from [KTVM], 3.14. However, especially for quadrangles the arguments in [KTVM] were quite involved, and it is worth giving a short geometric proof of the fact that G is isomorphic to one of the groups $\mathrm{PSL}_3(K)$, $\mathrm{PSp}_4(K)$ or $\mathrm{G}_2(K)$ which does not need the full classification of Moufang polygons.

As noted above, Knarr [Kn] proved for Moufang polygons that the group of even projectivities coincides with the group induced by the little projective group. The little projective group also acts transitively on ordinary n-gons, hence it induces 2-transitive groups on line pencils and point rows. Since $\mathrm{PSL}_2(K)$ (in its action on the projective line of K) and $K \rtimes K^*$ have no 2-transitive proper subgroups (see [KTVM], 4.9), the little projective group induces the same group on the line pencils and point rows as G.

For projective planes all projectivities are even, and the projectivity groups are always 3-transitive. Hence it follows immediately from von Staudt's Theorem that $\mathfrak{P}$ is the Pappian plane over K.

Let now $\mathfrak{P}$ be a Moufang quadrangle. Suppose first that at least one Levi-factor of G is sharply 2-transitive. Then it follows from [VM2] that $\mathfrak{P}$ is finite, which is a contradiction. If both Levi factors are isomorphic to $\mathrm{PSL}_2(K)$ (for possibly different fields K), then it follows from Case 2 of the proof above that G is transitive on ordinary pentagons contained in $\mathfrak{P}$. Since any Moufang quadrangle contains central elations (up to duality), see 5.4.7 of [VM1], it follows easily that all elations are central. Hence the points of the quadrangle are regular. But the projectivity groups on the line pencils are permutation equivalent to $\mathrm{PSL}_2(K)$, so by Thm. 1 of [BTVM], $\mathfrak{P}$ is the symplectic quadrangle over K. In this case it is well-known that both Levi factors are isomorphic to $\mathrm{PSL}_2(K)$ for the same field K.

The Moufang hexagons are quickly classified, as they belong to certain kinds of Jordan algebras, see Faulkner [Fau]. Over algebraically closed fields only the field itself is possible, showing that any infinite Moufang hexagon of finite Morley rank is the split Cayley hexagon over an algebraically closed field.

The Moufang octagons can be excluded right away, since, up to duality, the little projective group of the point rows induces neither $\mathrm{PSL}_2(K)$ nor $K^+ \rtimes K^*$ (cf. [VM1], Table 8.1).

Thus $\mathfrak{P}$ is Pappian, namely isomorphic to the projective plane, the symplectic quadrangle or the split Cayley hexagon over some algebraically closed field, and the classification shows that G acts transitively and thus regularly on ordered $n+1$-gons. Since the set of ordered $n+1$-gons has Morley degree 1 (see [KTVM], 1.9), it follows that G is connected and that $\mathrm{RM}(G) = 2n+2$ (see [KTVM], 2.8). By Proposition 3.1 and the remarks in 2.11, the little projective group $G(K)$ of $\mathfrak{P}$ is a definable subgroup of G. For the Pappian n-gons, $G(K)$ is a simple linear algebraic group over an algebraically closed field with $\mathrm{RM}\big(G(K)\big) = 2n+2$. Hence $\mathrm{RM}(G) = \mathrm{RM}\big(G(K)\big)$ and since G is connected, we must have $G = G(K)$. Thus, G is isomorphic to one of the groups $\mathrm{PSL}_3(K)$, $\mathrm{PSp}_4(K)$ or $\mathrm{G}_2(K)$ over an algebraically closed field K and as in [KTVM], 3.14 it follows that the isomorphism is definable.

This finishes the first part of Theorem 1.3 which suffices to prove Corollary 1.2.

To finish the proof of Theorem 1.3 now suppose that G contains a duality. Then the subgroup G^* of G of automorphisms of $\mathfrak{P}$ preserving the points (and hence the lines) is a definable group of finite Morley rank acting transitively and definably on the set of ordered ordinary n-gons contained in $\mathfrak{P}$. By the first part of Theorem 1.3, we know that $n \in \{3,4,6\}$ and $\mathfrak{P}$ is Pappian over some algebraically closed field K. Let $n = 4, 6$ and let ρ be any definable duality. Then we must have $\mathrm{char}(K) = 2$ if $n = 4$ and $\mathrm{char}(K) = 3$ if $n = 6$ (see 7.3.5 of [VM1]). By composing ρ with an element of G^* if necessary, we may assume that ρ stabilizes (but not pointwise) an ordinary n-gon $(x_0, \dots, x_{2n-1}, x_{2n})$ and switches x_0 and x_1. By choosing coordinates suitably (x_0 as (∞) and x_1 as $[\infty]$), we see that ρ induces two field automorphisms φ and ψ (one related to the point rows and the other related to the line pencils). Similarly as in [VM1] (4.6.1 and 4.6.6), one shows easily that $\psi\varphi^{-1}$ is the Frobenius. Let us sketch the calculations for $n = 4$, the case $n = 6$ being completely similar. We use the coordinatization of the symplectic quadrangle over K as given in 2.5. By the transitivity on ordinary 5-gons, we may assume that ρ also interchanges (1) and $[1]$. Hence there are permutations $\psi, \varphi_a, \psi_{a,l}$, for $a, l \in K$, and $\varphi, \psi_k, \varphi_{k,b}$, for $k, b \in K$, such that ρ maps (a, l, a') to $[a^\psi, l^{\varphi_a}, a'^{\psi_{a,l}}]$ and $[k, b, k']$ to $(k^\varphi, b^{\psi_k}, k'^{\varphi_{k,b}})$ (and all these permutations map 0 to 0). Noting that the characteristic of K is 2, we obtain by expressing that incidence is preserved by ρ:

$$a^\psi (k^\varphi)^2 + b^{\psi_k} = (ak+b)^{\psi_{a,l}},$$

and

$$a^{\psi}k^{\varphi} + l^{\varphi_a} = (a^2k + l)^{\varphi_{k,b}},$$

for all $a, b, k, l \in K$. Putting $a = 0$ respectively $k = 0$, we deduce immediately that $\psi_k \equiv \psi_{a,l}$, for all $a, l, k \in K$ and that $\varphi_a \equiv \varphi_{k,b}$, for all $a, b, k \in K$. Furthermore, putting $k = 1$ and $b = 0$ in the first equation and $a = 1$ and $l = 0$ in the second, we see that $\psi \equiv \psi_1$ and $\varphi \equiv \varphi_1$. Similarly, we see that $a^{\psi} = (a^2)^{\varphi}$, for all $a \in K$, hence $\psi\varphi^{-1}$ is the Frobenius. It is now easy to see that both ψ and φ are field automorphisms.

If we denote by θ the field automorphism associated to the semi-linear transformation corresponding to ρ^2, then $\theta = \psi\varphi$ (indeed, in the symplectic example above, (a) is mapped by ρ^2 onto $((a^{\psi})^{\varphi})$). Clearly $\rho^2 \in G^*$, hence $\theta = 1$ by the first part, implying that ψ is a square root of the Frobenius automorphism. But this does not exist in algebraically closed fields.

Now let $n = 3$ and consider a standard coordinatization of the Pappian projective plane over the algebraically closed field K. If G contains some duality ρ, then as before $\rho^2 \in G^*$, so the automorphism of K induced by ρ^2 is the identity. This implies that the field automorphism induced by ρ has order 1 or 2. On the other hand, the polarity τ defined by interchanging the points and lines with same coordinates is also definable and the automorphism of K induced by τ is the identity. If $\rho \neq \tau$, then $\rho\tau$ is a definable automorphism of the projective plane over K which induces a definable involutive field automorphism. But this is impossible in a structure of finite Morley rank. Hence τ is the only duality in G and G is isomorphic to $\mathrm{PSL}_3(K) \rtimes 2$.

This completes the proof of Theorem 1.3. □

Proof of Corollary 1.4. By Proposition 2.8 the BN-pair of G is spherical and of finite (Tits-) rank. If G has a BN-pair of rank 1, then 1.4 follows from Theorem 3.3, for BN-pairs of rank 2, this follows from Theorem 1.1. For BN-pairs of higher rank this follows either directly from [KTVM], Thm. 5.3 or as in loc. cit. from the fact that buildings of rank at least 3 are determined by their diagram and the corresponding rank 2 residues. □

As an alternative version of Theorem 1.2 we can state

3.4. Theorem. *Let G be an infinite group of finite Morley rank with an irreducible definable* BN-*pair of* (*Tits-*) *rank* 2. *If the Levi-factors have Morley rank at most* 3, *then* $G \cong \mathrm{PSL}_3(K)$, $\mathrm{PSp}_4(K)$ *or* $\mathrm{G}_2(K)$ *for some algebraically closed field K.*

Proof. If G has infinite Levi-factors of Morley rank at most 3, it follows immediately from the 2-transitivity of the Levi-factors that the panels have Morley rank 1. Conversely, it follows from Theorem 3.3 that the Levi-factors have Morley rank at most 3, if the panels have Morley rank 1. □

4. Appendix

In this section we treat the reducible case of finite Morley rank. Note that the connection 2.6 between groups and polygons now gives that a reducible BN-pair of Tits rank

2 is equivalent to a generalized digon with an automorphism group acting transitively on ordinary ordered digons.

4.1. Theorem. (i) *If G is an infinite group of finite Morley rank with a definable reducible* BN-*pair of* (*Tits-*) *rank* 2, *such that all panels of the associated building have Morley rank* 1, *then*

$$G \cong \mathrm{PSL}_2(K) \times \mathrm{PSL}_2(K'), \quad (K^+ \rtimes K^*) \times (K'^+ \rtimes K'^*)$$

or

$$(K^+ \rtimes K^*) \times \mathrm{PSL}_2(K')$$

for some algebraically closed fields K and K'.

(ii) *If $\mathfrak{P}$ is a generalized digon with strongly minimal point row and line pencil and G is a group of automorphisms of the incidence graph of $\mathfrak{P}$ (which is a complete bipartite graph) of finite Morley rank which acts transitively on the set of ordered ordinary 2-gons contained in $\mathfrak{P}$, then G is one of the groups above, or isomorphic to* $\mathrm{PSL}_2(K)$ wr 2 *or* $(K^+ \rtimes K^*)$ wr 2.

We need the following

4.2. Lemma. *Let G be a group of finite Morley rank acting definably and definably primitively on a definable set X. Then* $\deg(X) = 1$.

Proof. By definable primitivity it is easy to see that the connected component G^0 of G acts transitively on X. Hence X has Morley degree 1 as otherwise the setwise stabilizer in G^0 of a proper generic subset of X would be a proper subgroup of G^0 of finite index. □

Proof. As for the irreducible case, it suffices to prove (ii) assuming G does not contain elements interchanging points and lines. There are only two panels here—the set of points, and the set of lines. Since the Levi-factors are interpretable and act 2-transitively on the panels, the panels have Morley degree 1 and are thus strongly minimal. Let $\mathfrak{P} = (\mathscr{P}, \mathscr{L}, \mathscr{F})$ be the associated generalized digon. Let $\gamma = (p, L, q)$ be a root with p, q points, and L a line. Let r be any point distinct from both p, q. By the BN-pair property, the action of group $G_{p,q}$ on $\mathscr{L}$ is 2-transitive. Since $G_{p,q} \leqq G$ and since $\mathrm{PSL}_2(K)$ (for K algebraically closed) does not have a sharply 2-transitive subgroup, it follows from Theorem 3.3 that the restriction of the action of G to $\mathscr{L}$ coincides with the restriction of the action of $G_{p,q}$ to $\mathscr{L}$. Consequently, $\{p, L, q\}$ is a Moufang panel whenever the action of G on $\mathscr{P}$ is sharply 2-transitive. Hence if G acts sharply 2-transitively on both $\mathscr{P}$ and $\mathscr{L}$, then $\mathfrak{P}$ is a Moufang digon with respect to G. Suppose now that G acts 3-transitively on both $\mathscr{P}$ and $\mathscr{L}$. As in the proof of Proposition 3.1, we see that $G_{p,q,r,L}$ acts transitively on $\mathscr{L} \backslash \{L\}$. But, by Theorem 3.3, this group fixes $\mathscr{P}$ pointwise. Hence we again conclude (using the observation that $\mathrm{PSL}_2(K)$ does not have a sharply 2-transitive subgroup) that $\mathfrak{P}$ is Moufang with respect to G. Now suppose that G acts sharply 2-transitively on $\mathscr{P}$ and sharply 3-transitively on $\mathscr{L}$. As above, we can still conclude that the panel $\{p, L, q\}$ is a Moufang panel. Hence the pointwise stabilizer in G of $\mathscr{P}$ acts as $\mathrm{PSL}_2(K)$ on $\mathscr{L}$, for some algebraically closed field K. In order to show that $\mathfrak{P}$ is Moufang with respect to G, it suffices to prove that the pointwise stabilizer of $\mathscr{L}$ in G acts 2-transitively on $\mathscr{P}$. By the BN-pair property, there exists $g \in G$ mapping p, q to any desired $p', q' \in \mathscr{P}$, $p' \neq q'$. The action of g on $\mathscr{L}$ can be identified with an element of $\mathrm{PSL}_2(K)$. But now we consider an element $g' \in G$ fixing $\mathscr{P}$

pointwise and inducing the same action on $\mathscr{L}$ as g does. Then gg'^{-1} fixes $\mathscr{L}$ elementwise and maps p, q to p', q'. So $\mathfrak{P}$ is Moufang with respect to G.

It is now easy to see that G is a direct product of 2-transitive groups of finite Morley rank acting on strongly minimal sets. So we obtain the possibilities mentioned in the theorem. □

Finally, we remark that similarly to the previous theorem, one can classify all groups of finite Morley rank acting definably, effectively and transitively on the set of all apartments of any (not necessarily irreducible) spherical building with strongly minimal rank 1 residues. One obtains direct products of split simple algebraic groups over algebraically closed fields and groups as in Theorem 3.3(ii), possibly extended with some dualities or trialities, and possibly further extended by standard wreath products.

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