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Generalized polygons with valuation

By

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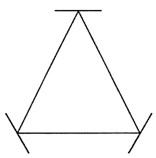
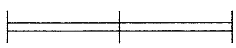
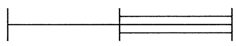
1. Introduction. The theory of affine buildings depends to a great extend on the theory of valuations on fields (in the classical case). Roughly speaking, there is a bijective correspondence between the class of (symmetric) affine buildings of rank $n \geqq 4$ and the class of spherical buildings of rank $n-1$ defined over a field with a valuation. This correspondence maps an affine building to its (spherical) building at infinity (see Tits [6]). The rank 3 case however is different since there exist non-classical examples, e.g. [3], [4], [7]. There are three classes of rank 3 affine buildings. They correspond to the following diagrams.


type $\tilde{A}_2$


type $\tilde{C}_2$


type $\tilde{G}_2$

The first class ($\tilde{A}_2$) was characterized in [7], [8] by defining the notion of a valuation on planar ternary rings (coordinatizing algebraic structure of projective planes). Recently, we also characterized the second class ($\tilde{C}_2$) by defining the notion of a valuation on quadratic quaternary rings (coordinatizing algebraic structure of generalized quadrangles [2]). But a quadratic quaternary ring with valuation can hardly be seen as a direct generalization of a planar ternary ring with valuation, hence it is not at all clear how one should extend these definitions to the $\tilde{G}_2$-case.
The idea of the present paper is to avoid the algebraic structures and to put a valuation directly on the geometry of the building at infinity of rank 3 affine buildings. Let us concentrate for a moment on the $\tilde{A}_2$-case to see where it comes from. So suppose Δ is a building of type $\tilde{A}_2$ with building at infinity $PG(\Delta)$ and call one set of vertices of $PG(\Delta)$ the set of points and the other the set of lines (by an arbitrary choice) to obtain a projective plane, also denoted by $PG(\Delta)$. Fix a vertex v of Δ. Consider any two points P_1 and P_2 of $PG(\Delta)$, then there exist unique "panels of quarters" ("pennels" in the terminology of [8]) p_1, resp. p_2 with source v and "direction" P_1, resp. P_2 (see [6]). Now let $u(P_1, P_2)$ be the number of panels of $p_1 \cap p_2$, and similarly for lines. This map u is called "the

partial valuation map" in [8] (see [8], §4.3.2). Note that different v implies different u. By [8], (RP), it follows readily that, if $P_1^* L_3^* P_2^* L_1^* P_3^* L_2^* P_1$ (see 2.1), then $u(P_1, P_2) + u(L_1, L_3) = u(L_1, L_2) + u(P_1, P_3)$. A similar thing works in $\tilde{C}_2$-case (see [9]) where we have: if $P_1^* L_1^* P_2^* L_2^* P_3^* L_3^* P_4^* L_4^* P_1$, then $u(P_1, P_2) + u(L_1, L_2) + u(P_2, P_3) = u(P_3, P_4) + u(L_3, L_4) + u(P_1, P_4)$. Now, in 2.2, we generalize axiomatically these properties to define generalized polygons with valuation with an eye to the following conjecture:

Conjecture. A generalized n-gon $n \geqq 3$, is isomorphic to the geometry at infinity of some rank 3 affine building if and only if it admits a valuation.

This would yield a geometric characterization of all rank 3 affine buildings. We can actually prove this conjecture for $n \neq 6$ (this exception corresponds to the $\tilde{G}_2$ case). In this paper, we show that generalized n-gons with valuation do not exist unless $n = 3, 4$ or 6. The proof of the case $n = 3, 4$ is too long to include here and will appear elsewhere. But a Dutch version of the proof is provided in [9], the "only if"-part for $n = 3$ being shown above!

2. Definitions and notation.

2.1. Generalized polygons. An incidence structure (of rank 2) is a triple (P, L, I) of non-empty sets with $I \subseteqq P \times L$ and $P \cap L = \emptyset$. Every incidence structure (P, L, I) gives rise to a graph $(V, *)$ with vertex set $V = P \cup L$ and adjacency relation $*$ defined by

$$x * y \Leftrightarrow (x = y \quad \text{or} \quad x I y \quad \text{or} \quad y I x);$$

hence $*$ is reflexive and symmetric (unlike I). Let $d: V^2 \to \mathbb{N} \cup \{+\infty\}$ be the path metric of $(V, *)$, i.e. $d(x, y)$ is the smallest integer $k \in \mathbb{N}$ such that there exists a chain of elements $x_0, x_1, \ldots, x_k \in V$ with $x = x_0 * x_1 * x_2 * \cdots * x_{k-2} * x_{k-1} * x_k = y$, and $d(x, y) = +\infty$ if x and y are not connected by such a path.

Let $n \geqq 2$ be an integer. A generalized n-gon is an incidence structure $S = (P, L, I)$ (with corresponding graph $(V, *)$) satisfying (GN.1) and (GN.2):

(GN.1) $n = \sup \{d(x, y) | x, y \in P \cup L\}$; in particular, $(V, *)$ is connected.

(GN.2) If $d(x, y) = k < n$, then there is a unique chain $x_0, x_1, \ldots, x_k$ such that $x = x_0 * x_1 * x_2 * \cdots * x_{k-1} * x_k = y$.

Moreover, a generalized n-gon is called thick if

(T) Every element of $P \cup L$ is incident with at least three elements.

A generalized polygon is a generalized n-gon for some integer $n \geqq 2$. All previous definitions are due to J. Tits [5]; see also W. M. Kantor [3].

Remarks. (1) A generalized 2-gon (digon) is a trivial incidence structure, every point being incident with every line; the corresponding graph is a complete bipartite graph.

(2) The generalized 3-gons are exactly the projective planes. Generalized 4- (resp. 6-) gons are also called generalized quadrangles (resp. hexagons).

(3) By Feit-Higman [1], finite thick generalized n-gons exist only for $n \in \{2, 3, 4, 6, 8\}$.

2.2. Generalized polygons with valuation. Let $S = (P, L, I)$ be any generalized n-gon, $n \geqq 2$, with corresponding graph $(V, *)$ and suppose $x, y \in P \cup L$, then we define $x \perp y$ if and only if there exists $z \in P \cup L$ such that $x * z * y$. In that case, x and y are called collinear. We denote $P_\perp = \{(x, y) \in P^2 \mid x \perp y\}$ and $L_\perp = \{(x, y) \in L^2 \mid x \perp y\}$. We call a map $u: P_\perp \cup L_\perp \to \mathbb{N} \cup \{+\infty\}$ a valuation on S if it satisfies (U.1) through (U.5) below. For $x \in P \cup L$, we denote $x_* = \{(y, z) \in V^2 \mid y * x * z\}$.

(U.1) u/x_* is surjective, for all $x \in P \cup L$; in particular S is thick.
(U.2) $u(x, y) = +\infty$ if and only if $x = y$.
(U.3) If $u(x, y) < u(y, z)$ and $x \perp z$, then $u(x, z) = u(x, y)$.

Putting $x = z$ in (U.3), (U.2) implies that u is symmetric.

(U.4) There exist $x, x_i \in P$, $y, y_i \in L$, $1 \leqq i \leqq n$, such that $x_i * y_i * x_{i+1}$, $i \pmod n$, $x * y_1$, $y * x_1$ and $u(x_1, x_{i+1}) = u(y_1, y_{i+1}) = u(x, x_1) = u(x, x_2) = u(y, y_1) = u(y, y_n) = 0$, for all $i \pmod n$.
(U.5) There exists a sequence $(a_1, a_2, \ldots, a_{n-1}, a_{n+1}, a_{n+2}, \ldots, a_{2n-1}) \in \mathbb{N}_0^{2n-2}$ such that, whenever $x_1 * x_2 * \cdots * x_{2n} * x_1$, with $x_1, x_3, \ldots \in P$ and $x_2, x_4, \ldots \in L$, one has

$$\sum_{i=1}^{n-1} a_1 \cdot u(x_{i-1}, x_{i+1}) = \sum_{i=n+1}^{2n-1} a_1 \cdot u(x_{i-1}, x_{i+1}).$$

Every non-negative sequence $(a_1, a_2, \ldots, a_{n-1}, a_{n+1}, a_{n+2}, \ldots, a_{2n-1})$ satisfying the condition of (U.5) and such that $a_1, a_2, \ldots, a_{2n-1}$ are relatively prime, is called a weight-sequence of S and (S, u) is called a generalized n-gon with valuation. In the next section, we show that in a generalized n-gon S with valuation u, the weight-sequence is, up to duallity, uniquely determined if $n = 3, 4, 6$. Excluding the trivial case $n = 2$, we will prove that, whenever $n \neq 3, 4, 6$, no valuation can live in S.

3. Calculation of the weight-sequences

3.1. The case $n \neq 3, 4, 6$. Suppose (S, u) is a generalized n-gon with valuation and $n = 5$ or $n \geqq 7$. Suppose first that n is even and put $n = 2m$. By permuting the indices in (U.5) and making linear combinations, we obtain a weight-sequence $(b_1, \ldots, b_{n-1}, b_{n+1}, \ldots, b_{2n-1})$, with $k \cdot b_i = a_i + a_{n-i} + a_{n+i} + a_{2n-i}$ for some $k \in \mathbb{N}_0$ and hence with $b_i = b_{n-i} = b_{n+i} = b_{2n-i}$. We call such a weight-sequence *symmetric.* Suppose now there exists some weight-sequence $(a'_1, \ldots, a'_{2n-1})$ with $a'_1 = 0$ and $a'_2 \neq 0$. One can check (starting from (U.4) and (U.1)) that there exists a sequence $z_1 * z_2 * \cdots * z_{2n} * z_1$, with $z_1, z_3, \ldots \in P$; $z_2, z_4, \ldots \in L$, such that $u(z_i, z_{i+2}) = 0$, for all $i \pmod{2n}$ except $i = 0, n$, and such that $u(z_{2n}, z_2) = u(z_n, z_{n+2}) = k > 0$, for some integer k. Now choose $y_3 \neq z_3$ and consider the unique sequence $z_{2n} * z_1 * z_2 * y_3 * y_4 * \cdots * y_{n+1} * z_{n+2} * \cdots * z_{2n}$. Since $(a'_1, \ldots, a'_{2n-1})$ is a weight-sequence, this implies $v(z_2, y_4) = 0$ and $v(y_1, y_{i+2}) = 0$ whenever $a'_{i+1} \neq 0$. But by the same token, $v(z_{n+2}, z_n) = 0$ by considering the n-gon $z_2 * y_3 * y_4 * \cdots * y_{n+1} * z_{n+2} * z_{n+1} * \cdots * z_3 * z_2$. This contradicts $k \neq 0$. Similarly, one shows in general that, if $(c_1, \ldots, c_{2n-1})$ is a weight-sequence with $c_i \geqq 0$, then $c_i \neq 0$. If $(c_1, \ldots, c_{2n-1}) \neq (b_1, \ldots, b_{2n-1})$, then a suitable linear combination yields a weight-sequence containing a zero, hence all weight-sequences are equal to $(a_1, \ldots, a_{2n-1}) = (b_1, \ldots, b_{2n-1})$. In particular $a_i = a_{n-i} = a_{n+i} = a_{2n-i}$.

Now, (U.5) implies that, whenever $x_1 * \cdots * x_{2n} * x_1$, with $x_1, x_3, \ldots \in P$ and $x_2, x_4, \ldots \in L$, then

$$\sum_{i=1}^{n-1} a_i \cdot u(x_{i-1}, x_{i+1}) = \sum_{i=n+1}^{2n-1} a_i \cdot u(x_{i-1}, x_{i+1})$$

and

$$\sum_{i=n+1}^{2n-1} a_i \cdot u(x_{n+i-5}, x_{n+i-3}) = \sum_{i=1}^{n-1} a_i \cdot u(x_{n+i-5}, x_{n+i-3}).$$

Adding these two equalities, we obtain

$$\begin{aligned}
&(a_3 - a_1) \cdot u(x_{n-2}, x_n) + a_4 \cdot u(x_{n-1}, x_{n+1}) \\
&\quad + \sum_{i=1}^{m-4} (a_i + a_{i+4}) \cdot u(x_{n+i-1}, x_{n+i+1}) \\
&\quad + (a_{m-3} + a_{m-1}) \cdot u(x_{n+m-4}, x_{n+m-2}) + 2\, a_{m-2} \cdot u(x_{n+m-3}, x_{n+m-1}) \\
&\quad + (a_{m-1} + a_{m-3}) \cdot u(x_{n+m-2}, x_{n+m}) \\
&\quad + \sum_{i=1}^{m-4} (a_{m-i+1} + a_{m-i-3}) \cdot u(x_{n+m+i-2}, x_{n+m+i}) \\
&\quad + a_4 \cdot u(x_{2n-5}, x_{2n-3}) + (a_3 - a_1) \cdot u(x_{2n-4}, x_{2n-2}) \\
&= (a_3 - a_1) \cdot u(x_{2n-2}, x_{2n}) + a_4 \cdot (x_{2n-1}, x_{2n+1}) + \cdots
\end{aligned}$$

(add n to the subscript of every x in the left hand side).

If $a_3 - a_1 < 0$ then a suitable linear combination with $(a_1, \ldots, a_{2n-1})$ yields a weight-sequence starting with a zero. Hence, $a_3 - a_1 \geqq 0$ and $(a_3 - a_1, a_4, a_1 + a_5, \ldots, a_i + a_{i+4}, \ldots, a_{m-3} + a_{m-1}, 2\,a_{m-2}, a_{m-3} + a_{m-1}, \ldots, a_4, a_3 - a_1, a_3 - a_1, a_4, \ldots)$ is proportional to the weight-sequence $(a_1, \ldots, a_{2n-1})$. By the symmetry of the weight-sequence, we have (for some positive integer k)

(1) $\qquad a_3 - a_1 = k \cdot a_1$

(2) $\qquad a_4 = k \cdot a_2$

$(i+2)$ $\qquad a_i + a_{i+4} = k \cdot a_{i+2}, \quad 1 \leqq i \leqq m-4$

$(m-1)$ $\qquad a_{m-3} + a_{m-1} = k \cdot a_{m-1}$

(m) $\qquad 2\,a_{m-2} = k \cdot a_m$

Case I. $m = 2p$ is even ($p \geqq 2$). We select (from the previous set of equations) the equations with even subscripts and concieve this as a system of p equations in the p unknowns $(a_2, a_4, \ldots, a_{2p})$. Since there must be a non-zero solution, the determinant of the system, say $D(k, p)$, must be zero. Hence k must satisfy

$$D(k,p) = \begin{vmatrix} -k & 1 & 0 & 0 & \dots & & & & \\ 1 & -k & 1 & 0 & \dots & & & 0 & \\ 0 & 1 & -k & 1 & \dots & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & -k & 1 & 0 \\ & 0 & & & \dots & 0 & 1 & -k & 1 \\ & & & & \dots & 0 & 0 & 2 & -k \end{vmatrix} = 0.$$

One easily shows

(A) $$D(k, p) = -k \cdot D(k, p-1) - D(k, p-2)$$

(B) $$= (k^2 - 1) \cdot D(k, p-1) - D(k, p-3), \quad p \geqq 3,$$

(when putting $D(k, 0) = 2$ and $D(k, 1) = -k$). Using (A) and (B), one shows inductively

$$\begin{aligned} D(2, p) &= 2 \cdot (-1)^p \\ D(1, p) &\in \{-1, 2\}, \\ D(0, p) &= 2 \quad \text{or} \quad -2, \quad p \text{ even}, \\ D(0, p) &= 0, \quad p \text{ odd}, \\ D(-1, p) &\in \{1, -1, 2, -2\}. \end{aligned}$$

Suppose p is even and $D(k, p) = 0$, $k > 0$. Then by (A), $D(k, p-2)$ is divisible by k. But since $D(0, p-2) = 2$ or -2, one has $k = 1$ or 2. But $D(1, p) \neq 0$ and $D(2, p) \neq 0$, a contradiction.

Suppose now p odd, then $p \geqq 3$ and by (B), $D(k, p-3)$ is divisible by $k^2 - 1$, hence $D(k, p-3)$ is divisible by $k + 1$. But $D(-1, p-3) \in \{1, -1, 2, -2\}$, hence $k + 1 = 2$ and $k = 1$, again contradicting $D(1, p) \neq 0$.
Hence if m is even, then there exists no valuation on S.

Case II. $m = 2p + 1$ is odd ($p \geqq 2$). Similarly as above, we consider the equations (2), (4), …, ($2p$) with even subscripts. We denote the determinant of the system of equations by $D^*(k, p)$ and one easily shows again

(C) $$D^*(k, p) = -k \cdot D^*(k, p-1) - D^*(k, p-2)$$

(with $D^*(k, 0) = 1$ and $D^*(k, 1) = 1 - k$). If $p = 2$, then $D^*(k, p) = k^2 - k - 1$ and this has no integer solution. Suppose now $p \geqq 3$. By (C), $D^*(k, p) = 0$ implies that $D^*(k, p-2)$ is divisible by k. But one can check that $D^*(k, p-2) = 1$ or -1 (use (C) and induction), hence $k = 1$. But then by (2), $a_2 = a_4$ and by (4), $a_6 = 0$, a contradiction.

Hence if n is even and $n \geqq 8$, no valuation can be put on S.

Suppose now n is odd and $n \geqq 5$. Let $n = 2m + 1$ ($m \geqq 2$). Similarly argueing as above, we obtain a contradiction (we find the same determinant $D^*(k, m)$ as in Case II above). This completes the case $n \neq 3, 4, 6$.

3.2. The cases $n = 3, 4, 6$. We first consider the case $n = 6$. So suppose (S, u) is a generalized hexagon with valuation. The very same arguments as for the even case above shows us now that the weight-sequence is of the form $(a_1, a_2, 2a_1, a_2, a_1, a_1, a_2, 2a_1, a_2, a_1)$ with a_1 and a_2 relatively prime. From (U.1) and (U.4) follows the existence of a sequence $x_1 * x_2 * \cdots * x_{12} * x_1$ (with $x_1, x_3, \ldots \in P$ and $x_2, x_4, \ldots \in L$) and a point x and a line x' such that $u(x_i, x_{i+2}) = 0$ for all $i \pmod{12}$, $x * x_2$, $x' * x_9$ and $u(x_1, x) = 1 = u(x_{10}, x')$. Hence, by (U.3), $u(x_8, x') = 0$. Consider the unique chain $x' * y_1 * y_2 * y_3 * y_4 * x$, then (U.5) implies in $x' * y_1 * \cdots * y_4 * x * x_2 * x_1 * x_{12} * \cdots * x_9 * x'$:

$$(1) \qquad \begin{aligned} &2a_1 + a_2 u(x_2, y_4) + a_1 u(x, y_3) \\ &\quad = 2a_1 u(x_9, y_1) + a_2 + a_2 u(x', y_2) + a_1 u(y_1, y_3) \end{aligned}$$

$$(2) \qquad \begin{aligned} &2a_1 u(x, y_3) + a_2 u(x_2, y_4) + a_2 u(y_2, y_4) + a_1 + a_1 u(y_1, y_3) \\ &\quad = a_2 + a_1 u(x_9, y_1) \end{aligned}$$

$$(3) \qquad 2a_1 u(y_1, y_3) + a_2 u(y_2, y_4) + a_2 u(x', y_2) + a_1 u(x, y_3) + a_1 u(x_9, y_1) = a_1 .$$

Since all terms are non-negative, (3) implies $u(y_1, y_3) = 0$ and $u(x, y_3) + u(x_9, y_1) \leqq 1$. There are three possibilities now.

(a) $u(x, y_3) = u(x_9, y_1) = 0$. Then $a_1 = a_2 \cdot (u(y_2, y_4) + u(x', y_2))$. Hence a_1 is divisible by a_2.

(b) $u(x, y_3) = 1$ and $u(x_9, y_1) = 0$. Then $u(y_2, y_4) = u(x', y_2) = 0$ (by (3)) and by (2), $a_2 = 2a_1 + a_2 u(x_2, y_4) + a_1 = 3a_1 + a_2 u(x_2, y_4)$. Hence $a_2 = 3a_1$.

We now make the following useful observation. Applying (U.5) twice (once as it stands and a second time by shifting the subscripts by two) and adding the resulting equalities, we see that (L, P, I^{-1}, u) is also a generalized hexagon with valuation and with a weight-sequence proportional to $(a_2, 3a_1, 2a_2, 3a_1, a_2, a_2, 3a_1, 2a_2, 3a_1, a_2)$.

(c) $u(x, y_3) = 0$ and $u(x_9, y_1) = 1$. Consider the unique chain $x * z_1 * z_2 * z_3 * z_4 * x_8$ and apply the dual of (U.5) in the n-gon $x_1 * x_2 * x * z_1 * \cdots * z_4 * x_8 * \cdots * x_{12} * x_1$ to obtain $u(x_8, z_3) = u(z_2, z_4) = u(z_1, z_3) = u(z_2, x) = u(x_2, z_1) = 0$ and $u(x_9, z_4) = 1$. Now apply (U.5) in the n-gon $z_4 * \cdots * z_1 * x * y_4 * \cdots * y_1 * x' * x_9 * x_8 * z_4$:

$$(4) \qquad 2a_1 + a_1 u(x_9, y_1) = 2a_1 u(x, y_3) + a_2 u(z_1, y_4) + a_2 u(y_2, y_4) + a_1 u(y_1, y_3).$$

By (3), we have $u(y_2, y_4) = 0$ again and by (2), $u(x_2, y_4) = 1$. By (U.3), this implies $u(z_1, y_4) = 0$. Hence (4) reduces to $3a_1 = 0$, a contradiction. So (c) cannot occur. By (a) and (b) and since a_1 and a_2 are relatively prime, we have (a) $a_2 = 1$ or (b) $a_2 = 3$ and $a_1 = 1$. Suppose $a_2 = 1$. By duallity, we have (a') a_2 is divisible by $3a_1$ (which is clearly impossible) or (b') $3a_1 = 3a_2 = 3$, hence $a_1 = a_2 = 1$. So there are only two possibilities: either $a_1 = a_2 = 1$ or $a_1 = 1$ and $a_2 = 3$. These two cases are mutually dual and hence, up to duallity, the weight-sequence of generalized hexagons with valuation is unique.

The cases $n = 3, 4$ are similar (and simpler) to $n = 6$ above. One obtains for $n = 3$ the unique weight-sequence $(1, 1, 1, 1)$ (which is also the weight-sequence of the dual projective plane); for $n = 4$, the mutually dual weight-sequences are $(1, 1, 1, 1, 1, 1)$ and $(1, 2, 1, 1, 2, 1)$. This unables us to conclude (with the aid of [9]) that the conjecture holds for $n = 3$ and $n = 4$. By (3.1), the conjecture is also true for $n = 5$ and $n \geqq 7$. So $n = 6$ is the only remaining open question. But by the above calculation, we know exactly the weight-sequence and hence, in view of the analogue for $n = 4$, one can write down the algebraic characterization of the generalized hexagons which are isomorphic to the building at infinity of some affine building of type $\tilde{G}_2$ (in terms of a valuation on "cubic sexternary rings", the coordinatizing algebraic structures of generalized hexagons), if there exists such a characterization. The advantage is, that we can now first look for (non-classical) examples before trying to prove the characterization in order to know if that work is worth while. Now, how do we derive from all this a definition of "cubic sexternary ring with valuation"? Besides the algebraic interpretation of the geometric defining properties of a generalized hexagon and some non-degeneracy conditions, the main axiom will be the algebraic expression of (U.5), with the right weight-sequence (as in the $\tilde{C}_2$-case, replace $u(x_{i-1}, x_{i+1})$ by $v(r_{i-1}, r_{i+1})$, where r_{i-1}, resp. r_{i+1}, is a suitable coordinate of x_{i-1}, resp. x_{i+1}, depending on the number of coordinates of x_i). The correspondence between u and v (= the valuation on the ring) will then be given by: $v(r, s) = u((r), (s)) - u((r), (\infty)) - u((s), (\infty))$, where r and s belong to the coordinatizing ring and (r), (s) and (∞) are well-defined points; similarly for lines (see [8], Proposition (4.5.1) and [9], Hoofdstuk 4, §2.1).

4. Remarks.

4.1. Similarly, one could try to define in general spherical buildings with valuation. The valuation map would then be defined on pairs of adjacent chambers. The axiom (U.5) would generalize to a linear equation in the valuation of pairs of adjacent chambers lying in an apartment (the left hand side, chambers in one half-apartment; the right hand side, chambers in the complementary half-apartment). This could possibly yield a common geometric characterization of all affine buildings.

4.2. One could include the case $n = 2$ in the conjecture by concieving buildings of reducible type $A_1 \times \tilde{A}_1$ as affine buildings of rank 3.

4.3. In stating the axioms (U.1) through (U.5), we had no intention to give the weakest possible axioms. E.g., (U.1) is certainly too strong, it is actually enough to ask surjectivety of u when restricted to one point, resp. line, of the n-gon in (U.4). In some cases (e.g. $n = 3, 4$) it is even enough to ask simply that $u/P_\perp$ and $u/L_\perp$ are surjective.

4.4. As we mentioned in the introduction, an affine building of rank 3 induces natural valuations on the generalized polygon at infinity. As for the converse, we can only say something in the cases $\tilde{A}_2$ and $\tilde{C}_2$, which are already investigated. In these cases, a generalized n-gon ($n = 3, 4$) with valuation determines a unique building of corresponding affine type (up to isomorphism). An explicit construction of this affine building is possible by introducing the coordinatizing rings (see [7], [9]), but it seems likely that this construction can be translated only in terms of the generalized n-gon with valuation. In view of the $\tilde{G}_2$-case, it would be very interesting not only to have such direct construction, but also a direct proof of it.

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