Moufang-Like Conditions for Generalized Quadrangles and Classification of All Finite Quasi-Transitive Generalized Quadrangles

Koen Thas and Hendrik Van Maldeghem

Ghent University, Department of Pure Mathematics and Computer Algebra
Galglaan 2, 9000 Gent, Belgium
E-mail: `kthas@cage.rug.ac.be`; `hvm@cage.rug.ac.be`

Abstract

We tell the history of Moufang generalized quadrangles. We review some Moufang-like conditions, and classify finite *quasi-transitive* generalized quadrangles. These are generalized quadrangles so that for any two non-concurrent lines U, V of the GQ, and for some $W \in \{U, V\}^\perp$, the group of generalized homologies with axes U and V acts transitively on the points of W not incident with U nor V.

As a by-product of the proof, we will show that a finite generalized quadrangle that admits a BN-pair of rank 2 is classical or dual classical if and only if it admits at least one nontrivial homology, without the classification of finite simple groups (!).

Key words: Moufang quadrangle, k-Moufang generalized quadrangle, elation generalized quadrangle, quasi-transitive generalized quadrangle
1991 MSC: 05B25, 05E20, 20B25, 20B27, 20E42, 51E12

1 Introduction

Recently, there has been a lot of activity in the theory of generalized quadrangles related to the Moufang condition for these structures. Indeed, in October 2002 the monograph of Tits & Weiss appeared that contains the full proof of the classification of all Moufang polygons, amongst which the case of the Moufang quadrangles is the most prominent (more than two thirds of the book is concerned with quadrangles). At the end of 2002, Katrin Tent finished a proof of the fact that any (finite or infinite) half Moufang quadrangle is automatically a Moufang quadrangle using a group theoretic lemma of Heineken. Soon afterwards, Fabienne Haot & the second author proved, using that same lemma, that the Moufang condition for quadrangles is equivalent with the 3-Moufang condition (for precise definitions, see below). When Richard Weiss pointed out a counter example to Heineken's lemma, the result of Haot and the second author could be easily saved by providing an alternative proof; the flaw in the proof of Tent was solved by the second author in February 2003. Also, in January 2003, the authors of the present paper finished the classification of finite half 2-Moufang quadrangles, resulting in another equivalent condition of the Moufang condition for finite generalized quadrangles. And last but not least, Jef Thas recently produced an entirely geometric proof of the fact that all finite translation generalized quadrangles all lines of which are regular, are classical. The latter result provides an alternative, geometric/combinatorial proof of the classification of finite Moufang generalized quadrangles, if taken together with some old results of Stanley Payne and Jef Thas. Alternative (but slightly less geometric) solutions of the last open part of that program were independently obtained by Bill Kantor in the beginning of the nineties, and by the first author in 2000.

In the present paper, we review the rich history of Moufang quadrangles, culminating in the above mentioned events. We also classify all finite quasi-transitive generalized quadrangles, thus characterizing a subclass of the class of finite Moufang generalized quadrangles. Some strong corollaries will also be obtained.

We remark that all results we review are independent of the classification of finite simple groups. (See [1] for results that can be proved *with* the classification of finite simple groups.)

2 Finite Generalized Quadrangles

A *generalized quadrangle* (GQ) of *order* (s,t), $s,t \geq 1$, is a point-line incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ in which $\mathcal{P}$ and $\mathcal{L}$ are disjoint (non-empty) sets of objects called 'points' and 'lines' respectively, and for which $\mathtt{I}$ is a symmetric point-line incidence relation satisfying the following axioms.

(1) Each point is incident with $t+1$ lines, and two distinct points are incident with at most one line.
(2) Each line is incident with $s+1$ points, and two distinct lines are incident with at most one point.
(3) If p is a point and L is a line not incident with p, then there is a unique point-line pair (q,M) such that $p\mathtt{I}M\mathtt{I}q\mathtt{I}L$.

If $s=t$, then $\mathcal{S}$ is also said to be 'of order s'. If both s and t are finite, then $\mathcal{S}$ is finite itself.

Note that there is a point-line duality for GQs of order (s,t) for which in any definition or theorem the words 'point' and 'line' are interchanged, and also the parameters.

Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p \sim q$ and say that p and q are *collinear*, provided that there is some line L so that $p\mathtt{I}L\mathtt{I}q$ (so $p \not\sim q$ means that p and q are *not* collinear). Dually, for $L, M \in \mathcal{L}$, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or *non-concurrent*. If $p \neq q \sim p$, the line incident with both is denoted by pq, and if $L \sim M \neq L$, the point which is incident with both is denoted by $L \cap M$. Two non-collinear points (non-concurrent lines) are sometimes called *opposite*. A pair of elements which are incident is sometimes called a *flag*. Two flags $\{p_1, L_1\}$ and $\{p_2, L_2\}$, with $p_1, p_2 \in \mathcal{P}$ and $L_1, L_2 \in \mathcal{L}$, are *opposite* if p_1 and p_2 are opposite and if L_1 and L_2 are opposite.

For $p \in \mathcal{P}$, put $p^\perp = \{q \in P \,|\, q \sim p\}$, and note that $p \in p^\perp$. For a pair of distinct points $\{p,q\}$, the *trace* of $\{p,q\}$ is defined as $p^\perp \cap q^\perp$, and we denote this set by $\{p,q\}^\perp$. Then $|\{p,q\}^\perp| = s+1$ or $t+1$, according as $p \sim q$ or $p \not\sim q$. A pair of opposite points p,q is called *regular* if every point collinear with at least two elements of $\{p,q\}^\perp$ is collinear with all elements of $\{p,q\}^\perp$. Dually, one defines regular pairs of opposite lines. If for a point x, all pairs of opposite points containing x are regular, then we say that x is *regular*.

A *collineation* or *automorphism* of a generalized quadrangle $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a permutation of $\mathcal{P} \cup \mathcal{L}$ which preserves $\mathcal{P}$, $\mathcal{L}$ and $\mathtt{I}$. By *Aut*$(\mathcal{S})$ we denote the full automorphism group of the GQ $\mathcal{S}$.
Two GQs $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ are said to be *isomorphic* if there are two bijective maps $\alpha : \mathcal{P} \mapsto \mathcal{P}'$ and $\beta : \mathcal{L} \mapsto \mathcal{L}'$ so that $p\mathtt{I}L$ in $\mathcal{S}$ if and only if $p^\alpha \mathtt{I}' L^\beta$ in $\mathcal{S}'$; the pair (α, β) is called an *isomorphism* of $\mathcal{S}$ (on)to $\mathcal{S}'$ (or *between* $\mathcal{S}$ and $\mathcal{S}'$). If $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, then we write $\mathcal{S} \cong \mathcal{S}'$.

We now define the finite classical GQs.
Consider a nonsingular quadric of Witt index 2, that is, of projective index 1, in $\mathbf{PG}(4,q)$ and $\mathbf{PG}(5,q)$, respectively. The points and lines of the quadric form a generalized quadrangle which is denoted by $\mathcal{Q}(4,q)$ and $\mathcal{Q}(5,q)$, respectively, and has order (q,q) and (q,q^2), respectively. Next, let $\mathcal{H}$ be a nonsingular Hermitian variety in $\mathbf{PG}(3,q^2)$, respectively $\mathbf{PG}(4,q^2)$. The points and lines of $\mathcal{H}$ form a generalized quadrangle $H(3,q^2)$, respectively $H(4,q^2)$, which has order (q^2,q), respectively (q^2,q^3). The points of $\mathbf{PG}(3,q)$ together with the totally isotropic lines with respect to a symplectic polarity of $\mathbf{PG}(3,q)$ form a GQ $W(q)$ of order q. The generalized quadrangles defined in this paragraph are the so-called *classical generalized quadrangles*, see Chapter 3 of [11].

3 The Moufang Condition and Related Conditions

In this section, we will define some conditions on collineation groups of generalized quadrangles. The general idea of the Moufang condition is that one hypothesizes the existence of a lot of collineation subgroups that fix many elements, and that act transitively on some given set. In the same spirit, we will define alternative conditions. We will review their (historical) connection in Section 4.

Throughout, we let $\mathcal{S}$ be a fixed GQ, and G is its full collineation group. An *apartment* in a GQ is a subGQ of order $(1,1)$. Fix $\ell \in \{2,3,4\}$, and let $\gamma = (x_1, \ldots, x_{\ell-1})$ be a sequence of distinct points and lines of $\mathcal{S}$ such that $x_1 \mathtt{I} \cdots \mathtt{I} x_{\ell-1}$; we call γ a *path of length* $\ell - 2$. We denote by $G_{[x_1,\ldots,x_{\ell-1}]}$, or by $G_{[\gamma]}$, the subgroup of G fixing every element incident with each of $x_1, \ldots, x_{\ell-1}$. Now choose $x_0 \mathtt{I} x_1$ and $x_\ell \mathtt{I} x_{\ell-1}$, with $x_0 \neq x_2$ and $x_\ell \neq x_{\ell-2}$. Then we say that γ is a *Moufang path* if $G_{[\gamma]}$ acts transitively on the set of apartments containing $x_0, x_1, \ldots, x_\ell$. It can be easily shown that this definition is independent of the chosen elements x_0 and x_ℓ.

If every path of length $\ell - 2$ is a Moufang path, then we say that $\mathcal{S}$ is an ℓ-Moufang GQ. For $\ell \in \{2,4\}$, there are two types of paths of length ℓ: those starting and ending with points, and those starting and ending with lines. If every path of one fixed type is a Moufang path, then we call $\mathcal{S}$ *half ℓ-Moufang*.

For $\ell = 4$, the elements of $G_{[x_1,x_2,x_3]}$ are called *root elations* (they are a special kind of elations, see below). Also, usually (half) 4-Moufang is simply referred to as (half) Moufang. It is straightforward to see that, for $\ell \in \{3,4\}$, the ℓ-Moufang condition implies the $(\ell-1)$-Moufang condition.

A special kind of root elation is one which fixes all lines not opposite a given line L, or all points not opposite a given point x. This is usually called a *symmetry* (about L or about x). The line L is called an *axis of symmetry* if for some distinct points u, v on L, the path (u, L, v) is Moufang and the group $G_{[u,L,v]}$ is a group of symmetries about L. The latter is known to be equivalent with L being a regular line. Dually, one defines *centers of symmetry*.

The following is clearly a generalization of the Moufang condition. Let (x_0, x_1, x_2, x_3) be a path of length 3. Then we say that the path (x_1, x_2, x_3) is a *pseudo-Moufang path* if the stabilizer G_{x_1,x_3} of x_1 and x_3 in G contains a normal subgroup $G(x_1, x_3) \lhd G_{[x_1]} \cap G_{[x_3]}$ acting regularly on the elements incident with x_0 but distinct from x_1. This definition is independent of the choice of x_0. We call $\mathcal{S}$ *pseudo-Moufang* if all paths of length 2 are pseudo-Moufang. We call $\mathcal{S}$ *half pseudo-Moufang* if all paths of a fixed type are pseudo-Moufang.

Let L, M be two opposite lines of $\mathcal{S}$. Then we say that $\mathcal{S}$ is (L, M)-*transitive* if, for some point x on L, the subgroup $G_{[L,M]}$ of G fixing every point on L and every point on M acts transitively on the set of lines through x distinct from L and opposite M. Clearly this definition is independent of the choice of x on L or M. Let N be the line through x concurrent with M; then we say that $\mathcal{S}$ is (L, M)-*quasi-transitive* if $G_{[L,M]}$ acts transitively on the set of points incident with N, but neither on L nor M. If $\mathcal{S}$ is (L, M)-quasi-transitive for all pairs of opposite lines L, M, then we say that $\mathcal{S}$ is *quasi-transitive* (w.r.t. lines). Dually, we have the obvious definitions and notations. The elements of $G_{[L,M]}$ are called *generalized homologies* with *axes* L and M. (Dually, one speaks of generalized homologies with 'centers'.)

Let x be a point of $\mathcal{S}$. An *elation about* x is a collineation in $G_{[x]}$ that either is the identity, or acts fixed point freely on the set of points opposite x. If there is a group E of elations about x acting transitively (and then regularly) on the set of points opposite x, then we call x an *elation point* (with respect to E). If E is moreover abelian, then we say that x is a *translation point*. In that case, every line through x is regular. If $\mathcal{S}$ contains two opposite elation points, then all points are elation points and $\mathcal{S}$ is called a *strong elation GQ*.

Finally, we consider the following almost purely group-theoretic condition. Suppose that G acts transitively on the set of all apartments of $\mathcal{S}$, and that the stabilizer $N := G_\Sigma$ of any apartment Σ acts transitively on the set of eight flags of Σ. Suppose moreover that the stabilizer G_F of some flag F contained in Σ has a normal subgroup U which acts transitively on the set of flags opposite F. If U is moreover nilpotent, then we say that $\mathcal{S}$ is a *split* GQ. This terminology stems from the fact that, in the literature, the corresponding group G is called a *split BN-pair of type* $\mathbf{B}_2$.

4 History and Background

The notion (or at least its origin) of generalized quadrangle and of Moufang quadrangle must of course be seen within the theory of generalized polygons. The origin of the notion of generalized quadrangle lies in its connection with so-called *Chevalley groups of rank 2*. It was Jacques Tits who wanted to construct new infinite classes of (finite) simple groups by considering them as automorphism group of certain geometries. In the "easiest" cases, namely the cases of "low rank", Tits [29] succeeded in constructing the known groups of type $\mathbf{G}_2$ (Dickson's groups) and the (at that moment new) groups of type ${}^3\mathbf{D}_4$ using a geometry which he called a *generalized hexagon*. This was in 1959 — or at least, then the paper appeared. The geometries that Tits had in mind were what later on would be called *(spherical) buildings*. Generalized polygons, and in particular generalized quadrangles, were examples of low rank.

Tits, however, observed that the situation was not completely satisfying; he was able to construct generalized quadrangles the automorphism group of which had nothing to do with simple groups (a situation comparable with the situation in the theory of projective planes, which can also be viewed as buildings — generalized triangles, in fact). These examples appeared in Dembowski's book [2]. At the same time, Tits was still convinced that, if the rank of a spherical building is at least three, then it should be essentially equivalent with the notion of certain simple Chevalley-like groups — more precisely, classical groups, algebraic groups and groups of mixed type — of the same rank. In fact, the classification of spherical buildings of rank ≥ 3 by Tits [30] established precisely this relationship.
Although the Lecture Notes [30] only appeared in 1974, Tits already finished that work long before that. In fact, he was already working on an alternative proof when [30] appeared. And this alternative proof was the motivation and the starting point of defining the notion of Moufang polygon.

First we remark that every building is made up of a lot of generalized polygons, which could be called the *bricks* of the building. Then, the central idea in Tits' classification of spherical buildings of rank ≥ 3 is his famous Extension Theorem that implies the existence of a lot of automorphisms of a certain type. In particular, it implies that every brick is a Moufang polygon. This observation is not an *ad hoc* one. It follows naturally from the theory of simple algebraic groups of relative type, where it was known that these are generated by certain subgroups ("root groups") that satisfy the so-called *Steinberg relations*. In the case of a quadrangle (or polygon), these root groups are nothing else than our groups of root elations, and the Steinberg relations, which are conditions on commutators (in particular stating that each root group is nilpotent of class at most 2), imply that the corresponding quadrangle (or polygon) is split. Tits could derive the Steinberg relations from the Moufang condition (see [33]; this paper appeared in 1994, but the result was known to Tits already in the sixties), and he conjectured that all Moufang polygons were related to

(1) either classical, algebraic or mixed type groups, or
(2) to the Ree groups in characteristic 2, which are the only rank ≥ 2 groups of Chevalley type not belonging to those three classes (others of rank one include the Suzuki groups, the Ree groups in characteristic 3, and a new class of groups discovered by Mühlherr and the second author (unpublished) — these three classes of rank 1 all arise from polarities in Moufang polygons).

An explicit list of all possibilities was given by Tits in [31]. The motivation to classify the Moufang polygons was a possible alternative proof for the classification of spherical buildings of rank ≥ 3. This alternative proof, now outlined in [34], starts with the list of Moufang polygons and uses some elementary arguments to decide whether a given Moufang polygon can be a brick in a higher rank spherical building. In this way, one can classify these buildings in a much shorter and elegant way than before.

1976

Conjecture (J. Tits [31]). *Every Moufang generalized quadrangle is related to a classical group, an algebraic group or a group of mixed type, and an explicit list can be given.*

1994

Theorem (J. Tits [33]). *Every Moufang quadrangle satisfies the Steinberg commutation relations and hence is a split quadrangle.*

Already in the sixties, Tits classified the Moufang hexagons. In the seventies, the octagons followed. Since the Moufang triangles — that is, the Moufang projective planes — were already dealt with long before the notion of generalized polygon was even introduced, only the case of Moufang GQs remained. Clearly, this was the hardest case.

In the meantime throughout the seventies, finite generalized quadrangles were becoming, under the impulse of Stanley Payne and Jef Thas, interesting and useful research objects in their own right. Hence there was a need for a list of all finite Moufang quadrangles, and for a proof that that list was complete. Tits [31] remarked that such a list and a proof followed from group-theoretic work of Fong & Seitz [4,5]. In the latter two (enormous) papers, the finite split BN-pairs of rank 2 are classified — no reference is made to the geometry. Translating the results to geometry yields a classification of finite Moufang GQs — they are exactly the classical GQs and their duals.

1973,1974

Theorem (P. Fong & G. M. Seitz [4,5]). *The following are equivalent for a finite generalized quadrangle $\mathcal{S}$:*
(1) *$\mathcal{S}$ is a Moufang generalized quadrangle;*
(2) *$\mathcal{S}$ is a split generalized quadrangle, and*
(3) *$\mathcal{S}$ is a classical or dual classical quadrangle.*

The finite case is so special here because Fong and Seitz could use the classification of finite split BN-pairs of rank 1, a group theoretic result which at the moment no-one believes to have an infinite counterpart. The proof of Fong & Seitz is a very untransparent one, certainly not satisfying finite geometers. Driven by success considering local Moufang-like conditions, finite geometers wanted to produce a geometric proof of the classification of finite Moufang GQs. An almost complete geometric proof of the classification was produced by Payne and Jef Thas, and written up in Chapter 9 of [11]. One of the essential ingredients is 'Property (H)', defined as follows. Suppose p and q are two non-collinear points of the GQ $\mathcal{S} = (P, B, \mathtt{I})$. Then we put $cl(p,q) = \{z \in \mathcal{S} \parallel z^\perp \cap \{p,q\}^{\perp\perp} \neq \emptyset\}$. A point x has *Property (H)* provided that $r \in cl(p,q)$ if and only if $p \in cl(q,r)$ whenever (p,q,r) is a triad of points in $x^\perp$.

Now suppose that $\mathcal{S}$ is a thick finite Moufang quadrangle. Then each point and each line satisfies Property (H). Now comes the crucial observation (cf. 5.6.2 of [11]), which was first observed by Jef Thas:

Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, for which each point satisfies Property (H). Then we have the following three possibilities for $\mathcal{S}$ (and conversely):

(1) $\mathcal{S} \cong H(4,s)$;
(2) *each span of non-collinear points has size* 2;
(3) *each point is regular.*

Since each finite Moufang quadrangle satisfies Property (H) for both its points and lines, the latter observation and its dual can then be applied, and each of the obtained cases is studied separately. The case where each span of non-collinear points has size 2 and each span of non-concurrent lines has size 2 can be excluded, so that up to duality and besides the possibilities $H(4,s)$ and its dual, one has to classify Moufang quadrangles with each line regular. This motivated Stanley Payne and Jef Thas to study the following more general type of GQs: GQs for which there is a point x so that each panel and each dual panel containing x is Moufang, and such that each line incident with x is regular. It can be shown that if $G(x)$ is the group generated by all these root elations, then $G(x)$ fixes x linewise, it acts regularly on the points opposite x and it is an abelian group, and the converse also holds. So x is a translation point. These GQs were called *translation generalized quadrangles* (TGQs), and soon became objects of study in their own right, because of their rich connections with certain combinatorial objects in projective spaces (called 'generalized ovoids' and 'generalized ovals'), with 'flocks' of the quadratic cone (these are partitions of the quadratic cone in $\mathbf{PG}(3,q)$ minus its vertex into disjoint irreducible conics) that arise from semifields, and with certain types of translation planes (and the corresponding spreads). Payne and Jef Thas almost completely finished their program before 1984, up to one open case:

PROBLEM. *Classify all finite Moufang GQs of order (s,s^2), $s > 1$, each line of which is regular.*

These GQs hence are TGQs for each of their points. Bill Kantor came up with a solution of the problem in 1991 [8], using the classification of finite split BN-pairs of rank 1 [13,7], and 4 B,C of Fong and Seitz [4]. The first author then found a second approach in 2000 (as a corollary of a much stronger result — he studied GQs which have two non-concurrent axes of symmetry), using *only* the classification of finite split BN-pairs of rank 1, and thus providing the first independent proof of the classification of finite Moufang quadrangles, see the appendix of [22] and also the monograph [27]. Recently, Jef Thas has also completed a solution of the problem, only using (advanced) projective geometry, cf. [18].

In the beginning of the nineties, Jef Thas and the second author put together a project that should have lead to the GQ counterpart of the Lenz-Barlotti classification in the theory of projective planes. The first question to handle was to see which configurations of Moufang paths of length 2 imply the Moufang condition (this is the Lenz-part of the program). Almost naturally, the half Moufang condition was born. Together with Stanley Payne, Jef Thas and the second author proved that, in the finite case, the half Moufang condition implies the Moufang condition. The starting point in the proof was again Property (H). The rest of the project was put in the freezer and eventually passed on to the first author, which resulted eventually in the research monograph [27] (for a survey on the description of that classification, see [26]).

> 1991
> **Theorem (J. A.Thas, S. E. Payne & H. Van Maldeghem [19]).** *Every finite half Moufang generalized quadrangle is a Moufang quadrangle.*

The second part of the program is the "Barlotti part" of the classification, consisting of determining the possible configurations of pairs of non-collinear points $\{x, y\}$ in a GQ $\mathcal{S}$ for which $\mathcal{S}$ is $\{x, y\}$-transitive. Jef Thas classified the finite GQs which are $\{x, y\}$-transitive for *each* such pair $\{x, y\}$.

> 1985, 1986
> **Theorem (J. A. Thas [16,17]).** *Every thick finite generalized quadrangle which is (x, y)-transitive for every pair of opposite points $\{x, y\}$, is a classical quadrangle, i.e. is isomorphic to one of $W(q)$, $\mathcal{Q}(4, q)$, $\mathcal{Q}(5, q)$, $H(3, q^2)$ or $H(4, q^2)$, and conversely.*

The method of proving this theorem consists of observing that each point and line has Property (H), and then handling all the combined cases separately. Every case is handled in a pure combinatorial fashion, except one: the case where $s = t^2$ and each point is regular. Here, it is shown that each point is a center of symmetry, so that the GQ is Moufang.

Around the nineties, the second author of this paper introduced the "k-Moufang condition" for generalized polygons. The motivation here was that for $k = 3$, the 3-Moufang condition implies a Desargues condition (a direct generalization of Desarguesian projective planes). Payne, Jef Thas and the second author prove in [37] the equivalence between finite 3-Moufang GQs and Desarguesian GQs, and also between finite 3-Moufang GQs and Moufang GQs, again using Property (H).

1992
Theorem (H. Van Maldeghem, J. A. Thas & S. E. Payne [37]). *Every finite 3-Moufang generalized quadrangle is a Moufang quadrangle.*

The fact that Property (H) was used in the above proofs and that there was no counterpart for it in the infinite case, discouraged some people to consider the infinite case regarding half Moufang GQs and 3-Moufang GQs.

In 1998, the second author showed that the 2-Moufang condition for GQs is equivalent to the 3-Moufang condition for GQs (finite or infinite). Consequently, finite 2-Moufang GQs are Moufang GQs. This, in turn, implies that every finite strong elation GQ which is also a dual strong elation GQ is classical or dual classical (and conversely). So at that moment, there were two problems left in the finite case (regarding the Moufang-like conditions stated above):

(1) *the classification of finite half 2-Moufang GQs (with as a particular case the strong elation GQs)*;
(2) *the classification of finite quasi-transitive GQs.*

For quite a while, problem (1) seemed to be too hard to tackle, while for problem (2) one even cannot apply Property (H).

1998
Theorem (H. Van Maldeghem [36]). *Every 2-Moufang generalized quadrangle is a 3-Moufang generalized quadrangle.*

Corollary. *Every finite 2-Moufang generalized quadrangle is a Moufang quadrangle. Every finite strong elation generalized quadrangle which is also a dual strong elation generalized quadrangle is a Moufang quadrangle.*

For infinite GQs, the situation was much more open. In fact, it was only in 1997 that all Moufang quadrangles were finally classified (in particular providing a third independent proof for the finite Moufang GQs!). The original conjecture of Tits — every Moufang quadrangle arises from a classical, algebraic or mixed type group — appeared to be true, but in his explicit list, an infinite class of examples was overlooked. Richard Weiss discovered and produced the precise Steinberg-type relations for the new example, and Bernhard Mühlherr and the second author [10] showed that it arises from a mixed group of exceptional type $\mathbf{F}_4$.

2002

Theorem (J. Tits & R. Weiss [34]). *Every Moufang generalized quadrangle is related to a classical group, an algebraic group or a group of mixed type, and an explicit list can be given.*

In [3], De Medts has introduced a uniform algebraic structure for Moufang quadrangles, and showed that every Moufang quadrangle can be parametrized by such a structure, called *quadrangular system*. He then classifies these systems without going back to the Moufang quadrangle from which they arise, thus obtaining an independent proof of the classification of Moufang quadrangles.

The explicit classification of all Moufang GQs was the start of a series of results generalizing all characterizations (considered in this paper) of the finite case. The first natural question was whether one could take the opposite direction and prove that every split GQ is a Moufang GQ. This would not only classify all split GQs (and all split BN-pairs of type $\mathbf{B}_2$) in the general case, it would also produce in the finite case a revision of the proof of Fong & Seitz. Katrin Tent and the second author settled this question in 2002 by showing that in the general case, "split" indeed implies "Moufang" for GQs.

2002

Theorem (K. Tent & H. Van Maldeghem [15]). *Every split generalized quadrangle is a Moufang quadrangle.*

The method of proof of the previous theorem inspired first Katrin Tent [14] at the end of 2002 to show that, in the general case, half Moufang implies Moufang for GQs, and then Fabienne Haot and the second author in the beginning of 2003, to prove that also for infinite GQs, 3-Moufang implies Moufang, which, together with the old result of the second author, also proves that 2-Moufang implies Moufang for GQs. However, both proofs (the one by Tent and the one by Haot and the second author) used a lemma by Heineken, which turned out to be wrong; Richard Weiss came up with a counter example in 2003 (private communication). Both proofs were repaired by the second author on the occasion of a course on Moufang quadrangles in Ghent with co-lecturers Jef Thas and the first author, cf. [20].

Hence, *except for the (L, M)-transitive GQs (for every pair of opposite lines L, M), the situation in the infinite case is levelled with the finite case:*

2003

Theorem (K. Tent [14]). *Every half Moufang generalized quadrangle is a Moufang quadrangle.*

2003

Theorem (F. Haot & H. Van Maldeghem [6]). *Every 3-Moufang generalized quadrangle is a Moufang quadrangle.*

Corollary. *For generalized quadrangles, the notions of 2-Moufang, 3-Moufang and Moufang are equivalent. In particular, every strong elation generalized quadrangle which is also a dual strong elation generalized quadrangle is a Moufang quadrangle, and every generalized quadrangle which is (x, y)-transitive for all opposite pairs of points and lines x, y is a Moufang quadrangle.*

However, in December 1999, the first author introduced the (half) pseudo-Moufang condition in the context of strong elation GQs [21], and some time later made a precise connection (together with first classification results) with split BN-pairs of rank 1. In 2001, the authors then managed to classify all finite half pseudo-Moufang GQs, and noticed that the classification of strong elation GQs follows almost immediately from the classification of finite half pseudo-Moufang GQs (after the observation of the first author in 1999). In fact, the classification of all finite half 2-Moufang GQs appeared also to be within reach, up to one case (where the GQ $\mathcal{S}$ in question contains a subGQ isomorphic to $\mathcal{Q}(5, q)$ and has itself order (s, q^2); every point of $\mathcal{S}$ is a Moufang path of length 0). It took quite some time to handle that case, but it was finally dealt with in January 2003. The global proof is a mixture of combinatorial, group-theoretic and geometric arguments covering a wide variety of techniques (and a large number of pages!). It makes us conjecture that maybe the classification of infinite half 2-Moufang GQs is not yet within reach.

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Theorem (K. Thas & H. Van Maldeghem [28]). (1) *Every finite half pseudo-Moufang generalized quadrangle is a Moufang quadrangle.*
(2) *Every finite half 2-Moufang generalized quadrangle is a Moufang quadrangle.*

Corollary. *Every finite strong elation generalized quadrangle is a Moufang quadrangle.*

5 Complete Elation Groups in GQs

We remarked earlier that the set of elations about a certain point is not necessarily a group of elation. This was for a long time an open problem, until Payne and the first author in [12] came up with a non-classical example of this situation; they show that the Kantor-Knuth semifield GQs of order (q^2, q) admit more than q^5 distinct elations about some special point (∞), and construct a concrete example of a nontrivial collineation θ which is an elation about (∞), so that there is some natural number n for which $\theta^n \neq \mathbf{1}$ is not an elation about (∞). This motivates the concept of 'standard elation' (in general GQs): a *standard elation* ϕ about x is an elation about x so that $\langle\phi\rangle$ is a group of elations (about x). Payne and the first author then show that for flock GQs of order (q^2, q), the set of standard elations about (∞) is always a group of size q^5.

We now show that already in the classical case, there are elations which are not contained in an elation group.

Indeed, consider the quadrangle $\mathcal{S}$ isomorphic to either $H(3, q^2)$ or the dual of $H(4, q^2)$, denote its order by (s, t) and its full automorphism group by G. Fix a point x. Then x is an elation point with respect to some group E. But the full automorphism group contains a nontrivial element fixing some subquadrangle of order (s', t) pointwise with $1 < s' < s$ (if $\mathcal{S} \cong H(3, q^2)$, this subGQ is isomorphic to $W(q)$ and this element has order 2; if $\mathcal{S}$ is isomorphic to the point-line dual of $H(4, q^2)$, then this subGQ is isomorphic to $\mathcal{Q}(5, q)$, and there is an automorphism group of size $q+1$ fixing $\mathcal{Q}(5, q)$ pointwise), and hence we may assume that this element belongs to $G_{[x]}$. Consequently, $|G_{[x]}| > s^2t$. Now a result of Burnside says that $|G_{[x]}|$ is equal to s^2t (the number of fixed points of the identity) plus $(|G_{[x]}| - s^2t)$ times the average number a of fixed points opposite x of an element of $G_{[x]} \setminus E$. Hence $a = 1$. But there is at least one element fixing a subquadrangle of order (s', t) pointwise, hence fixing $s'^2t > 1$ elements opposite x. As a result $G_{[x]} \setminus E$ contains at least $s'^2t - 1$ elations.

6 The Classification of Finite Quasi-Transitive Generalized Quadrangles

We now classify all finite quasi-transitive GQs:

Theorem I. *A thick finite generalized quadrangle of order (s, t) is quasi-transitive w.r.t. lines if and only if it is isomorphic to one of the following:*

(1) *$\mathcal{Q}(4, s)$ with s even;*

(2) $W(s)$ *for any* s;
(3) $\mathcal{Q}(5, s)$ *for any* s.

Proof. Let $\mathcal{S} = (P, B, \mathtt{I})$ be a quasi-transitive GQ (w.r.t. lines) of order (s, t), $s \neq 1 \neq t$. First of all, one notes that $Aut(\mathcal{S})$ acts transitively on the lines. For, let U and V be non-concurrent lines of $\mathcal{S}$. Let U', V' be distinct lines of $\{U, V\}^\perp$, and let W, W' be distinct lines of $\{U', V'\}^\perp \setminus \{U, V\}$. Then there is an element of $G_{[W,W']}$ mapping U onto V. Clearly, transitivity on lines follows. Now let L be an arbitrary but fixed line, and suppose $H = \langle G_{[L,L']} \parallel L' \notin L^\perp \rangle$. If $H \leq H_{[L]}$ acts transitively on $B \setminus L^\perp$ (that is, if H is an *axis of transitivity*), it follows that $\mathcal{S}$ is half 2-Moufang, and hence classical or dual classical by the result of the authors of [28]. Also, if for any point $x \not\mathtt{I} L$, H_x acts transitively on the lines incident with x and different from $proj_x L$, then L is an axis of transitivity, so we suppose that l is a point not on L for which this is not the case.

Let $M \mathtt{I} x$, $M \neq proj_x L$. Let p be a prime that divides $s - 1$. As $G_{[L,M]}$ acts transitively on the set X of points of $proj_x L$ different from $L \cap proj_x L$ and $M \cap proj_x L$, it follows that if K is a Sylow p-subgroup of $G_{[L,M]}$, K cannot fix all points of X. Hence, K cannot fix a line incident with x and different from $proj_x L$ and M. Whence $t \equiv 1 \mod p$.
We suppose H_x does not act transitively on the set Y of lines incident with x and different from $proj_x L$ and M, so that there are at least two distinct H_x-orbits in Y, say O_1 and O_2. Let $N \in O_1$. Then $G_{[L,N]}$ acts on O_2, and in the same way as above, $|O_2| \equiv 0 \mod p$. Now the latter clearly holds for each H_x-orbit in Y, so that $t \equiv 0 \mod p$, contradiction. Whence H_x acts transitively on Y, and L is an axis of transitivity. So $\mathcal{S}$ is half 2-Moufang, thus classical or dual classical by the result of the authors.

We now have a look at which classical and dual classical GQs are indeed quasi-transitive w.r.t. lines. First of all, suppose $\mathcal{S}$ is a quasi-transitive GQ (w.r.t. lines) of order (s, t), where $1 < t < s$. Let $L \not\sim M$ be arbitrary lines, and suppose $U \in \{L, M\}^\perp$. Let $G = Aut(\mathcal{S})$ be the automorphism group of $\mathcal{S}$. Then there is a nontrivial element θ of $G_{[L,M]}$ that fixes some line $V \mathtt{I} (U \cap M)$, $M \neq V \neq U$, and θ fixes some subGQ of $\mathcal{S}$ of order (s, t'), $t' > 1$, elementwise. This contradicts 2.2.2 of [11], so $s \leq t$. It follows that $H(3, q^2)$ and $H(4, q^2)^D$ are not quasi-transitive. We look at the remaining cases, and suppose w.l.o.g. that L and M are as above.

(1) $\mathcal{Q}(4, q)$.
If q is even, then L and M are axes of symmetry, and by [9,25], the group generated by the symmetries about L and M acts 3-transitively on $\{L, M\}^{\perp\perp}$. Whence $\mathcal{Q}(4, q)$ is quasi-transitive for q even. If $\mathcal{Q}(4, q)$ would be quasi-transitive for q odd, it would follow that the stabilizer of the $(q + 1) \times (q + 1)$-grid Γ defined by $\{L, M\}^\perp \cup \{L, M\}^{\perp\perp}$ in the full

automorphism group of $\mathcal{Q}(4,q)$, acts transitively on the triples of distinct mutually non-collinear points contained in Γ, contradicting 5.2.6 of [11] and the fact that $\mathcal{Q}(4,q) \not\cong W(q)$ if q is odd.

(2) $W(q)$.
If q is even, then $W(q)$ is quasi-transitive by the previous case (recall that as q is even, $W(q) \cong \mathcal{Q}(4,q)$). Suppose q is odd, and let $\{L,M\}$ be an arbitrary pair of non-concurrent lines of $W(q)$. Then the group of generalized homologies with axes L and M has size $q-1$ — this follows from the fact that $W(q)$ is $\{U,U'\}$-transitive for each such pair $\{U,U'\}$. Suppose that $W(q)$ is not quasi-transitive; then there must be a nontrivial element ϕ of $G_{[L,M]}$ fixing some point of U not on L nor M, where $U \in \{L,M\}^{\perp}$ is arbitrary. But then it easily follows that ϕ fixes $W(q)$ pointwise, as each span of non-concurrent lines has size 2. Thus $W(q)$ is quasi-transitive *for any* q.

(3) $\mathcal{Q}(5,q)$.
If q is even, it follows that $\mathcal{Q}(5,q)$ is quasi-transitive since $\mathcal{Q}(4,q)$ ($\subset \mathcal{Q}(5,q)$) is, and since the automorphism group of $\mathcal{Q}(4,q)$ extends to an automorphism group of $\mathcal{Q}(5,q)$. Let q be odd, and let $\{L,M\}$ be as usual. We remark that for each $\mathcal{Q}(4,q)$ in $\mathcal{Q}(5,q)$, there is a unique involution θ of $\mathcal{Q}(5,q)$ fixing $\mathcal{Q}(4,q)$ pointwise, and if L,M are lines of $\mathcal{Q}(4,q)$, then $\theta \in G_{[L,M]}$. There are $q+1$ distinct subGQs $\mathcal{Q}(4,q)$ containing $\{L,M\}^{\perp\perp}$. As $\mathcal{Q}(5,q)$ is $\{L,M\}$-transitive it now follows easily that $|G_{[L,M]}| = 2(q^2-1)$. Suppose by way of contradiction that $\mathcal{Q}(5,q)$ is not quasi-transitive. As $G_{L,M}$ contains the natural action of $\mathbf{PSL}(2,q)$ on $\{L,M\}^{\perp\perp}$ (see, e.g., [9,25] or [26]), we have that $|[G_{[L,M]}]_U| = 4(s+1)$, with $U \in \{L,M\}^{\perp\perp} \setminus \{L,M\}$. Suppose $\Pi = \mathbf{PG}(5,q)$ is the underlying space of $\mathcal{Q}(5,q)$. As $G_{[L,M]} \subset \mathbf{PGL}(6,q)$ (since L and M are pointwise fixed), $[G_{[L,M]}]_U$ fixes each line of $\{L,M\}^{\perp\perp}$, and hence each point on each line of $\{L,M\}^{\perp\perp}$. Suppose that x is a point of $\mathcal{Q}(5,q)$ not on a line of $\{L,M\}^{\perp\perp}$, and let Ω be the set of points on the lines of $\{L,M\}^{\perp\perp}$. Then each point of $x^{[G_{[L,M]}]_U}$ is collinear with each point of $x^{\perp} \cap \Omega$. It follows that $|x^{[G_{[L,M]}]_U}| \leq q+1$, as $\mathcal{Q}(5,q)$ is a GQ of order (q,q^2) — see [11, 1.2.4]. Hence the stabilizer K of x in $[G_{[L,M]}]_U$ has at least size 4. But K fixes a $\mathcal{Q}(4,q)$-subGQ pointwise, and there is only a unique involution fixing that $\mathcal{Q}(4,q)$ pointwise, contradiction. Hence $\mathcal{Q}(5,q)$ is quasi-transitive *for any* q.

(4) $H(4,q^2)$.
Let Γ be an arbitrary ordinary quadrangle in $H(4,q^2)$. Then there is a unique subGQ $H(3,q^2)$ which contains Γ. Suppose $L \not\sim M$ are lines of Γ, and suppose $H(4,q^2)$ is quasi-transitive. Then as $H(3,q^2)$ is not, it would follow that Γ is contained in more than one $H(3,q^2)$, contradiction. Thus $H(4,q^2)$ is not quasi-transitive.

The theorem follows. ■

The following theorem immediately follows from the proof of the previous theorem:

Theorem II. *Let $\mathcal{S}$ be a thick GQ of order (s,t). Suppose there is a prime p (that not necessarily divides $s-1$) so that for each two non-concurrent lines L, M of $\mathcal{S}$, some Sylow p-subgroup of $Aut(\mathcal{S})_{[L,M]}$ does not fix all points on some line of $\{L, M\}^{\perp}$. Then $\mathcal{S}$ is classical or dual classical, and conversely.* ■

A famous conjecture of Tits of 1974, see [30, p. 221], states that *all finite generalized polygons having a group acting transitively on the pairs (Σ, C), where Σ is an apartment and C a chamber contained in Σ, arise from absolutely simple algebraic groups over a finite field, or from Ree groups ${}^2\mathbf{F}_4(2^e)$, e an odd non-negative integer*. In other words,

CONJECTURE (J. TITS, 1974). *All generalized polygons associated with a finite BN-pair are classical or dual classical.*

In [1], Buekenhout and the second author solved that conjecture using the classification of finite simple groups. Without that classification, the question rested hopelessly wide open. Here we will 'almost completely' solve that conjecture for quadrangles *without* the classification of finite simple groups.

Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$. A collineation θ is said to be a *homology* with *centers* x and y, if it fixes the points x and y of $\mathcal{S}$ linewise, and if its set of fixed points is precisely $\{x, y\} \cup \{x, y\}^{\perp}$, or if it is the identity.

Theorem III. *A thick finite generalized quadrangle is classical or dual classical if and only if it admits a BN-pair and a nontrivial homology.*

Proof. If the GQ is classical or dual classical, the theorem is well-known to hold. Suppose conversely that $\mathcal{S}$ is as in the statement of the theorem. Then $Aut(\mathcal{S})$ acts transitively on the set of ordered ordinary subquadrangles of $\mathcal{S}$. Let Γ be such a quadrangle, and let $L \not\sim M$ be sides of Γ. We may assume w.l.o.g. that there is a nontrivial homology ϕ in $Aut(\mathcal{S})_{[L,M]}$. Then $\langle \phi \rangle$ does not fix all points of any line of $\{L, M\}^{\perp}$, and hence there is a prime p for which some Sylow p-subgroup of $\langle \phi \rangle$ has that same property. Then the assumptions of Theorem II are satisfied for p, and the theorem follows. ■

In view of Theorem II, it is clear that one could formulate Theorem III slightly more generally, but less elegantly.

Final Remark. The authors of this paper are presently trying to delete the extra assumption in Theorem III, to obtain the complete solution of Tits' conjecture for quadrangles.

Acknowledgements. The first author is a Postdoctoral Fellow of the Fund for Scientific Research — Flanders (Belgium).

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