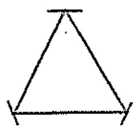
AUTOMORPHISMS OF NON-CLASSICAL TRIANGLE BUILDINGS

H. Van Maldeghem *

We study the relation between the automorphism group of a triangle building and those of its spherical building at infinity. As an application we exhibit a (locally finite) triangle building with non-classical residues and vertex-transitive automorphism group.

INTRODUCTION.

A triangle building is an affine building of type $\tilde{A}_2$ and has Buekenhout [4] diagram



A triangle building is said to be classical if it arises from an algebraic group over a local field (see Bruhat-Tits [3]). In fact, all affine buildings of rank ≥ 4 are classical in this sense (see Tits [10]). Non-classical examples of triangle buildings exist in large classes (see e.g. Ronan [8] and the author [11]). In [10], Tits introduces the (spherical) building at infinity for any affine building. In case of a triangle building Δ, this is the building associated to a pair of mutually dual projective planes. Denoting one of them (by an arbitrary choice) by PG(Δ), the latter is Moufang if Δ is classical. In general, PG(Δ) can be viewed as the inverse limit of an infinite sequence of epimorphic projective Hjelmslev planes $(V_n, \Pi_n^{n+1})_{n\in\mathbb{N}}$ (where V_n is of level n). In these planes, there is a notion of (P^n, L^n)-perspectivity and

* The research of the author is supported by the National Fund for Scientific Research (N.F.W.O.) of Belgium.

(P^n, L^n)-transitivity for point-line pairs (P^n, L^n) of V_n. We study the relation between these perspectivities and those of $PG(\Delta)$. We show that every perspectivity τ_∞ in $PG(\Delta)$ defines a unique automorphism τ in Δ inducing τ_∞ in $PG(\Delta)$. We investigate the action of τ on the set of vertices of Δ. Then we will be able to prove that the automorphism group of the triangle building Δ_D associated to the division ring $D((t))$ of formal Laurent series over an arbitrary division ring D acts vertex transitively on Δ_D.

1. TRIANGLE BUILDINGS.

1.1. The standard apartment.

Denote by A the real Euclidean plane provided with a tessellation $\mathcal{T}$ of congruent regular triangles. Suppose typ_A is a type map from the set of vertices of all these triangles to the set $\{1,2,3\}$ such that each triangle has vertices of each type (up to permutation of $\{1,2,3\}$, there is a unique way of doing this). Now let T be an arbitrary such triangle with vertices P_1, P_2, P_3 and suppose $typ_A(P_i) = i$, $i=1,2,3$. We consider T as the convex closure (in the usual Euclidean sense) of $\{P_1, P_2, P_3\}$. Denote by h_1 (resp. h_2) the closed half line bounded by P_3 and containing P_1 (resp. P_2). Let L_1 (resp. L_2) be the affine line supporting h_1 (resp. h_2). Let H_1 (resp. H_2) be the closed half plane bounded by L_1 (resp. L_2) and containing T. Denote by $\overline{W}$ the group of all orthogonal transformations of A stabilizing $\mathcal{T}$. Let $w \in \overline{W}$ be arbitrary. We call $w(P_1)$ (resp. $w([P_1, P_2])$, $w(T)$, $w(h_1)$, $w(L_1)$, $w(H_1 \;\; H_2)$, $w(H_1)$ a vertex (resp. a panel, a chamber, a panel of a quarter (with source $w(P_1)$), a wall, a quarter (with source $w(P_1)$), a half apartment). A face is a common name for vertex, panel and chamber. We abbreviate "panel of a quarter" to pennel, as in [13].

The tessellation $\mathcal{T}$ defines a triangulation $\mathcal{T}(A)$ of the topological space A in the natural way. The vertices of $\mathcal{T}(A)$ are the vertices of $\mathcal{T}$, a 1-

simplex is any set of vertices on a common panel and a 2-simplex is any set of vertices on a common chamber. In this manner, we obtain the Coxeter complex Σ of irreducible type $\tilde{A}_2$. A panel (resp. a chamber, a pennel (with source P), a wall, a quarter (with source P), a half apartment, a face) of Σ is by definition the set of vertices lying on some panel (resp. chamber, etc...) in A. The corresponding type map is now denoted by typ_{Σ}. Most of the above definitions are standard concepts (see N.Bourbaki [2]).

1.2. Buildings of type $\tilde{A}_2$.

Suppose Δ is a simplicial complex and suppose $\mathcal{A}$ is a set of subcomplexes of Δ. Suppose all elements of $\mathcal{A}$ are isomorphic copies of Σ. Then $(\Delta,\mathcal{A})$ is called a building of type $\tilde{A}_2$ (or a triangle building) and the elements of $\mathcal{A}$ are called apartments if $(\Delta,\mathcal{A})$ satisfies (B.1), (B.2) and (B.3).

(B.1) Δ is thick, i.e. every panel is contained in at least three distinct chambers.

(B.2) Every two simplices of Δ belong to a common apartment.

(B.3) If $\Sigma,\Sigma'\in\mathcal{A}$, there exists an isomorphism of Σ onto Σ' which fixes $\Sigma\cap\Sigma'$ elementwise. (see Tits [9]).

Let $(\Delta,\mathcal{A})$ be a triangle building. We call $\mathcal{A}$ a maximal set of apartments for Δ if (B.4) holds.

(B.4) If $(\Delta,\mathcal{A}')$ is also a triangle building, then $\mathcal{A}'\subseteq\mathcal{A}$.

Every building admits a maximal set of apartments and this maximal set of apartments is uniquely determined by Δ (see Tits [10],Theorème 1). This justifies the notation Δ for a triangle building with a maximal set of apartments.

Let $(\Delta,\mathcal{A})$ be any triangle building. A panel (resp. a chamber, a pennel (with source x), a quarter (with source x), a wall, a half apartment, a face) in $(\Delta,\mathcal{A})$ is the set of vertices S of Δ such that S is a panel (resp. a chamber, etc...) in some apartment of $(\Delta,\mathcal{A})$.

1.3. Notation.

Throughout this paper, Δ denotes a triangle building with a maximal set of apartments. If b is a vertex of Δ, the we will use the following notation (see [13]).

$\mathrm{Ve}(\Delta)$ = set of vertices of Δ,

$\mathrm{Pe}(\Delta)$ = set of pennels of Δ,

$\mathrm{Qu}(\Delta)$ = set of quarters of Δ,

$\mathrm{Ap}(\Delta)$ = set of apartments of Δ,

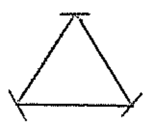
$\mathrm{Pe}(\Delta,b)$ = set of pennels of Δ with source b,

$\mathrm{Qu}(\Delta,b)$ = set of quarters of Δ with source b.

If $x,y \in \mathrm{Ve}(\Delta)$, then $d(x,y)$ denotes the minimal number of panels joining x to y (the graph distance). The map $d : \mathrm{Ve}(\Delta)^2 \to \mathbb{N}$ is a metric. Two vertices are called adjacent if they lie on a common panel.

1.4. The geometry at infinity.

1.4.1. Let $\Sigma \in \mathrm{Ap}(\Delta)$. A type map t_Σ on Σ is any mapping from the set of vertices of Σ onto $\{1,2,3\}$ such that t_Σ corresponds to $\mathrm{typ}_{\mathbf{\Sigma}}$ for at least one isomorphism $\Sigma \to \mathbf{\Sigma}$. Obviously, there are six distinct type maps on Σ. Also, there exists a type map $\mathrm{typ}_\Delta : \mathrm{Ve}(\Delta) \to \{1,2,3\}$ such that the restriction to any apartment Σ of Δ is a type map in Σ. In fact, there are again six possible type maps, but we fix one of them for once and for all (and denote it by typ_Δ as above). The type map typ_Δ turns Δ into a rank 3 geometry with Buekenhout diagram



in the classical way (see Bourbaki [2]). So the residue (see Buekenhout

[4], we will also define this later on) of any vertex of Δ is a projective plane. All residues have the same order and if they are finite, then we call Δ locally finite.

1.4.2. Let b be a vertex of Δ and consider the type of the vertices modulo 3. We define the point-line incidence geometry $V^b = (P(V^b), L(V^b), I)$ as follows. Denote for $p \in Pe(\Delta,b)$ by x_p the unique vertex on p adjacent to b. We define :

$$P(V^b) = \{p \in Pe(\Delta,b) \mid typ_\Delta(x_p) = typ_\Delta(b)+1\},$$
$$L(V^b) = \{\ell \in Pe(\Delta,b) \mid typ_\Delta(x_\ell) = typ_\Delta(b)-1\}.$$

Elements of $P(V^b)$ are called point-pennels (for every b) and elements of $L(V^b)$ are called line pennels (for every b). By definition, a point-pennel p is incident with a line-pennel ℓ if there is a quarter Q with source b containing $p \cup \ell$ as a subset.

Two pennels p,q are parallel if they are on bounded distance from one another (see Tits [10]). A germ of pennels (resp. quarters) is a class of pennels (resp. quarters) with respect to the following equivalence relation : X is equivalent to Y if $X \cap Y$ contains a pennel (resp. a quarter). Two pennels of the same germ are parallel, but there are pennels in distinct germs which are nevertheless parallel (for all of this, see Tits [10]). Note that "being parallel" is an equivalence relation.

We now state some known results on triangle buildings.

PROPOSITION 1 (Tits [10],proposition 5). For any vertex $b \in Ve(\Delta)$ and any pennel p (resp. quarter Q) of Δ, there exists a unique pennel p_b (resp. quarter Q_b) with source b and parallel to p (resp. in the same germ of Q).

LEMMA 1. A point-pennel is never parallel to some line-pennel.

PROOF. Suppose $b,c \in Ve(\Delta)$, $p_b \in P(V^b)$, $p_c \in Pe(\Delta,c)$ and p_b is parallel to p_c. It suffices to show that p_c is a point-pennel. By symmetry, the lemma will then follow. By Tits [10],17.3, there exists an apartment Σ containing pennels $p_b' \subseteq p_b$ and $p_c' \subseteq p_c$. So regarding Σ as a Euclidean plane p_b' and p_c' have parallel supporters. Now denote by x_n the vertex on p_b on distance n from b and y_n the vertex on p_c at distance n from c, defined for all non-negative integers n. But looking in any apartment containing p_b, one sees $typ_\Delta(x_n) = n(\text{mod } 3) + typ_\Delta(b)$. Assume for a moment $p_c \in L(V^c)$, then similarly $typ_\Delta(y_n) = -n(\text{mod } 3) + typ_\Delta(c)$. But if m is such that $y_m \in p_c'$, then looking in Σ, we obtain $typ_\Delta(y_{m+k}) = k(\text{mod } 3) + typ_\Delta(y_m)$ (since this is also true for p_b' and p_c' is parallel to p_b'). This contadicts the former formula for $typ_\Delta(y_n)$. Q.E.D.

LEMMA 2. Suppose $b,c \in Ve(\Delta)$, $Q_b \in Qu(\Delta,b)$, $Q_c \in Qu(\Delta,c)$, $p_b \in Pe(\Delta,b)$, $p_c \in Pe(\Delta,c)$, $p_b \subseteq Q_b$. If p_b and p_c are parallel and Q_b and Q_c are in the same germ, then $p_c \subseteq Q_c$.

PROOF. This is implicitly in Tits [10], but we provide a short proof. Without loss of generality, we can assume $p_b \in P(V^b)$. Let $Q \in Qu(\Delta)$ be such that $Q \subseteq Q_b \cap Q_c$ and suppose Q has source $d \in Ve(\Delta)$. Let $p_d \in P(V^d)$ be parallel to both p_b and p_c (possible since parallellism is an equivalence relation). Clearly (looking in any apartment through Q_b) $p_d \subseteq Q$. Similarly $p_c \subseteq Q_c$ by looking in any apartment containing Q_c. Q.E.D.

PROPOSITION 2. If $b,c \in Ve(\Delta)$, then V^b is isomorphic to V^c.

PROOF. Defining the isomorphism as the map which sends every pennel with source b to the unique parallel pennel with source c, the proposition follows directly from proposition 1 and lemma 2. Q.E.D.

It is easily seen that V^b is in fact isomorphic to one of the two projective planes corresponding to the building at infinity of Δ, as defined in Tits [10]. We denote that projective plane by $PG(\Delta)$. Note that $PG(\Delta)$ depends on the type map typ_Δ. Other type maps may give rise to the dual of $PG(\Delta)$. Recalling the definition by Tits [10], the points of $PG(\Delta)$ are the parallel classes of point-pennels and the lines of $PG(\Delta)$ are the parallel classes of line-pennels of Δ. If $p \in Pe(\Delta)$, then the unique parallel class containing p can be regarded as either a point or a line of $PG(\Delta)$ and is called the trace at infinity of p (see Tits [10]). If w is a wall in Δ, then there are exactly two parallel classes of pennels having representatives contained in w. The set of these two parallel classes is called the trace at infinity of w. It is a non-incident point-line pair in $PG(\Delta)$.

Let $\Sigma \in Ap(\Delta)$. Then the six germs of quarters of Σ define a unique triangle Σ_∞ in $PG(\Delta)$, i.e. a set of three points and three lines (all distinct from one another) forming a triangle. Any triangle is determined by its set of points (resp. lines).

PROPOSITION 3 (Tits [10],proposition 1). The map $\Sigma \to \Sigma_\infty$ above is a bijection from $Ap(\Delta)$ onto the set of all triangles of $PG(\Delta)$.

We denote that bijection by β and call Σ_∞ the trace at infinity of Σ.

PROPOSITION 4 (Van Maldeghem [13],lemma 4.1.5). Suppose P_1,P_2,P_3,P_4 are four points in $PG(\Delta)$ such that no three of them are collinear. Consider the subscripts of the above points modulo 4 and denote by T_i, $i = 1,2,3,4 \pmod 4$ the triangle in $PG(\Delta)$ determined by P_{i+1},P_{i+2},P_{i+3}. Then $\beta^{-1}(T_1) \cap \beta^{-1}(T_2) \cap \beta^{-1}(T_3) \cap \beta^{-1}(T_4)$ is a unique vertex which we denote by $s(P_1,P_2,P_3,P_4)$.

PROPOSITION 5 (Van Maldeghem [13],4.3.6). Suppose $\Sigma \in Ap(\Delta)$ and denote by

$Ve(\Sigma)$ the set of vertices of Σ. Suppose Σ_∞ is the trace at infinity of Σ and let Σ_∞ be determined by the set of points $\{P_1,P_2,P_3\}$. Suppose $s \in Ve(\Delta)$ and $p_i \in Pe(\Delta,s)$ has P_i as trace at infinity, i=1,2,3. Then $p_1 \cap p_2 = p_2 \cap p_3 = p_3 \cap p_1 = \{s\}$ if and only if $s \in Ve(\Sigma)$.

1.5. The n^{th} floor of Δ with basement $b \in Ve(\Delta)$.

1.5.1. We fix $b \in Ve(\Delta)$ and define for every non-negative integer n a point-line incidence geometry $V_n^b = (P(V_n^b), L(V_n^b), I)$, where $P(V_n^b)$ (resp. $L(V_n^b)$) is the point- (resp. line-) set and I is the symmetric incidence relation.

(V.1) $P(V_n^b) = \{P^n \in p \mid p \in P(V^b) \text{ and } d(P^n,b)=n\}$,

(V.2) $L(V_n^b) = \{L^n \in \ell \mid \ell \in L(V^b) \text{ and } d(L^n,b)=n\}$,

(V.3) For every $P^n \in P(V_n^b)$ and every $L^n \in L(V_n^b)$, we define $P^n I\ L^n$ if and only if there is some quarter $Q \in Qu(\Delta,b)$ containing both P^n and L^n.

For n=1, incidence is adjacency and we call V_1^b the residue of b in Δ. It is a projective plane. The geometry V_n^b is called the n^{th} floor of Δ with basement b.

1.5.2. Properties.

There is a natural surjective map $\Pi_{n,j}^b : V_n^b \to V_j^b$, $0 \leq j \leq n$, mapping a point P^n (resp. line L^n) to the point $P^j \in P(V_j^b)$ (resp. the line $L^j \in L(V_j^b)$) defined by $d(b,P^j) = j = n-d(P^n,P^j)$ (resp. $d(b,L^j) = j = n-d(L^n,L^j)$). This is well defined by a general convexity property of (affine) buildings, see [9]. In fact, P^j lies on the interval $[b,P^n]$. The map $\Pi_{n,j}^b$ preserves incidence and is therefore an epimorphism of incidence geometries (since it is clearly surjective). To shorten the notation slightly, we denote by Π_j^b the union of all the maps $\Pi_{n,j}^b$ for $n \geq j$. Note that Π_j^b can be extended to $PG(\Delta)$ via V^b in the obvious way. We denote that extension also by Π_j^b. So, if p $\in$

$Pe(\Delta,b)$, then $\Pi_n^b(c(p))$ (where $c(p)$ is the parallel class of p) is the unique vertex on p at distance n from b and we call it the n-trace of p.

We now define a "partial valuation map" u_b on V_n^b as follows (see also [13]). Let $P^n,Q^n \in P(V_n^b)$, $L^n,M^n \in L(V_n^b)$.

$$\text{(PV.1)} \quad u_b(P^n,Q^n) = \sup\{j \mid \Pi_j^b(P^n) = \Pi_j^b(Q^n)\},$$
$$\text{(PV.2)} \quad u_b(L^n,M^n) = \sup\{j \mid \Pi_j^b(L^n) = \Pi_n^b(M^n)\},$$
$$\text{(PV.3)} \quad u_b(P^n,L^n) = \sup\{j \mid \Pi_j^b(P^n) \text{ I } \Pi_n^b(L^n)\}.$$

Again, we can extend u_b to $PG(\Delta)$ and keep the same notation.

The neighbourhood of a point $P^n \in P(V_n^b)$ is by definition the set of points $\{Q^n \in P(V_n^b) \mid u_b(P^n,Q^n)>0\}$. Similarly for the neighbourhood of a line. A point P^n and a line L^n in V_n^b are called neighbouring if $u_b(P^n,L^n)>0$.

LEMMA 3. If $b \in Ve(\Delta)$ and $P_1^n,P_2^n,P_3^n \in P(V_n^b)$, then $u_b(P_1^n,P_2^n) \geq \inf\{u_b(P_1^n,P_2^n), u_b(P_2^n,P_3^n)\}$ and if $u_b(P_1^n,P_2^n) \neq u_b(P_2^n,P_3^n)$, then equality holds. Similarly for lines.

PROOF. This is obvious by definition. Q.E.D.

PROPOSITION 6 (Van Maldeghem [13],4.3.4). Suppose $P^n,Q^n \in P(V_n^b)$, $L^n,M^n \in L(V_n^b)$ and $k \leq \inf\{u_b(Q^n,L^n),u_b(P^n,L^n),u_b(P^n,M^n)\}$, then

(i) there exists a point incident with both L^n and M^n and there exists a line incident with both P^n and Q^n,

(ii) $u_b(Q^n,M^n) \geq k \iff u_b(P^n,Q^n) + u_b(L^n,M^n) \geq k$,

(for all $b \in Ve(\Delta)$).

This property allows us to conclude that V_n^b is a projective Hjelmslev plane, as shown in [6].

PROPOSITION 7 (Van Maldeghem [13], theorem 4.4.1). The inverse limit of the system $(V_n^b, \Pi_n^b)_{n\in\mathbb{N}}$ is canonically isomorphic to V^b and hence also to $PG(\Delta)$.

The canonical isomorphism of proposition 7 maps $(X_n)_{n\in\mathbb{N}}$ to the unique pennel with source b having $\{X_n \mid n\in\mathbb{N}\}$ as its set of vertices. The sequence $(V_n^b, \Pi_n^b)_{n\in\mathbb{N}}$ is called the Artmann-sequence related to b. This name is dedicated to B. Artmann, who was the first to study such sequences, although not in connection with buildings (see [1]) !

Proposition 7 allows us now to treat V^b as V_∞^b. In the next two lemmas, n is any natural number or ∞.

LEMMA 4 (Van Maldeghem [13], lemma 4.3.4). If $b\in Ve(\Delta)$, $P^n\in P(V_n^b)$, $L^n\in L(V_n^b)$ and $u_b(P^n,L^n)\geq j$, $j\leq n$, then there exists $Q^n\in P(V_n^b)$ such that Q^n I L^n and $u_b(Q^n,P^n)\geq j$.

LEMMA 5. If $b\in Ve(\Delta)$ and $L^n, M^n\in L(V_n^b)$, then

$$u_b(L^n,M^n) = \inf\{\sup\{u_b(P^n,Q^n) \mid Q^n \text{ I } M^n\} \mid P^n \text{ I } L^n\}$$
$$= \inf\{\sup\{u_b(P^n,Q^n) \mid P^n \text{ I } L^n\} \mid Q^n \text{ I } M^n\}.$$

PROOF. Let $k = \inf\{\sup\{u_b(P^n,Q^n) \mid Q^n \text{ I } M^n\} \mid P^n \text{ I } L^n\}$. Let P_0^n be any point incident with L^n. By assumption, there exists a point Q_0^n I M^n such that $u_b(P_0^n,Q_0^n)\geq k$. Hence $\Pi_k^b(P_0^n)$ I $\Pi_k^b(M^n)$ and since P_0^n was arbitrary, $\Pi_k^b(L^n) = \Pi_n^b(M^n)$, so $u_b(L^n,M^n)\geq k$. Now let P_1^n I L^n be such that $k = \sup\{u_b(P_1^n,Q^n) \mid Q^n \text{ I } M^n\}$. By lemma 4, $\Pi_{k+1}^b(P_1^n)$ is not incident with $\Pi_{k+1}^b(M^n)$ and hence $u_b(L^n,M^n) = k$. Q.E.D.

REMARK. Also the dual of lemmas 4 and 5 holds.

2. LOCAL PLANAR TERNARY RINGS.

2.1. Planar ternary rings.

This subsection is a brief summary of [7],Chapter V.

2.1.1. Suppose R is a set not containing the symbol ∞ and suppose T is a ternary operation on R. Then we call (R,T) a planar ternary ring, or PTR for short, if it satisfies (O),(A),(B),(C),(D),(E) below for all a,b,c,d in R.

(O) $0,1 \in R$,

(A) $T(a,0,c) = T(0,b,c) = c$,

(B) $T(a,1,0) = T(1,a,0) = a$,

(C) If $a \neq c$, there exists a unique $x \in R$ such that

$$T(x,a,b) = T(x,c,d),$$

(D) There exists a unique $x \in R$ such that $T(a,b,x) = c$,

(E) If $a \neq c$, there exists a unique $(x,y) \in R^2$ such that

$$T(a,x,y) = b \text{ and } T(c,x,y) = d.$$

PROPOSITION 8 (Hughes-Piper [7],theorem 5.2). If (R,T) is a PTR, the structure PG(R,T) defined as follows is a projective plane. The points of PG(R,T) are the ordered pairs (x,y) where $x,y \in R$ together with elements of the form (x) where $x \in R$ and (∞). Lines are represented by ordered pairs [m,k] where $m,k \in R$ together with elements of the form [m], where $m \in R$ and $[\infty]$. Incidence is defined in the following manner.

(x,y) is on $[m,k] \iff T(m,x,y) = k$,

(x,y) is on $[k] \iff x = k$,

(x) is on $[m,k] \iff x = m$,

(x) is on $[\infty]$ for all $x \in R$ and (∞) is on $[k]$ for all $k \in R$,

(∞) is on $[\infty]$.

Note that we introduced the notation PG(R,T) for any PTR (R,T). Now, consider PG(R,T) and denote $\mathbf{O}$ =(0,0), X = (0), Y = (∞) and E = (1,1), then ($\mathbf{O}$,X,Y,E) is a non-degenerate quadrangle in PG(R,T). We call (R,T) a coordinatizing PTR of PG(R,T) with respect to ($\mathbf{O}$,X,Y,E) and we also say that PG(R,T) is coordinatized by (R,T) with respect to ($\mathbf{O}$,X,Y,E). By Hughes-Piper [7],theorem 5.1, every projective plane -up to isomorphism- can be coordinatized by a PTR with respect to any non-degenarate quadrangle in the above way. However, distinct quadrangles may give rise to non-isomorphic PTRs. We now introduce some further notation.

Let V = (P(V),L(V),I) be a projective plane with point-set P(V), line-set L(V) and incidence relation I. If P and Q are distinct points, then we denote by PQ the unique line incident with both P and Q. Dually, for every distinct L,M $\in$ L(V), we denote by L $\cap$ M the unique point incident with both L and M (a line is viewed as the set of points incident with it).

2.1.2. If (R,T) is a PTR, one defines a multiplication in R by a.b = T(a,b,0) and an addition a+b = T(1,a,b), for all a,b $\in$ R. Recall that (denoting R^* = R-{0})

* (R,T) is called linear if T(a,b,c) = (a.b)+c,
* (R,T) is a quasifield if (R,T) is linear, (R,+) is a group and the left distributive law holds in (R,+,.). In this case, (R,+) is also abelian,
* (R,T) is a nearfield if (R,T) is a quasifield and (R^*,.) is a group,
* (R,T) is a division ring if (R,T) is a quasifield and also the right distributive law holds in (R,+,.),
* (R,T) is a skewfield if (R,T) is both a nearfield and a division ring,
* (R,T) is a field if (R,T) is a skewfield and (R^*,.) is abelian.

(see Hughes-Piper [7])

2.2. Local PTRs.

2.2.1. Suppose (R,T) is a PTR. If $v : R^2 \to \mathbb{Z}\cup\{+\infty\}$ is a map satisfying (L.1),(L.2), (L.3) and (L.4) below, then we call (R,T,v) a local PTR. The map v is called the valuation map. If (R,T,v) satisfies only (L.1),(L.2) and (L.3), we call (R,T,v) a PTR with valuation. In what follows, we assume $z+(+\infty) = +\infty = (+\infty)+z$, $z<(+\infty)$, for all $z \in \mathbb{Z}$.

(L.1) v is onto and $v(a,b) = +\infty \iff a=b$, for all $a,b \in R$,

(L.2) $v(a,b) \geq \inf\{v(a,c),v(b,c)\}$, and if $v(a,c)\neq v(b,c)$, then equality holds, for all $a,b,c \in R$,

(L.3) If $T(a_1,b_1,c_1) = T(a_1,b_2,c_2)$ and $T(a_2,b_1,c_1) = T(a_2,b_2,c_3)$, then $v(a_1,a_2) + v(b_1,b_2) = v(c_2,c_3)$, for all $a_1,a_2,b_1,b_2,c_1,c_2,c_3 \in R$,

(L.4) R is complete as a metric space with respect to the metric $\delta : R^2 \to \mathbb{R} : (a,b) \to \delta(a,b) = 2^{-v(a,b)}$.

We usualy write $v(x,0)$ as $v(x)$. By (v2) of the next proposition, $v(x) = v(x,0) = v(0,x)$.

PROPOSITION 9 (Van Maldeghem [11],properties 2.1). If (R,T,v) is a PTR with valuation, then (v1) through (v16) hold for all $a,b,c,a_1,a_2,\ldots \in R$.

(v1) v is onto,

(v2) $v(a,b) = v(b,a)$,

(v3) If $v(a,b)<v(b,c)$, then $v(a,c)=v(a,b)$,

(v4) $v(a,b)=+\infty \iff a=b$,

(v5) $v(1)=0$ and $v(0)= +\infty$,

For (v6) through (v11), we suppose $T(a_i,b_i,c_i)=d_i$, $i=1,2$.

(v6) If $a_1=a_2$ and $b_1=b_2$, then $v(c_1,c_2) = v(d_1,d_2)$,

(v7) If $a_1=a_2$ and $c_1=c_2$, then $v(b_1,b_2)+v(a_1) = v(d_1,d_2)$,

(v8) If $a_1=a_2$ and $d_1=d_2$, then $v(b_1,b_2)+v(a_1) = v(c_1,c_2)$,

(v9) If $b_1=b_2$ and $c_1=c_2$, then $v(a_1,a_2)+v(b_1) = v(d_1,d_2)$, (v10) If $b_1=b_2$ and $d_1=d_2$, then $v(a_1,a_2)+v(b_1) = v(c_1,c_2)$,

(v11) If $c_1=c_2$ and $d_1=d_2$, then $v(a_1,a_2)+v(b_1) = v(b_1,b_2)+v(a_2)$

and $v(a_1,a_2)+v(b_2) = v(b_1,b_2)+v(a_1)$

and in particular $v(a_1)+v(b_1) = v(a_2)+v(b_2)$,

(v12) If $T(a,b,c) = d$, then $v(a)+v(b) = v(c,d)$,

For (v13) through (v16), we suppose $T(a_1,b_1,c_1) = d_1 = T(a_1,b_2,c_2)$, $T(a_2,b_1,c_1) = d_2$ and $T(a_3,b_3,c_3) = d_3$.

(v13) If $a_2=a_3, b_2=b_3$ and $c_2=c_3$, then $v(a_1,a_2)+v(b_1,b_2) = v(d_2,d_3)$,

(v14) If $a_2=a_3, b_2=b_3$ and $d_2=d_3$, then $v(a_1,a_2)+v(b_1,b_2) = v(c_2,c_3)$,

(v15) If $a_2=a_3, c_2=c_3$ and $d_2=d_3$, then $v(a_1,a_2)+v(b_1,b_2) = v(b_2,b_3)+v(a_2)$

and $v(a_1,a_2)+v(b_1,b_3) = v(b_2,b_3)+v(a_1)$,

(v16) If $b_2=b_3, c_2=c_3$ and $d_2=d_3$, then $v(a_1,a_2)+v(b_1,b_2) = v(a_2,a_3)+v(b_2)$

and $v(a_1,a_3)+v(b_1,b_2) = v(a_2,a_3)+v(b_1)$.

2.2.2. REMARK.* If (R,T) is a PTR and $v : R^2 \to \mathbb{Z} \cup \{+\infty\}$ is a map satisfying (v3) and (v13) (or (v3) and (v14)) and if there is at least one pair $(a,b) \in R^2$ such that $v(a,b) = 1$, then one can show that (R,T,v) is a PTR with valuation.

* If (R,T) is a quasifield, then (L.3) can be replaced by

(Q) $v(a_1.b - a_2.b) = v(a_1 - a_2) + v(b)$.

* If (R,T) is a division ring, then (L.3) can be replaced by

(DR) $v(a.b) = v(a)+v(b)$.

For examples, we refer to [11].

2.2.3. THEOREM 2 (Van Maldeghem [13]). Suppose $\mathcal{P}$ is a projective plane, then the following statements are equivalent.

(i) $\mathcal{P}$ is isomorphic to the geometry at infinity of a triangle building Δ.

(ii) $\mathcal{P}$ can be coordinatized by a local PTR (R,T,v).

(iii) Every coordinatizing PTR of $\mathcal{P}$ is a local PTR.

In that case, v is completely determined by Δ, the choice of the non-degenerate quadrangle (O,X,Y,E) and the coordinates of the points on the line OY. Also, Δ is unique and for a given local PTR (R,T,v), we denote this corresponding triangle building by $\Delta(R,T,v)$.

2.2.4. NOTATION.* If (R,T,v) is a local PTR, then we write

$$R_0^+ = \{r \in R \mid v(r) \geq 0\} \text{ (this includes 0 !)},$$
$$R^- = R - R_0^+.$$

If $r \in R_0^+$, then we write r^+. Similarly, if $r \in R^-$, we write r^-. Writing just r leaves the possibility open, if no hypotheses were made before.

* For $a,b \in R$, we denote $w(a,b) = v(a,b)-v(a)-v(b)$.

2.2.5. THEOREM 3 (Van Maldeghem [13],5.1.3). Suppose that (R,T,v) is a local PTR. Let $x,y,a,b,m,k,p,q \in R$. If $x \neq 0$ (resp. $z \neq 0$), then we define z (resp. c) as $T(z,x,y) = 0$ (resp. $T(c,a,b) = 0$). Suppose $\Delta = \Delta(R,T,v)$ and denote $s = s(0,X,Y,E) \in Ve(\Delta)$ and let $u = u_s$ for short. Then we have :

$$u((x^+,y^+),(a^+,b^+)) = \inf\{v(x,a),v(y,b)\}$$
$$u((x^-,y^+),(a^-,b^+)) = \inf\{v(z,c),w(x,a)\}$$
$$u((x^+,y^-),(a^+,b^-)) = \inf\{w(z,c),w(y,b)\}$$
$$u((0,y^-),(0,b^-)) = w(y,b)$$
$$u((x^-,y^-),(a^-,b^-)) = \inf\{w(z,c),w(y,b)\} \text{ if } v(y)<v(x) \ \&\ v(b)<v(a)$$
$$u((x^-,y^-),(a^-,b^-)) = \inf\{v(z,c),w(x,a)\} \text{ if } v(y)\geq v(x) \ \&\ v(b)\geq v(a)$$
$$u((x^+),(a^+)) = v(x,a)$$
$$u((x^-),(a^-)) = w(x,a)$$
$$u((x^-,y^+),(a^-,b^-)) = \inf\{v(z,c),w(x,a)\} \quad \text{if } v(b)\geq v(a)$$
$$u((x^-,y^+),(a^+)) = \inf\{v(z,a),|v(x)|\}$$
$$u((x^+,y^-),(0,b^-)) = \inf\{w(y,b),|v(z)|\}$$
$$u((x^+,y^-),(a^-,b^-)) = \inf\{w(z,c),w(y,b)\} \quad \text{if } v(b)<v(a)$$
$$u((x^+,y^-),(a^-)) = \inf\{w(z,a),|v(y)|\}$$
$$u((x^+,y^-),(\infty)) = \inf\{|v(z)|,|v(y)|\}$$
$$u((0,y^-),(a^-,b^-)) = \inf\{|v(c)|,w(y,b)\} \quad \text{if } v(b)<v(a)$$

$$u((0,y^-),(a^-)) = \inf\{|v(a)|,|v(y)|\}$$

$$u((0,y^-),(\infty)) = |v(y)|$$

$$u((x^-,y^-),(a^-)) = \inf\{w(z,a),|v(y)|\} \quad \text{if } v(y)<v(x)$$

$$u((x^-,y^-),(\infty)) = \inf\{|v(z)|,|v(y)|\} \quad \text{if } v(y)<v(x)$$

$$u((x^-,y^-),(a^+)) = \inf\{v(z,a),|v(x)|\} \quad \text{if } v(y)\geq v(x)$$

$$u((x^-),(\infty)) = |v(x)|$$

In all other cases $u(P,Q)=0$ for P,Q points of $PG(\Delta)$.

Let l and r be defined as $T(m,l,0)=k$, resp. $T(p,r,0)=q$ if $m\neq 0$, resp. $p\neq 0$. Then we have :

$$u([m^+,k^+],[p^+,q^+]) = \inf\{v(m,p),v(k,q)\}$$

$$u([m^-,k^+],[p^-,q^+]) = \inf\{w(m,p),v(l,r)\}$$

$$u([m^+,k^-],[p^+,q^-]) = \inf\{w(k,q),w(l,r)\} \quad \text{if } m\neq 0 \ \&\ p\neq 0$$

$$u([0,k^-],[0,q^-]) = w(k,q)$$

$$u([m^-,k^-],[p^-,q^-]) = \inf\{w(k,q),w(l,r)\} \text{ if } v(k)<v(m) \ \&\ v(q)<v(p)$$

$$u([m^-,k^-],[p^-,q^-]) = \inf\{w(m,p),v(l,r)\} \text{ if } v(k)\geq v(m) \ \&\ v(q)\geq v(p)$$

$$u([k^+],[q^+]) = v(k,q)$$

$$u([k^-],[q^-]) = w(k,q)$$

$$u([m^-,k^+],[p^-,q^-]) = \inf\{w(m,p),v(l,r)\} \quad \text{if } v(q)\geq v(p)$$

$$u([m^-,k^+],[q^+]) = \inf\{v(l,q),|v(m)|\}$$

$$u([m^+,k^-],[0,q^-]) = \inf\{w(k,q),|v(l)|\}$$

$$u([m^+,k^-],[p^-,q^-]) = \inf\{w(k,q),w(l,r)\} \quad \text{if } v(q)<v(p)$$

$$u([m^+,k^-],[q^-]) = \inf\{w(l,q),|v(k)|\}$$

$$u([m^+,k^-],[\infty]) = \inf\{|v(l)|,|v(k)|\}$$

$$u([0,k^-],[p^-,q^-]) = \inf\{w(k,q),|v(r)|\} \quad \text{if } v(q)<v(p)$$

$$u([0,k^-],[q^-]) = \inf\{|v(q)|,|v(k)|\}$$

$$u([0,k^-],[\infty]) = |v(k)|$$

$$u([m^-,k^-],[q^-]) = \inf\{|v(k)|,w(l,q)\} \quad \text{if } v(k)<v(m)$$

$$u([m^-,k^-],[\infty]) = \inf\{|v(k)|,|v(l)|\} \quad \text{if } v(k)<v(m)$$

$$u([m^-,k^-],[q^-]) = \inf\{|v(m)|,v(l,q)\} \quad \text{if } v(k)\geq v(m)$$

$u([k^-],[\infty]) = |v(k)|$

In all other cases, $u(L,M)=0$ for L,M lines of $PG(\Delta)$.

2.2.6. Suppose Δ is a triangle building and $\Sigma \in Ap(\Delta)$. We concieve Σ as a Euclidean plane isomorphic to A (see 1.1). Suppose that s_0, s_1, s_2, s_3 and s are vertices of Δ in Σ. Then we can define the barycentric coordinates $(r_1,r_2,r_3) \in R^3$ of s with respect to $(s_0;s_1,s_2,s_3)$ as follows. Concieve Σ as the vector plane with origin s (called the barycenter), then

$$\vec{s} = r_1\vec{s}_1 + r_2\vec{s}_2 + r_3\vec{s}_3 \quad \text{with} \quad r_1+r_2+r_3 = 0.$$

2.2.7. THEOREM 4 (Van Maldeghem [13],4.5.5). Suppose (R,T,v) is a local PTR and suppose E^* is a point of $PG(R,T)$ such that (O,X,Y,E^*) is a non-degenerate quadrangle. Suppose we coordinatize $PG(R,T)$ with respect to (O,X,Y,E^*) by some local PTR (R^*,T^*,v^*), where v^* is the natural valuation on (R^*,T^*) induced by $\Delta(R,T,v)$. There are three maps b_∞, $b_{\overline{X}}$, $b_{\overline{Y}}$: $R \to R^*$ such that

$$T(m,a,b) = k \Longleftrightarrow T^*(b_\infty(m),b_{\overline{X}}(a),b_{\overline{Y}}(b)) = b_{\overline{Y}}(k),$$

and $[b_\infty(m),b_{\overline{Y}}(k)]$ (resp. $(b_{\overline{X}}(a),b_{\overline{Y}}(b))$ are the new coordinates of $[m,k]$ (resp. (a,b)). Denote $s = s(O,X,Y,E)$ and $s^* = s(O,X,Y,E^*)$ and let $\Pi_1^s(O) = O_1$, $\Pi_1^s(X) = X_1$, $\Pi_1^s(Y) = Y_1$. Let Σ be the apartment of $\Delta(R,T,v)$ of which the trace at infinity contains the points O,X,Y. Then O_1,X_1,Y_1 and s^* are vertices in Σ. Let (k_0,ℓ_0,m_0) be the barycentric coordinates of s^* in Σ, then we have (for all $x,y \in R$) :

$$v^*(b_\infty(x),b_\infty(y)) = v(x,y) + m_0 - \ell_0,$$

$$v^*(b_{\overline{X}}(x),b_{\overline{X}}(y)) = v(x,y) + \ell_0 - k_0,$$

$$v^*(b_{\overline{Y}}(x),b_{\overline{Y}}(y)) = v(x,y) + m_0 - k_0.$$

This completes the list of all known results that we will need to prove our assertions.

3. AUTOMORPHISMS OF TRIANGLE BUILDINGS.

In this section, we prove two general results about the relation between the automorphism group of a triangle building and the automorphism group of its building at infinity.

3.1. DEFINITION. An automorphism Ψ of a triangle building Δ is a bijection $\Psi : Ve(\Delta) \to Ve(\Delta)$ preserving adjacency. So any automorphism preserves the distance d and it is also readily seen that, since we have a maximal set of apartments and this set is completely determined by Δ, an automorphism maps apartments to apartments, pennels to pennels, quarters to quarters, etc... Note that with that definition, an automorphism does not necessarily preserve the type of a vertex, but it interchanges the types. Hence, if an automorphism preserves the types of the vertices of one chamber, then it preserves the type of all vertices. The set of all automorphisms of Δ is denoted by Aut(Δ).

An automorphism ψ of a projective plane $V = (P(V),L(V),I)$ is a bijection $\psi : P(V) \cup L(V) \to P(V) \cup L(V)$ mapping either P(V) (resp. L(V)) onto P(V) (resp. L(V)) or P(V) (resp. L(V)) onto L(V) (resp. P(V)) and preserving incidence. So with that definition, both collineations and correlations (in the sense of Hughes-Piper [7]) are automorphisms. The set of all automorphisms of V is denoted by Aut(V).

The following theorem is well known, but since there seems to be no proof of it in the liturature by our knowledge, we thought a short proof would suite here. Note that this result holds for every affine building!

THEOREM 5. For every triangle building Δ, Aut(Δ) is isomorphic to some subgroup of Aut(PG(Δ)).

PROOF. Suppose $\Psi \in \mathrm{Aut}(\Delta)$. Since Ψ preserves the distance, it maps parallel pennels to parallel pennels and germs of quarters to germs of quarters. Hence Ψ defines an automorphism Ψ_∞ of $PG(\Delta)$ in the obvious way. There remains to show that, if Ψ_∞ is the identity map, then Ψ is the identity map. But if Ψ_∞ is the identity map, then Ψ must stabilize every apartment. But every vertex of Δ is the intersection of four apartments (as in proposition 4). Hence every vertex is fixed and Ψ is the identity map.

Q.E.D.

If $\Psi \in \mathrm{Aut}(\Delta)$, then we denote by Ψ_∞ the unique automorphism of $PG(\Delta)$ defined as in the above proof.

THEOREM 6. Suppose $\psi \in \mathrm{Aut}(PG(\Delta))$, then the following statements are equivalent.

(i) There exists $\Psi \in \mathrm{Aut}(\Delta)$ such that $\Psi_\infty \equiv \psi$.

(ii) For all $s \in Ve(\Delta)$, there exists $s^* \in Ve(\Delta)$ such that $u_s(P,Q) = u_{s^*}(P^\psi,Q^\psi)$, for all points P and Q of $PG(\Delta)$.

(ii') For all $s \in Ve(\Delta)$, there exists $s^* \in Ve(\Delta)$ such that $u_s(L,M) = u_{s^*}(L^\psi,M^\psi)$, for all lines L and M of $PG(\Delta)$.

(iii) There exist $s,s^* \in Ve(\Delta)$ such that $u_s(P,Q) = u_{s^*}(P^\psi,Q^\psi)$, for all points P and Q of $PG(\Delta)$.

(iii') There exist $s,s^* \in Ve(\Delta)$ such that $u_s(L,M) = u_{s^*}(L^\psi,M^\psi)$, for all lines L and M of $PG(\Delta)$.

In that case, $\Psi(s) = s^*$ and consequently if $s = s(P_1,P_2,P_3,P_4)$, for some points P_1,P_2,P_3,P_4 of $PG(\Delta)$, then $s^* = s(P_1^\psi,P_2^\psi,P_3^\psi,P_4^\psi)$.

PROOF. (i) $\Rightarrow$ (ii). Evidently, since an automorphism of Δ preserves the distance.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Let $s,s^* \in Ve(\Delta)$ be as in the statement (iii).

Suppose that P^n is an arbitrary point of V_n^s. We define $(P^n)^\Psi$ as follows. Let p be any pennel through P^n with source s. Let P be the trace at infinity of p and let $(P^n)^\Psi$ be the n-trace of p* in V_v^{s*}, where p* is the unique pennel with source s* having P^ψ as trace at infinity. A similar definition holds for $(L^n)^\Psi$ with $L^n \in L(V_n^s)$. We show that this definition is independent of the pennel p. So suppose p' is another pennel with source s containing P^n. Suppose P' is the trace at infinity of p'. Then $u_s(P,P') \geq n$. Hence by assumption, $u_{s*}(P^\psi, P'^\psi) \geq n$ and hence the n-trace of $p'* \in Pe(\Delta, s*)$ (having P'^ψ as trace at infinity) is exactly $(P^n)^\Psi$. Now by lemma 4 or its dual, $u_{s*}(L^\psi, M^\psi) = \inf\{\sup\{u_{s*}(P^\psi, Q^\psi) \mid Q^\psi I\ M^\psi\} \mid P^\psi I\ L^\psi\}$ $= \inf\{\sup\{u_s(P,Q) \mid Q\ I\ M\} \mid P\ I\ L\} = u_s(L,M)$. Hence also $(L^n)^\Psi$ is well defined. Suppose P^n I L^n in V_n^s. By lemma 4, we can choose pennels $p, \ell \in Pe(\Delta, s)$ containing resp. P^n and L^n and being incident in V^s. Hence their respective traces at infinity P and L are incident in PG(Δ), and so $(P^n)^\Psi$ is incident with $(L^n)^\Psi$ in V_n^{s*}. Hence Ψ is an isomorphism from V_n^s onto V_n^{s*} or its dual. From the reconstruction of Δ from the n^{th} floors in [11], it follows easily that Ψ extends to all vertices of Δ (since they are all defined in terms of points and lines of V_n^s, resp. V_n^{s*}) and preserves adjacency, hence Ψ becomes an automorphism of Δ. By the definition of Ψ, clearly $\Psi_\infty \equiv \psi$.

Similarly (and dually) one shows (i) ⇒ (ii') ⇒ (iii') ⇒ (i). Q.E.D.

DEFINITION. Suppose $\psi \in Aut(PG(\Delta))$, $s, s* \in Ve(\Delta)$ and $n \in \mathbb{N}$. Then we say that ψ induces an isomorphism $\psi_n : V_n^s \to V_n^{s*}$ if the map Ψ, defined as in the proof above, is well-defined on V_n^s and we set $\Psi/V_n^s = \psi_n$.

COROLLARY 1. Let $b \in Ve(\Delta)$ and suppose $\psi \in Aut(PG(\Delta))$, then ψ induces an automorphism in V_n^b for all $n \in \mathbb{N}$ if and only if ψ preserves the partial valuation map u_b on the pairs of points if and only if ψ preserves the partial valuation u_b on pairs of lines.

PROOF. Setting s=s* in the proof of theorem 6, the result follows directly from that proof.

Q.E.D.

4. PERSPECTIVITIES OF PG(Δ).

4.1. DEFINITIONS. Let $b \in Ve(\Delta)$. Suppose $P^n \in P(V_n^b)$, $L^n \in L(V_n^b)$, then we call an automorphism α_n of V_n^b a (P^n,L^n)-perspectivity, if it fixes L^n pointwise and if it stabilizes all lines through P^n. If P^n I L^n, then α_n is also called a (P^n,L^n)-elation. If P^n and L^n are neighbouring, but not incident, then α_n is also called a central quasi elation. If P^n and L^n are not neighbouring, then α_n is also called a (P^n,L^n)-homology. Similar definitions hold in projective planes, replacing "neighbouring" by "incident" (so no central quasi elations are defined in projective planes). Suppose now $P^n,Q_0^n,Q_1^n \in P(V_n^b)$, $L^n \in L(V_n^b)$ with Q_i^n and L^n (resp. P^n) not neighbouring (resp. not in the same neighbourhood), i=0,1 and P^n,Q_0^n,Q_1^n collinear. If for all such pairs (Q_0^n,Q_1^n) and for fixed P^n,L^n, there exists a (P^n,L^n)-perspectivity mapping Q_0^n to Q_1^n, then V_n^b is said to be (P^n,L^n)-transitive. Similar definition again for projective planes. Note that these definitions are standard ones in the literature, thinking of V_n^b as a projective Hjelmslev plane of level n.

PROPOSITION 10. Let $b \in Ve(\Delta)$. Suppose $P^n \in P(V_n^b)$, $L^n \in L(V_n^b)$ and α_n is a (P^n,L^n)-perspectivity mapping Q_0^n to Q_1^n, $Q_i^n \in P(V_n^b)$, i=0,1. If $u_b(Q_i^n,P^n) = 0 = u_b(Q_i^n,L^n)$, i=0,1, then α_n is completely determined by $\alpha_n(Q_0^n)=Q_1^n$.

PROOF. This is completely similar to the projective plane case (see e.g. Hughes-Piper [7]).

Q.E.D.

LEMMA 6. Suppose $\alpha \in Aut(PG(\Delta))$ is a (P,L)-elation for some point-line pair (P,L). Suppose there exist $s \in Ve(\Delta)$ and a point Q of PG(Δ) such that $u_s(Q,L) = u_s(Q^\alpha,L) = 0$. Then α preserves the partial valuation u_s

on pairs of points.

PROOF. Let Q_1 and Q_2 be two arbitrary points of PG(Δ). We have to show that $u_s(Q_1,Q_2) = u_s(Q_1^\alpha,Q_2^\alpha)$. There are several possibilities now.

(1). Suppose $u_s(Q_1,L) = u_s(Q_2,L) = 0$. We coordinatize PG($\Delta$) with respect to the non-degenerate quadrangle $(\mathbf{O},X,Y,E)$ by a local PTR (R,T,v), where $\mathbf{O}=Q$, $Y=P$, $X \text{ I } L$ and X and E are chosen such that $(\Pi_1^s(\mathbf{O}),\Pi_1^s(X),\Pi_1^s(Y),\Pi_1^s(E))$ is non-degenerate in V_1^s (this is possible since $u_s(Q,L) = 0$ and V_1^s is a non-degenerate projective plane). As a consequence of proposition 5, $s = s(\mathbf{O},X,Y,E)$. Suppose $Q_1 = (x_1,y_1)$ and $Q_2 = (x_2,y_2)$, then $x_1,x_2,y_1,y_2 \in R_0^+$ (follows from the definition of v in [13], but can also be gotten by theorem 3 and lemma 4). By theorem 3, $Q^\alpha = (0,q)$ with $q \in R_0^+$. Now one can check that $(x_i,y_i)^\alpha = (x_i,T(y_i,1,q))$, i=1,2 and since $v(T(y_1,1,q),T(y_2,1,q)) = v(y_1,y_2)$ by (v5) and (v9), the result follows from $v(T(y_i,1,q)) \geq 0$ and theorem 3.

(2). Suppose $u_s(Q_1,L)>0$ and $u_s(Q_2,L)>0$; $u_s(Q_1,P) = u_s(Q_2,P) = 0$. We now coordinatize PG(Δ) with respect to the quadrangle (and we forget the notation of (1)) $(\mathbf{O},X,Y,E)$ by a local PTR (R,T,v), where $Y=P$, $X=Q$, $\mathbf{O} \text{ I } L$ and $\mathbf{O}$ and E are such that $(\Pi_1^s(\mathbf{O}),\Pi_1^s(X),\Pi_1^s(Y),\Pi_1^s(E))$ is non-degenerate in V_1^s. Again we have $s = s(\mathbf{O},X,Y,E)$. Suppose $Q_i = (x_i,y_i)$, i=1,2. By theorem 3, $v(x_i)>0$ and $v(y_i)\geq 0$, i=1,2. Also $Q^\alpha = (q)$ with $v(q)\geq 0$. From now on, we assume throughout this proof i=1,2. One can easily check that

$$(x_i,y_i)^\alpha = (x_i,\overline{T}(x_i,q,y_i)) =: (x_i,y_i^*),$$
$$(x_1,y_2)^\alpha = (x_1,\overline{T}(x_1,q,y_2)) =: (x_1,y),$$

where $\overline{T}(a,m,k) = b$ is defined as the unique $b \in R$ such that $T(m,a,b) = k$ (well defined by (D) for all $a,m,k \in R$). By (v3) and (v11), $v(y_i^*)\geq 0$ and $v(y)\geq 0$. Hence by theorem 3,

$$u_s((x_1,y_1),(x_2,y_2)) = \inf\{v(x_1,x_2),v(y_1,y_2)\},$$

$$u_s((x_1,y_1)^\alpha,(x_2,y_2)^\alpha) = \inf\{v(x_1,x_2),v(y_1^*,y_2^*)\}.$$

Now $v(y_1^*,y_2^*) \geq \inf\{v(y_1^*,y),v(y,y_2^*)\}$ and by resp. (v8) and (v6),

$$v(y_1^*,y) = v(x_1,x_2) + v(q),$$
$$v(y,y_2^*) = v(y_1,y_2).$$

There are two possibilities.

(i) If $v(y_1,y_2) < v(x_1,x_2) + v(q)$, then by (v3), $v(y_1^*,y_2^*) = v(y_1,y_2)$ and the result follows.

(ii) If $v(y_1,y_2) \geq v(x_1,x_2) + v(q)$, then $v(y_1,y_2) \geq v(x_1,x_2)$ and $v(y_1^*,y_2^*) \geq v(x_1,x_2)$ and hence $u_s(Q_1,Q_2) = v(x_1,x_2) = u_s(Q_1^\alpha,Q_2^\alpha)$.

(3). Suppose $u_s(Q_1,P)>0$ and $u_s(Q_2,P)>0$. We recoordinatize PG(Δ) (and forget the notation above) by a local PTR (R,T,v) with respect to the quadrangle (O,X,Y,E), where now O=P , X=Q , Y I L and Y and E are such that $(\Pi_1^s(O),\Pi_1^s(X),\Pi_1^s(Y),\Pi_1^s(E))$ is a non-degenerate quadrangle in V_1^s. Again we have $s = s(O,X,Y,E)$. Suppose $Q_i = (x_i,y_i)$, then by theorem 3, $v(x_i)>0$ and $v(y_i)>0$. We also have $Q^\alpha = (q,0)$ with $v(q)\leq 0$. One can easily check that both Q_1 and Q_2 are not incident with XY and that the first coordinate of $(x,y)^\alpha$ is independent of y, for all $x,y \in R$. We again consider some possibilities.

(3.I) Suppose first $x_i \neq 0$ and $y_i \neq 0$. Let $(x_i,y_i)^\alpha = (x_i^*,y_i^*)$. Since $(x_i,y_i),(x_i,y_i)^\alpha$ and (0,0) are collinear, there exists $m_i \in R$, such that

$$T(m_i,x_i,y_i) = 0 \qquad (1.i)$$
$$T(m_i,x_i^*,y_i^*) = 0 \qquad (2.i)$$

Let $[0,y_i]^\alpha = [m_i^*,y_i]$ with $m_i^* \in R$, then

$$T(m_i^*,q,0) = y_i \qquad (3.i)$$
$$T(m_i^*,x_i^*,y_i^*) = y_i \qquad (4.i)$$

Similarly considering (x_2,y_1) and $(x_2,y_1)^\alpha = (x_2^*,y_1^{**})$, we obtain $r \in R$ such that

$$T(r,x_2,y_1) = 0 \quad (5)$$

$$T(r,x_2^*,y_1^{**}) = 0 \quad (6)$$

and since (x_2,y_1) I $[0,y_1]$, we also have

$$T(m_1^*,x_2^*,y_1^{**}) = y_1 \quad (7)$$

By (1.1),(5) and (v11), $v(r,m_1) = v(x_1,x_2) + v(m_1) - v(x_2)$. (8)

By (4.1),(7) and (v8), $v(y_1^*,y_2^*) = v(x_1^*,x_2^*) + v(m_1^*)$. (9)

By (3.i) and (v12), $v(y_i) = v(m_i^*) + v(q)$. (10.i)

By (1.i),(2.i) and (v12), $v(x_i) - v(y_i) = v(x_i^*) - v(y_i^*)$. (11.i)

By (1.i),(3.i) and (v12), $v(m_i) + v(\ddot{x}_i) = v(y_i) = v(m_i^*) + v(q)$ $\leq v(m_i^*)$. Since $v(x_i) > 0$, $v(m_i) < v(m_i^*)$ and hence by (v3), $v(m_i)$ $= v(m_i,m_i^*)$, so $v(m_i,m_i^*) + v(x_i) = v(y_i)$. (12.i)

By (2.i),(4.i) and (v9), $v(m_i,m_i^*) + v(x_i^*) = v(y_i)$ and by (12.i), $v(x_i) = v(x_i^*)$. Hence by (11.i), $v(y_i) = v(y_i^*)$. (13.i)

By (2.2),(6) and (v8), $v(r,m_2) = v(y_2^*,y_1^{**}) - v(x_2^*)$. (14)

By (1.2),(5) and (v9), $v(y_1,y_2) = v(r,m_2) + v(x_2)$ and so by (13.1) and (14), $v(y_1,y_2) = v(y_2^*,y_1^{**})$. (15)

By (2.1),(4.1),(6),(7) and (V16), $v(m_1,r) = v(m_1,m_1^*)+v(x_1^*,x_2^*)-v(x_2^*)$ and hence by (8) and (12.1), $v(x_1,x_2) = v(x_1^*,x_2^*)$. (16)

Now $v(y_1^*,y_2^*) \geq \inf\{v(y_1^*,y_1^{**}),v(y_1^{**},y_2^*)\}$. By (9) and (12.1), $v(y_1^*,y_1^{**}) =$ $v(x_1^*,x_2^*) + v(y_1) - v(q)$. So by (15), $v(y_1^*,y_2^*) \geq \inf\{v(y_1,y_2),\ v(x_1^*,x_2^*)+$ $v(y_1)-v(q)\}$. There are two possibilities.

(3.I.1) If $v(y_1,y_2) < v(x_1^*,x_2^*)+v(y_1)-v(q)$, then $v(y_1^*,y_2^*) = v(y_1,y_2)$ by (v3) and the result follows by theorem 3.

(3.I.2) If $v(y_1,y_2) \geq v(x_1^*,x_2^*)+v(y_1)-v(q)$, then, since $v(y_1)>0$ and $v(q)<0$, $v(y_1,y_2) > v(x_1^*,x_2^*) = v(x_1,x_2)$ (by (16)) and also $v(y_1^*,y_2^*) \geq$ $v(x_1^*,x_2^*)+v(y_1)-v(q) > v(x_1^*,x_2^*) = v(x_1,x_2)$. So $u_s(Q_1,Q_2) = v(x_1,x_2) =$ $v(x_1^*,x_2^*) = u_s(Q_1^\alpha,Q_2^\alpha)$.

(3.II) Suppose $x_1 = 0$ and $x_2, y_1, y_2 \neq 0$. Let $(x_2,y_2)^\alpha = (x_2^*,y_2^*)$ as in (3.I). We still have $v(x_2) = v(x_2^*)$ and we can assume the equalities (1.2), (2.2),(3.2) and (4.2) of (3.I). We have

$$u_s(Q_1,Q_2) = u_s((0,y_1),(x_2,y_2)) = \inf\{v(x_2),v(y_1,y_2)\},$$
$$u_s(Q_1^\alpha,Q_2^\alpha) = u_s((0,y_1),(x_2^*,y_2^*)) = \inf\{v(x_2^*),v(y_1,y_2^*)\}.$$

(3.II.1) Suppose $v(y_1,y_2) \leqq v(x_2)$, then $u_s(Q_1,Q_2) = v(y_1,y_2)$. Now by (3.2) and (v12), $v(m_2^*) = v(y_2)-v(q) > 0$ and hence by (4.2) and (V12), $v(y_2,y_2^*) > v(x_2^*) = v(x_2) \geq v(y_1,y_2)$. So by (v3), $v(y_1,y_2^*) = v(y_1,y_2)$ $v(x_2) = v(x_2^*)$ and thus $u_s(Q_1^\alpha,Q_2^\alpha) = v(y_1,y_2^*) = v(y_1,y_2) = u_s(Q_1,Q_2)$.

(3.II.2) suppose $v(y_1,y_2) > v(x_2)$, then $u_s(Q_1,Q_2) = v(x_2)$. We still have $v(y_2,y_2^*) > v(x_2^*) = v(x_2)$, hence by (L.2), $v(y_1,y_2^*) > v(x_2) = v(x_2^*)$ and $u_s(Q_1^\alpha,Q_2^\alpha) = v(x_2^*) = v(x_2) = u_s(Q_1,Q_2)$.

(3.III) Suppose $x_1=y_1=0$. Let $(x_2,y_2)^\alpha = (x_2^*,y_2^*)$. If $x_2=0$, then the result is trivial. If $x_2\neq0$, then as in (3.I), $v(x_2)=v(x_2^*)$ and $v(y_2)=v(y_2^*)$. But then $u_s(Q_1,Q_2) = \inf\{v(x_2),v(y_2)\} = \inf\{v(x_2^*),v(y_2^*)\} = u_s(Q_1^\alpha,Q_2^\alpha)$.

(3.IV) If $x_1=x_2=0$, then $Q_i^\alpha = Q_i$.

(3.V) Suppose $x_1=y_2=0$. Let $(x_2,0)^\alpha = (x_2^*,0)$. If $x_2=0$, the assertion is trivial, so suppose $x_2\neq0$. Considering $(x_2,x_2)^\alpha = (x_2^*,x_2^{**})$, for some x_2^{**}, we can use the results of (3.I) and get $v(x_2)=v(x_2^*)$. Hence $u_s(Q_1,Q_2)$ $= \inf\{v(x_2),v(y_1)\} = \inf\{v(x_2^*),v(y_1)\} = u_s(Q_1^\alpha,Q_2^\alpha)$.

(3.VI) If $y_1=0$ and $X_1,x_2,y_2\neq0$, then we apply the same trick as in (3.V) and get $v(x_1,x_2) = v(x_1^*,x_2^*)$ and $v(y_2) = v(y_2^*)$, where $(x_1,0)^\alpha = (x_1^*,0)$ and $(x_2,y_2)^\alpha = (x_2^*,y_2^*)$. The result follows immediately.

(3.VII) If $y_1=y_2=0$, then again $v(x_1,x_2) = v(x_1^*,x_2^*)$, where $(x_i,0)^\alpha = (x_i^*,0)$, and the result follows.

(3.VIII) All other cases can be obtained by interchanging the rôles of Q_1 and Q_2.

(4). Suppose $u_s(Q_1,P) = 0$ and $u_s(Q_2,P) > 0$. By (1) and (2), $u_s(Q_1^\alpha,P) = 0$ and by (3), $u_s(Q_2^\alpha,P) > 0$, hence $u_s(Q_1,Q_2) = 0 = u_s(Q_1^\alpha,Q_2^\alpha)$ by lemma 3.

(5). Suppose $u_s(Q_1,L) = 0$ and $u_s(Q_2,L) > 0$. By (1), $u_s(Q_1^\alpha,L) = 0$ and by (2) and (3), $u_s(Q_2^\alpha,L) > 0$, hence $u_s(Q_1,Q_2) = 0 = u_s(Q_1^\alpha,Q_2^\alpha)$ by lemmas 3 and 5.

(6). All other cases can be obtained by interchanging Q_1 and Q_2 in (4) or (5).

Q.E.D.

THEOREM 7. If $\alpha \in \mathrm{Aut}(PG(\Delta))$ is a (P,L)-elation for some incident point-line pair (p,L) in $PG(\Delta)$, there exists $\nabla \in \mathrm{Aut}(\Delta)$ such that $\nabla_\infty \equiv \alpha$ and ∇ has at least one fixed vertex in Δ.

PROOF. Suppose Q is a point of $PG(\Delta)$ not incident with L. We coordinatize $PG(\Delta)$ with respect to a non-degenerate quadrangle $(\mathbf{O},X,Y,E)$ by a local PTR (R,T,v), where we choose $\mathbf{O}=Q$, $(0,1) = Q^\alpha$, $Y=P$ and $L=XY$. By theorem 3, $u_s(Q,L) = u_s(Q^\alpha,L) = 0$ for $s=s(\mathbf{O},X,Y,E)$. The result follows from lemma 6 and theorem 6.

Q.E.D.

COROLLARY 2. Suppose $\nabla \in \mathrm{Aut}(\Delta)$ and ∇_∞ is a (P,L)-elation of $PG(\Delta)$ for some incident point-line pair (P,L) of $PG(\Delta)$. Then ∇ acts type-preserving. For any positive integer n and any vertex s fixed by ∇, ∇ induces a $(\Pi_n^s(P),\Pi_n^s(L))$-elation in V_n^s.

PROOF. The last assertion follows from the reconstruction of ∇ from ∇_∞ as in the proof of theorem 6. Now, if s is such a fixed vertex, then $\{s,\Pi_1^s(P),\Pi_1^s(L)\}$ is a fixed chamber and hence ∇ acts type-preserving on Δ.

Q.E.D.

LEMMA 7. Suppose $\lambda \in \mathrm{Aut}(PG(\Delta))$ is a (P,L)-homology for some non-incident point-line pair (P,L) of $PG(\Delta)$. Let $s=s(P,P_1,P_2,Q)$, where P_1 and P_2 are both incident with L and Q is such that s is well-defined. If $s^* = s(P,P_1,P_2,Q^\lambda)$, then $u_s(Q_1,Q_2) = u_s(Q_1^\lambda,Q_2^\lambda)$, for all points Q_1,Q_2 of $PG(\Delta)$.

PROOF. The proof is similar to that of lemma 6, using additionally theorem 4 if $s \neq s^*$. Since the proof is uninformative, we restrict ourselves to show one particular case, which is representative for all cases. We coordinatize $PG(\Delta)$ by a local PTR (R,T,v) (resp. (R^*,T^*,v^*)) with respect to (P,P_1,P_2,Q) (resp. (P,P_1,P_2,Q^λ)). We will always use (R,T,v)-coordinates. So let $Q^\lambda = (q_1,q_2)$, then, since $Q = (1,1)$, there exists $q \in R$ such that $T(q,1,1) = 0 = T(q,q_1,q_2)$. By (v12), $v(q)+v(1) = v(1)$ (hence $v(q) = 0$) and $v(q)+v(q_1) = v(q_2)$, hence $v(q_1) = v(q_2)$. We will assume $v(q_1) > 0$. Suppose $Q_i = (x_i,y_i)$ and $Q_i^\lambda = (x_i^*,y_i^*)$, i=1,2 (and throughout this proof we will always assume i=1,2), and $u_s(Q_1,L) = u_s(Q_2,L)$. We show the assertion in this particular case. We already deduce $v(x_i) \geq 0$ and $v(y_i) \geq 0$. We now use the same notation $b_{\overline{X}}$ and $b_{\overline{Y}}$ as in theorem 4. By that theorem, $v(x,y) - v^*(b_{\overline{X}}(x),b_{\overline{X}}(y))$ is a constant for variable $x,y \in R$. Putting $x=q_1$ and $y=0$, we get $v(q_1)-v^*(1) = v(q_1) = v(x,y)-v^*(b_{\overline{X}}(x),b_{\overline{X}}(y))$, for all $x,y \in R$. Similarly $v(q_2) = v(x,y)-v^*(b_{\overline{Y}}(x),b_{\overline{Y}}(y))$, for all $x,y \in R$. So we must show

$$\begin{aligned}\inf\{v(x_1,x_2),v(y_1,y_2)\} &= \inf\{v^*(b_{\overline{X}}(x_1^*),b_{\overline{X}}(x_2^*)),v^*(b_{\overline{Y}}(y_1^*),b_{\overline{Y}}(y_2^*))\}\\ &= \inf\{v(x_1^*,x_2^*)-v(q_1),v(y_1^*,y_2^*)-v(q_2)\}.\end{aligned}$$

We will actually show $v(x_1,x_2) = v(x_1^*,x_2^*)-v(q_1)$ and the same equality for y_1,y_2 will follow similarly. Note first that x_i^* is independent on y_i, so $(x_i,0)^\lambda = (x_i^*,0)$ and $(0,1)^\lambda = (0,q_2)$, since similarly, the second coordinate of $(x,y)^\lambda$ is independent on x, for all $x,y \in R$. Hence there exists $m_i \in R$ such that

$$T(m_i,x_i,0) = 1 \qquad (1.i)$$

$$T(m_i,x_i^*,0) = q_2 \qquad (2.i)$$

By (1.1),(1.2) and (v11), $v(x_1,x_2) = v(m_1,m_2)+v(x_2)-v(m_1)$ and by (2.1), (2.2) and (V11), $v(x_1^*,x_2^*) = v(m_1,m_2)+v(x_2^*)-v(m_1)$. Hence, eliminating $v(m_1,m_2)$, we get $v(x_1,x_2) = v(x_1^*,x_2^*)+v(x_2)-v(x_2^*)$. By (1.2),(2.2) and (v12), $v(x_2) = -v(m_2)$ and $v(x_2^*) = -v(m_2)+v(q_2) = -v(m_2)+v(q_1)$. Hence the result follows.

Q.E.D.

THEOREM 8. If $\lambda \in \mathrm{Aut}(PG(\Delta))$ is a (P,L)-homology for some non-incident point-line pair (P,L) of $PG(\Delta)$, then there exists $\Lambda \in \mathrm{Aut}(\Delta)$ such that $\Lambda_\infty \equiv \lambda$.

PROOF. This follows directly from lemma 7 and theorem 6. Q.E.D.

COROLLARY 3. Suppose $\Lambda \in \mathrm{Aut}(\Delta)$ and Λ_∞ is a (P,L)-homology of $PG(\Delta)$ for some non-incident point-line pair (P,L) of $PG(\Delta)$. For every positive integer n and every vertex s fixed by Λ, Λ induces a $(\Pi_n^s(P),\Pi_n^s(L))$-perspectivity in V_n^s.

PROOF. Similarly to corollary 2. Q.E.D.

In section 6, we will see that Λ (where Λ_∞ is a homology) does not necessarily fix any vertex and that Λ is not necessarily type-preserving. We will construct examples of this in section 7.

DEFINITION. Suppose $\phi \in \mathrm{Aut}(PG(\Delta))$. We say that ϕ is induced by a PTR-automorphism if there exists a coordinatizing PTR (R,T) for $PG(\Delta)$ and a PTR-automorphism ρ such that $(x,y)^\phi = (x^\rho,y^\rho)$, $(x)^\phi = (x^\rho)$, $(\infty)^\phi = (\infty)$, $[m,k]^\phi = [m^\rho,k^\rho]$, $[k]^\phi = [k^\rho]$, $[\infty]^\phi = [\infty]$ for all $x,y,m,k \in R$. An automorphism of (R,T) is a permutation of R preserving T.

THEOREM 9. Suppose $\phi \in \mathrm{Aut}(PG(\Delta))$ is induced by a PTR-automorphism $\rho : R \to R$ for some local PTR (R,T,v). There exists $\Phi \in \mathrm{Aut}(\Delta)$ such that $\Phi_\infty \equiv \phi$ if and only if ρ preserves the valuation.

PROOF. If ρ preserves the valuation, then the result follows from theorems 3 and 6. Suppose now there exists $\Phi \in \mathrm{Aut}(\Delta)$ such that $\Phi_\infty \equiv \phi$. Consider the quadrangle (O,X,Y,E) (see 2.1.1). Since all of the four points O,X,Y,E are fixed by ϕ, the vertex s(O,X,Y,E) is fixed by Φ. Hence ϕ preserves the partial valuation $u_{s(O,X,Y,E)}$ by theorem 6. It is now straight forward to check (in view of theorem 3) that ρ preserves the valuation v. Q.E.D.

5. PERSPECTIVITIES IN THE n^{TH} FLOORS OF Δ.

We now relate the (P,L)-transitivity of $PG(\Delta)$ to the (P^n,L^n)-transitivity of the n^{th} floors of Δ.

5.1. DEFINITIONS. Suppose $b \in Ve(\Delta)$ and $n \in \mathbb{N}$. If $L^n, M^n \in L(V_n^b)$, then we call V_n^b (M^n,L^n)-transitive if V_n^b is (P^n,L^n)-transitive for all P^n I M^n. A similar definition holds in $PG(\Delta)$ (in fact the classical definition, see Hughes-Piper [7]). If V_n^b is (L^n,L^n)-transitive, then L^n is called a translation line and V_n^b is called a translation floor. Dually, one can define translation point and dual translation floor. A division ring floor is an n^{th} floor having a translation line L^n and a translation point P I L. If every line of V_n^b is a translation line, then V_n^b is called a Moufang floor. If V_n^b is (P^n,L^n)-transitive for all points P^n and all lines L^n of V_n^b, then V_n^b is called Desarguesian.

Since the n^{th} floor of Δ with basement b is a projective Hjelmslev plane of level n, we have the following well known result.

THEOREM 10 (Dugas [5]). If $b \in Ve(\Delta)$, $n \in \mathbb{N}^*$, V_n^b is Moufang and V_1^b is a finite projective plane not of order 2, then V_n^b is Desarguesian.

We also have

PROPOSITION 11 (Van Maldeghem [12]). If Δ is locally finite and $PG(\Delta)$ is Moufang, then $PG(\Delta)$ is Desarguesian and hence Δ is classical.

5.2. The effect of (P,L)-transitivity of $PG(\Delta)$ on the floors.

PROPOSITION 12. Suppose $PG(\Delta)$ is (P,L)-transitive and let $b \in Ve(\Delta)$ and $n \in \mathbb{N}^*$. Then V_n^b is $(\Pi_n^b(P),\Pi_n^b(L))$-transitive.

PROOF. If P I L, the result follows from lemmas 4,6 and theorem 6. So suppose $P \not I L$. First assume $u_b(P,L) = 0$. Let $Q_0^n, Q_1^n \in P(V_n^b)$ be such that Q_0^n, Q_1^n and $P^n = \Pi_n^b(P)$ are collinear and $u_b(Q_i^n,P^n) = u_b(Q_i^n,L^n) = 0$, $i=0,1$.

Let $P_1^n, P_2^n \in P(V_n^b)$ be incident with $L^n = \Pi_n^b(L)$ and such that $u_b(P_1^n, Q^n) = u_b(P_2^n, Q^n) = 0$, where Q^n is the intersection point of $Q_0^n P^n$ and L^n, well defined by proposition 6. By lemma 4, there exist points P_1, P_2, Q_0, Q_1 of $PG(\Delta)$ such that P_1 and P_2 are incident with L; Q_0, Q_1 and P are collinear and $\Pi_n^b(P_1) = P_1^n$, $\Pi_n^b(P_2) = P_2^n$, $\Pi_n^b(Q_0) = Q_0^n$, $\Pi_n^b(Q_1) = Q_1^n$. Consequently $b = s(P, P_1, P_2, Q_0) = s(P, P_1, P_2, Q_1)$. Let λ be the (P,L)-homology of $PG(\Delta)$ mapping Q_0 to Q_1 and let $\Lambda \in Aut(\Delta)$ be such that $\Lambda_\infty \equiv \lambda$. By theorem 6, $b^\Lambda = b$ and the result follows from corollary 3 (and the construction of Λ in the proof of theorem 6). Now suppose $u_b(P,L) > 0$, then the result will follow by an analogous argument if we show that, for every two points Q_0, Q_1 in $PG(\Delta)$ satisfying $u_b(Q_0, L) = u_b(Q_1, L) = 0$ and Q_0, Q_1, P collinear, the (P,L)-homology in $PG(\Delta)$ mapping Q_0 to Q_1 induces an automorphism of Δ fixing b. Well, since $u_b(Q_0, L) = 0$, one can find P_1 I L and P_2 I L and a point E such that $b = s(Q_0, P_1, P_2, E)$. Let μ be the (P,L)-homology of $PG(\Delta)$ mapping Q_0 to Q_1. Then it is a consequence of propositions 5 and 6 (for $n=\infty$) and the construction of E^μ from $Q_0^\mu = Q_1$ that $b = s(Q_1, P_1, P_2, E^\mu)$. Q.E.D.

COROLLARY 4. Suppose $b \in Ve(\Delta)$ and $n \in \mathbb{N}$. If $PG(\Delta)$ is a translation (resp. dual translation, division ring, Moufang, Desarguesian) plane, then V_n^b is a translation (resp. dual translation, division ring, Moufang, Desarguesian) floor.

PROOF. Follows directly from proposition 12. Q.E.D.

5.3. The effect of (P^n, L^n)-transitivity of floors on $PG(\Delta)$.

PROPOSITION 13. Suppose (P,L) is some point-line pair in $PG(\Delta)$ and $b \in Ve(\Delta)$. Suppose Q_0 and Q_1 are two points collinear with P and $u_b(Q_i, P) = u_b(Q_i, L) = 0$, $i=0,1$. Suppose $\psi_n : V_n^b \to V_n^b$ is a $(\Pi_n^b(P), \Pi_n^b(L))$-perspectivity mapping $\Pi_n^b(Q_0)$ to $\Pi_n^b(Q_1)$, for all $n \in \mathbb{N}^*$. Then there exists a (P,L)-perspectivity $\Psi \in Aut(PG(\Delta))$ mapping Q_0 to Q_1.

PROOF. By proposition 10, $\psi_n(\Pi^b_{n+1,n}(X^{n+1})) = \Pi^b_{n+1,n}(\psi_{n+1}(X^{n+1}))$, for all

$X^{n+1} \in P(V^b_{n+1}) \cup L(V^b_{n+1})$. The result follows by taking the inverse limit.

Q.E.D.

THEOREM 11. Suppose (P,L) is an incident point-line pair of $PG(\Delta)$ and let p (resp. ℓ) be any pennel having P (resp. L) as trace at infinity. If V^b_n is $(\Pi^b_n(P),\Pi^b_n(L))$-transitive for all $n \in \mathbb{N}^*$ and all b on p (resp. ℓ), then $PG(\Delta)$ is (P,L)-transitive.

PROOF. Let b_0 be the source of p. Choose $X \text{ I } L$, O and E such that $b_0 = s(O,X,P,E)$ and denote by Σ the apartment of Δ corresponding to the triangle $\{O,X,P\}$ in $PG(\Delta)$. Let $O^* \text{ I } OP$ be arbitrary. If we show that there exists a (P,L)-elation mapping O to O^*, then we are done. If $u_{b_0}(O^*,L) = 0$, then the result follows from proposition 13. Suppose now $u_{b_0}(O^*,L) > 0$. By theorems 3 and 4, there exists a vertex b on p such that $u_b(O^*,L) = 0$ (namely, b has barycentric coordinates $(k_0,l_0,m_0) = (\frac{1-k}{3},\frac{1-k}{3},\frac{1+2k}{3})$ with respect to $(b_0;\Pi_1^{b_0}(O),\Pi_1^{b_0}(X),\Pi_1^{b_0}(P))$ in , where $k = u_{b_0}(O^*,L) = u_{b_0}(O^*,P)$). Hence, the result again follows from proposition 13. Dually for L and ℓ.

Q.E.D.

COROLLARY 5. Suppose ℓ (resp. p) is a pennel of Δ with trace at infinity a line L (resp. point P) of $PG(\Delta)$. If $\Pi^b_n(L)$ (resp. $\Pi^b_n(P)$) is a translation line (resp. point) of V^b_n for all $n \in \mathbb{N}^*$ and for all vertices b on ℓ (resp. p), then L (resp. P) is a translation line (resp. point) of $PG(\Delta)$.

COROLLARY 6. * Suppose ℓ (resp. p) is a pennel of Δ with trace at infinity a line L (resp. point P) of $PG(\Delta)$. Then $PG(\Delta)$ is a translation plane (resp. dual translation plane) with translation line L (resp. point P) if and only if V^b_n is a translation floor (resp. dual translation floor) with translation line $\Pi^b_n(L)$ (resp. point $\Pi^b_n(P)$) for all b on ℓ (resp. p) and all $n \in \mathbb{N}^*$.

* Suppose Q is a quarter with trace at infinity $\{P,L\}$. Then $PG(\Delta)$ is a division ring plane with translation line L and translation point P if and only if V^b_n is a division ring floor with translation line $\Pi^b_n(L)$ and translation point $\Pi^b_n(P)$ for all b on the bounding pennels of Q and all $n \in \mathbb{N}^*$.

* Suppose p is an arbitrary pennel of Δ. Then PG(Δ) is Moufang if and only if V_n^b is Moufang for all b on p and all $n \in \mathbb{N}^*$.

PROOF. The first two statements follow directly from corollary 5. We show now the last assertion. If PG(Δ) is Moufang, then V_n^b is Moufang for all vertices b by corollary 4. Suppose now V_n^b is Moufang for all b on p and all $n \in \mathbb{N}^*$. By corollary 5, the trace at infinity X of p is a translation line or a translation point. But by proposition 13, Aut(PG(Δ)) contains elations not fixing X. Hence PG(Δ) has at least two translation lines or points and therefore it is Moufang (see Hughes-Piper [7],§VI,6).

Q.E.D.

It is an open question wether V_n^b Moufang for all $n \in \mathbb{N}^*$ and fixed $b \in Ve(\Delta)$ implies PG(Δ) Moufang.

The following result is well known although we could not find a precise reference. One can prove it by taking the inverse limit of local rings coordinatizing projective Hjelmslev planes.

THEOREM 12. Suppose $b \in Ve(\Delta)$. Then PG(Δ) is Desarguesian if and only if V_n^b is Desarguesian for all $n \in \mathbb{N}^*$.

COROLLARY 7. Suppose $b \in Ve(\Delta)$, Δ is locally finite and the residues do not have order 2. If V_n^b is Moufang for all $n \in \mathbb{N}^*$, then PG(Δ) is Desarguesian.

6. A THEOREM OF FIXED VERTICES.

THEOREM 13. (1). Suppose $\nabla \in Aut(\Delta)$ with ∇_∞ an elation of PG(Δ). For every $n \in \mathbb{N}^*$, there exists a vertex $b \in Ve(\Delta)$ such that all vertices at distance at most n from b are fixed by ∇.

(2). Suppose $\Phi \in Aut(\Delta)$, Φ_∞ is induced by a PTR-automorphism $\rho : (R,T,v) \to (R,T,v)$ and (R,T,v) coordinatizes PG(Δ) with respect to (O,X,Y,E). Then Φ fixes all vertices of the apartments of Δ corresponding to every triangle in the set {O,X,Y,E,(1,0),(0,1),(1)}.

(3). There exists a building Γ and an automorphism Λ (resp. Λ^*) of Γ such that Λ_∞ (resp. Λ^*_∞) is a homology in PG(Γ) and Λ has (resp. Λ^* has no) fixed vertices in Γ.

PROOF. (1). Suppose ∇_∞ is a (P,L)-elation, P I L and $n \in \mathbb{N}^*$. Choose any apartment Σ in Δ such that the trace at infinity of Σ contains P and L. Suppose Σ_∞ contains furthermore the points $O \not{I}$ L and X I L, $X \neq P$. One can choose a point E such that $u_s(O, O^{\nabla_\infty}) = n$ for $s = s(O,X,P,E) \in \Sigma$ (cp. theorem 4). But by lemma 6, ∇_∞ preserves u_s and by theorem 6, $s^\nabla = s$. By corollary 2 and proposition 10, ∇ acts trivially on V_j^s for all $j \leq n$. Hence ∇ fixes all vertices of Δ corresponding to V_j^s, $j \leq n$. But these are exactly the vertices at distance $j \leq n$ from s (see the construction of Δ [11]).

(2). We show e.g. Φ fixes all vertices of Σ, where Σ_∞ is determined by {O,X,Y}, pointwise. By theorem 6, Φ fixes s(O,X,Y,E) in Σ. Hence also all pennels o,x,y with trace at infinity resp. O,X,Y and source s(O,X,Y,E) are fixed pointwise. Hence Σ is fixed pointwise.

(3). We consider any classical building Γ. Suppose PG(Γ) is coordinatized by a local field F,+,.,v with respect to the quadrangle (O,X,Y,E). Let Λ_∞ be the (O,XY)-homology of PG(Γ) mapping (1,1) to (t,t), with v(t) = 0, $t \neq 1$. Then $s(O,X,Y,E) = s(O,X,Y,E^{\Lambda_\infty})$ and hence by theorem 6, s(O,X,Y,E) is a fixed vertex. Let Λ^* be the (O,XY)-homology of PG(Γ) mapping (1,1) to (t*,t*), with v(t*) > 0. Since $u_{s(O,X,Y,E)}(O,(t^*,t^*)) > 0$, $s(O,X,Y,E) \neq s(O,X,Y,E^{\Lambda^*_\infty})$, and hence s(O,X,Y,E) is not fixed. Suppose some vertex b is fixed by Λ. Then Λ induces a $(\Pi_n^b(P),\Pi_n^b(L))$-perspectivity in V_n^b, for all $n \in \mathbb{N}^*$. This perspectivity fixes $\Pi_n^b(X)$ and $\Pi_n^b(L)$ and hence Λ fixes the quarter Q with source b and trace at infinity {X,L}. If we denote by Σ the apartment of Γ determined by {O,X,Y} in PG(Γ), then clearly Λ stabilizes Σ. But since $\Sigma \cap Q$ contains a quarter fixed by Λ, Λ must fix Σ pointwise, contradicting the fact that Λ does not fix s(O,X,Y,E).

Q.E.D.

7. A CLASS OF TRIANGLE BUILDINGS WITH NON-CLASSICAL RESIDUES AND VERTEX-TRANSITIVE AUTOMORPHISM GROUP.

DEFINITION. In this section, we denote by $(R,+,.)$ a fixed division ring (finite or infinite). We define the set of formal Laurent series over R as

$$R((t)) = \{\sum_{i=k}^{\infty} a_i t^i \mid k \in \mathbb{Z},\ a_i \in R \text{ and } a_k \neq 0\}.$$

We extend the addition and multiplication of R to $R((t))$ in the usual way by

$$\sum_{i=k_1}^{\infty} a_i t^i + \sum_{i=k_2}^{\infty} b_i t^i = \sum_{i=\inf\{k_1,k_2\}}^{\infty} (a_i+b_i)t^i$$

(where $a_i=0$ if $i<k_1$ and $b_i=0$ if $i<k_2$) and

$$\sum_{i=k_1}^{\infty} a_i t^i \cdot \sum_{i=k_2}^{\infty} b_i t^i = \sum_{i=k_1+k_2}^{\infty} \left(\sum_{j=k_1}^{i-k_2} a_j b_{i-j}\right) t^i.$$

Defining $v(\sum_{i=k}^{\infty} a_i t^i)=k$ (for $a_k \neq 0$) and $v(a-b)=v(a,b)$, for all $a,b \in R((t))$, one easily verifies that $(R((t)),+,.,v)$ is a local division ring. We denote the corresponding triangle building by $\Delta(R)$.

THEOREM 14. $\mathrm{Aut}(\Delta(R))$ acts vertex-transitivily on $\Delta(R)$.

PROOF. We coordinatize $PG(\Delta(R))$ by means of $R((t))$ with respect to some quadrangle (O,X,Y,E) and denote $s = s(O,X,Y,E) \in Ve(\Delta(R))$. Let $b \in Ve(\Delta(R))$ be arbitrary. We show that there exists $\Psi \in \mathrm{Aut}(\Delta(R))$ such that $b^{\Psi} = s$. Note that $L=XY$ is a translation line for $PG(\Delta(R))$. Suppose $\ell_b \in Pe(\Delta(R),b)$ has L as trace at infinity. Considering any apartment containing ℓ_b, we obtain a wall W containing ℓ_b. Suppose $\{P,L\}$ is the trace at infinity of W. Let $(\Psi_1)_\infty$ be the elation (fixing L pointwise) mapping P to O. Then W^{Ψ_1} is a wall lying in some apartment Σ' where Σ'_∞ is determined by $\{O,Y,P'\}$, P' I L (by Tits [10], proposition 4 and 17.3). Hence b is mapped by Ψ_1 to a vertex b_1 lying in Σ'. Since $R((t))$ is a division ring, there exists a (Y,OY)-elation $(\Psi_2)_\infty$ mapping P' to X. Hence Ψ_2 maps Σ' to the apartment Σ, where Σ_∞ is determined by $\{O,X,Y\}$, and it maps b_1 to a vertex b_2 in Σ. Suppose b_2 has barycentric coordinates $(\frac{1+k+m}{3},\frac{1+k-2m}{3},\frac{1-2k+m}{3})$ in Σ with respect to $(s;\Pi_1^s(O),\Pi_1^s(X),\Pi_1^s(Y))$ (one easily verifies that every vertex of Σ has barycentric coordinates of that form). Consider the following automorphism $(\Psi_3)_\infty$ of PG(Δ(R)), given by the action on the coordinates.

$$
\begin{aligned}
(x,y) &\to (t^{-k}x, t^{-m}y),\\
(x) &\to (t^{k-m}x),\\
(\infty) &\to (\infty),\\
[x,y] &\to [t^{k-m}x, t^{-m}y],\\
[x] &\to [t^{-k}x],\\
[\infty] &\to [\infty].
\end{aligned}
$$

This is well defined since multiplication by a power of t is commutative and associative in R((t)). Now, Ψ_3 is well defined since $(\Psi_3)_\infty$ is the juxtaposition of a (**O**,XY)-homology and a (X,**O**Y)-homology. Note that $(\Psi_3)_\infty$ maps (t^k,t^m) to E=(1,1). By theorem 4, s(**O**,X,Y,(t^k,t^m)) = b_2 (since with the notation of theorem 4, the barycentric coordinates of s(**O**,X,Y,(t^k,t^m)) must satisfy $m_0-l_0=-k$ and $l_0-k_0=-m$ because recoordinatizing PG(Δ(R)) with respect to (**O**,X,Y,(t^k,t^m)), the new point (1,0) has old coordinates $(t^k,0)$ and the new point (0,1) has old coordinates $(0,t^m)$). But $(\Psi_3)_\infty$ fixes **O**,X, Y, hence Ψ_3 maps b_2 to s.

Q.E.D.

REMARK. The map Ψ_3 in the above proof is a concrete example of an automorphism not preserving the types of the vertices, if $typ_\Delta(b_2) \neq typ_\Delta(s)$.

COROLLARY 8. Every residue in Δ(R) is isomorphic to the projective plane coordinatized by the division ring (R,+,.).

PROOF. The projective plane V_1^s (where $s \in Ve(\Delta(R))$ is as above) is coordinatized by (R,+,.) by [11],2.7.2. The result follows from theorem 14.

Q.E.D.

REFERENCES

1. B. Artmann, Existenz und projektive Limiten von Hjelmslev-Ebenen n-ter Stufe, Atti del Convegno di Geometria Combinatoria e sue Applicazioni, Perugia (1971), 27-41.

2. N. Bourbaki, Groupes et algèbres de Lie, chapters IV,V and VI, Paris, Hermann, 1966.

3. F. Bruhat and J. Tits, Groupes réductifs sur un corps local, I. Données radicielles valuées, Publ. Math. Inst. Hautes Etudes Scientifiques 41 (1972), 5-251.

4. F. Buekenhout, Diagrams for geometries and groups, J. Comb. Theory (A) 27 (1972), 121-151.

5. M. Dugas, Moufang Hjelmslev-Ebenen, Arch. Math. 28 (1977), 318-322.

6. G. Hanssens and H. Van Maldeghem, On projective Hjelmslev planes of level n, Glasgow Math.J. 31 (1989), 257-261.

7. D.R. Hughes and F.C. Piper, Projective planes, Springer-Verlag (1972).

8. M.A. Ronan, A universal construction of buildings with no rank 3 residue of spherical type, in "Buildings and the geometry of diagrams", Lect. Notes 1181, Springer-Verlag (1986), 242-248.

9. J. Tits, Buildings of spherical type and finite BN-pairs, Lect. Notes 386, Springer-Verlag (1974).

10. J. Tits, Immeubles de type affine, in "Buildings and the geometry of diagrams", Lect. Notes 1181, Springer-Verlag (1986), 159-190.

11. H. Van Maldeghem, Non-classical triangle buildings, Geom. Ded. 24 (1987), 123-206.

12. H. Van Maldeghem, On locally finite alternative division rings with valuation, J. Geometry 30 (1987), 42-48.

13. H. Van Maldeghem, Valuations on PTRs induced by triangle buildings, Geom. Dedicata 26 (1988), 29-84.

Address of the author :

H. Van Maldeghem

Rijksuniversiteit Gent
Seminarie voor Meetkunde en Kombinatoriek

Krijgslaan 281

B - 9000 Gent

BELGIUM