MITTEILUNGEN

aus dem

MATHEM. SEMINAR GIESSEN

Herausgegeben von den Professoren
des Mathematischen Instituts der Universität Giessen

Geschäftsführung: D. Gaier, F. Timmesfeld

Sonderdruck aus Heft 198

GIESSEN 1990

SELBSTVERLAG DES MATHEMATISCHEN INSTITUTS
ISSN 0373-8221 CODEN: MMUGAU

AN ALGEBRAIC CHARACTERIZATION OF AFFINE BUILDINGS OF TYPE $\tilde{C}_2$

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We show that every (symmetric) affine building of type $\tilde{C}_2$ is uniquely and completely determined by any coordinatizing quadratic quaternary ring with valuation of its generalized quadrangle at infinity.

INTRODUCTION

This paper is the last one in a sequence of four which have as purpose to characterize the class of affine buildings of type $\tilde{C}_2$, the first three papers of this sequence being [9],[10] and [20].

Let us briefly describe the situation. In 1984, J.Tits classified all affine buildings of rank ≥4 by showing they all arise from algebraic groups over a local field (see [3],[16]). But counter examples to this statement were known for affine buildings of rank 3 (see e.g. [11],[13], [17] and recently [20]). They were called *non classical* buildings. Now there are three classes of affine buildings of rank 3. They correspond to the following diagrams.

* The author was supported by the National Fund for Scientific Research (N.F.W.O.) of Belgium.

type $\tilde{A}_2$

type $\tilde{C}_2$

type $\tilde{G}_2$

An algebraic characterization of affine buildings of type $\tilde{A}_2$ is given in [17] and [18], by means of the notion of a *planar ternary ring with valuation*. The present paper presents the analogue for affine buildings of type $\tilde{C}_2$. In [20], we introduced the notion of a *quadratic quaternary ring with valuation* and showed that every such algebraic structure $\mathcal{R}$ gives rise to an explicitly defined (symmetric) affine building Δ of type $\tilde{C}_2$. In this paper, we prove that there is a reverse procedure and that both operations are mutually inverse. In fact, $\mathcal{R}$ coordinatizes the generalized quadrangle at infinity of Δ. This leads us to the main result of this paper :

MAIN THEOREM. *Any coordinatizing quadratic quaternary ring of the generalized quadrangle at infinity of a (symmetric) affine building of type* $\tilde{C}_2$ *is naturally endowed with a valuation, which determines the building up to isomorphism.*

The proof of this theorem will be completed in section 4.

The motivation for our research, as explained in [10], lies beyond the pure construction of new explicitly defined affine buildings of type $\tilde{C}_2$. What we want in the first place is a deeper geometrical insight in affine buildings of rank 3. The case $\tilde{A}_2$ was "easy" enough (because of the symmetric diagram and the fact that there are only projective planes around) to get us started and we believe that the case $\tilde{C}_2$ is general enough to gain the desired insight (that is the reason why we consider the case $\tilde{G}_2$ as uninteresting). In our opinion, a better geometrical understanding of affine buildings of rank 3 could be helpful to approach other types of buildings (starting with rank 3) which are not understood well yet. Of course we also aim at direct applications : the explicit description of certain buildings of types $\tilde{A}_2$ and $\tilde{C}_2$ might e.g. lead to new GABs or to another description of existing GABs.

The paper is organised as follows. In a first section, we recall the definitions of *Hjelmslev quadrangle of level n*, a quadratic quaternary ring with valuation and a symmetric affine building of type $\tilde{C}_2$. In section 2, we state some main results which are proved in section 3. In fact, by previous results in [9],[10] and [20], it will basically suffice to show how one can put a valuation on any coordinatizing quadratic quaternary ring of the generalized quadrangle at infinity of an arbitrary affine building of type $\tilde{C}_2$. In section 4, we finish off the proof of our main theorem. Section 5 finally deals with some examples.

REMARK. One will find most definitions in section 1 quite long and rather difficult to work with when one is not accostumed to them. It will be surprising that relatively simple examples can be found. But in order not to overload this paper, we will only give examples of the most important notion : the quadratic quaternary rings.

ACKNOWLEDGEMENT

I am very grateful to J. Tits for some helpful remarks on the subject of the paper.

1. DEFINITIONS AND NOTATION

1.1. Definition of a Hjelmslev quadrangle of level n.

1.1.1. Notation.

Suppose $X = (\mathcal{P}(X), \mathcal{L}(X), I)$ is a point-line incidence geometry with point set $\mathcal{P}(X)$ and line set $\mathcal{L}(X)$. We denote the set of points incident with a given line $\mathcal{L}$ by $\sigma(\mathcal{L})$ and call it the *shadow* (*of* $\mathcal{L}$) (see Buekenhout[4]). Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are two distinct lines of X. If there is a point incident with both $\mathcal{L}_1$ and $\mathcal{L}_2$, then we call $\mathcal{L}_1$ and $\mathcal{L}_2$ *concurrent* and we denote "$\mathcal{L}_1 \perp \mathcal{L}_2$" . Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are two distinct points of X. If there is a line incident with both $\mathcal{P}_1$ and $\mathcal{P}_2$, then we call $\mathcal{P}_1$ and $\mathcal{P}_2$

collinear and we denote "$\mathcal{P}_1 \perp \mathcal{P}_2$". A *flag* in X is an incident point-line pair of X. The set of flags of X is denoted by $\mathcal{F}(X)$. A *morphism* from X to some other point-line incidence geometry $X' = (\mathcal{P}(X'),\mathcal{L}(X'),I)$ maps $\mathcal{P}(X)$ to $\mathcal{P}(X')$, $\mathcal{L}(X)$ to $\mathcal{L}(X')$ and the map induced on $\mathcal{F}(X)$ maps $\mathcal{F}(X)$ to $\mathcal{F}(X')$. An *epimorphism* is a morphism which is surjective on the set of flags. The geometry X is called *thick* if every line is incident with at least three points and every point is incident with at least three lines.

Suppose $\mathcal{A}$ is an arbitrary set and $\mathbf{P}_1(\mathcal{A})$ and $\mathbf{P}_2(\mathcal{A})$ are two arbitrary partitions of $\mathcal{A}$. then we say that $\mathbf{P}_1(\mathcal{A})$ is *properly finer than* $\mathbf{P}_2(\mathcal{A})$ if *every* class of $\mathbf{P}_2(\mathcal{A})$ is the union of at least *two* classes of $\mathbf{P}_1(\mathcal{A})$. In that case, we denote

$$\mathbf{P}_2(\mathcal{A})/\mathbf{P}_1(\mathcal{A}) = \{\{\mathcal{C} \in \mathbf{P}_1(\mathcal{A}) \mid \mathcal{C} \subseteq \mathcal{D}\} \mid \mathcal{D} \in \mathbf{P}_2(\mathcal{A})\},$$

which is a partition of $\mathbf{P}_1(\mathcal{A})$. If $\mathcal{D} \in \mathbf{P}_2(\mathcal{A})$, then we call $\{\mathcal{C} \in \mathbf{P}_1(\mathcal{A}) \mid \mathcal{C} \subseteq \mathcal{D}\}$ the *canonical image of* $\mathcal{D}$ *in* $\mathbf{P}_2(\mathcal{A})/\mathbf{P}_1(\mathcal{A})$.

1.1.2. Generalized quadrangles.

Let $\mathcal{S} = (\mathcal{P}(\mathcal{S}),\mathcal{L}(\mathcal{S}),I)$ a be point-line incidence geometry. Then we call $\mathcal{S}$ a *generalized quadrangle* if there exist positive integers $s \geq 1$ and $t \geq 1$ (s and/or t may also be infinite) such that the following axioms hold.

(Q1) *Every point of* $\mathcal{S}$ *is incident with* $1+t$ *lines and two distinct points are incident with at most one line.*

(Q2) *Every line of* $\mathcal{S}$ *is incident with* $1+s$ *points and two distinct lines are incident with at most one point.*

(Q3) *If* $\mathcal{P} \in \mathcal{P}(\mathcal{S})$ *and* $\mathcal{L} \in \mathcal{L}(\mathcal{S})$ *are not incident, then there exists a unique flag* $(\mathcal{Q},\mathcal{M}) \in \mathcal{F}(\mathcal{S})$ *such that* $\mathcal{P}\ I\ \mathcal{M}\ I\ \mathcal{Q}\ I\ \mathcal{L}$.

Generalized quadrangles were introduced by J.Tits in his celebrated paper [14]. More information about generalized quadrangles can be found in e.g. Payne-Thas [12] or in the survey paper of W.M.Kantor [11]. In this paper, we will always assume that every generalized quadrangle is thick.

1.1.3. Definition of a Hjelmslev quadrangle of level n.
Throughout, n, i, j and k denote positive integers.
We define a Hjelmslev quadrangle of level n by induction on n. The induction will start with $n=1$. We give a separate definition for level 0. We abbreviate "Hjelmslev quadrangle of level n" by "level n HQ".

A *level* 0 *HQ* is any trivial geometry $\mathcal{V}_0 = (\{*\},\{*\},=)$, where $*$ is any arbitrary (but twice the same) symbol.

A *level* 1 *HQ* is any 6-tuple

$$\mathcal{V}_1 = (\mathcal{P}(\mathcal{V}_1),\mathcal{L}(\mathcal{V}_1),I,(\mathbf{P}_i(\mathcal{V}_1))_{i\le 1},(\mathbf{L}_i(\mathcal{V}_1))_{i\le 1},(\mathcal{W}_0(\mathcal{V}_1,\{\mathcal{P}\}),\{\mathcal{P}\})_{\mathcal{P}\in\mathcal{P}(\mathcal{V}_1)}),$$

where $(\mathcal{P}(\mathcal{V}_1),\mathcal{L}(\mathcal{V}_1),I)$ is an arbitrary generalized quadrangle ; $\mathbf{P}_0(\mathcal{V}_1)$ is the partition of $\mathcal{V}_1$ determined by: every class is a singleton ; $\mathbf{P}_1(\mathcal{V}_1)$ is the partition of $\mathcal{P}(\mathcal{V}_1)$ consisting of one class ; similar for $(\mathbf{L}_i(\mathcal{V}_n))_{i\le 1}$, and for every $\mathcal{P}\in\mathcal{P}(\mathcal{V}_1)$, $\mathcal{W}_0(\mathcal{V}_1,\{\mathcal{P}\}) = (\{\mathcal{P}\},\{\mathcal{P}\},=)$. The last three elements of $\mathcal{V}_1$ do not add more structure to the generalized quadrangle, but they are necessary to start the induction. So in fact, a level 1 HQ "is" a generalized quadrangle.

Now suppose $n\ge 2$. Let $(\mathcal{P}(\mathcal{V}_n),\mathcal{L}(\mathcal{V}_n),I)$ be a thick incidence geometry. Suppose $(\mathbf{P}_i(\mathcal{V}_n))_{i\le n}$, resp. $(\mathbf{L}_i(\mathcal{V}_n))_{i\le n}$ is a family of partitions of $\mathcal{P}(\mathcal{V}_n)$, resp. $\mathcal{L}(\mathcal{V}_n)$ satisfying :

(PS1) $\mathbf{P}_0(\mathcal{V}_n) = \{\{\mathcal{P}\}\mid\mathcal{P}\in\mathcal{P}(\mathcal{V}_n)\}$; $\mathbf{P}_n(\mathcal{V}_n) = \{\mathcal{P}(\mathcal{V}_n)\}$,

(PS2) $\mathbf{L}_0(\mathcal{V}_n) = \{\{\mathcal{L}\}\mid\mathcal{L}\in\mathcal{L}(\mathcal{V}_n)\}$; $\mathbf{L}_n(\mathcal{V}_n) = \{\mathcal{L}(\mathcal{V}_n)\}$,

(PS3) $\mathbf{P}_i(\mathcal{V}_n)$ is properly finer than $\mathbf{P}_{i+1}(\mathcal{V}_n)$, for all $i<n$,

(PS4) $\mathbf{L}_i(\mathcal{V}_n)$ is properly finer than $\mathbf{L}_{i+1}(\mathcal{V}_n)$, for all $i<n$,

The elements of $\mathbf{P}_i(\mathcal{V}_n)$, resp $\mathbf{L}_i(\mathcal{V}_n)$ are called *i-point-neighbourhoods, resp. i-line-neighbourhoods (of their elements)*. An i-point-neighbourhood is also called a *point-neighbourhood*, an *i-neighbourhood* or briefly a *neighbourhood*. Similar definitions for i-line-neighbourhoods. If $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$ and $\mathcal{L} \in \mathcal{L}(\mathcal{V}_n)$, then we denote by $\mathcal{O}^i(\mathcal{P})$, resp. $\mathcal{O}^i(\mathcal{L})$ the unique i-point-neighbourhood of $\mathcal{P}$, resp. i-line- neighbourhood of $\mathcal{L}$.

Suppose for every $\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n)$, we have a level $(n-1)$ HQ, denoted by $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ (this is an element of a well-defined class of objects by induction) and select in every $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ an $(n-2)$-point-neighbourhood $\mathcal{N}_{\mathcal{C}}$. Then we call the 6-tuple

$$\mathcal{V}_n = (\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I, (\mathbf{P}_i(\mathcal{V}_n))_{i \le n}, (\mathbf{L}_i(\mathcal{V}_n))_{i \le n}, (\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}), \mathcal{N}_{\mathcal{C}})_{\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n)})$$

a *level n HQ* if $\mathcal{V}_n$ satisfies the axioms (IS), (GQ) and (NP) below. The geometry $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$ is called the *base geometry* of $\mathcal{V}_n$. Before stating the actual axioms, we need some preliminaries.

We first define the canonical $(n-1)$-image of $\mathcal{V}_n$ by induction on n. The *canonical 0-image* of a level 1 HQ $\mathcal{V}_1$ is by definition the trivial geometry $(\{\mathcal{P}(\mathcal{V}_1)\}, \{\mathcal{P}(\mathcal{V}_1)\}, =)$. Now let $n \ge 2$. Define the geometry $(\mathbf{P}_1(\mathcal{V}_n), (\mathbf{L}_1(\mathcal{V}_n), I)$ as follows. If $\mathcal{C} \in \mathbf{P}_1(\mathcal{V}_n)$ and $\mathcal{D} \in (\mathbf{L}_1(\mathcal{V}_n)$, then $\mathcal{C}\ I\ \mathcal{D}$ if and only if there exist $\mathcal{P} \in \mathcal{C}$ and $\mathcal{L} \in \mathcal{D}$ which are incident in $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$. Furthermore, denote by $\mathcal{W}_{n-2}(\mathcal{V}_n, \mathcal{C})$ the canonical $(n-2)$-image of $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ (well-defined by the induction hypothesis). Denote by $\mathcal{N}'_{\mathcal{C}}$ the canonical image of $\mathcal{N}_{\mathcal{C}}$ in $\mathbf{P}_{n-2}(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}))/\mathbf{P}_1(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}))$ if $n > 2$ and $\mathcal{N}'_{\mathcal{C}} = \{\mathcal{P}(\mathcal{W}_1(\mathcal{V}_2, \mathcal{C}))\}$ if $n = 2$. Obviously, there is a bijective correspondence between $\mathbf{P}_{n-1}(\mathcal{V}_n)$ and $\mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ and the unique element of $\mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ corresponding with the element $\mathcal{C}$ of $\mathbf{P}_{n-1}(\mathcal{V}_n)$ is denoted by $\mathcal{C}^*$. In particular, all elements of $\mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ are denoted with a *. We define the *canonical (n-1)-image* of $\mathcal{V}_n$ as the 6-tuple

$$\mathcal{V}_{n-1} = (\mathbf{P}_1(\mathcal{V}_n), (\mathbf{L}_1(\mathcal{V}_n), I, (\mathbf{P}_{i+1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n))_{i \le n-1}, (\mathbf{L}_{i+1}(\mathcal{V}_n)/\mathbf{L}_1(\mathcal{V}_n))_{i \le n-1}, (\mathcal{W}_{n-2}(\mathcal{V}_n, \mathcal{C}), \mathcal{N}'_{\mathcal{C}})_{\mathcal{C}^* \in \mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)}).$$

We can now state the very natural axiom (IS).

(IS) *The canonical* $(n-1)$*-image* $\mathcal{V}_{n-1}$ *of* $\mathcal{V}_n$ *is a level* $n-1$ *HQ.*

Using a similar notation for $\mathcal{V}_{n-1}$ as for $\mathcal{V}_n$, (IS) implies e.g. $\mathbf{P}_i(\mathcal{V}_{n-1}) = \mathbf{P}_{i+1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ and similarly for the line-partitions.

Define inductively the *canonical* $(n-j)$*-image* of $\mathcal{V}_n$ $(0 < j \le n)$ as the canonical $(n-j)$-image $\mathcal{V}_{n-j}$ of the canonical $(n-j+1)$-image $\mathcal{V}_{n-j+1}$ of $\mathcal{V}_n$, or as $\mathcal{V}_n$ (for $j = 0$). Note that σ^1 defines a mapping from $\mathcal{P}(\mathcal{V}_n)$ to $\mathcal{P}(\mathcal{V}_{n-1})$ and from $\mathcal{L}(\mathcal{V}_n)$ to $\mathcal{L}(\mathcal{V}_{n-1})$. By the definition of the incidence relation in $\mathcal{V}_{n-1}$, we can see that this mapping is an epimorphism from $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$ onto $(\mathbf{P}_1(\mathcal{V}_n), (\mathbf{L}_1(\mathcal{V}_n), I)$. We denote this epimorphism by Π^n_{n-1}. By the induction hypothesis, a similar epimorphism exists from the base geometry of $\mathcal{V}_{n-j+1}$ onto the base geometry of $\mathcal{V}_{n-j}$ and we denote it by Π^{n-j+1}_{n-j}. By induction, we can put

$$\Pi^n_{n-j} = \Pi^{n-j+1}_{n-j} \circ \Pi^n_{n-j+1}.$$

From now on, we denote the canonical j-image $\mathcal{V}_j$ of $\mathcal{V}_n$ by

$$(\mathcal{P}_i(\mathcal{V}_j), \mathcal{L}_i(\mathcal{V}_j), I, (\mathbf{P}_i(\mathcal{V}_j))_{i \le j}, (\mathbf{L}_i(\mathcal{V}_j))_{i \le j}, (\mathcal{W}_{j-1}(\mathcal{V}_j, \mathcal{C}), \mathcal{N}_{\mathcal{C}})_{\mathcal{C} \in \mathbf{P}_{j-1}(\mathcal{V}_j)}),$$

for all j, $0 < j \le n$. The epimorphism Π^n_j is called the *projection*. We define the *valuation map* $u : (\mathcal{P}(\mathcal{V}_n) \cup \mathcal{L}(\mathcal{V}_n)) \times (\mathcal{P}(\mathcal{V}_n) \cup \mathcal{L}(\mathcal{V}_n)) \to \mathbf{N}$ as follows. Let x, y be either both points or both lines of $\mathcal{V}_n$, then

$$u(x,y) = \sup\{j \le n \mid \Pi^n_j(x) = \Pi^n_j(y)\}$$

If $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$ and $\mathcal{L} \in \mathcal{L}(\mathcal{V}_n)$, then

$$u(\mathcal{P},\mathcal{L}) = u(\mathcal{L},\mathcal{P}) = (u_1(\mathcal{P},\mathcal{L}), u_2(\mathcal{P},\mathcal{L}))$$

with

$$u_1(\mathcal{P},\mathcal{L}) = u_1(\mathcal{L},\mathcal{P}) = \sup\{j \le n \mid \exists Q \, I \, \mathcal{L} \text{ such that } \Pi^n_j(Q) = \Pi^n_j(\mathcal{P}),\ Q \in \mathcal{P}(\mathcal{V}_n)\},$$

$$u_2(\mathcal{P},\mathcal{L}) = u_2(\mathcal{L},\mathcal{P}) = \sup\{j \le n \mid \exists \mathcal{M} \, I \, \mathcal{P} \text{ such that } \Pi^n_j(\mathcal{M}) = \Pi^n_j(\mathcal{L}),\ \mathcal{M} \in \mathcal{L}(\mathcal{V}_n)\}.$$

We now write down the axiom (GQ), consisting of two statements (GQ1) and (GQ2).

(GQ1) *If* $\mathcal{P}, Q \in \mathcal{P}(\mathcal{V}_n)$, $\mathcal{L}, \mathcal{M} \in \mathcal{L}(\mathcal{V}_n)$, $Q \; I \; \mathcal{L} \; I \; \mathcal{P} \; I \; \mathcal{M}$, $u(\mathcal{P}, Q)=0$ *and* $\mathcal{L} \neq \mathcal{M}$, *then*

$$\sigma^{n-j}(Q) \cap \sigma(\mathcal{M}) \neq \emptyset \iff 2.j \leq u(\mathcal{L}, \mathcal{M})$$

(GQ2) *If* $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$, $\mathcal{L} \in \mathcal{L}(\mathcal{V}_n)$ *and* $u(\mathcal{P}, \mathcal{L}) = (k, 2k)$ *for some* $k \leq \frac{n}{2}$, *then there exists a unique* $\mathcal{M} \in \mathcal{L}(\mathcal{V}_n)$ *such that* $\mathcal{P} \; I \; \mathcal{M} \perp \mathcal{L}$. *Moreover,* $u(\mathcal{L}, \mathcal{M}) = 2k$ *and* $u(\mathcal{P}, Q) = 0$, *for all* $Q \in \sigma(\mathcal{L}) \cap \sigma(\mathcal{M})$. *If* $k=0$, *then* $u(Q_1, Q_2) \geq \frac{n}{2}$, *for all* $Q_1, Q_2 \in \sigma(\mathcal{L}) \cap \sigma(\mathcal{M})$.

We now define an affine structure on level j HQs. Suppose X_j is a level j HQ, $0<j<n$, with

$$X_j = (\mathcal{P}_i(X_j), \mathcal{L}_i(X_j), I, (\mathbf{P}_i(X_j))_{i \leq j}, (\mathbf{L}_i(X_j))_{i \leq j}, (\mathcal{W}_{j-1}(X_j, \mathcal{C}), \mathcal{N}_{\mathcal{C}})_{\mathcal{C} \in \mathbf{P}_{j-1}(X_j)}).$$

Let $X_1 = (\mathcal{P}(X_1), \ldots)$ be its canonical 1-image. Let $\mathfrak{D} \in \mathbf{P}_{j-1}(X_j)$ be arbitrary. We denote:

* $\mathcal{L}_{\mathfrak{D}}^{\infty} = \{\mathcal{L} \in \mathcal{L}(X_j) \mid \sigma(\mathcal{L}) \cap \mathfrak{D} \neq \emptyset\}$,
* $\mathcal{P}_{\mathfrak{D}}^{\infty} = \{\mathcal{P} \in \mathcal{P}(X_j) \mid \exists \mathcal{L} \in \mathcal{L}_{\mathfrak{D}}^{\infty}$ such that $\mathcal{P} \; I \; \mathcal{L}\}$,
* $\mathcal{AP}(X_j, \mathfrak{D}) = \mathcal{P}(X_j) - \mathcal{P}_{\mathfrak{D}}^{\infty}$,
* $\mathcal{AL}(X_j, \mathfrak{D}) = \mathcal{L}(X_j) - \mathcal{L}_{\mathfrak{D}}^{\infty}$.

We call the elements of $\mathcal{AP}(X_j, \mathfrak{D})$ the *affine points* (*of* $(X_j, \mathfrak{D})$, if there is confusion possible) and the elements of $\mathcal{AL}(X_j, \mathfrak{D})$ the *affine lines* (*of* $(X_j, \mathfrak{D})$). The elements of $\mathcal{P}_{\mathfrak{D}}^{\infty} - \mathfrak{D}$, resp. of $\mathcal{L}_{\mathfrak{D}}^{\infty}$ are called the *points*, resp. the *lines at infinity* (*of* $(X_j, \mathfrak{D})$). The elements of $\mathfrak{D}$ are the *hyperpoints* (*of* $(X_j, \mathfrak{D})$). The pair $(X_j, \mathfrak{D})$ is called an *affine HQ* (*of level* n). The following lemma is proved in [9].

LEMMA(1.1.3.1). *Let* $(X_j, \mathfrak{D})$ *be as above. Every element of the* $(j-1)$-*point-neighbourhood of any affine point is again an affine point. Hence every element of the* $(j-1)$-*point-neighbourhood of any point at infinity, resp. hyperpoint, is again a point at infinity, resp. hyperpoint.*

This lemma will give sense to axiom (NP) below.

We now introduce the notion of a "strip of width i" in an affine HQ $(X_j,\mathcal{D})$. Suppose $\mathcal{P}\in\mathcal{P}(X_j)$ is a point at infinity of $(X_j,\mathcal{D})$ and $\mathcal{L}\in\mathcal{L}(X_j)$ is an affine line incident with $\mathcal{P}$. If $i<j$, then we call the set

$$\{Q\in \mathcal{AP}(X_j,\mathcal{D}) \mid Q\ I\ \mathcal{M}\ I\ \mathcal{P} \text{ for some } \mathcal{M}\in \mathcal{O}^i(\mathcal{L})\}$$

a *strip of width* i *(in* $(X_j,\mathcal{D})$*)*. If $i\geq j$, then the set

$$\{Q\in \mathcal{AP}(X_j,\mathcal{D}) \mid Q\perp \mathcal{P}\}$$

is called a *strip of width* i *(in* $(X_j,\mathcal{D})$*)*. In every case, we call $\mathcal{P}$ a *base point (of the strip)*. It is not necessarily unique, even if the strip has width > 0 (cp.[9],property(2.26)).

We can now state the first part of (NP).

(NP1) *If* $\mathcal{C}\in \mathbf{P}_{n-1}(\mathcal{V}_n)$, *then* $\mathcal{AP}(\mathcal{W}_{n-1}(\mathcal{V}_n,\mathcal{C}),\mathcal{N}_{\mathcal{C}})=\mathcal{C}$. *Moreover, the* i*-point-neighbourhood of any point* $\mathcal{P}\in\mathcal{C}$ *in* $\mathcal{W}_{n-1}(\mathcal{V}_n,\mathcal{C})$ *coincides with the* i*-point-neighbourhood of* $\mathcal{P}$ *in* $\mathcal{V}_n$, *for all* $i\leq n-2$.

Suppose $\mathcal{C}_{n-j}\in \mathbf{P}_{n-j}(\mathcal{V}_n)$ and let $\mathcal{C}_{n-k}$ be the unique element of $\mathbf{P}_{n-k}(\mathcal{V}_n)$ containing $\mathcal{C}_{n-j}$ as a subset, $0\leq k\leq j<n$. By (NP1),

$$\mathcal{C}_{n-2}\in \mathbf{P}_{n-2}(\mathcal{W}_{n-1}(\mathcal{V}_n,\mathcal{C}_{n-1})),$$
$$\mathcal{C}_{n-3}\in \mathbf{P}_{n-3}(\mathcal{W}_{n-2}(\mathcal{W}_{n-1}(\mathcal{V}_n,\mathcal{C}_{n-1}),\mathcal{C}_{n-2})), \text{ etc....}$$

This way, we justify the following notation.

$$\mathcal{W}_{n-j}(\mathcal{V}_n,\mathcal{C}_{n-j})=\mathcal{W}_{n-j}(\mathcal{W}_{n-j+1}(\ldots(\mathcal{W}_{n-1}(\mathcal{V}_n,\mathcal{C}_{n-1}),\ldots),\mathcal{C}_{n-j+1}),\mathcal{C}_{n-j}).$$

Moreover, $\mathcal{C}_{n-j}=\mathcal{AP}(\mathcal{W}_{n-j}(\mathcal{V}_n,\mathcal{C}_{n-j}),\mathcal{N}_{\mathcal{C}_{n-j}})$.

The axiom (NP1) was about points of the point-neighbourhoods. The last axiom, (NP2), which we call the *strip axiom*, says something about the lines in the affine HQs corresponding to these neighbourhoods.

(NP2) *If* $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$, $\mathcal{L} \in \mathcal{L}(\mathcal{V}_n)$, $0<j<n$ *and* $\sigma(\mathcal{L}) \cap \mho^{n-j}(\mathcal{P}) \neq \emptyset$, *then the set*

$$S_j^n(\mathcal{P},\mathcal{L}) = \sigma(\mathcal{L}) \cap \mho^{n-j}(\mathcal{P})$$

is a strip of width j *in* $(\mathcal{W}_{n-j}(\mathcal{V}_n, \mho^{n-j}(\mathcal{P})), \mathcal{N}_{\mho^{n-j}(\mathcal{P})})$. *Every strip of width* 1 *in* $(\mathcal{W}_{n-1}(\mathcal{V}_n, \mho^{n-1}(\mathcal{P})), \mathcal{N}_{\mho^{n-1}(\mathcal{P})})$ *can be obtained in this way (putting* $j=1$)

This completes our list of axioms for a level n HQ.

We keep the same notation as above. Suppose $\mathcal{M}$ is an affine line of $(\mathcal{W}_{n-j}(\mathcal{V}_n, \mho^{n-j}(\mathcal{P})), \mathcal{N}_{\mho^{n-j}(\mathcal{P})})$ such that the set of affine points of $\mathcal{M}$ is a subset of $\sigma(\mathcal{L}) \cap \mho^{n-j}(\mathcal{P})$ (with the notation of (NP2) above), then we call $\mathcal{M}$ a *component of* $\mathcal{L}$, or a *component of the strip* $S_j^n(\mathcal{P},\mathcal{L})$ and we denote $\mathcal{M} < \mathcal{L}$. The set of all components of $S_j^n(\mathcal{P},\mathcal{L})$ is denoted by $C_j^n(\mathcal{P},\mathcal{L})$. The set of affine points of $\mathcal{M}$ is called the *affine shadow of* $\mathcal{M}$. As an extension, we call every point of $\mathcal{V}_n$ incident with $\mathcal{L}$ a *component of* $\mathcal{L}$.

Now let $\mathcal{V}'_n = (\mathcal{P}(\mathcal{V}'_n), \mathcal{L}(\mathcal{V}'_n), \ldots)$ be a second level n HQ and suppose

$$\Psi : (\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I) \rightarrow (\mathcal{P}(\mathcal{V}'_n), \mathcal{L}(\mathcal{V}'_n), I)$$

is an isomorphism of incidence geometries mapping the affine shadow of every component of any line $\mathcal{L}$ onto the affine shadow of a component of $\Psi(\mathcal{L})$ and mapping i-neighbourhoods onto i-neighbourhoods, for all i, $0 < i \leq n$, then we call $\mathcal{V}_n$ and $\mathcal{V}'_n$ *equivalent*. This way, we can extend Ψ to the set of all components of all lines of $\mathcal{V}_n$ and this extended map, which we also denote by Ψ, preserves "being component of". We call Ψ an *equivalence*.

We now define by induction the notion of an isomorphism between $\mathcal{V}_n$ and $\mathcal{V}'_n = (\mathcal{P}(\mathcal{V}'_n), \mathcal{L}(\mathcal{V}'_n), \ldots)$. If $n=1$, then $\mathcal{V}_1$ and $\mathcal{V}'_1$ are called *isomorphic* if their base geometries are isomorphic generalized quadrangles. Now let

$n \geq 2$, then we call $\mathcal{V}_n$ and $\mathcal{V}'_n$ *isomorphic* if they are equivalent (denote in that case the corresponding equivalence by Ψ) and if for all $\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n)$, $\mathcal{W}_{n-1}(\mathcal{V}_n,\mathcal{C})$ is isomorphic with $\mathcal{W}_{n-1}(\mathcal{V}'_n,\Psi(\mathcal{C}))$ and this isomorphism $\Psi_{\mathcal{C}}$ coincides with $\Psi/\mathcal{C}$ over $\mathcal{C}$. We can now extend Ψ with every $\Psi_{\mathcal{C}}$ and if we denote this extension still by Ψ, then we call Ψ an *isomorphism*. Obviously, isomorphic level n HQs are also equivalent.

Recall that Π'_{n-1} is the projection mapping the base geometry of $\mathcal{V}_n$ onto the base geometry of the canonical $(n-1)$-image $\mathcal{V}_{n-1} = (\mathcal{P}(\mathcal{V}_{n-1}),\ldots)$. We can extend Π'_{n-1} to all $\mathcal{W}_{n-j}(\mathcal{V}_n,\mathcal{C})$, $\mathcal{C} \in \mathbf{P}_{n-j}(\mathcal{V}_n)$ and $0<j<n$, with the projection of $\mathcal{W}_{n-j}(\mathcal{V}_n,\mathcal{C})$ onto $\mathcal{W}_{n-j-1}(\mathcal{V}_{n-1},\Pi'_{n-1}(\mathcal{C}))$. We denote that extension still by Π'_{n-1}. Suppose now that $\mathcal{V}_{n-1}$ is isomorphic with some level $n-1$ HQ X_{n-1} and call the corresponding isomorphism Ψ. Then we call $\Psi \circ \Pi'_{n-1}$ a *HQ-epimorphism*. Suppose now that $(X_n,\nabla_n^{n+1})_{n\in\mathbf{N}}$ is an infinite sequence with X_n a level n HQ and ∇_n^{n+1} an HQ-epimorphism from X_{n+1} onto X_n, then we call $(X_n,\nabla_n^{n+1})_{n\in\mathbf{N}}$ an *HQ-Artmann-sequence*. This name is inspired by the work of Artmann [1], who studied similar sequences of level n Hjelmslev planes, giving rise to affine buildings of type $\tilde{A}_2$ (by [8],[17]).

If Z_n is the base geometry of X_n, for all $n \in \mathbf{N}$, then we call the sequence $(Z_n,\nabla_n^{n+1}/Z_{n+1})_{n\in\mathbf{N}}$ the *base sequence of* $(X_n,\nabla_n^{n+1})_{n\in\mathbf{N}}$. The inverse limit of the base sequence of an HQ-Artmann-sequence $\mathcal{H}$ is also called the *inverse limit of* $\mathcal{H}$.

THEOREM(1.1.3.2) (Hanssens-Van Maldeghem [9]). *Every HQ-Artmann-sequence gives rise to an explicitly defined affine building of type* $\tilde{C}_2$.

We now mention some more results of [9] that we will need in the proof of our main results. Therefore, suppose $\mathcal{V}_n = (\mathcal{P}(\mathcal{V}_n),\mathcal{L}(\mathcal{V}_n),I,\ldots)$ is a level n HQ, $n \in \mathbf{N}^*$. We denote by $\mathcal{V}_j = (\mathcal{P}_i(\mathcal{V}_j),\mathcal{L}_i(\mathcal{V}_j),I,(\mathbf{P}_i(\mathcal{V}_j))_{i\leq j},\ldots)$ a level j HQ, isomorphic to the canonical j-image of $\mathcal{V}_n$, $0 < j \leq n$. The corresponding projection map is denoted by Π'_j. Furthermore, we denote the valuation map in $\mathcal{V}_n$, resp. $\mathcal{V}_j$, $\mathcal{W}_k(\mathcal{V}_n,\mathcal{O}^k(\mathcal{P}))$ (for arbitrary $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$ and k, $0 < k \leq n$) by u, resp. $u[j,j],u[n,k]$ (to be in conformity with [9]). Throughout, we assume that $j,k,\ell \in \mathbf{N}$, all smaller than or equal to n and $\mathcal{P}, \mathcal{Q} \in \mathcal{P}(\mathcal{V}_n)$, $\mathcal{L}, \mathcal{M} \in \mathcal{L}(\mathcal{V}_n)$.

PROPERTY(1.1.3.3). $u_1(\mathcal{P},\mathcal{L}) \leq u_2(\mathcal{P},\mathcal{L}) \leq 2.u_1(\mathcal{P},\mathcal{L})$.

PROPERTY(1.1.3.4). *If $\mathcal{P}$ and $\mathcal{Q}$ lie in one anothers j-point-neighbourhood, then*

$$j - u[n,j](\mathcal{P},\mathcal{Q}) = n - u(\mathcal{P},\mathcal{Q}).$$

PROPERTY(1.1.3.5). *If $\mathcal{Q}$ I $\mathcal{L}$ I $\mathcal{P}$ I $\mathcal{M}$, $u(\mathcal{P},\mathcal{Q}) = 0$ and $\mathcal{L} \neq \mathcal{M}$, then*

$$u(\mathcal{Q},\mathcal{M}) = (\frac{u(\mathcal{L},\mathcal{M})}{2}, u(\mathcal{L},\mathcal{M}))$$

PROPERTY(1.1.3.6). *If $\mathcal{P}$ and $\mathcal{Q}$ are both incident with both lines $\mathcal{L}$ and $\mathcal{M}$, and $u(\mathcal{P},\mathcal{Q}) = 0$, then $\mathcal{L} = \mathcal{M}$.*

PROPERTY(1.1.3.7). *Suppose $\sigma(\mathcal{L}) \cap \sigma(\mathcal{M}) \cap \mathcal{O}^j(\mathcal{P}) \neq \emptyset$ and $C^n_{n-j}(\mathcal{P},\mathcal{L}) \cap C^n_{n-j}(\mathcal{P},\mathcal{M}) \neq \emptyset$, then $u(\mathcal{L},\mathcal{M}) \geq j$ and if j is odd, equality does not occur.*

PROPERTY(1.1.3.8). *If $u[j,j]_1(\Pi^n_j(\mathcal{P}),\Pi^n_j(\mathcal{L})) \leq j \leq n$, then $u(\mathcal{P},\mathcal{L}) = u[j,j](\Pi^n_j(\mathcal{P}),\Pi^n_j(\mathcal{L}))$.*

PROPERTY(1.1.3.9). If $2.u_1(\mathcal{P},\mathcal{L}) < j \leq n$, *then* $u[j,j](\Pi^n_j(\mathcal{P}),\Pi^n_j(\mathcal{L})) = u(\mathcal{P},\mathcal{L})$.

PROPERTY(1.1.3.10). $u_2(\mathcal{P},\mathcal{L}) = \sup\{ i \leq n \mid \Pi^n_i(\mathcal{P}) \text{ I } \Pi^n_i(\mathcal{L})\}$.

PROPERTY(1.1.3.11). *Suppose $u(\mathcal{P},\mathcal{L}) = (k,\ell)$ with $2k \leq n$. For every component $\mathcal{L}'$ of $\mathcal{L}$ in $\mathcal{W}_{n-i}(\mathcal{V}_n,\mathcal{O}^{n-i}(\mathcal{P}))$, with $i \leq 2k-\ell$, we have $(k-i,\ell-i) = u[n,n-i](\mathcal{P},\mathcal{L}')$.*

PROPERTY(1.1.3.12). *If $u(\mathcal{L},\mathcal{M}) \geq j$ and $\mathcal{L}$ I $\mathcal{P}$ I $\mathcal{M}$, then the lines $\mathcal{L}$ and $\mathcal{M}$ have a common component in $\mathcal{W}_j(\mathcal{V}_n,\mathcal{O}^j(\mathcal{P}))$ incident with $\mathcal{P}$.*

PROPERTY(1.1.3.13). *Suppose $u(\mathcal{P},\mathcal{L}) = (k,\ell)$ with $2k < n$. Then there exists a line $\mathcal{M}$ and a point $\mathcal{Q}$ such that $\mathcal{P}$ I $\mathcal{M}$ I $\mathcal{Q}$ I $\mathcal{L}$. Moreover, there holds*

(1) $u(\mathcal{L},\mathcal{M}) = 2\ell-2k$,

(2) $u(\mathcal{P},Q) = 2k-\ell$,

(3) *If* $\mathcal{M}' \in \mathcal{L}(\mathcal{V}_n)$ *and* $Q' \in \mathcal{P}(\mathcal{V}_n)$ *and* $\mathcal{P}\ I\ \mathcal{M}'\ I\ Q'\ I\ \mathcal{L}$,

then : (3a) $u(\mathcal{M},\mathcal{M}') \geq n-4k+2\ell$,

(3b) $u(Q,Q') \geq \frac{n}{2}+k-\ell$.

1.2. Quadratic quaternary rings with valuation.

Quadratic quaternary rings are in fact the coordinatizing algebraic structures of a generalized quadrangle (see [6]). But for the purpose of this paper, we do not need the full background of this theory.

1.2.1. Quadratic quaternary rings.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two sets intersecting in the set $\{0,1\}$ of distinct elements 0,1 and both not containing the symbol ∞. Let Q_1 and Q_2 be two quaternary operations with

$$Q_1 : \mathcal{R}_2 \times \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_1 \to \mathcal{R}_1,$$
$$Q_2 : \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_1 \times \mathcal{R}_2 \to \mathcal{R}_2.$$

The quadruple $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2)$ is called a *quadratic quaternary ring* if it satisfies (0), $(\overline{0})$, (1), $(\overline{1})$, (A), $(\overline{A})$, (B), $(\overline{B})$ and (C) below.

(0) $Q_1(k,0,0,a') = a' = Q_1(0,a,k,a')$.

$(\overline{0})$ $Q_2(a,0,0,k') = k' = Q_2(0,k,a,k')$.

(1) $Q_1(1,a,0,0) = a$.

$(\overline{1})$ $Q_2(1,k,0,0) = k$.

(A) *There exists exactly one* $x \in \mathcal{R}_1$ *such that* $Q_1(k,a,\ell,x) = b$.

$(\overline{A})$ *There exists exactly one* $p \in \mathcal{R}_2$ *such that* $Q_2(a,k,b,p) = \ell$.

(B) *If* $k \neq \ell$, *there exists exactly one pair* $(x,y) \in \mathcal{R}_1 \times \mathcal{R}_1$ *such that*

$$Q_1(k,x,Q_2(x,k,a,k'),y) = a,$$
$$Q_1(\ell,x,Q_2(x,k,a,k'),y) = b.$$

($\overline{\text{B}}$) *If* $a \neq b$, *there exists exactly one pair* $(p,q) \in \mathcal{R}_2 \times \mathcal{R}_2$ *such that*

$$Q_2(a,p,Q_1(p,a,k,a'),q) = k,$$
$$Q_2(b,p,Q_1(p,a,k,a'),q) = \ell.$$

(C) *If*

$$Q_1(k,a,\ell,a') \neq b \qquad \text{(C1)}$$
$$Q_2(a,k,b,k') \neq \ell \qquad \text{(C2)}$$

then there exists a unique quadruple $(x,x',p,p') \in \mathcal{R}_1 \times \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_2$ *such that*

$$Q_1(k,x,Q_2(x,k,b,k'),x') = b,$$
$$Q_1(p,x,Q_2(x,k,b,k'),x') = Q_1(p,a,\ell,a'),$$
$$Q_2(a,p,Q_1(p,a,\ell,a'),p') = \ell,$$
$$Q_2(x,p,Q_1(p,a,\ell,a'),p') = Q_2(x,k,b,k').$$

If exactly one of the conditions (C1) *or* (C2) *holds, then there exists no quadruple* (x,x',p,p') *having the above properties.*

We abbreviate the term *quadratic quaternary ring* by *QQR*. Examples can be found in [5] and [7] and also in section 5.

THEOREM(1.2.1.1) (Hanssens-Van Maldeghem [6]). *Let* $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2)$ *be a QQR. The following point-line geometry* $\mathcal{S}$ *is a generalized quadrangle. The points of* $\mathcal{S}$ *are the elements of* $\mathcal{R}_1 \cup \mathcal{R}_2 \times \mathcal{R}_1 \cup \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_1$, *denoted with round brackets, together with the symbol* (∞). *The lines of* $\mathcal{S}$ *are the elements of* $\mathcal{R}_2 \cup \mathcal{R}_1 \times \mathcal{R}_2 \cup \mathcal{R}_2 \times \mathcal{R}_1 \times \mathcal{R}_2$, *denoted with square brackets, together with the symbol* $[\infty]$. *Incidence is defined as follows.*

$(a,\ell,a')\ I\ [k,b,k']$ *if and only if* $Q_1(k,a,\ell,a') = b$
and $Q_2(a,k,b,k') = \ell$,

$(a,\ell,a')\ I\ [a,\ell]$,

$(k,a)\ I\ [k,a,k']$,

$(k,a)\ I\ [k]$,

$(a)\ I\ [a,k]$,

$(a)\ I\ [\infty]$,

$(\infty)\ I\ [k]$,

$(\infty)\ I\ [\infty]$, *for all* $a,a',b \in \mathcal{R}_1$; $k,k',\ell \in \mathcal{R}_2$.

There are no other incidences.

Let $\mathcal{S}$ be any generalized quadrangle. We now show (following [6]) how $\mathcal{S}$ may be coordinatized by a QQR.

Suppose (s,t) is the order of $\mathcal{S}$ and let $\mathcal{R}_1$ resp. $\mathcal{R}_2$ be a set of cardinallity s resp. t containing the symbols 0 and 1 but not the symbol ∞. We choose any ordinary quadrangle in $\mathcal{S}$ and starting with one point of that quadrangle, we coordinatize its elements by $(\infty)\ I\ [\infty]\ I\ (0)\ I\ [0,0]\ I\ (0,0,0)\ I\ [0,0,0]\ I\ (0,0)\ I\ [0]\ I\ (\infty)$. We choose an arbitrary bijection from $\mathcal{R}_1$ to the set of points incident with $[\infty]$ and distinct from (∞) with the only restriction that 0 is mapped to (0). This way, every point on $[\infty]$ gets a coordinate (a), where $a \in \mathcal{R}_1 \cup \{\infty\}$. Similarly, we assign to every line through (∞) a coordinate $[k]$, $k \in \mathcal{R}_2 \cup \{\infty\}$. The unique point incident with $[0,0,0]$ and collinear with (a), $a \in \mathcal{R}_1$, is given the coordinates $(a,0,0)$. We assign to the unique point incident with $[1]$ and collinear with $(a,0,0)$ the coordinates $(1,a)$. Next, the unique point incident with $[0,0]$ and collinear with $(1,a)$ is given the coordinates $(0,0,a)$ and finally we assign to the unique point incident with $[k]$ and collinear with $(0,0,a)$ the coordinates (k,a). In a complete dual way, we obtain lines with coordinates $[k,0,0],[1,k]$, $[0,0,k],[a,k]$, $a \in \mathcal{R}_1, k \in \mathcal{R}_2$. Now, the unique point incident with $[a,\ell]$ and collinear with $(0,a')$ gets the coordinates (a,ℓ,a'), $a,a' \in \mathcal{R}_1, \ell \in \mathcal{R}_2$ and dually, the unique line incident with (k,b) and concurrent with $[0,k']$ gets the coordinates $[k,b,k']$, $b \in \mathcal{R}_1, k,k' \in \mathcal{R}_2$. Now every point and every line has recieved coordinates and every n-tuple of the form $(\infty),[\infty],(a),[k],(k,a),[a,k],(a,\ell,a')$ or $[k,b,k']$, $a,a',b \in \mathcal{R}_1, k,k',\ell \in \mathcal{R}_2$ corresponds to exactly one point or line in $\mathcal{S}$. We can define two quaternary operations Q_1 and Q_2 as follows. Let $a,a',b \in \mathcal{R}_1, k,k',\ell \in \mathcal{R}_2$, then

$$Q_1(k,a,\ell,a') = b \iff (a,\ell,a') \perp (k,b),$$

$$Q_2(a,k,b,k') = \ell \iff [k,b,k'] \perp [a,\ell].$$

One can check that this implies

$$\begin{cases} Q_1(k,a,\ell,a') = b \\ Q_2(a,k,b,k') = \ell \end{cases} \iff (a,\ell,a')\ I\ [k,b,k'].$$

THEOREM(1.2.1.2) (Hanssens-Van Maldeghem [6]). *Let $\mathcal{S}$ any generalized quadrangle and suppose we coordinatize $\mathcal{S}$ by the quadruple $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ as described above. Then $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ is a quadratic quaternary ring.*

The quadrangle $(\infty)\ I\ [\infty]\ I\ (0)\ I\ [0,0]\ I\ (0,0,0)\ I\ [0,0,0]\ I\ (0,0)\ I\ [0]\ I\ (\infty)$ will be called the *base quadrangle of coordinatization*. The point (1), resp. the line [1] will be called the *unit point*, resp. *unit line*.

1.2.2. Quadratic quaternary rings with valuation.

Let $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ be a QQR and v a map from $\mathcal{R}_1 \times \mathcal{R}_1 \cup \mathcal{R}_2 \times \mathcal{R}_2$ to $\mathbf{Z} \cup \{+\infty\}$. Then we call $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ a *QQR with valuation* if $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ and v satisfy

(v1) $v(x,y) = \infty$ *if and only if* $x = y$, *for all suitable* (x,y).

(v2) *For all suitable* x, y *and* z, $v(x,z) \geq \inf\{v(x,y), v(z,y)\}$ *and if* $v(x,y) \neq v(z,y)$, *then equality holds.*

(v3) $v/\mathcal{R}_1 \times \mathcal{R}_1$ *and* $v/\mathcal{R}_2 \times \mathcal{R}_2$ *are both surjective.*

(v4) *If*

$$\begin{aligned}
Q_1(k_1, a_1, \ell_1, a_1') &= Q_1(k_1, a_2, \ell_2, a_2') = b_1, \\
Q_2(a_1, k_1, b_1, k_1') &= Q_2(a_1, k_2, b_2, k_2') = \ell_1, \\
Q_1(k_2, a_1, \ell_1, a_1') &= Q_1(k_2, a_3, \ell_3, a_3') = b_2, \\
Q_2(a_3, k_2, b_2, k_2') &= Q_2(a_3, k_3, b_3, k_3') = \ell_3, \\
Q_1(k_3, a_3, \ell_3, a_3') &= Q_1(k_3, a_2, \ell_2, a_2') = b_3, \\
Q_2(a_2, k_3, b_3, k_3') &= Q_2(a_2, k_1, b_1, k_4') = \ell_2,
\end{aligned}$$

then

$$v(k_1, k_2) + v(k_1', k_4') = v(k_1, k_3) + v(k_2, k_3) + v(a_2, a_3).$$

(v5) *If*

$$Q_1(k_1, a_1, \ell_1, a_1') = Q_1(k_1, a_2, \ell_2, a_2') = b_1,$$
$$Q_2(a_1, k_1, b_1, k_1') = Q_2(a_1, k_2, b_2, k_2') = \ell_1,$$
$$Q_1(k_2, a_1, \ell_1, a_3') = Q_1(k_2, a_2, \ell_2, a_2') = b_2,$$
$$Q_2(a_2, k_2, b_2, k_2') = Q_2(a_2, k_1, b_1, k_3') = \ell_2,$$

then

$$v(k_1', k_3') = v(k_1, k_2) + v(a_1, a_2),$$
$$v(a_1', a_3') = v(a_1, a_2) + 2.v(k_1, k_2).$$

(v6) *If*

$$Q_1(k, a, \ell, a_1') = b_1 \quad \textit{and} \quad Q_1(k, a, \ell, a_2') = b_2,$$

then

$$v(a_1', a_2') = v(b_1, b_2).$$

(v7) *If*

$$Q_2(a, k, b, k_1') = \ell_1 \quad \textit{and} \quad Q_2(a, k, b, k_2') = \ell_2,$$

then

$$v(k_1', k_2') = v(\ell_1, \ell_2).$$

We call (v2) sometimes *the triangle inequality*.
We abbreviate *QQR with valuation* by V-QQR. Let $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ be a V-QQR, the we can define the following metric in $\mathcal{R}_1$.

$$\delta_1 : \mathcal{R}_1 \times \mathcal{R}_1 \to \mathbf{R} : (x, y) \to \mathbf{e}^{-v(x,y)},$$

where $\mathbf{e} \in \mathbf{R}$ denotes the natural exponential base number. Similarly, one can define

$$\delta_2 : \mathcal{R}_2 \times \mathcal{R}_2 \to \mathbf{R} : (x, y) \to \mathbf{e}^{-v(x,y)}.$$

We call $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ *complete* if $(\mathcal{R}_1, \delta_1)$ en $(\mathcal{R}_2, \delta_2)$ are *complete metric spaces*, i.e. every *Cauchy-sequence* converges. We abbreviate a *complete V-QQR* to *CV-QQR*.

We usually write $v(x, 0)$ as $v(x)$. It is shown in [20] that v is symmetric and hence $v(x) = v(0, x) = v(x, 0)$, for all $x \in \mathcal{R}_1 \cup \mathcal{R}_2$. We postpone our examples to section 5.

THEOREM(1.2.2) (The author in [20]). *Let $\mathcal{S}$ be a generalized quadrangle coordinatized by a V-QQR $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2,\nu)$. Then $\mathcal{S}$ defines in an explicit way an HQ-Artmann-sequence $\mathcal{H}$ and hence an affine building of type $\tilde{C}_2$ by theorem(1.1.3.2). Moreover, $\mathcal{S}$ is isomorphic to the inverse limit of $\mathcal{H}$, provided that $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2,\nu)$ is complete. If not, $\mathcal{S}$ is isomorphic to a subquadrangle of this inverse limit.*

1.3. Affine buildings of type $\tilde{C}_2$.

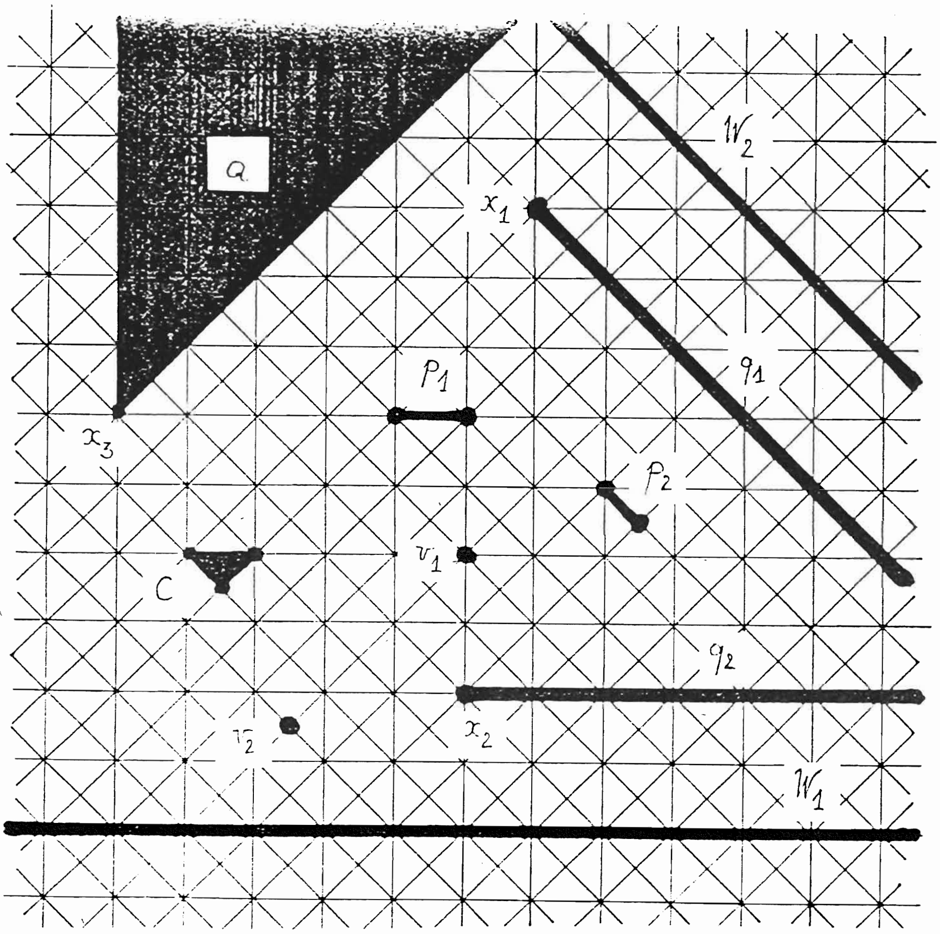
1.3.1. The standard apartment.

Let **A** be the real Euclidean plane provided with the usual distance map $d_{\mathbf{A}}$. Denote by $\mathcal{T}$ a solid triangle with respective angles 45°, 45° and 90°. The length of the two shortest sides is 1. We denote the lines supporting the sides of $\mathcal{T}$ by $\mathcal{L}_i$, i=1,2,3. Let $\mathcal{W}$ be the group of automorphisms of **A** generated by the reflections about the lines $\mathcal{L}_i$, i=1,2,3. The group $\mathcal{W}$ is called *the Weylgroup of type* $\tilde{C}_2$. The image of $\mathcal{T}$ under an arbitrary element of $\mathcal{W}$ is called a *chamber*. The set of all chambers determines a tessellation τ of **A** in congruent isosceles right triangles. We call $(\mathbf{A},\tau)$ the *standard apartment of type* $\tilde{C}_2$. The vertices of the triangles of τ are briefly called *vertices* and the sides of these triangles are called *panels*. Vertices, panels and chambers are also called *simplices*. Two vertices are called *adjacent* if they lie on a common panel. Panels of length 1 are called *short* panels, the other ones are called *long*. The lines supporting the panels are called *walls*. A vertex x is called *special* if for every wall $\mathcal{M}$, there exists a wall $\mathcal{M}^*$ parallel to $\mathcal{M}$ and incident with x. So x is special if and only if it lies on exactly four walles or eight panels. Now note that $\mathcal{W}$ defines two orbits in the set of walls. One orbit consists of all walls containing only special vertices and long panels. We call such walls *straight*. The walls of the other orbit, called *diagonal* walls, contain both special and non special vertices, but they contain only short panels. Now let x be a special vertex and denote by $\mathfrak{L}_x$ the set of all walls through x. The closure of any connected component of $\mathbf{A}-\cap\mathfrak{L}_x$ (where **A** and the elements of $\mathfrak{L}_x$ are considered as sets of points) is called a *sector (with source x)*. The closure of any connected component

of $\cap\mathfrak{L}_x - \{x\}$ is called a *sectorpanel* (*with source* x).
All definitions (most of them are standard concepts ; see Bourbaki [2]) above are illustrated in figure 1, where we picture the standard apartment (it is in fact the geometric realization of the *Coxeter complex of type* $\tilde{C}_2$).

1.3.2. Discrete systems of apartments of type $\tilde{C}_2$.

An *affine building of type* $\tilde{C}_2$, also called a *discrete system of apartments of type* $\tilde{C}_2$, is by definition a pair $(\Delta,\mathfrak{F})$, where Δ is a set and $\mathfrak{F}$ is a family of injections from **A** in Δ satisfying the axioms (SA1), (SA2),(SA3) and (SA4) below. The building $(\Delta,\mathfrak{F})$ is called *symmetric* (resp. *complete*) if it also satisfies (SA5) (resp. (SA6)). The image of **A** under an arbitrary element of $\mathfrak{F}$ is called an *apartment*. We suppose that **A** is provided with the tessellation τ and we call the image of a chamber, panel, vertex, wall, etc... under the action of any element of $\mathfrak{F}$ also a *chamber*, resp. *a panel, a vertex*, etc... In particular, the elements of Δ are called *points* (just like the elements of **A**). Here are the first four axioms.



Q = Sector with source x_3.
W_1= Straight wall.
W_2= Diagonal wall.
p_1= Long panel.
p_2= Short panel.
q_1= Diagonal sectorpanel with source x_1.
q_2= Straight sectorpanel with source x_2.
C = Chamber.
v_1= Special vertex.
v_2= Non special vertex.

FIGURE 1

(SA1) $\mathcal{F} \cdot \mathcal{W} = \mathcal{F}$.

(SA2) *Let* $\ell, \ell' \in \mathcal{F}$. *The set* $\mathcal{B} = (\ell^{-1} \cdot \ell)(\mathbf{A})$ *is a (not necessarily finite) union of simplices, it is closed and convex (with respect to* $d_{\mathbf{A}}$ *and the topology induced by* $d_{\mathbf{A}}$*) and there exists* $w \in \mathcal{W}$ *such that* $\ell/\mathcal{B} = \ell' \cdot w/\mathcal{B}$.

(SA3) *Every two points of* Δ *lie in a commom apartment.*

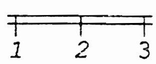
(SA4) *If* $\ell \in \mathcal{F}$ *and* x *is an arbitrary point of* $\ell(\mathbf{A})$*, then there exists a retraction (i.e. an idempotent surjection)* ρ : $\Delta \to \ell(\mathbf{A})$ *such that the restriction to every apartment diminishes distances (i.e.* $\ell^{-1} \cdot \rho \cdot \ell'$ *diminishes distances* $\mathbf{A}$*, for every* $\ell' \in \mathcal{F}$*) and such that* $\rho^{-1}(x) = \{x\}$.

A *germ of sectors* is an equivalence class in the set of all sectors of Δ with respect to the equivalence relation "*Q_1 and Q_2 are equivalent if $Q_1 \cap Q_2$ contains a sector*". We say that a germ of sectors *lies in an apartment* if at least one (and hence infinitly many) representative does.

(SA5) *Every two germs of sectors lie in a common apartment.*

This set of axioms was introduced by J.Tits in [16]. Note that by the saying "x lies in the apartement Σ" we really mean "there exists $\ell \in \mathcal{F}$ such that x is an element or a subset of $\Sigma = \ell(\mathbf{A})$".

J.Tits [16] shows that, in view of our slightly modified axiom (SA2), this definition is equivalent to the definition of an abstract building of type $\tilde{C}_2$. The way to go from a discrete system of apartments to an abstract building is symply by ignoring all points which are not vertices. Hence there exists a type map *typ* from the set of vertices of any affine building Δ of type $\tilde{C}_2$ to $\{1,2,3\}$ turning Δ into a *Buekenhout-Tits geometry of rank 3* with *Buekenhout-diagram* (see [4]) :



There exists a similar type map $typ_{\mathbf{A}}$ defined over the set of vertices of $(\mathbf{A},\tau)$ and typ can be considered as the image of $typ_{\mathbf{A}}$ under the action of the maps $f \in \mathcal{F}$. This is well defined by (SA2). The vertices of type 1 and 3 are special. The *residue* (cp. Buekenhout [4]) of such vertex is a generalized quadrangle and the residue of a non special vertex (a vertex of type 2) is a *generalized digon*.

Now, the set of apartments of an affine building of type $\tilde{C}_2$ is not uniquely determined, but Tits proves that it always contains a *maximal set of apartments* (see Tits [16],théorème 1). Hence the following axiom :

(SA6) *The set* $\{f(\mathbf{A}) \mid f \in \mathcal{F}\}$ *is a maximal set of apartments for* Δ.

By Tits [16], every complete building is symmetric.

Throughout, we will always assume that every affine building of type $\tilde{C}_2$ is *thick*, i.e. every panel is contained in at least three chambers. This conforms to the notion of building in Tits [15].

1.3.3. The geometry at infinity.

In this paragraph, we define for every symmetric affine building of type $\tilde{C}_2$ a geometry at infinity. Our definition is equivalent to Tits' building at infinity (cp.[16]).

Let Δ be a symmetric affine building of type $\tilde{C}_2$. By (SA2) and (SA3), $d_{\mathbf{A}}$ induces a well defined metric d_{Δ} in Δ (in the obvious way, see also Tits [16]). Now, two sectorpanels p and q are called *parallel* if they are at bounded distance from one another, i.e. the sets $\{d_{\Delta}(x,q) \mid x \in p\}$ and $\{d_{\Delta}(y,p) \mid y \in q\}$ are bounded (where $d_{\Delta}(x,q) = inf\{d_{\Delta}(x,y) \mid y \in q\}$ and similarly for $d_{\Delta}(y,p)$). This relation is apperently an equivalence relation and we denote the equivalence class of a sectorpanel p by $c(p)$. One can easily see that such a class contains either straight or diagonal sectorpanels (see also [19]).

We can now define the following point-line incidence geometry $\Delta_{\infty} = (\mathcal{P}(\Delta_{\infty}),\mathcal{L}(\Delta_{\infty}),I)$: the points (elements of $\mathcal{P}(\Delta_{\infty})$) are the parallel classes of straight sectorpanels ; the lines (elements of $\mathcal{L}(\Delta_{\infty})$) are the parallel classes of diagonal sectorpanels and a point $c(p)$ is incident with a line $c(\ell)$ if there exists a sector containing at least one representative of both $c(p)$ and $c(\ell)$. By [16],proposition 5, we can identify the set of incident point-line pairs (the flags) with the set of germs of sectors.

In every generalized quadrangle, we call two points (resp. lines) *opposite* if they are distinct and not collinear (resp. concurrent).

THEOREM(1.3.3) (Tits [16]). *The geometry Δ_∞ as defined above is a generalized quadrangle. The eight germs of sectors in an arbitrary apartment Σ define eight flags in Δ_∞ which determine a customary non degenerate quadrangle Σ_∞ in Δ_∞. The map $\Sigma \to \Sigma_\infty$ is a bijection from the set of apartments of Δ to the set of customary non degenerate quadrangles in Δ_∞. The "trace at infinity" of a straight (resp. diagonal) wall of Δ is a pair of opposit points (resp. lines) in Δ_∞.*

We call Δ_∞ the *geometry at infinity of* Δ or also the *generalized quadrangle at infinity of* Δ.

1.3.4. The n^{th} floor of Δ with basement $\mathfrak{s}$.

Let Δ be an affine building of type $\tilde{C}_2$ and $\mathfrak{s}$ a special vertex of Δ. We denote

$\mathcal{T}\sigma(\Delta)$ = de verzameling van alle toppen van Δ.

$\mathcal{SP}(\Delta,\mathfrak{s})$ = de verzameling van alle sektorpanelen van Δ met bron $\mathfrak{s}$.

We define the point-line incidence geometry $\mathcal{W}_n = (\mathcal{P}(\mathcal{W}_n),\mathcal{L}(\mathcal{W}_n),I)$ for every $n \in \mathbf{N}$ and every special vertex $\mathfrak{s}$ of Δ as follows.

$$\mathcal{P}(\mathcal{W}_n) = \{\mathcal{P} \in \mathcal{T}\sigma(\Delta) \mid \mathcal{P} \in p \in \mathcal{SP}(\Delta,\mathfrak{s}),\ p \text{ is straight and } d_\Delta(\mathcal{P},\mathfrak{s})=n\}.$$

$$\mathcal{L}(\mathcal{W}_n) = \{\mathcal{L} \in \mathcal{T}\sigma(\Delta) \mid \mathcal{L} \in \ell \in \mathcal{SP}(\Delta,\mathfrak{s}),\ p \text{ is diagonal and } d_\Delta(\mathcal{P},\mathfrak{s}) = \sqrt{2}.n\},$$

$$\forall \mathcal{P} \in \mathcal{P}(\mathcal{W}_n),\ \forall \mathcal{L} \in \mathcal{L}(\mathcal{W}_n):\ \mathcal{P}\ I\ \mathcal{L} \iff \mathcal{P},\mathcal{L},\mathfrak{s} \text{ lie is a common apartment.}$$

We call $\mathcal{W}_n$ the *n^{th} floor of Δ with basement $\mathfrak{s}$*, or briefly the *n^{th} floor of* $(\Delta,\mathfrak{s})$. Note that the first floor of $(\Delta,\mathfrak{s})$ is nothing other than the residue $\mathcal{R}(\mathfrak{s})$ of $\mathfrak{s}$ with a predestinated prescription of what will be called points or lines. It is a generalized quadrangle.

Suppose $\mathcal{P} \in \mathcal{P}(\mathcal{W}_n)$, then the vertex $\mathcal{P}'$ adjacent to $\mathcal{P}$ at distance n-1 from $\mathfrak{s}$ (assuming $n \geq 1$) is a point of $\mathcal{W}_{n-1}$ and we denote $\mathcal{P}' = \Pi^n_{n-1}(\mathcal{P})$. A similar definition for $\Pi^n_{n-1}(\mathcal{L})$ holds if $\mathcal{L} \in \mathcal{L}(\mathcal{W}_n)$. Apparently, Π^n_{n-1} defines an epimorphism from $\mathcal{W}_n$ onto $\mathcal{W}_{n-1}$ in the sense of (1.1.1).

1.4. Some more known results.

THEOREM(1.4.1) (Hanssens-Van Maldeghem [10]). *Given Δ and $\mathfrak{s}$ as in the previous paragraph, then the sequence $(\mathcal{W}_n, \Pi^{n+1}_n)_{n \in \mathbf{N}}$ is the base sequence of some canonically defined HQ-Artmann-sequence $\mathcal{H}(\Delta, \mathfrak{s})$. The building defined by $\mathcal{H}(\Delta, \mathfrak{s})$ as in theorem(1.1.3.2) is canonically isomorphic to Δ and the geometry at infinity of Δ is isomorphic to a subquadrangle of the inverse limit of $\mathcal{H}(\Delta, \mathfrak{s})$. Conversely, given an HQ-Artmann-sequence $\mathcal{H}$, denote by Δ the corresponding building (cp. theorem(1.1.3.2)). There is a well defined special vertex $\mathfrak{s}$ (corresponding to the level 0 HQ of $\mathcal{H}$) in Δ such that the sequence $(\mathcal{W}_n, \Pi^{n+1}_n)_{n \in \mathbf{N}}$ defined above is the base geometry of $\mathcal{H}$ and moreover $\mathcal{H}$ is equivalent to $\mathcal{H}(\Delta, \mathfrak{s})$.*

THEOREM(1.4.2) (Van Maldeghem [20]). *Let $\mathcal{S}$ be a generalized quadrangle coordinatized by a V-QQR. Let Δ be the corresponding building as in theorem(1.2.2) (provided with a suitable set of apartments ; see [0]). Then $\mathcal{S}$ is isomorphic to the generalized quadrangle at infinity of Δ.*

2. MAIN RESULTS

THEOREM(2.1). *Every coordinatizing QQR* $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ *of the generalized quadrangle at infinity of any symmetric affine building* Δ *of type* $\tilde{C}_2$ *is a V-QQR. Moreover, if* Δ *is complete, then* $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ *is a CV-QQR.*

THEOREM(2.2). *Suppose* $\mathcal{S}$ *is a generalized quadrangle. Then the following statements are equivalent.*

(2.2.1) $\mathcal{S}$ *can be coordinatized by at least one CV-QQR.*

(2.2.2) *All coordinatizing QQRs of* $\mathcal{S}$ *can be structured in a natural way to CV-QQRs.*

(2.2.3) $\mathcal{S}$ *is isomorphic to the generalized quadrangle at infinity of some complete affine building of type* $\tilde{C}_2$.

If one of these conditions is satisfied, then there is a natural and unique way to recover the valuation from the building and vice versa.

THEOREM(2.3). *Suppose* $\mathcal{S}$ *is a generalized quadrangle. Then the following statements are equivalent.*

(2.3.1) $\mathcal{S}$ *can be coordinatized by at least one V-QQR.*

(2.3.2) *All coordinatizing QQRs of* $\mathcal{S}$ *can be structured in a natural way to V-QQRs.*

(2.3.3) $\mathcal{S}$ *is isomorphic to the generalized quadrangle at infinity of some symmetric affine building of type* $\tilde{C}_2$.

If one of these conditions is satisfied, then there is a natural and unique way to recover the valuation from the building and vice versa.

The proofs of these theorems will be given in the next section. They will unable us to show our main theorem, stated in the introduction.

3. PROOFS

3.1. Definition of the valuation and first properties.

3.1.1. Definition of the valuation map.

Throughout, Δ is a symmetric affine building of type $\tilde{C}_2$, Δ_∞ is the generalized quadrangle at infinity of Δ and $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ is a coordinatizing QQR of Δ_∞.

Let $\mathfrak{s}$ be a special vertex in Δ, then we can project down the points (∞), (0), $(0,0)$, $(0,0,0)$, (1) and the lines $[\infty]$, $[0]$, $[0,0]$, $[0,0,0]$, $[1]$ onto $\mathcal{R}(\mathfrak{s})$, the residue of $\mathfrak{s}$ in Δ by intersecting $\mathcal{R}(\mathfrak{s})$ with the sectorpanels with source $\mathfrak{s}$ corresponding to these points and lines. Recall that $\mathcal{R}(\mathfrak{s})$ is a generalized quadrangle. If all points and lines above are mapped onto pairwise distinct varieties, we call $\mathfrak{s}$ a *basement for* $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$, or briefly a *basement*.

THEOREM(3.1.1.1) *With the above notation, there exists a unique basement* $\mathfrak{s}$ *for* $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$.

PROOF. Suppose $\mathfrak{s}$ is a basement. By [10],corollary(2.2.11), $\mathfrak{s}$ must lie in the apartement Σ of Δ determined by the basequadrangle of coordinatization. By the same token, $\mathfrak{s}$ must lie in the apartement $\Sigma_{(1)}$ determined by the quadrangle (∞) I $[\infty]$ I (1) I $[1,0]$ I $(1,0,0)$ I $[0,0,0]$ I $(0,0)$ I $[0]$ I (∞). But Σ meets $\Sigma_{(1)}$ in a half apartement bounded by a wall $\mathfrak{M}$ with trace at infinity $\{[\infty],[0,0,0]\}$ (see Tits[16],§8). The apartment $[(\Sigma \cup \Sigma_{(1)}) - (\Sigma \cap \Sigma_{(1)})] - \mathfrak{M}$ corresponds to the quadrangle (1) I $[\infty]$ I (0) I $[0,0]$ I $(0,0,0)$ I $[0,0,0]$ I $(1,0,0)$ I $[1,0]$ I (1) and hence $\mathfrak{s}$ must also lie in this apartement. These three apartments meet in $\mathfrak{M}$. So $\mathfrak{s}$ must lie on $\mathfrak{M}$. Note that $\mathfrak{M}$ is a straight wall. Now let $\Sigma_{[1]}$ be the apartement of Δ determined by the quadrangle (∞) I $[\infty]$ I (0) I $[0,0]$ I $(0,0,0)$ I $[1,0,0]$ I $(1,0)$ I $[1]$ I (∞) and let $\Sigma \cap \Sigma_{[1]}$ be bounded by the wall $\mathfrak{M}'$. Then similarly as above, $\mathfrak{s}$ must lie on $\mathfrak{M}'$. But $\mathfrak{M}'$ is a diagonal wall. Both walls $\mathfrak{M}$ ans $\mathfrak{M}'$lie in Σ and hence they meet. So $\mathfrak{s}$ must be the meeting vertex of $\mathfrak{M}$ and $\mathfrak{M}'$. Conversely, the meeting point of $\mathfrak{M}$ and $\mathfrak{M}'$ satisfies the required conditions. □

Throughout, we denote by $\mathfrak{s}$ the unique basement of $(\mathcal{R}_1, \mathcal{R}_2, \mathcal{Q}_1, \mathcal{Q}_2)$. We denote the set of points of Δ_∞ by $\mathcal{P}(\Delta_\infty)$ and the set of lines of Δ_∞ by $\mathcal{L}(\Delta_\infty)$. Consider $\mathcal{H}(\Delta, \mathfrak{s})$ (with the notation of theorem(1.4.1)) and denote the level n HQ of $\mathcal{H}(\Delta, \mathfrak{s})$ by

$$\mathcal{W}_n = (\mathcal{P}(\mathcal{W}_n), \mathcal{L}(\mathcal{W}_n), I, (\mathrm{P}_i(\mathcal{W}_n))_{i \le n}, (\mathrm{L}_i(\mathcal{W}_n))_{i \le n}, (\mathcal{Z}_{n-1}(\mathcal{W}_n, \mathcal{C}), \mathcal{N}_{\mathcal{C}})_{\mathcal{C} \in \mathrm{P}_{n-1}(\mathcal{W}_n)}),$$

for all positive integers n. Let u be the valuation map in $\mathcal{W}_n$ (for all n) and extend u to the inverse limit $\mathcal{W}_\infty$ of $\mathcal{H}(\Delta, \mathfrak{s})$ in the obvious way. Note that $\mathfrak{b}$ is a subquadrangle of $\mathcal{W}_\infty$ by theorem(1.4.1) and hence μ is also defined in $\mathfrak{b}$. Denote by Π_n the projection of $\mathfrak{b}$ onto $\mathcal{W}_n$ (it is the restriction of the natural projection of the inverse limit onto one of its constituents). Then we define the valuation v in $(\mathcal{R}_1, \mathcal{R}_2, \mathcal{Q}_1, \mathcal{Q}_2)$ as follows.

Let $a, b \in \mathcal{R}_1$, then $v(a,b) =$

$$u((a),(b)) \quad \text{if} \begin{cases} \Pi_1((a)) \neq \Pi_1((\infty)) \\ \Pi_1((b)) \neq \Pi_1((\infty)) \end{cases},$$

$$-u((a),(\infty)) \quad \text{if} \begin{cases} \Pi_1((a)) = \Pi_1((\infty)) \\ \Pi_1((b)) \neq \Pi_1((\infty)) \end{cases},$$

$$-u((b),(\infty)) \quad \text{if} \begin{cases} \Pi_1((a)) \neq \Pi_1((\infty)) \\ \Pi_1((b)) = \Pi_1((\infty)) \end{cases},$$

$$u((a),(b)) - u((a),(\infty)) - u((b),(\infty)) \quad \text{if} \begin{cases} \Pi_1((a)) = \Pi_1((\infty)) \\ \Pi_1((b)) = \Pi_1((\infty)) \end{cases}.$$

And dually for elements $k, \ell \in \mathcal{R}_2$, we define $2.v(k,\ell) =$

$$u([k],[\ell]) \quad \text{if} \begin{cases} \Pi_1([k]) \neq \Pi_1([\infty]) \\ \Pi_1([\ell]) \neq \Pi_1([\infty]) \end{cases},$$

$$-u([k],[\infty]) \quad \text{if} \begin{cases} \Pi_1([k]) = \Pi_1([\infty]) \\ \Pi_1([\ell]) \neq \Pi_1([\infty]) \end{cases},$$

$$-u([\ell],[\infty]) \qquad \text{if} \begin{cases} \Pi_1([k]) \neq \Pi_1([\infty]) \\ \Pi_1([\ell]) = \Pi_1([\infty]) \end{cases},$$

$$u([k],[\ell]) - u([k],[\infty]) - u([\ell],[\infty]) \qquad \text{if} \begin{cases} \Pi_1([k]) = \Pi_1([\infty]) \\ \Pi_1([\ell]) = \Pi_1([\infty]) \end{cases}.$$

Note that the valuation of a pair of concurrent lines is even and hence $v(k,\ell) \in \mathbf{Z}$ is well defined.

3.1.2. Properties of the valuation map.

THEOREM(3.1.2.1). *The map v as defined above satisfies the axioms* (v1), (v2) *and* (v3).

PROOF. The proof is a copy of the one in the $\tilde{A}_2$-case, see [18],4.5.4. □

LEMMA(3.1.2.2). *Suppose $\mathcal{P}, \mathcal{Q} \in \mathcal{P}(\Delta_\infty)$, $\mathcal{L}, \mathcal{M} \in \mathcal{L}(\Delta_\infty)$ and $\mathcal{P}\ I\ \mathcal{M}\ I\ \mathcal{Q}\ I\ \mathcal{L}$. Put $u(\mathcal{P},\mathcal{L}) = (k,\ell)$, $k,\ell \in \mathbf{N}$. Then we have :*

$$\text{(a)} \quad u(\mathcal{P},\mathcal{Q}) + u(\mathcal{L},\mathcal{M}) = \ell,$$

$$\text{(b)} \quad \Pi_i(\mathcal{P})\ I\ \Pi_i(\mathcal{L}) \iff i \leq \ell.$$

PROOF. Let $n > 2k$. Similarly to property(1.1.3.9), $u(\Pi_n(\mathcal{P}),\Pi_n(\mathcal{L})) = (k,\ell)$. By property(1.1.3.13), $u(\Pi_n(\mathcal{P}),\Pi_n(\mathcal{Q})) = 2k-\ell$ and $u(\Pi_n(\mathcal{L}),\Pi_n(\mathcal{M})) = 2\ell-2k$ and hence $u(\mathcal{P},\mathcal{Q}) = 2k-\ell$ and $u(\mathcal{L},\mathcal{M}) = 2\ell-2k$. Hence **(a)**. Claim **(b)** follows from property(1.1.3.10). □

PROPOSITION(3.1.2.3). *Let $\mathcal{P}_i, \mathcal{Q}_i \in \mathcal{P}(\Delta_\infty)$ and $\mathcal{L}_i, \mathcal{M}_i \in \mathcal{L}(\Delta_\infty)$, $i=1,2$. Suppose $\mathcal{P}_1\ I\ \mathcal{M}_2\ I\ \mathcal{Q}_2\ I\ \mathcal{L}_1\ I\ \mathcal{P}_2\ I\ \mathcal{M}_1\ I\ \mathcal{Q}_1\ I\ \mathcal{L}_2\ I\ \mathcal{P}_1$ (non-degenerate quadrangle) and $u(\mathcal{P}_2,\mathcal{Q}_1) = u(\mathcal{P}_2,\mathcal{Q}_2) = u(\mathcal{P}_1,\mathcal{Q}_1) = 0$. Then :*

$$\text{(a)} \quad u(\mathcal{L}_1,\mathcal{M}_1) = u(\mathcal{L}_2,\mathcal{M}_2) + u(\mathcal{P}_1,\mathcal{Q}_2),$$

$$\text{(b)} \quad u(\mathcal{L}_2,\mathcal{M}_1) = u(\mathcal{P}_1,\mathcal{Q}_2) + u(\mathcal{L}_1,\mathcal{M}_2).$$

PROOF. Let the left, resp. right hand side of **(a)** be equal to k, resp. ℓ. By the preceding lemma, $\Pi_\ell(\mathcal{Q}_2)\ I\ \Pi_\ell(\mathcal{L}_2)$. But by (GQ2) and property (1.1.3.6), $\Pi_\ell(\mathcal{L}_1) = \Pi_\ell(\mathcal{M}_1)$. Hence $k \geq \ell$. We project everything down onto

$\mathcal{W}_k$. If $\Pi_k(Q_2)$ is not incident with $\Pi_k(\mathcal{L}_2)$, then $\Pi_k(\mathcal{L}_1), \Pi_k(\mathcal{L}_2)$ and $\Pi_k(\mathcal{M}_2)$ are pairwise distinct. But by (GQ2), property(1.1.3.5) and the fact that $u(\Pi_k(\mathcal{P}_1), \Pi_k(Q_1)) = 0$, we have $\Pi_k(\mathcal{L}_1) = \Pi_k(\mathcal{L}_2)$, a contradiction, hence $\Pi_k(Q_2)$ I $\Pi_k(\mathcal{L}_2)$ and by the preceding lemma $k \leq \ell$, hence $k = \ell$. This shows **(a)**. The proof of **(b)** is similar. Note that **(b)** can also be derived from **(a)** by interchanging $\mathcal{L}_1$ and $\mathcal{L}_2$, Q_1 and $\mathcal{P}_2$ and $\mathcal{P}_1$ and Q_2. □

PROPOSITION(3.1.2.4). *Let $\mathcal{P}, Q \in \mathcal{P}(\Delta_\infty)$, $\mathcal{L} \in \mathcal{L}(\Delta_\infty)$, $u(\mathcal{P}, Q) = j < n$ and suppose $\mathcal{P}$ I $\mathcal{L}$ I Q. Then, in $\mathcal{W}_n$, the component of $\Pi_n(\mathcal{L})$ through $\Pi_n(\mathcal{P})$ in $\mathcal{O}^{n-j}(\mathcal{P})$ is the same as the one through $\Pi_n(Q)$ in $\mathcal{O}^{n-j}(\mathcal{P}) = \mathcal{O}^{n-j}(Q)$.*

PROOF. Let s' be the vertex in Δ corresponding to $\Pi_j(\mathcal{P}) = \Pi_j(Q)$, then the line structure in $\mathcal{O}^{n-j}(\mathcal{P})$ is by definition determined by the projection Π'_{n-j} of Δ_∞ onto the $(n-j)^{\text{th}}$ floor of (Δ, s'). It even is trivial to see (look in the sector with source s and "trace at infinity" $\{\mathcal{P}, \mathcal{L}\}$) that this projection maps $\mathcal{L}$ onto a component of $\Pi_n(\mathcal{L})$. The latter is certainly incident with $\Pi_n(\mathcal{P}) = \Pi'_{n-j}(\mathcal{P})$ and $\Pi_n(Q) = \Pi'_{n-j}(Q)$. □

PROPOSITION(3.1.2.5). *Let $\mathcal{P}_i, Q_i \in \mathcal{P}(\Delta_\infty)$ and $\mathcal{L}_i, \mathcal{M}_i \in \mathcal{L}(\Delta_\infty)$, $i=1,2$. Suppose $\mathcal{P}_1$ I $\mathcal{M}_2$ I Q_2 I $\mathcal{L}_1$ I $\mathcal{P}_2$ I $\mathcal{M}_1$ I Q_1 I $\mathcal{L}_2$ I $\mathcal{P}_1$ (non-degenerate quadrangle) and $u(\mathcal{P}_2, Q_1) = u(\mathcal{P}_2, Q_2) = 0$. Then :*

(a) $u(\mathcal{L}_1, \mathcal{M}_1) = u(\mathcal{L}_2, \mathcal{M}_2) + u(\mathcal{P}_1, Q_2) + u(\mathcal{P}_1, Q_1)$,

(b) $u(\mathcal{L}_2, \mathcal{M}_1) + u(\mathcal{P}_1, Q_1) = u(\mathcal{P}_1, Q_2) + u(\mathcal{L}_1, \mathcal{M}_2)$.

PROOF. By lemma(3.1.2.2), $u(\mathcal{L}_2, \mathcal{M}_1) + u(\mathcal{P}_1, Q_1) \geq j$ if and only if $\Pi_j(\mathcal{P}_1)$ I $\Pi_j(\mathcal{M}_1)$. But since $u(\mathcal{P}_2, Q_2) = 0$, we have either $\Pi_j(\mathcal{L}_1) = \Pi_j(\mathcal{M}_2)$, or we apply (GQ1), get $2.u_1(\mathcal{P}_2, \mathcal{M}_2) = u_2(\mathcal{P}_2, \mathcal{M}_2)$ and conclude with (GQ2) that $\Pi_j(\mathcal{M}_1) = \Pi_j(\mathcal{L}_1)$. In both cases $\Pi_j(\mathcal{P}_1)$ I $\Pi_j(\mathcal{L}_1)$ and hence $u(\mathcal{P}_1, Q_2) + u(\mathcal{L}_1, \mathcal{M}_2) \geq j$. Similarly (or by a suitable permutation of the indices) also the converse holds, proving **(b)**.
The proposition is symmetric in $u(\mathcal{P}_1, Q_1)$ and $u(\mathcal{P}_1, Q_2)$. So we can assume without loss of generality that $u(\mathcal{P}_1, Q_1) \leq u(\mathcal{P}_1, Q_2)$ and we put $k = u(\mathcal{P}_1, Q_1)$. Also denote $s' = \Pi_k(\mathcal{P}_1)$ and let $\mathcal{W}'_n$ be the level n HQ determined by the n^{th} floor of (Δ, s'). Denote by Π'_n the corresponding projection map from Δ_∞ onto $\mathcal{W}'_n$. Choose n arbitrary but bigger than 0.

We have $\mathcal{W}'_n = \mathcal{Z}_n(\mathcal{W}_{n+k}, \mathcal{O}^n(\Pi_{n+k}(\mathcal{P}_1)))$ by construction. By assumption on k, $\Pi'_n(\mathcal{P}_1), \Pi'_n(Q_1)$ and $\Pi'_n(Q_2)$ are affine points of $\mathcal{W}'_n$ (with the natural definition of 'affine'). Now since $u(\mathcal{P}_2, Q_i) = 0 < k$, $i=1,2$, the point $\Pi'_n(\mathcal{P}_2)$ cannot be affine. Hence it is a point at infinity. So in $\mathcal{W}'_n$, we have a quadrangle $\Pi'_n(\mathcal{P}_1)\ I\ \Pi'_n(\mathcal{M}_2)\ I\ \Pi'_n(Q_2)\ I\ \Pi'_n(\mathcal{L}_1)\ I\ \Pi'_n(\mathcal{P}_2)\ I\ \Pi'_n(\mathcal{M}_1)\ I\ \Pi'_n(Q_1)\ I\ \Pi'_n(\mathcal{L}_2)\ I\ \Pi'_n(\mathcal{P}_1)$. If u' denotes the valuation map in $\mathcal{W}'_n$, by construction and since $\Pi'_n(\mathcal{P}_2)$ is a point at infinity, we have that $u'(\Pi'_n(\mathcal{P}_2), \Pi'_n(Q_1)) = u'(\Pi'_n(\mathcal{P}_2), \Pi'_n(Q_2)) = u'(\Pi'_n(\mathcal{P}_1), \Pi'_n(Q_1)) = 0$. Now choose n big enough such that the quadrangle above is non-degenerate in $\mathcal{W}'_n$. By proposition(3.1.2.4) and its proof, $\Pi'_n(\mathcal{L}_1)$ is a component of $\Pi_{n+k}(\mathcal{L}_1)$ and similar for the other lines. Hence by property(1.1.3.4), $u'(\Pi'_n(\mathcal{P}_1), \Pi'_n(Q_2)) = u(\Pi_{n+k}(\mathcal{P}_1), \Pi_{n+k}(Q_2)) - k$ and so :

$$u'(\mathcal{P}_1, Q_2) = u(\mathcal{P}_1, Q_2) - k \tag{1}$$

By properties (1.1.3.7) and (1.1.3.12), it follows immediately that $u'(\Pi'_n(\mathcal{L}_2), \Pi'_n(\mathcal{M}_2)) = u(\Pi_{n+k}(\mathcal{L}_2), \Pi_{n+k}(\mathcal{M}_2))$ and so :

$$u'(\mathcal{L}_2, \mathcal{M}_2) = u(\mathcal{L}_2, \mathcal{M}_2) \tag{2}$$

Consider the sequence $\Pi_{n+k}(Q_2)\ I\ \Pi_{n+k}(\mathcal{L}_1)\ I\ \Pi_{n+k}(\mathcal{P}_2)\ I\ \Pi_{n+k}(\mathcal{M}_1)$. There holds $u(\Pi_{n+k}(\mathcal{P}_2), \Pi_{n+k}(Q_2)) = 0$ and hence $u(\Pi_{n+k}(\mathcal{L}_1), \Pi_{n+k}(\mathcal{M}_1)) = 2.u_1(\Pi_{n+k}(Q_2), \Pi_{n+k}(\mathcal{M}_1))$. By property(1.1.3.11) (all conditions are satisfied), $u_1(\Pi_{n+k}(Q_2), \Pi_{n+k}(\mathcal{M}_1)) - k = u'_1(\Pi'_n(Q_2), \Pi'_n(\mathcal{M}_1))$. Since also $\Pi'_n(Q_2)\ I\ \Pi'_n(\mathcal{L}_1)\ I\ \Pi'_n(\mathcal{P}_2)\ I\ \Pi'_n(\mathcal{M}_1)$ and $u'(\Pi'_n(Q_2), \Pi'_n(\mathcal{P}_2)) = 0$, we have by (GQ1), $u'(\Pi'_n(\mathcal{L}_1), \Pi'_n(\mathcal{M}_1)) = 2.u'_1(\Pi'_n(Q_2), \Pi'_n(\mathcal{M}_1))$. Combining the last three equalities, we get $u'(\Pi'_n(\mathcal{L}_1), \Pi'_n(\mathcal{M}_1)) = u(\Pi_{n+k}(\mathcal{L}_1), \Pi_{n+k}(\mathcal{M}_1)) - 2k$. Hence :

$$u'(\mathcal{L}_1, \mathcal{M}_1) = u(\mathcal{L}_1, \mathcal{M}_1) - 2k \tag{3}$$

Since $u'(\mathcal{P}_1, Q_1) = 0$, we have by proposition(3.1.2.3) :

$$u'(\mathcal{L}_1, \mathcal{M}_1) = u'(\mathcal{L}_2, \mathcal{M}_2) + u'(\mathcal{P}_1, Q_2) \tag{4}$$

Substituting (1),(2) and (3) into (4), we abtain **(a)**. □

PROPOSITION(3.1.2.6). *Suppose* $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{L}(\Delta_\infty)$, $\mathcal{P}_1, \mathcal{P}_2, Q_1, Q_2 \in \mathcal{P}(\Delta_\infty)$ *and* $\mathcal{P}_i$ I $\mathcal{L}_i$, Q_i I $\mathcal{L}_i$, $\mathcal{P}_1 \perp \mathcal{P}_2$, $Q_1 \perp Q_2$, $i=1,2$. *If* $\Pi_1(\mathcal{L}_1)$ *and* $\Pi_1(\mathcal{L}_2)$ *have no point in common, then* $u(\mathcal{P}_1, Q_1) = u(\mathcal{P}_2, Q_2)$.

PROOF. First note that $u(\mathcal{L}_1, \mathcal{L}_2) = 0$. Put $n = u(\mathcal{P}_1, Q_1)$ and consider $\mathcal{W}_n$. If $u_1(\Pi_n(\mathcal{P}_1), \Pi_n(\mathcal{L}_2)) \neq 0$, then $\Pi_1(\mathcal{P}_1)$ I $\Pi_1(\mathcal{L}_2)$, contradicting our assumptions. Hence $u(\Pi_n(\mathcal{P}_1), \Pi_n(\mathcal{L}_2)) = (0,0)$ (cp. property(1.1.3.3). Consequently, by (GQ2), the lines joining $\Pi_n(\mathcal{P}_1)$ and $\Pi_n(\mathcal{P}_2)$, resp. $\Pi_n(Q_2)$ coincide. Call that unique line $\mathcal{M}_n$. Suppose $j = u(\mathcal{P}_2, Q_2) < n$. Consider $\mathcal{Z}_{n-j}(\mathcal{W}_n, \mathcal{O}^{n-j}(\Pi_n(\mathcal{P}_2)))$ and let $\mathcal{M}^{\mathcal{P}}_{n-j}$, resp. $\mathcal{M}^{Q}_{n-j}$ be the unique component of $\mathcal{M}_n$ in $\mathcal{Z}_{n-j}(\mathcal{W}_n, \mathcal{O}^{n-j}(\Pi_n(\mathcal{P}_2)))$ through $\Pi_n(\mathcal{P}_2)$, resp. $\Pi_n(Q_2)$. Let $\mathcal{L}_{\mathcal{P}Q}$ be the unique component of $\Pi_n(\mathcal{L}_2)$ through both $\Pi_n(\mathcal{P}_2)$ and $\Pi_n(Q_2)$ in $\mathcal{Z}_{n-j}(\mathcal{W}_n, \mathcal{O}^{n-j}(\Pi_n(\mathcal{P}_2)))$ (cp. proposition(3.1.2.4)). Denote by $\mathcal{T}_n$ a base point of the strip determined by $\mathcal{M}_n$ and $\mathcal{Z}_{n-j}(\mathcal{W}_n, \mathcal{O}^{n-j}(\Pi_n(\mathcal{P}_2)))$, then $\mathcal{T}_n$ I $\mathcal{M}^{\mathcal{P}}_{n-j}$ I $\Pi_n(\mathcal{P}_2)$ I $\mathcal{L}_{\mathcal{P}Q}$ I $\Pi_n(Q_2)$ I $\mathcal{M}^{Q}_{n-j}$ I $\mathcal{T}_n$. Since $u[n,n-j](\mathcal{T}_n, \Pi_n(\mathcal{P}_2)) = 0$, by (GQ) we have $\mathcal{M}^{\mathcal{P}}_{n-j} = \mathcal{M}^{Q}_{n-j}$ ($u[n,n-j]$ is the valuation map in the HQ $\mathcal{Z}_{n-j}(\mathcal{W}_n, \mathcal{O}^{n-j}(\Pi_n(\mathcal{P}_2))))$. But $u[n,n-j](\Pi_n(\mathcal{P}_2), \Pi_n(Q_2)) = 0$, hence by property(1.1.3.6), $\mathcal{M}^{\mathcal{P}}_{n-j} = \mathcal{M}^{Q}_{n-j} = \mathcal{L}_{\mathcal{P}Q}$. By property(1.1.3.7), we have then $u(\Pi_n(\mathcal{L}_2), \mathcal{M}_n) > 0$ and so $\Pi_1(\mathcal{P}_1)$ I $\Pi_1(\mathcal{L}_2)$, a contradiction. Hence $u(\mathcal{P}_2, Q_2) \geq u(\mathcal{P}_1, Q_1)$. Similarly or by symmetry, $u(\mathcal{P}_1, Q_1) \geq u(\mathcal{P}_1, Q_2)$. □

PROPOSITION(3.1.2.7). *Suppose* $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{P}(\Delta_\infty)$, $\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}_1, \mathcal{M}_2 \in \mathcal{L}(\Delta_\infty)$, $\mathcal{L}_i$ I $\mathcal{P}_i$, $\mathcal{M}_i$ I $\mathcal{P}_i$, $\mathcal{L}_1 \perp \mathcal{L}_2$ *and* $\mathcal{M}_1 \perp \mathcal{M}_2$, $i=1,2$. *If* $\Pi_1(\mathcal{P}_1)$ *and* $\Pi_1(\mathcal{P}_2)$ *are not collinear, then* $u(\mathcal{L}_1, \mathcal{M}_1) = u(\mathcal{L}_2, \mathcal{M}_2)$.

PROOF. Let Q_1 resp. Q_2 be the intersection point of $\mathcal{L}_1$ and $\mathcal{L}_2$ resp. $\mathcal{M}_1$ and $\mathcal{M}_2$. By the assumption, $u(\mathcal{P}_1, Q_1) = u(\mathcal{P}_1, Q_2) = u(\mathcal{P}_2, Q_1) = u(\mathcal{P}_2, Q_2) = 0$ and so by proposition(3.1.2.3), $u(\mathcal{L}_1, \mathcal{M}_1) = u(\mathcal{L}_2, \mathcal{M}_2)$. □

As usual, we denote $v(x,0) = v(0,x)$ as $v(x)$, for all x in $\mathcal{R}_1$ or $\mathcal{R}_2$. We also define :

$$\mathcal{R}_1^+ = \{a \in \mathcal{R}_1 \mid v(a) \geq 0\} \text{ and } \mathcal{R}_1^- = \mathcal{R}_1 - \mathcal{R}_1^+,$$
$$\mathcal{R}_2^+ = \{k \in \mathcal{R}_2 \mid v(k) \geq 0\} \text{ and } \mathcal{R}_2^- = \mathcal{R}_2 - \mathcal{R}_2^+.$$

PROPOSITION(3.1.2.8). *If* $a, b \in \mathcal{R}_1^+$*, then*

$$v(a,b) = \begin{cases} u((a),(b)), & (1) \\ u((a,0,0),(b,0,0)), & (2) \\ u((p,a),(p,b)), & (3) \\ u((x,p,a),(x,p,b)), & (4) \end{cases}$$

for all $x \in \mathcal{R}_1^+$ *and all* $p \in \mathcal{R}_2^+$.

PROOF. By definition, $v(a,b) = u((a),(b))$. Now $u([\infty],[0,0,0]) = 0$ by the construction of $\mathcal{S}$, so from proposition(3.1.2.6) follows (2). Similarly, $v(a,b) = u((1,a),(1,b)) = u((0,0,a),(0,0,b))$. But by definition, $\Pi_1([\infty]) \neq \Pi_1([p])$ and so from proposition(3.1.2.6) follows $v(a,b) = u((p,a),(p,b))$ since $(0,0,a) \perp (p,a)$ and $(0,0,b) \perp (p,b)$. If $\Pi_1([x,p]) \perp \Pi_1([0])$, then $\Pi_1((\infty))\ I\ \Pi_1([x,p])$ since $\mathcal{W}_1$ is a generalized quadrangle and $[x,p]$ meets $[\infty]$. But then $\Pi_1((x)) = \Pi_1((\infty))$ since $\Pi_1([x,p]) \neq \Pi_1([\infty])$ (after all, $\Pi_1([x,p]) = \Pi_1([\infty])$ would imply $\Pi_1([0,0,p]) = \Pi_1([0,0])$, thus $\Pi_1([1,p]) = \Pi_1([\infty])$, thus $\Pi_1([p,0,0]) = \Pi_1([0,0])$, hence $\Pi_1([p]) = \Pi_1([\infty])$ contradicting $v(p) \geq 0$) which contradicts $v(x) \geq 0$. Hence $\Pi_1([x,p])$ is not concurrent with $\Pi_1([0])$. Since $(0,a) \perp (x,p,a)$ and $(0,b) \perp (x,p,b)$, (4) follows from proposition(3.1.2.6). ☐

PROPOSITION(3.1.2.9). *If* $k, \ell \in \mathcal{R}_2^+$*, then*

$$2.v(k,\ell) = \begin{cases} u([k],[\ell]), & (1) \\ u([k,0,0],[\ell,0,0]), & (2) \\ u([x,k],[x,\ell]), & (3) \\ u([p,x,k],[p,x,\ell]), & (4) \end{cases}$$

for all $x \in \mathcal{R}_1^+$ *and all* $p \in \mathcal{R}_2^+$.

PROOF. Similarly and dually to the preceding proposition, now using proposition(3.1.2.7) in stead of proposition(3.1.2.6). ☐

PROPOSITION(3.1.2.10). *Let* $\mathcal{L} \in \mathcal{L}(\Delta_\infty)$ *and suppose* $\mathcal{L} = [k,b,k']$ *with* $v(k)$, $v(b), v(k') \geq 0$. *Then* $\Pi_1(\mathcal{L})$ *and* $\Pi_1([\infty])$ *are not concurrent. If moreover* $\mathcal{P}_i = (a_i, \ell_i, a_i')\ I\ \mathcal{L}$ *with* $v(a_i), v(\ell_i), v(a_i') \geq 0$, $i=1,2$, *then* $u(\mathcal{P}_1,\mathcal{P}_2) = v(a_1,a_2)$.

PROOF. From the proof of proposition(3.1.2.9), dual to the one of proposition(3.1.2.8), there follows $\Pi_1((k,b)) \neq \Pi_1((\infty))$. Since $\Pi_1([k']) \neq \Pi_1([\infty])$, we have successively $\Pi_1([k',0,0]) \neq \Pi_1([0,0])$, $\Pi_1([1,k']) \neq$

$\Pi_1([\infty])$, $\Pi_1([0,0,k'])\neq\Pi_1([0])$, $\Pi_1([0,k'])\neq\Pi_1([\infty])$ (by the construction of the coordinates, the fact that $\mathcal{W}_1$ is a generalized quadrangle and the definition and construction of $\mathcal{S}$). Hence $\Pi_1(\mathcal{L})$ can not meet $\Pi_1([\infty])$, nor these two lines can coincide. Now $(a_i)\perp\mathcal{P}_i$, i=1,2, and hence the result follows from proposition(3.1.2.6). ☐

PROPOSITION(3.1.2.11). *Let $\mathcal{P}\in\mathcal{P}(\Delta_\infty)$ and suppose $\mathcal{P}=(a,\ell,a')$ with $v(a)$, $v(\ell),v(a')\geq 0$. Then $\Pi_1(\mathcal{P})$ and $\Pi_1((\infty))$ are not collinear. If moreover $\mathcal{L}_i=(k_i,b_i,k_i')$ I $\mathcal{P}$ with $v(k_i),v(b_i),v(k_i')\geq 0$, i=1,2, then $u(\mathcal{L}_1,\mathcal{L}_2)$ = $v(k_1,k_2)$.*

PROOF. Similarly and dually to proposition(3.1.2.10) ☐

THEOREM(3.1.2.12). *The 5-tuple $(\mathcal{R}_1^+,\mathcal{R}_2^+,Q_1,Q_2,v)$ satisfies* (v4),(v5),(v6) *and* (v7).

PROOF. We show (v4). Given :

$$\begin{aligned}
Q_1(k_1,a_1,\ell_1,a_1') &= Q_1(k_1,a_2,\ell_2,a_2') = b_1,\\
Q_2(a_1,k_1,b_1,k_1') &= Q_2(a_1,k_2,b_2,k_2') = \ell_1,\\
Q_1(k_2,a_1,\ell_1,a_1') &= Q_1(k_2,a_3,\ell_3,a_3') = b_2,\\
Q_2(a_3,k_2,b_2,k_2') &= Q_2(a_3,k_3,b_3,k_3') = \ell_3,\\
Q_1(k_3,a_3,\ell_3,a_3') &= Q_1(k_3,a_2,\ell_2,a_2') = b_3,\\
Q_2(a_2,k_3,b_3,k_3') &= Q_2(a_2,k_1,b_1,k_4') = \ell_2.
\end{aligned}$$

Put $\mathcal{P}_i=(a_i,\ell_i,a_i')$, $\mathcal{L}_i=[k_i,b_i,k_i']$, i=1,2,3, $\mathcal{P}_4=(k_1,b_1)$ and finally $\mathcal{L}_4$ = $[k_1,b_1,k_4']$. Then by assumption and the coordinatization of $\mathcal{S}$, $\mathcal{P}_4$ I $\mathcal{L}_1$ I $\mathcal{P}_1$ I $\mathcal{L}_2$ I $\mathcal{P}_3$ I $\mathcal{L}_3$ I $\mathcal{P}_2$ I $\mathcal{L}_4$ I $\mathcal{P}_4$. Since $\Pi_1(\mathcal{P}_1)$ is not collinear with $\Pi_1((\infty))$ by proposition(3.1.2.11), we have $u(\mathcal{P}_1,\mathcal{P}_4)=0$. By the same token, $u(\mathcal{P}_2,\mathcal{P}_4)=0$. Hence by proposition(3.1.2.5) (except if the quadrangle above degenerates in which case the assertion is trivial since both sides of the equality to prove are $+\infty$) :

$$u(\mathcal{L}_1,\mathcal{L}_4) = u(\mathcal{L}_2,\mathcal{L}_3)+u(\mathcal{P}_1,\mathcal{P}_3)+u(\mathcal{P}_2,\mathcal{P}_3) \qquad (1)$$

$$u(\mathcal{L}_1,\mathcal{L}_2)+u(\mathcal{P}_1,\mathcal{P}_3) = u(\mathcal{L}_3,\mathcal{L}_4)+u(\mathcal{P}_2,\mathcal{P}_3) \qquad (2)$$

But by proposition(3.1.2.9), $u(\mathcal{L}_1,\mathcal{L}_4) = 2.v(k_1',k_4')$; by proposition (3.1.2.11), $u(\mathcal{L}_2,\mathcal{L}_3) = 2.v(k_2,k_3)$, $u(\mathcal{L}_1,\mathcal{L}_2) = 2.v(k_1,k_2)$ and $u(\mathcal{L}_3,\mathcal{L}_4) = 2.v(k_3,k_1)$; by proposition(3.1.2.10), $u(\mathcal{P}_1,\mathcal{P}_3) = v(a_1,a_3)$ and $u(\mathcal{P}_2,\mathcal{P}_3) = v(a_2,a_3)$. Substituting all these values in (1) and (2), we obtain :

$$2.v(k_1',k_4') = 2.v(k_2,k_3) + v(a_1,a_3) + v(a_2,a_3) \quad (3)$$

$$2.v(k_1,k_2) + v(a_1,a_3) = 2.v(k_3,k_1) + v(a_2,a_3) \quad (4)$$

Adding (3) and (4) side by side and dividing by 2, we get :

$$v(k_1,k_2) + v(k_1',k_4') = v(k_1,k_3) + v(k_2,k_3) + v(a_2,a_3),$$

proving the assertion. Similarly, one shows (v5),(v6) and (v7), completing the proof of the theorem. □

The next theorem can be proved similarly to the case $\tilde{A}_2$ (see [18], 4.5.4). We will therefore only scetch the proof.

THEOREM(3.1.2.13). *The 5-tuple $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2,v)$ is a V-QQR.*

PROOF. Consider the apartment Σ of Δ corresponding to the quadrangle (∞) I $[\infty]$ I (0) I $[0,0]$ I $(0,0,0)$ I $[0,0,0]$ I $(0,0)$ I $[0]$ I (∞) in Δ_∞. We concieve Σ as the real Euclidean plane and define coordinates as follows. The origin is the point $\mathfrak{s}$. The first base vector has its end point in $\Pi_1((\infty))$, the second base vector has its end point in $\Pi_2([\infty])$. Note that these points are special vertices and every special vertex of Σ can be obtained by an integer linear combination of the base vectors. Now let $\mathfrak{s}^*$ be an arbitrary special vertex in Σ. Suppose $\mathfrak{s}^*$ has coordinates (α,β). As in the case $\tilde{A}_2$, one shows that there exist a point (x) and a line $[p]$ such that, if we recoordinatize Δ_∞ with respect to the same base quadrangle of coordinatization, but taking as unit point, resp. line, the point (x) resp. the line $[p]$, then $\mathfrak{s}^*$ is the basement for the new coordinatizing QQR $(\mathcal{R}_1^*,\mathcal{R}_2^*,Q_1^*,Q_2^*)$ of Δ_∞. Denote the corresponding valuation map as defined in 3.1.1 (replacing $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2)$ by $(\mathcal{R}_1^*,\mathcal{R}_2^*,Q_1^*,Q_2^*)$) by v^*. Then theorem(3.1.2.12) is valid for $(\mathcal{R}_1^*,\mathcal{R}_2^*,Q_1^*,Q_2^*,v^*)$. Denoting the new coordinates between two stars, we can define the following bijections.

$$\pi_1 : \mathcal{R}_1 \to \mathcal{R}_1^* : a \to a^{\pi_1} \quad \text{where } (a) = *(a^{\pi_1})*,$$
$$\pi_2 : \mathcal{R}_1 \to \mathcal{R}_1^* : a \to a^{\pi_2} \quad \text{where } (0,a) = *(0,a^{\pi_2})*,$$
$$\lambda_1 : \mathcal{R}_2 \to \mathcal{R}_2^* : k \to k^{\lambda_1} \quad \text{where } [k] = *[k^{\lambda_1}]*,$$
$$\lambda_2 : \mathcal{R}_2 \to \mathcal{R}_2^* : k \to k^{\lambda_2} \quad \text{where } [0,k] = *[0,k^{\lambda_2}]*.$$

There holds :

$$Q_1(k,a,\ell,a')^{\pi_2} = Q_1^*(k^{\lambda_1},a^{\pi_1},\ell^{\lambda_2},a'^{\pi_2}), \qquad (QQ^*)$$
$$Q_2(a,k,b,k')^{\lambda_2} = Q_1^*(a^{\pi_1},k^{\lambda_1},b^{\pi_2},k'^{\lambda_2}). \qquad (QQ^*)$$

And as in the case $\tilde{A}_2$, one can show here that

$$v^*(a^{\pi_1},b^{\pi_1}) = v(a,b) + \alpha, \qquad (VV^*)$$
$$v^*(k^{\lambda_1},\ell^{\lambda_1}) = v(k,\ell) + \beta, \qquad (VV^*)$$
$$v^*(a^{\pi_2},b^{\pi_2}) = v(a,b) + \alpha + 2\beta, \qquad (VV^*)$$
$$v^*(k^{\lambda_2},\ell^{\lambda_2}) = v(k,\ell) + \alpha + \beta. \qquad (VV^*)$$

Suppose now

$$Q_1(k_1,a_1,\ell_1,a_1') = Q_1(k_1,a_2,\ell_2,a_2') = b_1,$$
$$Q_2(a_1,k_1,b_1,k_1') = Q_2(a_1,k_2,b_2,k_2') = \ell_1,$$
$$Q_1(k_2,a_1,\ell_1,a_1') = Q_1(k_2,a_3,\ell_3,a_3') = b_2,$$
$$Q_2(a_3,k_2,b_2,k_2') = Q_2(a_3,k_3,b_3,k_3') = \ell_3,$$
$$Q_1(k_3,a_3,\ell_3,a_3') = Q_1(k_3,a_2,\ell_2,a_2') = b_3,$$
$$Q_2(a_2,k_3,b_3,k_3') = Q_2(a_2,k_1,b_1,k_4') = \ell_2.$$

Let $j = \inf\{v(a_1), v(a_2), v(a_3), v(a_1'), \ldots, v(\ell_3)\}$. If $j \geq 0$, then (v4) follows from the previous theorem. Suppose now $j < 0$. We choose $(\alpha,\beta) = (|j|,|j|)$ and remark that $0^{\pi_1} = 0^{\pi_2} = 0^{\lambda_1} = 0^{\lambda_2} = 0$. Hence the formulas (VV^*) yield

$$v^*(a^{\pi_1}) = v(a) + |j|,$$
$$v^*(k^{\lambda_1}) = v(k) + |j|,$$
$$v^*(a^{\pi_2}) = v(a) + 3|j|,$$
$$v^*(k^{\lambda_2}) = v(k) + 2|j|.$$

We rewrite the assumptions of (v4) by means of the formulas (QQ^*) :

$$Q_1^*(k_1^{\lambda_1}, a_1^{\pi_1}, \ell_1^{\lambda_2}, a_1'^{\pi_2}) = Q_1^*(k_1^{\lambda_1}, a_2^{\pi_1}, \ell_2^{\lambda_2}, a_2'^{\pi_2}) = b_1^{\pi_2},$$

$$Q_2^*(a_1^{\pi_1}, k_1^{\lambda_1}, b_1^{\pi_2}, k_1'^{\lambda_2}) = Q_2^*(a_1^{\pi_1}, k_2^{\lambda_1}, b_2^{\pi_2}, k_2'^{\lambda_2}) = \ell_1^{\lambda_2},$$

$$Q_1^*(k_2^{\lambda_1}, a_1^{\pi_1}, \ell_1^{\lambda_2}, a_1'^{\pi_2}) = Q_1^*(k_2^{\lambda_1}, a_3^{\pi_1}, \ell_3^{\lambda_2}, a_3'^{\pi_2}) = b_2^{\pi_2},$$

$$Q_2^*(a_3^{\pi_1}, k_2^{\lambda_1}, b_2^{\pi_2}, k_2'^{\lambda_2}) = Q_2^*(a_3^{\pi_1}, k_3^{\lambda_1}, b_3^{\pi_2}, k_3'^{\lambda_2}) = \ell_3^{\lambda_2},$$

$$Q_1^*(k_3^{\lambda_1}, a_3^{\pi_1}, \ell_3^{\lambda_2}, a_3'^{\pi_2}) = Q_1^*(k_3^{\lambda_1}, a_2^{\pi_1}, \ell_2^{\lambda_2}, a_2'^{\pi_2}) = b_3^{\pi_2},$$

$$Q_2^*(a_2^{\pi_1}, k_3^{\lambda_1}, b_3^{\pi_2}, k_3'^{\lambda_2}) = Q_2^*(a_2^{\pi_1}, k_1^{\lambda_1}, b_1^{\pi_2}, k_4'^{\lambda_2}) = \ell_2^{\lambda_2}.$$

But now all elements in these expressions have positive v^*-valuation. Hence we can apply theorem(3.1.2.12) and obtain :

$$v^*(k_1^{\lambda_1}, k_2^{\lambda_1}) + v^*(k_1'^{\lambda_2}, k_4'^{\lambda_2}) = v^*(k_1^{\lambda_1}, k_3^{\lambda_1}) + v^*(k_2^{\lambda_1}, k_3^{\lambda_1}) + v^*(a_2^{\pi_1}, a_3^{\pi_1})$$

Using (VV*) we get the result. Similarly, one shows (v5),(v6) and (v7). This completes the proof of the theorem. □

3.2. Proof of theorem(2.1).

We use the notation of the preceding paragraph.
In Δ, a convex (with respect to the metric d_Δ) subset of a straight wall bounded by vertices s_1 and s_2 is called a *straight interval* and is denoted by $[s_1, s_2]$.
There clearly only remains to show that, if Δ is a complete building, then $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ is complete. We show that $\mathcal{R}_1$ is complete with respect to v. Things are similar for $\mathcal{R}_2$. So let $(a_i)_{i \in \mathbf{N}}$ be a Cauchy-sequence in $\mathcal{R}_1$. Hence there exists a $p \in \mathbf{N}$ such that, for all $q > p$, $v(a_p, a_q) \geq 0$. There are two possibilities now.

(1) $v(a_p) \geq 0$.
By the triangle inequality, $v(a_q) \geq 0$ for all $q \geq p$. We define a vertex s_n, for $n \in \mathbf{N}$, as follows. Let j be such that for all $k \geq j$, $v(a_j, a_k) \geq n$. Then $s_n = \Pi_n((a_j))$. This is independent from the choice of a_j since another choice a_j' yields a point (a_j') with $v(a_j, a_j') \geq n$ and hence $u((a_j), (a_j')) \geq n$. So if $m \leq n$, then also $s_m = \Pi_m((a_j))$ and hence $[s, s_n]$ is

a straight interval containing s_m for all $m \le n$. By [10],corollary(2.2.8) $\cup\{[s, s_n] \mid n \in \mathbf{N}\}$ is a sectorpanel $\mathfrak{P}$. Let $\mathcal{P}$ be the point of Δ_∞ determined by $\mathfrak{P}$. If $\mathcal{P}$ is not incident with $[\infty]$, then $u_2(\mathcal{P}, [\infty]) = \ell$ is finite. If $n \ge 2\ell$, then $u_2(\Pi_n(\mathcal{P}), \Pi_n([\infty])) = \ell$ (cp. property(1.1.3.8)). But $\Pi_n(\mathcal{P}) = s_n$, a contradiction, hence $\mathcal{P}$ is incident with $[\infty]$. Set $\mathcal{P} = (a)$. Then $\Pi_n((a)) = s_n$ and hence there exists j such that $v(a, a_k) \ge n$ for all $k \ge j$. Hence a is the limit of $(a_i)_{i \in \mathbf{N}}$.

(2) $v(a_p) < 0$.
In this case, $v(a_q) = v(a_p) = c$ constant for all $q \ge p$. We define s_n exactly the same way as in **(1)**. Then the result follows similarly, taking into account $u((a_i),(a_j)) = w(a_i, a_j) = v(a_i, a_j) - v(a_i) - v(a_j) = v(a_i, a_j) - 2c > v(a_i, a_j)$, for all $i, j \ge p$.

This completes the proof of the theorem □

3.3. Proof of theorem(2.2).

(2.2.3) ⇒ (2.2.2). This is theorem(2.1).
(2.2.2) ⇒ (2.2.1). This is trivial.
(2.2.1) ⇒ (2.2.3). This is theorem(1.4.2).

3.4. Proof of theorem(2.3).

(2.3.3) ⇒ (2.3.2). Follows from theorem(2.2) by completing the set of apartments of the given building to a complete set of apartments. So given generalized quadrangle $\mathcal{B}$ is a subquadrangle of a generalized quadrangle which can be coordinatized by means of a CV-QQR. This ring restricts to a coordinatizing V-QQR of $\mathcal{B}$.

(2.3.2) ⇒ (2.3.1). This is trivial.

(2.3.1) ⇒ (2.3.3). Similar to the the case of buildings of type $\tilde{A}_2$ (see [18],Remark(6.2.3)). □

4. PROOF OF THE MAIN THEOREM

We first proof the result for complete buildings :

THEOREM. *Every complete affine building of type* $\tilde{C}_2$ *is uniquely and completely determined by any coordinatizing quadratic quaternary ring with complete valuation of its generalized quadrangle at infinity.*

PROOF. By theorem(1.4.1), it suffices to show that, if $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ is an arbitrary CV-QQR coordinatizing Δ_∞ and $\mathfrak{s}$ is a basement for $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$, then the level n HQ $\mathcal{V}_n$ determined by the n-th floor of $(\Delta, \mathfrak{s})$ is isomorphic to the level n HQ $\mathcal{W}_n$ determined by $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ as meant in theorem(1.2.2) (see [20]). We assume here that the reader is familiar with that construction.

By the definition of the valuation map v, it is straight forward to verify that one can identify the sets $\mathcal{P}(\mathcal{V}_n)$ and $\mathcal{P}(\mathcal{W}_n)$, resp. $\mathcal{L}(\mathcal{V}_n)$ and $\mathcal{L}(\mathcal{W}_n)$ as follows. Suppose $\mathcal{P}_n \in \mathcal{P}(\mathcal{V}_n)$ and let $\mathcal{P}$ be any sectorpanel with source $\mathfrak{s}$ containing $\mathcal{P}_n$. Then $\mathcal{P}$ can be viewed as a point of Δ_∞ and since $\mathcal{W}_n$ is defined as a quotient geometry of Δ_∞ by identifying suitable points and lines, it is clear that we will identify $\mathcal{P}_n$ with the class of points containing $\mathcal{P}$. We denote that class (which is a point of $\mathcal{W}_n$ called *the projection of* $\mathcal{P}$ *onto* $\mathcal{W}_n$) by $\Phi(\mathcal{P})$. Similarly for lines. If $\mathcal{P}_n \; I \; \mathcal{L}_n$ in $\mathcal{V}_n$, consider a sector $\mathfrak{Q}$ with source $\mathfrak{s}$ containing both $\mathcal{P}_n$ and $\mathcal{L}_n$. The *trace at infinity* of $\mathfrak{Q}$ is an incident point-line pair $(\mathcal{P}, \mathcal{L})$ of Δ_∞. The projection onto $\mathcal{W}_n$ of this pair is $(\Phi(\mathcal{P}_n), \Phi(\mathcal{L}_n))$ and this is by construction an incident point-line pair in $\mathcal{W}_n$. Similarly for the converse. Hence Φ is an isomorphism from the the base geometry of $\mathcal{V}_n$ to the base geometry of $\mathcal{W}_n$.

Now consider an $(n-1)$-point-neighbourhood $\mathcal{C}$ of $\mathcal{V}_n$ and denote $\mathcal{D} = \Phi(\mathcal{C})$. In fact, $\mathcal{C}$ can be identified in an obvious way with a vertex of Δ adjacent to $\mathfrak{s}$. We recoodinatize Δ_∞ as done in [20] to define $\mathcal{I}_{n-1}(\mathcal{W}_n, \mathcal{D})$ (see 3.1.1). By projecting the new base quadrangle of coordinatization a nd the new unit point and unit line onto the residue of $\mathcal{C}$, we see that $\mathcal{C}$ is a basement for the new coordinatizing CV-QQR. With an inductive argument, we see that $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ is isomorphic to $\mathcal{L}_{n-1}(\mathcal{W}_n, \mathcal{D})$ and the isomorphism coincides with Φ over $\mathcal{C}$ in view of the connection between the partial valuation (see [20]). With a second inductive argument, this is also true for every j-point-neighbourhood, $0 < j < n$. Hence $\mathcal{V}_n$ and $\mathcal{W}_n$ are isomorphic. □

In the same way as theorem(2.3) follows from theorem(2.2), the main theorem follows from the above theorem.

5. EXAMPLES

All examples are taken from [20], where one also can find proofs and further comments.

Example 1.

Let $\mathcal{R}_1 = \mathcal{R}_2 = \mathrm{GF}(q)((t))$ with $q = 2^h$, $h > 1$. Let $h_1, h_2 \in \mathbf{N}$ be such that $q-1$ and $2^{1+h_1+h_2}-1$ are relatively prime (e.g. $h=3$, $h_1=1$, $h_2=0$; $h=4$, $h_1=1$, $h_2=1$ etc...) with $(h_1, h_2) \neq (0,0)$. We first define the finite QQR $(\mathrm{GF}(q), \mathrm{GF}(q), Q_1, Q_2)$ as follows. Put $\theta_i = 2^{h_i}$ en define

$$Q_1(k,a,\ell,a') = k^{2\theta_1}.a + a',$$
$$Q_2(a,k,b,k') = a^{\theta_2}.k + k'.$$

We extend θ_i to $\mathcal{R}_i$ as follows :

$$(\Sigma x_n t^n)^{\theta_i} = \Sigma x_n^{\theta_i} t^n \quad , \quad i=1,2.$$

Then θ_i is a field automorphism of $\mathcal{R}_i$. We define new multiplication laws as follows :

$$x \otimes_1 y = x^{2\theta_1}.y.$$
$$x \otimes_2 y = x^{\theta_2}.y,$$

for all $x, y \in GF(q)((t))$. Let v be the standard valuation of $GF(q)((t))$, then $(\mathcal{R}_1, \mathcal{R}_2, Q_1^*, Q_2^*, v)$, where

$$Q_1^*(k,a,\ell,a') = k \otimes_1 a + a',$$
$$Q_2^*(a,k,b,k') = a \otimes_2 k + k',$$

is a CV-QQR.

Example 2.
Let $\mathcal{R}_1 = \mathcal{R}_2 = GF(q)((t))$ with $q = 2^h$, $h > 1$. We define $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ as follows :

$$Q_1(k,a,\ell,a') = k^2 a + a',$$
$$Q_2(a,k,b,k') = a^\theta k + k',$$

where $(\sum x_r t^r)^\theta = \sum x_r (\frac{t}{1+t})^r$. Similarly as above $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ is a CV-QQR.

Example 3.
Let $\mathbf{F}$ be a field with valuation v and suppose $\mathbf{F}$ is complete w.r.t. v. Let θ be an arbitrary element of $\mathbf{F}$ with valuation 1. Let ψ be any field automorfisme leaving invariant the valuation v. We put $\mathcal{R}_1 = \mathbf{F} \times \mathbf{F}$, $\mathcal{R}_2 = \mathbf{F}$ and (with $x = (x_1, x_2) \in \mathcal{R}_1$) define :

$$Q_1(k,a,\ell,a') = (a_1 k + a_1', \ a_2 k^\psi + a_2'),$$
$$Q_2(a,k,b,k') = a_1^2 k + \theta a_2^2 k^\psi + k' - 2a_1 b_1 - 2\theta a_2 b_2 .$$

We extend the valuation v to $\mathcal{R}_1$ by defining for all $a = (a_1, a_2) \in \mathcal{R}_1$:

$$v(a) = v(a_1^2 + \theta a_2^2).$$

This defines again a class of CV-QQRs. In the case $\mathbf{F} = GF(q)((t))$, one can omit the codition $v(\theta) = 1$ and replace it by : $-\theta$ is a non-square in $GF(q)$. This way, one obtaines buildings with Kantor's bad eggs as residues.

Example 4.

Let $\mathbf{K}$ be an arbitrary field of characteristic 2 and put $\mathbf{F}=\mathbf{K}((t))$. Let θ_i, $i=1,2,3$, be arbitrary elements of $\mathbf{F}((t))$ such that $v(\theta_i)=1$ and $v(\theta_i+t)>1$ (v is again the natural valuation on $\mathbf{F}((t))$). Define the field automorfisme

$$\psi_i : \mathbf{F}\to\mathbf{F} : (\Sigma x_n t^n)^{\psi_i} = \Sigma x_n \theta_i^n , \quad i=1,2,3.$$

We define three new multiplication laws as follows : for all $a,b\in\mathbf{F}^2$,

$$a\otimes_i b = (a_1,a_2)\otimes_i(b_1,b_2) = (a_1.b_1 + t.a_2^{\psi_i}.b_2^{\psi_i} , a_1.b_2 + a_2.b_1),$$

$i=1,2,3$. Finally, the quaternary operations are :

$$Q_1(k,a,\ell,a') = k\otimes_1 a + a',$$
$$Q_2(a,k,b,k') = (a\otimes_2 a)\otimes_3 k + k'.$$

This defines again a large class of CV-QQRs and correspondingly a large class of non-classical affine buildings of type $\tilde{C}_2$.

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