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# A Configurational Characterization of the Moufang Generalized Polygons

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We generalize Baer's theorem on Desargues configurations in projective planes to all generalized polygons, thus obtaining a common geometric characterization of all (finite and infinite) Moufang generalized polygons.

## 1. INTRODUCTION

The notion of a *generalized polygon* was introduced by J. Tits in his celebrated paper on trialities [14]. Ever since, they have become the research object of several mathematicians (especially the finite generalized polygons). A generalized polygon is a generalized $n$-gon for some $n \geq 2$ (a point–line geometry where every two elements are contained in a cycle of length $2n$ and there are no shorter cycles) and an important theorem of Feit and Higman [1] states that, if the generalized $n$-gon is finite, then $n \in \{2, 3, 4, 6, 8\}$, provided that all lines contain at least three points and all points are incident with at least three lines (the so-called *thick* generalized polygons). Another important theorem, proved by Tits [15, 16], states that if a generalized polygon satisfies the *Moufang condition* (i.e. it has 'enough' certain automorphisms, see below for the exact formulation), then it is known. Such generalized polygons are called *Moufang*. Finite Moufang generalized $n$-gons are the projective planes over a finite field ($n = 3$), the geometries naturally associated with the classical Chevalley groups $PSp(4, q)$, $PSU(4, q)$, $PSU(5, q)$ ($n = 4$), $G_2(q)$, ${}^3D_4(q)$ ($n = 6$), ${}^2F_4(2^e)$ ($n = 8$), $q$ a prime power and $e$ a positive odd integer, and their duals (see e.g. [8] for an excellent survey on that matter).

Now, a generalized 3-gon is nothing other than an ordinary projective plane, where Baer's theorem (see e.g. [7]) states a connection between a purely group-theoretical property and a purely geometric property ($(P, l)$-transitivity being equivalent to $(P, l)$-Desarguesian). As a corollary, one obtains a geometric characterization of all Moufang projective planes. Recently, a similar theorem for generalized quadrangles (generalized 4-gons) was proved by J. A. Thas and the author (see [12]). In the present paper, we present a generalization of that result valid in all generalized polygons (excluding the trivial generalized digons). As a corollary, this yields a common configurational characterization of all Moufang generalized polygons providing in particular the first geometric characterization of the known generalized octagons (see [8]).

## 2. DEFINITIONS, NOTATION AND RESULTS

### 2.1. Generalized Polygons

A generalized $n$-gon, $n \in \{3, 4, 5, \ldots\}$ of order $(s, t)$ is a point-line incidence geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ satisfying (GP1), (GP2) and (GP3):

(GP1) Two distinct lines have at most one point in common and there are exactly $1 + s$ points incident with every line.

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(GP2) Two distinct points lie on at most one common line and there are exactly $1+t$ lines incident with every point.

(GP3) Given two elements $x, y \in \mathcal{P} \cup \mathcal{L}$, there exists at least one chain $xIz_1Iz_2I \ldots Iz_jIy$ of minimal length such that $j \leq n-1$ and there exists at most one such chain of minimal length such that $j < n-1$.

With the notation of (GP3), we call $j+1$ the *distance* between $x$ and $y$ and we denote it by $d(x, y)$. If $j < n-1$, then (GP3) says that $z_k$ is uniquely defined and we denote $z_k$ by $\Pi_x^k(y)$. It is convenient to define also $\Pi_x^{j+1}(y) = y$ and $\Pi_x^1(x) = \phi$. If $d(x, y) = n$, then we call $x$ and $y$ *opposite*. The parameters $s$ and $t$ can be infinite, but there are no examples of generalized polygons with $s$ (resp. $t$) finite and larger than or equal to 2 and at the same time $t$ (resp. $s$) infinite. In this paper, we will always assume $s, t \geq 2$. In that case, if $n$ is odd, then $s = t$ and if $n$ is even, then there are some strong restrictions on $s$ and $t$ in the finite case ($s \leq t^2$ and $t \leq s^2$ if $n = 4$ or 8, $s \leq t^3$ and $t \leq s^3$ if $n = 6$, $st$ is a square integer if $n = 6$ and $2st$ is a square integer if $n = 8$ are the most important and shortest to write down, see [1, 3, 6]). Also, if $n = 4$ and $s$ (resp. $t$) equals 2 or 3, then $t$ (resp. $s$) is finite by unpublished work of P. Cameron (case $s$ or $t = 2$) and A. Brouwer (case $s$ or $t = 3$).

Note that, if $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ is a generalized polygon, then $\mathcal{S}^* = (\mathcal{L}, \mathcal{P}, I)$ is also a generalized polygon, called the dual of $\mathcal{S}$.

### 2.2. *Further Definitions and Notation*

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ be a generalized $n$-gon as defined above. If $x$ and $y$ are points or lines in $\mathcal{S}$ at distance 2 from each other, then we say that $x$ and $y$ are *collinear* (if they are points) or *concurrent* (if they are lines) and we write $x \perp y$. If $d(x, y) = 3$, then we write $x \sim y$. We will also use combinations of these symbols and their meaning should be clear, e.g. $x \perp\perp y$ means $d(x, y) = 4$; $x \perp\sim y$ means $d(x, y) = 5$, etc.

A *polylateral* in $\mathcal{S}$ is a sub-$n$-gon of order (1, 1) in $\mathcal{S}$. We have chosen that word in anology to *quadrilaterals* in generalized quadrangles and to clearly mark the difference between *generalized* $n$-gons and usual $n$-gons. Also, if $n = 6$, then a polylateral is called a *hexalateral*. We will denote polylaterals by lower case Greek letters.

Suppose $x_1Ix_2I \ldots Ix_{n-1}$, $x_i \in \mathcal{P} \cup \mathcal{L}$. If $d(x_1, x_{n-1}) = n-2$, then we call the $(n-1)$-tuple $(x_1, x_2, \ldots, x_{n-1})$ a root. We usually denote roots by upper case German letters.

### 2.3. *Generalized Desargues Configurations*

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ be a generalized $n$-gon, $n \geq 3$ and suppose $\mathfrak{R} = (x_1, x_2, \ldots, x_{n-1})$ is a root in $\mathcal{S}$. Let $\pi = (y_1Iy_2I \ldots Iy_{2n}Iy_1)$ and $\pi' = (y_1'Iy_2'I \ldots Iy_{2n}'Iy_1')$ be two polylaterals in $\mathcal{S}$. We call $\pi$ and $\pi'$ *in perspective from* $\mathfrak{R}$ if for every $i \in \{1, 2, \ldots, n-1\}$ and every $j \in \{1, 2, \ldots, 2n\}$:

(GDC1) $d(x_i, y_j) = d(x_i, y_j')$,

(GDC2) $\Pi_{x_i}^1(y_j) = \Pi_{x_i}^1(y_j')$ if $d(x_i, y_j) < n$.

In this case, we call $y_j$ and $y_j'$ *corresponding elements* and the whole configuration of $\mathfrak{R}$, $\pi$, $\pi'$ and all shortest chains (if they are unique) from the elements of $\mathfrak{R}$ to the elements of $\pi$ and $\pi'$ is called a *generalized Desargues configuration*. For $n = 3$, this amounts to the usual Desargues configuration in a projective plane, for $n = 4$, this notion is the same as defined in [12]. Our main result now states that the existence of 'enough' such configurations will imply the existence of 'many' automorphisms of a certain type. We now explain the terms 'enough', 'many' and 'of certain type'.

*2.4. Root-Desarguesian*

We keep the same notation as in the previous paragraph.

The generalized polygon $\mathcal{S}$ is called *$\mathfrak{R}$-Desarguesian* if for every polylateral $\pi = (y_1 I y_2 I \ldots I y_{2n} I y_1)$ (where without loss of generality we can assume that $d(x_1, y_1) < n$) and every $y \in \mathcal{P} \cup \mathcal{L}$, $x_1 \neq y I \Pi^1_{x_1}(y_1)$, there exists a polylateral $\pi' = (y'_1 I y'_2 I \ldots I y'_{2n} I y'_1)$ in perspective with $\pi$ from $\mathfrak{R}$ such that $y_i$ corresponds to $y'_i$ (for all $i \in \{1, 2, \ldots, 2n\}$) and $\Pi^2_{x_1}(y'_1) = y$. In this case, we say that $\pi'$ is in perspective with $\pi$ from $\mathfrak{R}$ via $(y_1, y)$.

If $\mathcal{S}$ is $\mathfrak{R}$-Desarguesian for all roots $\mathfrak{R}$ in $\mathcal{S}$, then we call $\mathcal{S}$ *root-Desarguesian* (for more on terminology, see Section 5).

*2.5. Root-Elations*

An automorphism $\theta$ of $\mathcal{S}$ is called an $\mathfrak{R}$-elation, $\mathfrak{R}$ a root as above, if $\theta$ fixes all elements incident with $x_i$, $i = 1, 2, \ldots, n-1$. It is easy to see that, if an $\mathfrak{R}$-elation $\theta$ fixes an element at distance 2 from $x_1$, then $\theta$ is the identity. So if $A$ is the set of all elements distinct from $x_1$ and incident with any given element incident with $x_1$, then the group of all $\mathfrak{R}$-elations acts semi-regularly on $A$. If that group is transitive on $A$, then we say that $\mathcal{S}$ is *$\mathfrak{R}$-transitive.*

Now, if $\mathcal{S}$ is $\mathfrak{R}$-transitive for all roots $\mathfrak{R}$ contained in a given polylateral (and hence for all roots), and if not at the same time $n = 4$ and $\mathcal{S}$ is infinite, then $\mathcal{S}$ is known ([2, 15, 16]). In that case, $\mathcal{S}$ is called *Moufang*. Every finite Moufang generalized polygon is classical or dual classical, and a complete list is given in the Introduction.

## 3. Main Result and Consequences

Main Theorem. *The generalized polygon $\mathcal{S}$ is $\mathfrak{R}$-Desarguesian for some fixed root $\mathfrak{R}$ iff $\mathcal{S}$ is $\mathfrak{R}$-transitive.*

Proof. This will be given in the next section.

Corollary 1. *A generalized polygon $\mathcal{S}$ is root-Desarguesian iff $\mathcal{S}$ is $\mathfrak{R}$-Desarguesian for all $\mathfrak{R}$ contained in a given polylateral in $\mathcal{S}$.*

Proof. If $\mathcal{S}$ is $\mathfrak{R}$-Desarguesian for all $\mathfrak{R}$ in a given polylateral, then by the main theorem, $\mathcal{S}$ is $\mathfrak{R}$-transitive for all $\mathfrak{R}$ in the same polylateral, hence $\mathcal{S}$ is $\mathfrak{R}$-transitive for all $\mathfrak{R}$ in $\mathcal{S}$ and so $\mathcal{S}$ is root-Desarguesian by the main theorem again. The converse is trivial. □

Corollary 2. *A generalized polygon $\mathcal{S}$ is Moufang iff $\mathcal{S}$ is root-Desarguesian.*

Proof. This follows immediately from the main theorem. □

Corollary 3. *A finite generalized n-gon, $n = 3, 4, 6, 8$, is classical or dual classical iff it is root-Desarguesian.*

Proof. This follows from Corollary 2 and the classification of all finite Moufang generalized polygons (see [2, 15]). □

In particular, Corollary 3 yields a geometric characterization for the known finite generalized hexagons and octagons.

All these results are known in case $n=3$ (following from Baer's theorem) and $n=4$ (see Thas-Van Maldeghem [12]).

In the last corollary, we combine our results with another beautiful characterization of the Moufang generalized hexagons due to M. A. Ronan (see [11]). Note beforehand that a generalized $n$-gon, $n$ even but fixed, contains two different types of roots, namely roots containing an odd number of points and roots containing an even number of points. The next result characterizes a class of Moufang generalized hexagons by using only one specific type of roots plus an extra condition.

COROLLARY 4. *Suppose a generalized hexagon $\mathcal{S}$ is $\mathfrak{R}$-Desarguesian for all roots $\mathfrak{R}=(x_1, x_2, \ldots, x_5)$ containing three lines ($x_1, x_3$ and $x_5$) and two points ($x_2$ and $x_4$). Suppose that every such $\mathfrak{R}$ and every pair of hexalaterals $\eta=(y_1Iy_2I\ldots Iy_{12}Iy_1)$ and $\eta'=(y_1'I\ldots)$ which are in perspective from $\mathfrak{R}$ satisfy the following two conditions:*
(i) *$\Pi^2_{x_i}(y_j)=\Pi^2_{x_i}(y_j')$, for all $i\in\{2,3,4\}$ and all $j\in\{1,2,\ldots,12\}$ whenever this is well defined,*
(ii) *$\Pi^3_{x_3}(y_j)=\Pi^3_{x_3}(y_j')$, for all $j\in\{1,2,\ldots,12\}$ whenever this is well, defined, then $\mathcal{S}$ is Moufang. If moreover $\mathcal{S}$ is finite, then $\mathcal{S}$ arises naturally from one of the classical groups $G_2(q)$ or ${}^3D_4(q)$, $q$ a prime power.*

PROOF. By the main theorem, $\mathcal{S}$ is $\mathfrak{R}$-transitive for all roots $\mathfrak{R}$ containing three lines. It is now readily seen that conditions (*i*) and (*ii*) imply that every $\mathfrak{R}$-elation ($\mathfrak{R}$ containing three lines) $\theta$ is in fact an *axial automorphism* (with the terminology of Ronan [11]), i.e. $\theta$ fixes all elements at distance two or three from the middle element of $\mathfrak{R}$. The result now follows directly from [11]. □

## 4. PROOF OF THE MAIN THEOREM

A complete geometric proof of the main theorem in the spirit of the proof of Baer's result on projective planes (see e.g. [7]) is out of question in the general case (it is too complicated). On the other hand, a complete algebraic proof would be too abstract and almost unreadable. Therefore, one has to compromise between geometrical arguments and algebraic ones. The algebra is brought in by abstract coordinatization along the lines of Hanssens and Van Maldeghem [4] and Thas and Van Maldeghem [12]. We will spend a few words on abstract coordinatization of generalized $n$-gons, $n\geq 5$, in the next paragraph.

IMPORTANT REMARK. For the sake of clarity, we will put $n=6$ in the main part of the proof. By adding some dots in the middle of all chains and coordinate-tuples, one can easily write down the general arguments, but going through the case $n=6$ (which is already fairly general) first helps a lot. Moreover, $n=6$ is the first and (together with $n=8$) most important case after $n=4$. Since coordinatization plays a crucial role here, we will expose that part in full generality (but restricting ourselves to the things we need for the actual proof).

### 4.1. *Coordination of generalized polygons*

Throughout, we abbreviate $0, 0, \ldots, 0$ ($k$ zero's) by $0_k$. The method given below is a generalization of the coordinatization method of generalized quadrangles of Hanssens and Van Maldeghem [4].

Suppose that $\mathcal{S}=(\mathcal{P}, \mathcal{L}, I)$ is a generalized $n$-gon, $n\geq 3$, of order $(s,t)$, $s,t\geq 2$ and possibly infinite. Let $R_1$ and $R_2$ be two arbitrary sets of elements (called the *coordinates*) satisfying the conditions:

(C1) $|R_1| = s, \quad |R_2| = t,$
(C2) $R_1 \cap R_2 = \{0, 1\}$ if $n$ is even and $R_1 = R_2$ if $n$ is odd,
(C3) neither $R_1$ nor $R_2$ contains the symbol $\infty$.

In the coordinatization process, we use round brackets for coordinates of points and square brackets for coordinates of the lines. If a coordinate-tuple represents some element (point or line), then we write the coordinates between the signs $\langle\langle \ldots \rangle\rangle$.

We choose in $\mathcal{S}$ an arbitrary polylateral $\pi$ and label its elements $(\infty)I[\infty]I(0)I[0, 0]I(0_3)I[0_4]I \ldots I[0_{n-1}]I \ldots I[0]I(\infty)$ (with the above conventions). Note that the point $(0_{n-1})$ comes either on the left ($n$ even) or on the right ($n$ odd) of the line $[0_{n-1}]$ in the above chain. We now label the points on the line $[\infty]$ (resp. $[0_k]$) except for the point $(0)$ (resp. $(0_{k-1})$) by $(a)$ (resp. $(0_l, a)$, $l = \inf\{k, n-2\}$), $a \in R_1$, in a bijective (i.e. every element of $R_1$ is used exactly once) and consistent (i.e. the new $(0_l, 0)$ coincides with the old $(0_{l+1})$) manner. Similarly, one coordinatizes the lines through the points $(\infty)$, $(0_k)$, $k < n$, by using elements of $R_2$. We now impose the following conditions on the previous assignments of coordinates:

(C4) If $x$ and $y$ are opposite elements of $\pi$ in $\mathcal{S}$ and $zIx$ is not contained in $\pi$, then $z$ has the same unique non-zero coordinate as $\Pi_y^1(z)$.
(C5) Let $x$ be the unique element of $\pi$ opposite to $(\infty)$ in $\mathcal{S}$ and let $zIx$ be such that its unique non-zero coordinate equals 1. Then $\Pi_z^1((a)) = \Pi_z^1((0, a))$, for all $a \in R_1$.

Condition (C4) is necessary for the further coordinatization, whereas condition (C5) is only needed to symplify the algebra we need (it is a kind of *normalization*, see e.g. [5] for the case of generalized quadrangles).

Points at distance $2k+1$ from $[\infty]$ are coordinatized by a $(2k+1)$-tuple except if they lie closer to $(\infty)$, in which case they are at distance $2k$ from $(\infty)$ and they are coordinatized by a $2k$-tuple. Dually for the lines. Inductively, we define the element $\langle\langle a_1, a_2, \ldots, a_k \rangle\rangle$, $2 \leq k \leq n-1$, as follows. Let $x = \langle\langle a_1, \ldots, a_{k-1} \rangle\rangle$ and let $y$ be the unique element of $\pi$ opposite to $\langle\langle 0_{k-1} \rangle\rangle$ in $\mathcal{S}$. One can easily check that $y$ is also opposite to $x$ (since it will follow that $xI\langle\langle a_1, \ldots, a_{k-2} \rangle\rangle I \langle\langle a_1, \ldots, a_{k-3} \rangle\rangle I \ldots I \langle\langle a_1 \rangle\rangle I \langle\langle \infty \rangle\rangle$). Denote by $u$ the unique element incident with $y$ not having less coordinates than $y$ has and having as last coordinate exactly $a_k$. Then $\langle\langle a_1, a_2, \ldots, a_k \rangle\rangle = \pi_x^1(u)$. By condition (C4), this is consistent with the earlier given coordinates. By this method, all elements of $\mathcal{S}$ are coordinatized. Note that a coordinate-tuple of a point or a line consists alternately of elements of $R_1$ and $R_2$. Suppose $x \in \mathcal{P}$, $y \in \mathcal{L}$ or $x \in \mathcal{L}$, $y \in \mathcal{P}$ and suppose at least one of them is not opposite to $(\infty)$ or to $[\infty]$. Then $xIy$ iff $x$ has the coordinates $\langle\langle a_1, a_2, \ldots, a_k \rangle\rangle$ and $y$ has the coordinates $\langle\langle a_1, \ldots, a_{k-1} \rangle\rangle$ or vice versa, or one of them is $\langle\langle \infty \rangle\rangle$ and the other one has only one coordinate. If $x$ and $y$ both have $n-1$ coordinates, then it depends on the geometrical structure of $\mathcal{S}$ whether or not $x$ is incident with $y$. A way to put this incidence relation in an algebraic form is by introducing the following $n-2$ algebraic operations $M_1, \ldots, M_{n-2}$:

$$M_j(a_1, a_2, \ldots, a_j, k_1, k_2, \ldots, k_{n-j}) = a_{j+1}$$

if $d(\langle\langle a_1, \ldots, a_{j+1} \rangle\rangle, \langle\langle k_1, \ldots, k_{n-j} \rangle\rangle) = n-2$, $1 \leq j \leq n-3$ and $\langle\langle a_1, \ldots, a_j \rangle\rangle$ is a line when $j$ is even and a point if $j$ is odd. Alternatively, this is equivalent to

$$\Pi^1_{\langle\langle a_1, \ldots, a_j \rangle\rangle}(\langle\langle k_1, \ldots, k_{n-j} \rangle\rangle) = \langle\langle a_1, \ldots, a_{j+1} \rangle\rangle.$$

For $j = n-2$, we define

$$M_{n-2}(a_1, k_1, k_2, \ldots, k_{n-1}) = a_{n-1}$$

if $\langle\langle k_1, k_2, \ldots, k_{n-1}\rangle\rangle$ is incident with some element having coordinates $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle$ ($n-1$ coordinates), where $\langle\langle k_1, k_2, \ldots, k_{n-1}\rangle\rangle$ is a line if $n$ is even and a point if $n$ is odd. This is equivalent to

$$\Pi^1_{\langle\langle k_1, k_2, \ldots, k_{n-1}\rangle\rangle}((a_1)) = \Pi^1_{\langle\langle k_1, k_2, \ldots, k_{n-1}\rangle\rangle}((0, a_{n-1})).$$

It is easy to check that a point $(a_1, a_2, \ldots, a_{n-1})$ is incident with a line $[k_1, k_2, \ldots, k_{n-1}]$ iff

$$M_j(a_1, a_2, \ldots, a_j, k_1, k_2, \ldots, k_{n-j}) = a_{j+1}, \qquad \text{for all } j \in \{1, 2, \ldots, n-3\},$$
$$M_{n-2}(a_1, k_1, k_2, \ldots, k_{n-1}) = a_{n-1}.$$

For $n = 3$, there is only one, ternary, operation. The resulting structure is weaker than a *planar ternary ring* (see e.g. [7]) since we only used the element $(1, 0)$ to coordinatize the axes and we did not use the point $(1)$ as required.

For $n = 4$, the resulting structure is again weaker than a (normalized) *quadratic quaternary ring* (see e.g. [5]), since we did not use the line $[1, 0, 0]$ to ask something similar to condition (C5).

Let $a, b \in R_1$, then we define

$$a + b = M_{n-2}(a, 1, b, 0, 0, \ldots, 0) \in R_1.$$

By conditions (C4) and (C5), this addition has the property

$$\text{(ADD)} \quad a + 0 = a = 0 + a.$$

We can also define for $a \in R_1$ and $k \in R_2$ a multiplication

$$k \cdot a = M_{n-2}(a, k, 0, 0, \ldots, 0) \in R_1.$$

Again by (C4) and (C5), we have the properties

(MU1) $0 \cdot a = 0 = k \cdot 0$,
(MU2) $1 \cdot a = a$.

### 4.2. *Proof of the Converse of the Main Theorem*

Suppose $\mathscr{S}$ is a generalized $n$-gon which is $\mathfrak{R}$-transitive for some root $\mathfrak{R} = (x_1, x_2, \ldots, x_{n-1})$. Let $\pi = (y_1 I y_2 I \ldots I y_{2n} I y_1)$ be any polylateral and suppose that $d(x_1, y_1) < n$. Let $y \neq x_1$ be incident with $\Pi^1_{x_1}(y_1)$. Consider the $\mathfrak{R}$-elation $\theta$ mapping $\Pi^2_{x_1}(y_1)$ to $y$, then clearly (by definition) $\pi^\theta$ is in perspective with $\pi$ from $\mathfrak{R}$ and $\Pi^2_{x_1}(y_1^\theta) = y$, which was arbitrary. Hence $\mathscr{S}$ is $\mathfrak{R}$-Desarguesian.

### 4.3. *Explicit Form of an $\mathfrak{R}$-Elation in Coordinates*

From now on we put $n = 6$ (see the comment in the beginning of this section). We denote the elements of $R_1$ by $a, b, a', B, c, \ldots$ and those of $R_2$ by $k, l, k', \ldots$ and this will be understood in the sequel. Hence an expression such as 'for all $a, k$' means 'for all $a \in R_1$ and all $k \in R_2$'. Suppose $\mathfrak{R}$ is a root in $\mathscr{S}$ and we coordinatize $\mathscr{S}$ in such a way that $\mathfrak{R} = ((\infty), [\infty], (0), [0, 0], (0, 0, 0))$ (possibly by considering the dual of $\mathscr{S}$). Suppose that $\mathscr{S}$ admits an $\mathfrak{R}$-elation $\theta$ mapping $(0, 0)$ to $(0, B)$. Then one checks consecutively (similarly to the case $n = 3$ [7] or $n = 4$ [5]):

$$(0, 0, 0, 0, 0)^\theta = (0, 0, 0, 0, B),$$
$$[k]^\theta = [k],$$
$$(k, 0)^\theta = (k, B),$$
$$[1, 0, 0, 0, 0]^\theta = [1, B, 0, 0, 0],$$
$$[k, 0, k']^\theta = [k, B, k'],$$
$$(k, 0, k', b')^\theta = (k, B, k', b'),$$
$$[k, 0, k', b', k'']^\theta = [k, B, k', b', k''],$$
$$(0, b)^\theta = (0, b + B),$$
$$(0, 0, 0, 0, b)^\theta = (0, 0, 0, 0, b + B),$$

and hence

$$\boxed{[k, b, k', b', k'']^\theta = [k, b + B, k', b', k'']}\ .$$

We now define some abbreviations of expressions we will frequently use. Put

$$F_1(a, b, k) = M_1(a, 0, b, 0, 0, k),$$
$$F_2(a, b, k, c) = M_2(a, F_1(a, b, k), 0, b, 0, c),$$
$$F_3(a, b, k, c, l) = M_3(a, F_1(a, b, k), F_2(a, b, k, c), 0, b, l).$$

One can check the following equations step-by step:

$$(a)^\theta = (a),$$
$$(0, 0, 0, 0)^\theta = (o, B, 0, 0),$$
$$[0, l]^\theta = (0, l],$$
$$[0, 0, 0, 0, l]^\theta = [0, B, 0, 0, l],$$
$$[a, l]^\theta = [a, F_1(a, B, l)],$$
$$(a, l, a')^\theta = (a, F_1(a, B, l), F_2(a, B, l, a')),$$
$$[a, l, a', l']^\theta = [a, F_1(a, B, l), F_2(a, B, l, a'), F_3(a, B, l, a', l')],$$

and hence

$$\boxed{(a, l, a', l', a'')^\theta = (a, F_1(a, B, l), F_2(a, B, l, a'), F_3(a, B, l, a', l'), a'' + B)}\ .$$

So if $(a, l, a', l', a'')I[k, b, k', b', k'']$, then

(*) $(a, F_1(a, B, l), F_2(a, B, l, a'), F_3(a, B, l, a', l'), a'' + B)I[k, b + B, k', b', k'']$,

for all $a, a', a'', b, b', k, k', k'', l, l'$. Conversely, it is almost trivial that, if (*) holds whenever $(a, l, a', l', a'')I[k, b, k', b', k'']$, then there exists an $\mathfrak{R}$-elation $\theta$ mapping $(0, 0)$ to $(0, B)$. So from here on, we assume that $\mathcal{S}$ is $\mathfrak{R}$-Desarguesian and we will show in nine steps that, whenever $(a, l, a', l', a'')I[k, b, k', b', k'']$, then (*) holds for every $B$. This will imply that $\mathcal{S}$ is $\mathfrak{R}$-transitive. For a given hexalateral $\pi$, we will always denote a hexalateral in perspective with $\pi$ from $\mathfrak{R}$ by $\pi'$ and we abbreviate the sentence 'let $\pi'$ be the unique hexalateral in perspective with $\pi$ from $\mathfrak{R}$ via $(x, y)$' by 'define $\pi'$ by $(x, y)$'.

### 4.4. Step 1

Suppose $(a, l, a', l', a'')I[k, 0, 0, 0, 0]$, $k \neq 0$ and $k \neq 1$. By definition of the coordinates and the multiplication, $a'' = k \cdot a$. Consider the hexalateral $\pi = ((a, l, a', l', a'') \perp\perp (0, a'') \perp\perp (a'', M_1(a, 1, 0_4), M_2(a, 1, 0_4), M_3(a, 1, 0_4), a'')I[1, 0_4]I(0_5)I(k, 0_4]I(a, l, a', l', a''))$ and let $\pi'$ be defined by $((0_5), (0_4, \mathrm{B}))$. By constructing one by one the elements of $\pi'$ using the definition of corresponding elements, one sees that $\pi'$ looks like $(0_4, B)I\ [1, B, 0_3]I\ (a'', \ldots, a' + B) \perp\perp (0, a'' + B) \perp\perp (a, \ldots, a'' + B) = (a, \ldots, M_4(a, k, B, 0, 0, 0))I[k, B, 0_3]$. Hence

$$a'' + B = M_4(a, k, B, 0, 0, 0) \qquad \text{or} \qquad k \cdot a + B = M_4(a, k, B, 0, 0, 0).$$

So $M_4$ is *linear* in the first three arguments in the sense of a linear planar ternary ring (see e.g. [7]). Of course, this is trivial if $k = 0$ or $k = 1$. Geometrically, this shows that, *whenever a hexalateral* $\pi$ *contains a point–line pair* $(a, \ldots, a'')I[k, 0, 0, 0, 0]$, *then* $\pi'$ *defined by* $([k, 0, 0, 0, 0], (0, 0, 0, 0, B))$ *contains* $(a, \ldots, a'' + B)$ *corresponding to* $(a, \ldots, a'')$.

### 4.5. Step 2

Suppose $p = (a, l, a', l', a'') \perp\perp (0_5)$ and $[0_4] \neq \Pi_p^3((0_5)) = [k, 0]$. First assume $k \neq 0$. Consider the hexalateral $\pi = (p \perp\perp (0, a'') \perp\sim [k, 0_4] \sim p)$ and define $\pi'$ by $([k, 0_4], (0_4, B))$. By step 1, $\pi'$ looks like $[k, B, 0_3] \perp\sim (0, a'' + B) \perp\perp (a, \ldots, a'' + B) \sim [k, B, 0_3]$. Now assume $k = 0$. Let $p \perp (0, a'', \ldots) = q \perp (0, a'')$. Consider the hexalateral $\pi$ defined by $[0_5] \sim p \perp q \perp q' \sim [1, 0_4]I(0_5)I[0_5]$, then $q'$ is well defined and in fact $q' = \Pi_q^2([1, 0_4]) = (\ldots, a'')$. In $\pi'$ defined by $((0_5), (0_4, B))$, $q'$ corresponds to a point $(\ldots, a'' + B)$ by the case $k \neq 0$ above. Hence the point in $\pi'$ corresponding to $p$ has also as last coordinate $a'' + B$. So we have shown that *whenever a hexalateral* $\pi$ *contains the chain* $(a, \ldots, a') \sim [k, 0_4]$, *the hexalateral* $\pi'$ *defined by* $([k, 0_4], (0_4, B))$ *contains* $(a, \ldots, a'' + B)$ *corresponding to* $(a, \ldots, a'')$ (this is true if $(a, \ldots, a'') \perp (0_5)$ by step 1). But now it is easy to see that *whenever a hexalateral* $\pi$ *contains the chain* $(a, \ldots, a'')\ IL$, *where* $d(L, (0_5)) = 3$ *and* $d(L, [0_4]) = 4$, *then* $\pi'$ *defined by* $(L, (0_4, B))$ *contains* $(a, \ldots, a'' + B)$ *corresponding to* $(a, \ldots, a'')$.

### 4.6. Step 3

Suppose $p = (a, l, a', l', a'')$ and assume first $a \neq 0$. Suppose also that $p$ and $(0_5)$ are opposite. Let $k$ be arbitrary and consider the chain $p \sim L \perp [k, 0_4]$. Let $p'IL$ such that $p' = (\ldots, a'')$ and consider the hexalateral $\pi$ defined by $p' \perp\perp (0, a') \perp\perp p \perp\perp p'$. By step 2, the hexalateral $\pi'$ defined by $(L', (0_4, B))$ contains $(0, a' + B)$ corresponding to $(0, a'')$ and hence to $p$ there corresponds $(a, \ldots, a'' + B)$. Note that the construction of $(a, \ldots, a' + B)$ only depends on the chain $L \sim p$. Hence *whenever a hexalateral* $\pi$ *contains a chain* $(a, \ldots, a'') \sim L$, *where* $a \neq 0$ *and* $d(L, (0_5)) = 3$, $d(L, [0_4]) = 4$, *then the hexalateral* $\pi'$ *defined by* $(L, (0_4, B))$ *contains* $(a, \ldots, a'' + B)$ *corresponding to* $(a, \ldots, a'')$. Suppose now $a = 0$. Denote $M = \Pi_p^1((0))$ and $N = \Pi_{(0_5)}^1(p)$. If $N$ has five coordinates, then we denote the first one by $BAD(p)$, so $N = [BAD(p), 0_4]$. If $\pi$ is any hexalateral containing the chain $N \perp\sim p$, and $\pi'$ is arbitrary, then it is impossible to construct the element corresponding to $p$ in $\pi'$ only by using the chain $N \perp\sim p$, in other words, the element corresponding to $p'$ in $\pi'$ *a priori* depends on $\pi$ and not only on the chain $N \perp\sim p$. But if $N' \neq N$ and $(0_5)IN$, then the element corresponding to $p$ in any $\pi'$ where $\pi$ contains $N' \perp\sim p$ only depends on the latter chain. Hence, *in the* *above property, one can drop the assumption* $a \neq 0$ *except in the case* $BAD((a, \ldots, a''))$ *is well defined and* $L \perp [BAD((a, \ldots, a'')), 0_4]$. It is clear how to generalize the properties in steps 1, 2 and 3 to $n \neq 6$.

### 4.7. *Step 4*

Suppose $p = (a, l, a', l', a'')$, $l \neq 0$, $p$ opposite to $(0_5)$. Let $k \neq BAD(p)$ (if well defined) and suppose $p \perp p_1 \perp p_2 I[k, 0_4]$. Furthermore, we denote

$$p \perp p_3 I[a, l],$$
$$p_3 \perp p_4 \perp p_5 I[0_5],$$
$$p \perp p_6 \perp p_7 I[0_5],$$
$$(0_4) \perp p_8 \perp p_9 I[a, l].$$

In the general case, all the points we just defined are distinct and the three hexalaterals

$(\pi_1)$ $(0_5) \perp p_2 \perp p_1 \perp p \perp p_6 \perp p_7 \perp (0_5)$,

$(\pi_2)$ $p_7 \perp p_6 \perp p \perp p_3 \perp p_4 \perp p_5 \perp p_7$,

$(\pi_3)$ $p_5 \perp p_4 \perp p_3 \perp p_9 \perp p_8 \perp (0_4) \perp p_5$

are also distinct and do not collapse. In $\pi'_3$ defined by $([0_5], (0_4, B))$, $(0_4)$ corresponds to $(0, B, 0, 0)$ and $[0_4, l]$ corresponds to $[0, B, 0, 0, l]$; hence $[a, l]$ corresponds to $[a, M_1(a, 0, B, 0, 0, l)] = [a, F_1(a, B, l)]$. Hence in $\pi'_2$ (defined by $([0_5], (0_4, B))$, $p_3$ corresponds to a point $(a, F_1(a, B, l), .)$ and so $p$ corresponds to a point $p' = (a, F_1(a, B, l), \ldots, a'' + B)$. Hence this is also true in $\pi'_1$ defined by $((0_5), (0_4, B))$. Since $k \neq BAD(p)$ if $BAD(p)$ is well defined, we deduce from this that in the general situation, *whenever a hexalateral* $\pi$ *contains the chain* $p = (a, l, \ldots, a'') \sim L$, *where* $d(L, (0_5)) = 3$, $d(L, [0_4]) = 4$ *and* $L$ *does not meet* $[BAD(p), 0_4]$ *if* $BAD(p)$ *is well defined, then the hexalateral* $\pi'$ *defined by* $(L, (0_4, B))$ *contains* $(a, F_1(a, b, l), \ldots, a'' + B)$ *corresponding to* $p$. In the special cases, such as $p_4 = p_6$, and in the case $l = 0$, the same property can be proved (in an easier way since some of the hexalaterals $\pi_1$, $\pi_2$, $\pi_3$ collapse or coincide). Hence the above property, printed in italics, holds in full generality.

### 4.8. *Step 5*

We keep the same notation and assumptions as in the beginning of step 4, except for the fact that we no longer ask that $l = 0$. We define some more points by

$$p_3 \perp p_{10} \perp p_{11} I[0, 0, 0],$$
$$p \perp p_{12} \perp p_{13} I[0, 0, 0].$$

In the general case, all of the points $(0_5)$, $(0_4)$, $p$, $p_1$, $p_2$, $p_3$, $p_6$, $p_7$, $p_9$, $p_{10}$, $p_{11}$, $p_{12}$, $p_{13}$ are distinct and also the following hexalaterals are distinct:

$(\pi_1)$ $(0_5) \perp p_2 \perp p_1 \perp p \perp p_6 \perp p_7 \perp (0_5)$.

$(\pi_4)$ $p_7 \perp p_6 \perp p \perp p_{12} \perp p_{13} \perp (0_4) \perp p_7$,

$(\pi_5)$ $p_{13} \perp p_{12} \perp p \perp p_3 \perp p_{10} \perp p_{11} \perp p_{13}$.

Define $\pi'_1$ as in step 4, $\pi'_4$ by $([0_3]), (0, B))$ (note that $\pi'_4$ is also defined by $([0_5], (0_4, B))$ and $\pi'_5$ by $[(0_3], (0, B))$. It is easy to check that a point lying in different hexalaterals corresponds to the same point in the respective corresponding hexalaterals in perspective, e.g. $p$ corresponds to a point $p'$ in $\pi'_1$, $\pi'_4$ and $\pi'_5$. By the previous step, $p'=(a, F_1(a, B, l), \ldots, a''+B)$. The point $p_{11}=(0_3, a)$ corresponds in $\pi'_5$ to $p'_{11}=(0, B, 0, a)$. By step 4, $p_3$ corresponds to a point $p'_3$ with coordinates $(a, F_1(a, B, l), a^*)$. But since $p'_{11} \perp\perp p'_3$, we have by definition

$$a^* = M_2(a, F_1(a, b, l), 0, B, 0, a') = F_2(a, B, l, a')$$

Hence $p'=(a, F_1(a, B, l), F_2(a, B, l, a'), \ldots, a''+B)$. This can also be shown in all special cases, such as $k=0$; $p_{10} \perp p$; $p_6 \perp p_3$, $a=0$ and $k \neq BAD(p)$, etc. Hence *whenever a hexalateral* $\pi$ *contains the chain* $p=(a, l, \ldots, a'') \sim L$, *where* $d(L, (0_5))=3$, $d(L, [0_4])=4$ *and* $L$ *does not meet* $[BAD(p), 0_4]$ *if* $BAD(p)$ *is well defined, then the hexalateral* $\pi'$ *defined by* $(L, (0_4, B))$ *contains* $(a, F_1(a, B, l), F_2(a, B, l, a'), \ldots, a''+B)$ *corresponding to* $p$.

### 4.9. Step 6

From now on, we can be more sketchy because all forthcoming arguments are similar to one of the arguments of steps 1–5. So, similar to the property in step 5, the reader can check that *whenever a hexalateral* $\pi$ *contains the chain* $p=(a, l, \ldots, a'') \sim L$, *where* $d(L, (0_5))=3$, $d(L, [0_4])=4$ *and* $L$ *does not meet* $[BAD(p), 0_4]$ *if* $BAD(p)$ *is well defined, then the hexalateral* $\pi'$ *defined by* $(L, (0_4, B))$ *contains*

$$(a, F_1(a, B, l), F_2(a, B, l, a'), F_3(a, B, l, a', l'), a''+B)$$

*corresponding to* $p$.

### 4.10. Step 7

We sketch the proof of the fact that *whenever a hexalateral* $\pi$ *contains the chain* $p=(k, b) \sim L$, *where* $d(L, (0_5))=3$, $d(L, [0_4])=4$ *and* $L$ *does not meet* $[k, 0_4]$, *then the hexalateral* $\pi'$ *defined by* $(L, (0_4, B))$ *contains* $(k, b+B)$ *corresponding to* $p$. One can reconstruct the proof by using the following chains (in the case $k \neq 0$ and $b \neq 0$): $p \perp p_1 \perp (o_4, b) \sim L_1 I(0, b) \perp\!\sim L I p_2 \perp p$; $p_2 \sim L_2 \perp L_1$; $p_1 \perp\!\sim L_2$. The union of all these chains form some hexalaterals (call a general one $\pi$). Now consider $\pi'$ defined by $(\ldots, (0_4, B))$ and use step 3. It shows similarly as all proofs above, our claim. If $k=0$, then the property follows directly from step 3 and if $b=0$, then the property is obvious.

### 4.11. Step 8

Suppose that $p=(a, l, a', l', a'')$, $d(p, (0_5))=3$ and $pI\,L=[k, b, k', b', k'']$. Suppose first that there exists $k^* \neq k$, $k^* \neq BAD(p)$ if $BAD(p)$ exists. This is certainly true if $t \geq 3$. Considering the hexalaterals $pILI(k, b, k', b') \perp\!\sim [k^*, 0_4] \perp\!\sim p$ and $(k, b, k', b') \perp (k, b) \perp\!\sim [k^*, 0_4] \perp\!\sim (k, b, k', b')$ and their perspective ones defined by $([k^*, 0_4], (0_4, B))$, one can see that, since $L$ corresponds to $(k, b+B, k', b', k'')$ by construction and by step 6.

$$(a, F_1(a, B, l), F_2(a, B, l, a'), F_3(a, B, l, a', l'), a''+B) I [k, b+B, k', b', k''].$$

Now suppose $t=2$ and $k \neq BAD(p)$. Then, similarly to before, we can prove the above equality by considering the hexalaterals

$$pIL \sim (k, b) \sim L_1 \sim p_1 \perp p, \qquad \text{where } L_1 = \Pi^2_{[k^*, 0_4]}((k, b)),$$
$$pI[a, l, a', l'] \perp\perp [k^*, 0_4] \perp L_1 \perp\!\sim p, \qquad \text{where } k^* = BAD(p),$$
$$[k^*, 0_4] \perp L_1 \sim p_1 \perp\!\sim [k, 0_4] \perp [k^*, 0_4],$$
$$[k, 0_4] \perp\!\sim p_1 \perp p \perp\!\sim [k, 0_4].$$

Note that the chain $p \perp\!\sim [k, 0_4]$ contains $L$ since $t=2$.

*4.12. Step 9*

Suppose that $p = (a, l, a', l', a'')IL = [k, b, k', b', k'']$, with $d(p, (0_5)) < 6$, then one can show in the same way as above that $(a, F_1(\ldots), \ldots)I[k, b + B, \ldots]$, although we would like to focus on on particular case which is not so straightforward, namely when $p \perp [0_4]$. If that is the case, then for all $k$, one can always find a chain $p \perp p_0 \sim L \perp [k, 0_4]$ such that $k \neq BAD(p_0)$ if the latter exists. Now one has to prove similar results to steps 3, 4, 5 and 6 for the chain $p \perp\!\sim L$ and then combine this with step 7 as done in step 8. There is no separate argument for $t = 2$ since $k$ is now arbitrary. But in all arguments, there is an extra hexalateral since the chain connecting $p$ to $L$ is one 'unit' longer now.

So we showed that, *whenever* $(a, l, a', l', a'')I[k, b, k', b', k'']$, *then also* $(a, F_1(a, B, l), F_2(a, B, l, a'), F_3(a, B, l, a', l'), a'' + B)I\ [k, b + B, k', b', k'']$. Hence by the discussion preceding the first step, the generalized hexagon $\mathcal{S}$ is $\mathfrak{R}$-transitive. This concludes the proof of the main theorem.

## 5. Some Remarks

*5.1. Flag-Desarguesian*

Suppose we denote by $\mathfrak{F}$ a flag of the generalized polygon $\mathcal{S}$, i.e. an incident point–line pair. Then we can define when two polylaterals are in perspective from $\mathfrak{F}$ just in the same way as for roots, only, $\mathfrak{F}$ has less elements than a root. We could call the whole configuration of two polylaterals in perspective from a flag a *flag-Desargues-configuration.* Then $\mathcal{S}$ is called $\mathfrak{F}$-Desarguesian, $\mathfrak{F} = \{x_1, x_2\}$, if for every polylateral $\pi = (xI\ldots)$, where we can assume $d(x, x_1) = n - 1$, and every element $y$ such that $d(y, x_i) = d(x, x_i)$, $i = 1, 2$, and such that $\Pi^1_{x_1}(x) = \Pi^1_{x_1}(y)$, there exists a polylateral $\pi' = (y, \ldots)$ in perspective with $\pi$ from $\mathfrak{F}$. And we can call $\mathcal{S}$ *flag-Desarguesian* if $\mathcal{S}$ *is* $\mathfrak{F}$-Desarguesian for all flags $\mathfrak{F}$ in $\mathcal{S}$. Of course, for projective planes these definitions coincide with the root-case. In [9], Gy. Kiss proves that every finite generalized polygon contains a flag-Desargues configuration and there exist (infinite free-constructed) $n$-gons (for each $n$) containing no flag-Desargues configuration (and hence no generalized Desargues-configuration).

*5.2. Generalized Quadrangles*

In the case $n = 4$, a lot more is known than the main result of the present paper. In fact, S. E. Payne, J. A. Thas and the author show in [13] that a finite generalized quadrangle $\mathcal{S}$ is root-Desarguesian iff $\mathcal{S}$ is $\mathfrak{R}$-Desarguesian for all roots $\mathfrak{R}$ of one arbitrary type (recall the definition of *type* in Section 3, discussion preceding Corollary 4). In [17], the same three authors show that $\mathcal{S}$ is root-Desarguesian iff $\mathcal{S}$ is flag-Desarguesian. Other (local) results on the Moufang condition in generalized quadrangles may be found in the monograph [10].

*5.3. Other Generalizations*

Note that a generalized Desargues configuration in a projective plane is nothing else than a usual Desargues configuration (see e.g. [7]) where 'center' and 'axis' are incident. A similar definition holds in projective planes for non-incident 'center'–'axis'-pairs. Call this for a moment an *anti-flag-Desargues configuration.* A similar definition for generalized quadrangles would lead to generalized quadrangles which are $\mathfrak{U}$-Desarguesian or $\mathfrak{U}$-transitive with $\mathfrak{U}$ an antiflag. It is easy to show that the only collineation of a generalized quadrangle fixing all points on a given line $L$ and all lines through a given point not incident with $L$ is the identity. Hence, there is no straightforward generalization of being $\mathfrak{U}$-Desarguesian, with $\mathfrak{U}$ an antiflag, for generalized polygons.

### 5.4. Weaker Hypotheses

In the definition of $\mathfrak{R}$-Desarguesian, we ask a property for *all* polylaterals. In fact, this is asking too much. Indeed, our main result is also true if we require the condition only for polylaterals for which the element nearest to $\mathfrak{R} = (x_1 I x_2 I \ldots)$ is at least on distance $l = 2$ from $x_1$ (in the proof, we do not use other polylaterals). In fact, for $n = 4$ and $n = 6$, we can proof that it is sufficient to require the property only for polylaterals for which the above defined distance $l$ is exactly equal to 2, but this might not be the case for $n = 8$. In any case, we think of these variants as minor details and therefore we did not try to write down exactly the weakest possible hypotheses.

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