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CHARACTERIZATION OF THE FREE σ-HOMOTOPY GROUPS OF COXETER COMPLEXES

H. Van Maldeghem *
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J.Tits introduced in his work on local characterisations of buildings [4] the notion of a σ-homotopy group of an arbitrary chamber system. In [6], we gave a method to calculate some σ-homotopy groups for arbitrary Coxeter complexes. The purpose of this paper is to give a creterium whether such a group is free.

INTRODUCTION

We can concieve any Coxeter complex as a connected chamber system Σ of rank n, for some $n \in \mathbb{N}^*$ (for general definitions see Tits [4] and Ronan [2]). Let $\Delta=\{1,2,...,n\}$ and $\sigma \subseteq 2^\Delta$, then we call two galleries γ and γ' elementary σ-homotopic (notation : $\gamma \overset{e}{\simeq} \gamma'$) if and only if they can be written as the juxtaposition of three galleries $\gamma=\alpha\delta\beta$ and $\gamma'=\alpha\delta'\beta$ where α and δ , resp. α and δ' are i-adjacent and δ and β, resp. δ' and β are j-adjacent ($i,j\in\Delta$,α and/or possibly empty) and δ and δ' lie in a same cell of cotype J, $J\in\sigma$ and have the same extremity chambers. Two galleries

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γ and γ' are called σ-homotopic if and only if they can be connected by a sequence of elementary σ-homotopics (notation $\gamma \underset{\sigma}{=} \gamma'$).

Let us denote by $[\gamma]_\sigma$ the σ-homotopy class of galleries containing γ. Then by definition $\pi^\sigma(\Sigma)$, the σ-homotopy group of Σ, is the group whose elements are all σ-homotopy classes of galleries of Σ based at a chamber c of Σ (which means : galleries with extremity chambers the chamber c) and with binary operation $[\alpha]_\sigma[\beta]_\sigma=[\alpha\beta]_\sigma$ ($\alpha\beta$ is the juxtaposition of α and β). This group is un-independant on the choice of the chamber c (Σ is connected) (see also [2] and [4]).

DEFINITIONS AND NOTATIONS

We denote always the cardinality of a set A by $|A|$

By Σ_n, we denote always a Coxeter complex of rank n, $n\in N^\star$. Its diagram is in fact a set $\Delta_n=\{1,2,...,n\}$ together with a map $f:\binom{\Delta_n}{2}\to N^\star$. We denote this diagram also by Δ_n. If $J\subseteq\Delta_n$, we denote by J the set J, and by $J\cap\Delta_n$ the diagram with set J and map $f/_{\binom{J}{2}}:\binom{J}{2}\to N^\star$.

If Σ_n is a Coxeter complex (reducible or not) of rank n, $n\in N^\star$, with diagram Δ_n, then Σ_n^J is the set of flags of type J of Σ_n, $J\subseteq\Delta_n$ and $\Sigma_n^{(i)}$ is the set of flags of rank i, $1\leqslant i\leqslant n$. Hence $|\Sigma_n^{\Delta n}|=|\Sigma_n^{(n)}|$ is the number of chambers of Σ_n. Since a Coxetercomplex Σ_n of rank n defines a triangulation of the hypersphere $S^{n-1} = \{(x_1,\dots,x_n)\in R^n \| \sum_{k=1}^{n} x_k^2 = 1\}$, we have by the Euler-Poincaré formula the equation

$$\sum_{k=0}^{n} (-1)^k |\Sigma_n^{(n-k)}| = \sum_{J\subseteq\Delta_n} (-1)^{|J|} |\Sigma_n^{\Delta_n - J}| = 1$$

If $J\subseteq\Delta_n$, then we denote by Σ_J the Coxetercomplex of rank $|J|$ and with diagram $J\cap\Delta_n$ and we notice the equation (see also [6]) :

$$\text{(E)} \qquad |\Sigma_n^{\Delta_n - J}| . |\Sigma_J^{J-L}| = |\Sigma_n^{\Delta_n - L}| \qquad L\subseteq J\subseteq\Delta_n$$

Suppose now $\Delta_n\subseteq\sigma'\subseteq 2^{\Delta_n}$, then it follows by [6] (remark 2) that there exists σ, $\Delta_n\subseteq\sigma\subseteq\Delta_n\cup\binom{\Delta_n}{2}$ and $\pi^{\sigma'}(\Sigma_n) = \pi^{\sigma}(\Sigma_n)$. $(\{i,j\}\in\sigma \Rightarrow \{i,j\}\subseteq K\in\sigma')$. So we consider only σ with $\Delta_n\subseteq\sigma\subseteq\Delta_n\cup\binom{\Delta_n}{2}$. But then we can concieve σ as a linear graph $(\Delta_n, \sigma-\Delta_n)$, which we denote by $\overline{\sigma}$, and we define $\overline{\sigma}^{\star} = \text{flag}\sigma$. If E is the set of simplices of $\overline{\sigma}^*$, then we denote $\sigma^{\star} = E\cup\phi$.

If $\sigma=\Delta_n$, then $\pi^{\sigma}(\Sigma_n)$ is a free group with rank

$$1 - \Sigma_n^{(n)} + \Sigma_n^{(n-1)}$$

(see [6]) and we denote this rank by $\chi(\Sigma_n)$, which is in fact the Euler-Poincarré characteristic of the chambergraph of Σ_n (see [6]).

Finally , we define the family $\overline{S}$ of linear graphs with set of vertices Δ_n, for some $n\in N^{\star}$ inductively as follows :

1) Each complete linear graph belongs to $\overline{S}$

2) If two lineair graphs of the class $\overline{S}$ have a common complete subgraph as intersection, or have an empty intersection, then their union belongs to $\overline{S}$. From theorem 2 in [6], it follows that if $\overline{\sigma}$ is a complete subgraph, then $\pi^{\sigma}(\Sigma_n)$ is trivial, (and hence free). From theorem 4 in [6], it follows that if $\sigma=\sigma_1\cup\sigma_2$ and $\overline{\sigma}_1,\overline{\sigma}_2\in\overline{S}$, $\overline{\sigma_1\cap\sigma_2}$ is a complete subgraph, than $\pi^{\sigma}(\Sigma_n)$ is a free group. Hence $\overline{\sigma}\in\overline{S}$ is a sufficient condition in order that $\pi^{\sigma}(\Sigma_n)$ is a free group. The purpose of this paper is to prove that it is also necessary. Thus we have to prove :

<u>THEOREM</u> : $\overline{\sigma}\in\overline{S}\Longleftrightarrow\pi^{\sigma}(\Sigma_n)$ <u>is a free group</u>. <u>In this case, the rank of the free group is given by</u>

$$1-\sum_{J\in\sigma^+}(-1)^{|J|}|\Sigma_n^{\Delta_n-J}| \qquad (1)$$

Suppose we have proved the first part of the theorem. Then by remark 3 of [6] we know that the rank of $\pi^{\sigma}(\Sigma_n)$, $\overline{\sigma}\in\overline{S}$, is

$$\chi(\Sigma_n)-\sum_{\substack{J\in\sigma^+\\|J|\geqslant 2}}(-1)^{|J|}\,|\Sigma_n^{\Delta_n-J}|$$

Applying the definition of $\chi(\Sigma_n)$, we obtain (1). So the second part of the theorem follows from the first part. To prove the theorem, we need some lemma's.

LEMMA 1. *Suppose* $\Delta_n \subseteq \sigma \subseteq \Delta_n \cup \binom{\Delta_n}{2}$ *and suppose* $\nexists J \subseteq \Delta_n$ *with* $|J|>3$ *such that* $\binom{J}{2} \cap \sigma$ *is a cycle of length* $|J|$ (*or* $\overline{\sigma}$ *contains no minimal circuit of length* $m>3$). *Then we have* $\overline{\sigma} \in \overline{S}$.

PROOF.

If $n=1$, the result follows immediately (also for $n=2,3$) since there exist no cycles of length m, $m>3$ in this case, and each linear graph with set of vertices Δ_1, Δ_2 or Δ_3 belongs to S.

We complete the proof by induction.

Let $n>1$ be arbitrary. Assume $\overline{\sigma}$ is not connected . Then $\sigma = \sigma_1 \cup \sigma_2$ with $(\cup \sigma_i) \neq \phi$, $i=1,2$ and $(\cup \sigma_1) \cap (\cup \sigma_2) = \phi$. It is clear that neither $\overline{\sigma}_1$ nor $\overline{\sigma}_2$ contains a minimal circuit of length $m>3$. By induction we have $\overline{\sigma}_1, \overline{\sigma}_2 \in \overline{S}$ and since their sets of vertices are disjoint, $\overline{\sigma_1 \cup \sigma_2} = \overline{\sigma} \in \overline{S}$.

Let σ be connected . We denote by G_K the complete graph with as set of vertices $K \subseteq \Delta_n$. Let G_J be a maximal complete subgraph of $\overline{\sigma}$. We define for each $X \subseteq J$

$$Q_X = \{t \in \Delta_n - J \,\|\, t \sim x,\ \forall x \in X \text{ and } t \not\sim y, \forall y \in J-X\}$$

where $\sim$ denotes the adjacency relation in the graph $\overline{\sigma}$.

Notice that $Q_J = \phi$, otherwise $G_{J \cup Q_J}$ is a greater complete subgraph than G_J. Suppose $X_1 - X_2 \neq \phi$, $X_2 - X_1 \neq \phi$, $X_1, X_2 \subseteq J$, and $Q_{X_1}, Q_{X_2} \neq \phi$. Take $x_1 \in Q_{X_1}$, $x_2 \in Q_{X_2}$ and suppose $x_1 \sim x_2$. Then $x_1 \sim t_1 \in X_1 - X_2$ and $x_2 \sim t_2 \in X_2 - X_1$, so $x_1 \sim t_1 \sim t_2 \sim x_2 \sim x_1$ is a cycle of length 4 since $t_1 \not\sim x_2, t_2 \not\sim x_1$. Hence $x_1 \not\sim x_2$

We define now for each $X \subseteq J$ (1)

$$R_X = \begin{cases} \phi, \text{ if } (\exists Y \subseteq J)((X \subset Y) \text{ and } (Q_Y \neq \phi)) \\ Q_X \cup \{t \in \Delta_n - J \Vert (\exists Y \subset X)(t \in Q_Y) \text{ and } ((\exists x \in Q_X)(t \sim x)\}, \text{ otherwise .} \end{cases}$$

From that definition, it follows that if $Q_X=\phi$, then $R_X=\phi$. Suppose $R_{X_1}, R_{X_2} \neq \phi$ and $X_1 \subsetneq X_2$, then $Q_{X_1} \neq \phi$ and so $R_{X_2}=\phi$ and we conclude that $X_1 - X_2 \neq \phi$ and $X_2 - X_1 \neq \phi$. Suppose now $R_{X_1} \cap R_{X_2} \neq \phi$. Then $\exists z \in R_{X_1} \cap R_{X_2}$. If $z \in Q_{X_1}$, then $z \notin R_{X_2}$ since $X_1 \not\subset X_2$, thus $z \in Q_Y$ with $Y \subset X_1 \cap X_2$ and $\exists x_1, x_2$ with $x_1 \in Q_{X_1}$ and $x_2 \in Q_{X_2}$ such that $x_1 \sim z \sim x_2$. Take $t_1 \sim x_1, t_2 \sim x_2$; $t_1 \in X_1 - Y$, $t_2 \in X_2 - Y$, then $t_1 \sim x_1 \sim z \sim x_2 \sim t_2 \sim t_1$ is a cycle of length 5, since $t_1 \not\sim z \not\sim t_2$ $(t_1, t_2 \notin Y)$, $x_1 \not\sim x_2$ (by (1)), $x_1 \not\sim t_2$ and $x_2 \not\sim t_1$. Thus $R_{X_1} \cap R_{X_2} = \phi$. Suppose now that $x_1 \in R_{X_1}$, $x_2 \in R_{X_2}$ and $x_1 \sim x_2$. Assume first $x_1 \in Q_{X_1}$, then $x_2 \notin Q_{X_2}$ and thus $(\exists Y \subseteq X_2)(Y \not\subseteq X_1)(x_2 \in Q_Y)$. If $t_1 \in X_1 - X_2$, $t_2 \in Y - X_1$, then $t_1 \sim x_1 \sim x_2 \sim t_2 \sim t_1$ is a cycle of length 4, since $t_1 \not\sim x_2$ and $x_1 \not\sim t_2$. Assume now $x_1 \notin Q_{X_1}$, then for the same reason $x_2 \notin Q_{X_2}$ and we have $(\exists y_1 \sim x_1)(y_1 \in Q_{X_1})$ and $(\exists y_2 \sim Q_2)(y_2 \in Q_{X_2})$. If $t_1 \in X_1 - X_2$, $t_2 \in X_2 - X_1$, then $t_1 \sim y_1 \sim x_1 \sim x_2 \sim y_2 \sim t_2 \sim t_1$ is a cycle of length 6. We conclude $x_1 \not\sim x_2$ (2).

We define : $\lambda = \{X \subseteq J \Vert R_X \neq \phi\}$ and for each $X \in \lambda$

$S_X = \{t \in \Delta_n - J \Vert \exists$ gallery γ who's extremity vertices are t and t_0 for some $t_0 \in R_X$ and with $\gamma \subseteq \Delta_n - J\}$

Notice that $J \notin \lambda$.

Suppose $X_1, X_2 \in \lambda$, then we prove that no element of S_{X_1} is adjacent to any element of S_{X_2} (or that $S_{X_1} \neq S_{X_2}$). Suppose therefore $x_1 \in S_{X_1}, x_2 \in S_{X_2}$, $x_1 \sim x_2$. Let γ_i be a gallery of minimal length between R_{X_i} and x_i, $i=1,2$. We proceed by induction on $k=|\gamma_1|+|\gamma_2|-2$. If $k=0$, then $x_i \in R_{X_i}$, $i=1,2$ and we proved this in (2).

Suppose now $k>0$. By induction, no element of γ_1 is adjacent to any element of γ_2, except $x_1 \sim x_2$. Let $t_1 \in X_1 - X_2$, $t_2 \in X_2 - X_1$ then $t_1 \sim \gamma_1 \sim \gamma_2 \sim t_2 \sim t_1$ is a cycle of length $k+4>4$, so $x_1 \not\sim x_2$ (3)

To complete the proof, we concieve two cases :

1) $|\lambda|=0$ Then $\sigma = \Delta_n \cup \binom{\Delta_n}{2}$ since $\overline{\sigma}$ is connected and $\overline{\sigma} \in \overline{S}$.

2) $|\lambda|>0$. Suppose $X_0 \in \lambda$, $X_0 \neq J$ since $J \notin \lambda$. We denote by $\overline{\sigma}/_P$, where $P \subseteq \Delta_n$, the restriction of $\overline{\sigma}$ to P (Thus $\overline{\sigma}/_P$ is the graph $(P, \sigma \cap \binom{P}{2}))$). If $\overline{\sigma}_1 = \overline{\sigma}/_{X_0 \cup S_{X_0}}$ and $\overline{\sigma}_2 = \overline{\sigma}/_{\Delta_n - S_{X_0}}$. Since $X_0 \neq J$, $X_0 \cup S_{X_0} \neq \Delta_n$ and since $S_{X_0} \neq \phi$,$\Delta_n - S_{X_0} \neq \Delta_n$, so we can apply induction : $\overline{\sigma}_1 \in \overline{S}$, $\overline{\sigma}_2 \in \overline{S}$. But we have : $\sigma_1 \cap \sigma_2 = \binom{X_0}{2}$ and $\sigma_1 \cup \sigma_2 = \sigma$ since no element of S_{X_0} is adjacent to any element of $\Delta_n - (X_0 \cup S_{X_0})$. We conclude $\overline{\sigma} \in \overline{S}$. ∎

<u>LEMMA 2. Let</u> v <u>be an idempotent endomorphism of the Coxetercomplex</u> Σ_n <u>with diagram</u> Δ_n, <u>then for each</u> σ, $\Delta_n \subseteq \sigma \subseteq \Delta_n \binom{\Delta_n}{2}$ <u>we have</u> :

$$\pi^\sigma(v(\Sigma_n)) \leqslant \pi^\sigma(\Sigma_n)$$

PROOF. It is sufficient to prove that ($[\gamma]_\sigma=[C]_\sigma$ in $v(\Sigma)$) $\Longleftrightarrow$ ($[\gamma]_\sigma=[C]_\sigma$ in Σ_n), with γ a gallery in $v(\Sigma_n)$ based at some chamber C in $v(\Sigma_n)$.

- If $[\gamma]_\sigma=[C]_\sigma$ in $v(\Sigma_n)$, then of course $[\gamma]_\sigma=[C]_\sigma$ in Σ_n
- If $[\gamma]_\sigma=[C]_\sigma$ in Σ_n, then there is a sequence

$$[\gamma]\underset{\sigma}{\overset{e}{\cong}}[\delta_1]\underset{\sigma}{\overset{e}{\cong}}[\delta_2]\underset{\sigma}{\overset{e}{\cong}}\cdots\underset{\sigma}{\overset{e}{\cong}}[\delta_l]\underset{\sigma}{\overset{e}{\cong}}[C].$$

But then we have

$$[\gamma]\underset{\sigma}{=}[v(\gamma)]\overset{e}{\cong}[v(\delta_1)]\underset{\sigma}{\overset{e}{\cong}}\cdots\underset{\sigma}{\overset{e}{\cong}}[v(\delta_l)]\underset{\sigma}{\overset{e}{\cong}}[v(c)]=[c].$$

and so $[\gamma]\underset{\sigma}{\cong}[c]$ in $v(\Sigma_n)$ or $[\gamma]_\sigma=[c]_\sigma$ in $v(\Sigma_n)$. ∎

PROPOSITION 1. If $J\subseteq\Delta_n$, $\Delta_n\subseteq\sigma\subseteq\Delta_n\cup\binom{\Delta_n}{2}$, and $\bar\sigma'=\bar\sigma/_J$, then we have

$$\pi^{\sigma'}(\Sigma_J)\leqslant\pi^{\sigma}(\Sigma_n).$$

PROOF. Σ_J is a convex subcomplex of Σ_n and contains at least one chamber. So by theorem 2.19 pp.24-25 [5], Σ_J is the image of an idempotent endomorphism. But, since no i-adjacent chambers appear in Σ_J, for $i\in\Delta_n-J$, we can restrict $\bar\sigma$ to J and so the result follows by lemma 2. ∎

LEMMA 3. If Σ_n is a Coxeter complex of rank n with diagram Δ_n, and $\bar\sigma$ is a circuit of length n, $n\geqslant3$, then $\pi^\sigma(\Sigma_n)$ is the fundamental group of an orientable surface without boundary. Moreover, $\sigma^*=\sigma\cup\{\phi\}$ for $n>3$, and we have

$n=3$: $\pi^{\sigma}(\Sigma_3)$ is the trivial group of one element

$n>3$: $\pi^{\sigma}(\Sigma_n)$ is the fundamental group of an orientable surface with $1-\frac{1}{2}(\sum_{J\in\sigma^*}(-1)^{|J|}|\Sigma_n^{\Delta_n - J}|)$ $(\neq 0)$ handles.

PROOF. We define the 2-dimensional complex $\Gamma_\sigma(\Sigma_n)$ (for general definition see Ronan [2]). The vertices of $\Gamma_\sigma(\Sigma_n)$ are the chambers of Σ_n, the cells of codimension 1, and the cells of cotype $\{i,j\}$, where $\{i,j\}\in\sigma$. In this proof c and c_i, $i\in\mathbb{N}$ denote always chambers; s and s_i, $i\in\mathbb{N}$ denote always cells of codimension 1, and t and t_i, $i\in\mathbb{N}$ denote always cells of cotype $\{i,j\}$, $\{i,j\}\in\sigma$. Then we define :

$$\{c,s\}\in\Gamma_\sigma(\Sigma_n)\Leftrightarrow c\in s$$
$$\{c,t\}\in\Gamma_\sigma(\Sigma_n)\Leftrightarrow c\in t$$
$$\{t,s\}\in\Gamma_\sigma(\Sigma_n)\Leftrightarrow s\subseteq t$$
$$(c,s,t)\in\Gamma_\sigma(\Sigma_n)\Leftrightarrow c\in s\subseteq t$$

We claim that $\Gamma_\sigma(\Sigma_n)$ is a triangulation of an orientable surface. We first prove that every 1-simplex (i.e. every edge) lies in exactly two 2-simplices. There are three cases :

1) $\{c,s\}\in\Gamma_\sigma(\Sigma_n)$ with $c\in s$, $\text{cotyp}\,s=i\in\Delta_n$. Then, there are exactly two 2-sets of σ which contain i. So we have $\{i,j_1\}$, $\{i,j_2\}\in\sigma$. By definition of a chamber system , s is in exactly one cell of each cotype.

2) $\{c,t\}\in\Gamma_\sigma(\Sigma_n)$, thus $c\in t$. Suppose $\text{cotyp}\,t=\{i,j\}\in\sigma$. There is exactly one cell s_i of cotype i for which $c\in s_i$. Similar $c\in s_j$, $\text{cotyp}\,s_j=j$. By definition of a chamber

system $\{c,s_i,t\},\{c,s_j,t\}\in\Gamma_\sigma(\Sigma_n)$. If s_k is a cell of cotype k, $k\neq i$, $k\neq j$ then $s_k\not\subset t$.

3) $\{s,t\}\in\Gamma_\sigma(\Sigma_n)$ and $s\subset t$. Since a Coxeter complex is a thin chamber complex, there are exactly two chambers c_1 and c_2 with $c_1,c_2\in s$.

So, $\Gamma_\sigma(\Sigma_n)$ is a triangulation of a surface without boundary. We prove now that every orientation of this complex is a coherent orientation. Let C_i, i=1,2, be the free $\mathbb{Z}$-module with base the set of i-simplices. Then we have to define a map $\partial:C_2\to C_1:w\mapsto \sum_{v_i\subseteq w} (-1)^{k^w_{v_i}} v_i$ such that

if $w_1,w_2\in C_2$ and $v\in C_1$ with $v\subseteq w_1$, $v\subseteq w_2$, then v appears with different signs in ∂w_1 and ∂w_2 (*). To define ∂, we take an arbitrary chamber c_0 and we define $m_c=d(c_0,c)$ for c a chamber of Σ_n. If $c_1\sim c_2$, then $|d(c_0;c_1)-d(c_0,c_2)|=1$ (see Carter [1]) and so we have $m_{c_1}=m_{c_2}\pm 1$ or $(-1)^{m_{c_1}}(-1)^{m_{c_2}}=-1$. We now write Δ_n as $\Delta_n=\{i_1,i_2,\ldots,i_n\}$ with the property that $\{i_k,i_{k+1}\}\in\sigma$ $k=1,\ldots,n-1$, and $\{i_1,i_n\}\in\sigma$. Then we define

$$k^{\{c,s,t\}}_{\{c,s\}} = \begin{cases} +1 & \text{if cotyp } s=i_k \text{ and cotyp } t=\{i_k,i_{k+1}\} \\ & \text{or cotyp } s=i_n \text{ and cotyp } t=\{i_n,i_1\} \\ -1 & \text{otherwise} \end{cases}$$

$$k^{\{c,s,t\}}_{\{c,t\}} = k^{\{c,s,t\}}_{\{c,s\}}$$

$$k^{\{c,s,t\}}_{\{s,t\}} = m_c$$

It is clear that the property (*) holds with that definition of ∂. So, we only have still to determine the genus of the orientable surface. But if m is the genus, then $2m-1=X$ is the Euler-Poincaré characteristic of $\Gamma_\sigma(\Sigma_n)$. Thus we only have to compute this characteristic and it follows that the genus m is $1+\frac{1}{2}X$. We first count the 2 simplices (c,s,t). There are $\sum_{\{i,j\}\in\sigma} |\Sigma_n^{\Delta_n-\{i,j\}}|$

cells t of codimension 2. Each cell t of cotype{i,j}

contains $|\Sigma_{\{i,j\}}^{\{i\}}|+|\Sigma_{\{i,j\}}^{\{j\}}|$ cells of codimension 1. Each cell of cotype i contains $|\Sigma_{\{i\}}^{\{i\}}|=2$ chambers. Thus the number of 2-simplices k_3 of $\Gamma_\sigma(\Sigma_n)$ is

$$k_3 = \sum_{\substack{\{i,j\}\in\sigma \\ i\neq j}} |\Sigma_n^{\Delta_n-\{i,j\}}| (|\Sigma_{\{i,j\}}^{\{i\}}| . |\Sigma_{\{j\}}^{\{j\}}| + |\Sigma_{\{i,j\}}^{\{j\}}| . |\Sigma_{\{i\}}^{\{i\}}|)$$

Similar, there are $k_{2,1}$ 1-simplices of type {s,t}; $k_{2,2}$ 1-simplices of type {c,t} and $k_{2,3}$ 1-simplices of type {c,s}with

$$k_{2,1} = \sum_{\substack{\{i,j\}\in\sigma \\ i\neq j}} \Sigma_n^{\Delta_n-\{i,j\}} . (|\Sigma_{\{i,j\}}^{\{i\}} + |\Sigma_{\{i,j\}}^{\{j\}}|)$$

$$k_{2,2} = \sum_{\substack{\{i,j\}\in\sigma \\ i\neq j}} \Sigma_n^{\Delta_n-\{i,j\}} . |\Sigma_{\{i,j\}}^{\{i,j\}}|$$

$$k_{2,3} = \sum_{i\in\Delta_n} | \Sigma_n^{\Delta_n-\{i\}} | . |\Sigma_{\{i\}}^{\{i\}}|$$

Finally , the number of vertices is

$$k_1=|\Sigma_n^{\Delta_n}|+\sum_{i\in\Delta_n}|\Sigma_n^{\Delta_n-\{i\}}|+\sum_{\substack{\{i,j\}\in\sigma\\ i\neq j}}|\Sigma_n^{n-\{i,j\}}|$$

and if $k_2=k_{2,1}+k_{2,2}+k_{2,3}$, then $\chi=1-k_1+k_2-k_3$.

Using the equation (E), we have :

$$k_3=2n|\Sigma_n^{\Delta_n}|$$

$$k_2=2|\Sigma_n^{(n-1)}|+n|\Sigma_n^{(n)}|+n|\Sigma_n^{(n)}|$$

$$=1-|\Sigma_n^{(n)}-\Sigma_n^{(n-1)}|-\sum_{\substack{\{i,j\}\in\sigma\\ i\neq j}}|\Sigma_n^{n-\{i,j\}}|+2|\Sigma_n^{(n-1)}|+2n|\Sigma_n^{(n)}|$$

$$-2n|\Sigma_n^{(n)}|$$

$$=1-|\Sigma_n^{(n)}|+|\Sigma_n^{(n-1)}|-\sum_{\substack{\{i,j\}\in\sigma\\ i\neq j}}|\Sigma_n^{\Delta_n-\{i,j\}}|$$

$$=1-\sum_{J\in\sigma\cup\{\phi\}}(-1)^{|J|}|\Sigma_n^{\Delta_n-J}|$$

- If $n=3$, then $\sum_{J\in\sigma\cup\{\phi\}}(-1)^{|J|}|\Sigma_3^{\Delta_3-J}|=1-(-1)^{\Delta_3}|\Sigma_3^{\Delta_3-\Delta_3}|=2$

  since $\sigma\cup\{\phi\}=2^{\Delta_3}-\{\Delta_3\}$.

  Thus $\chi=-1$ and $m=0$ (This is in conformity with theorem 2 [6], after all $\overline{\sigma}\in\overline{S}$).

- If $n>3$, then $\sigma^*=\sigma\cup\{\phi\}$ and $\chi=1-\sum_{J\in\sigma^*}(-1)^{|J|}|\Sigma_n^{\Delta_n-J}|$

$$m=1-\frac{1}{2}\sum_{J\in\sigma}(-1)^{|J|}|\Sigma_n^{\Delta_n-J}|.$$

But since $n>3$, there exist $i,j\in\Delta_n$ with $\{i,j\}\notin\sigma$. Applying proposition 1, we have

$$\pi^{\{\{i\},\{j\}\}}(\Sigma_{\{i,j\}}) \leqslant \pi^{\sigma}(\Sigma_n).$$

Since $Z,+\cong\pi^{\{\{i\},\{j\}\}}(\Sigma_{\{i,j\}}).,\pi^{\sigma}(\Sigma_n)$ is not trivial and $m\geqslant 1$. ∎

DEFINITION. Let F_k be the free group generated by a_1, $a_2,\ldots,a_k$. If $w=w(a_1,\ldots,a_k)$ is a word (reduced or not), then $L_w(a_i)$ denotes the sum of the exponents of a_i in w, $i\in\{1,2,\ldots,k\}$.

LEMMA 4. If G is a one relator-group, $G=\langle a_1,\ldots,a_k \| R(a_1,\ldots,a_k)\rangle$ and if $(\forall i\in\{1,\ldots,k\})(L_R(a_i)\in 2Z)$, then G is not a free group.

PROOF. It has been proved by J.M.C. Whitehead [8] that a one relator group is free if and only if the cyclic word R (which is the only relation) is simple. We recall that a word R is simple if and only if R can be derived from a letter (a generator) by a sequence of simple transformations [7].

Finally, a simple transformation is one of the three following

1) Replacing x by xy and x^{-1} by $y^{-1}x^{-1}$ $(x\neq y)$, x and y are generators
2) insert xx^{-1}, x generator
3) omit xx^{-1}, x generator

see [7].

Now let l be the length of a minimal sequence of simple transformations to obtain $R(a_1,\dots,a_k)$ from a certain letter a_i, $i\in\{1,\dots,k\}$. If $l>0$, then we denote by $R'(a_1,\dots,a_k)$ the word we obtain by applying exactly the same sequence of simple transformations on a_i, except the last, which is dropped.

We prove by induction on l that for every simple word $w(a_1,\dots,a_k)$ there is a $j\in\{1,\dots,k\}$ for which $L_w(a_j)\in 2Z+1$. Then it follows that R is not simple and G not free

If $l=0$, $l_w(a_i)=1\in 2Z+1$

If $l>0$, then suppose for $j\in\{1,\dots,k\}$, $L_{w'}(a_j)\in 2Z+1$. An insertion or an omission of xx^{-1} changes nothing at the $L_w(a_m)$ $\forall m\in\{1,\dots,k\}$. Suppose in w', we replace a_p by a_pa_q and a_p^{-1} by $a_q^{-1}a_p^{-1}$ for $j\neq p,q$. Then $L_w(a_j)=L_{w'}(a_j)\in 2Z+1$.

If $p=j$, then again $L_w(a_j)=L_{w'}(a_j)\in 2Z+1$.

If $q=j$, then $L_w(a_j)=L_w(a_p)+\dots$. If $L_{w'}(a_p)\in 2Z$, then again $L_w(a_j)\in 2Z+1$. If $L_{w'}(a_p)\in 2Z+1$, then $L_w(a_p)=L_{w'}(a_p)\in 2Z+1$ and so every simple word has some letter a_m for which $L_w(a_m)\in 2Z+1$, $m\in\{1,\dots,k\}$.

COROLLARY 1. The fundamental group of a surface is either trivial or not free. In particular, if $\overline{\sigma}$ is a circuit of length n, $\pi^{\sigma}(\Sigma_n)$ is not free for $n>3$.

PROOF. The fundamental group of a surface (without boundary) is either trivial or is of the following type [3]:

$$G=\langle a_1,b_1,\dots,a_m,t_m \| a_1b_1a_1^{-1}b_1^{-1}\dots a_mb_ma_m^{-1}b_m^{-1}\rangle$$

or $G=\langle a_1,\dots,a_m \| a_1^2a_2^2\dots a_m^2\rangle$

In both last cases, the sum of the exponents of an arbitrary generator in the generator is even. Applying lemma 4 gives us the first part of the corollary. The second part follows immediately with lemma 3.

PROOF OF THE THEOREM.

Suppose $\overline{\sigma}\notin\overline{S}$. Then by lemma 1, $\overline{\sigma}$ contains a circuit of length m>3. Suppose $\overline{\tau}$ is such a circuit with set of vertices $J\subseteq\Delta_n$. By proposition 1, $\pi^{\tau}(\Sigma_J)\leqslant\pi^{\sigma}(\Sigma_n)$. By lemma 3 $\pi^{\tau}(\Sigma_J)$ is the fundamental group of an orientable surface and not trivial since $|J|>3$)which is, by corollary 1, not free. Since a subgroup of a free group is always free, $\pi^{\sigma}(\Sigma_n)$ cannot be free. Hence the theorem is proved.

REMARK. One can easily prove directly from the definition, that if $\overline{\sigma}\in\overline{S}$, than $\overline{\sigma}$ contains no circuits of length n>3, but using our lemma's , the proof can be very short. If $\overline{\sigma}\in\overline{S}$, then $\pi^{\sigma}(\Sigma_n)$ is free and admits no non-free subgroup, hence by corollary 1 and proposition 1 $\overline{\sigma}$ cannot contain a circuit of length m>3 . So the conversion of lemma 1 is also true.

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Seminar of Geometry and Combinatorics
Krijgslaan 281
B-9000 Gent
Belgium

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