Graphs, defined by Weyl distance or incidence, that determine a vector space

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Abstract

We study to which extent the family of pairs of subspaces of a vector space related to each other via intersection properties determines the vector space. In another language, we study to which extent the family of vertices of the building of a projective space related to each other via several natural respective conditions involving the Weyl distance and incidence determines the building. These results can be seen as generalizations of and variations on the Fundamental Theorem of Projective Geometry.

1   Introduction

Let $\mathsf{PG}(n, \mathbb{L})$ be an n-dimensional projective space over the skew field $\mathbb{L}$, i.e., the geometry of all nontrivial subspaces of an $(n + 1)$-dimensional vector space $V_{n+1}(\mathbb{L})$ over $\mathbb{L}$. *The Fundamental Theorem of Projective Geometry* (see e.g. [5]) states that, in graph-theoretical terms, every automorphism of the incidence graph of the point-line geometry associated to $\mathsf{PG}(n, \mathbb{L})$, $n \geq 2$, is induced by a semi-linear permutation of the underlying vector space, or a duality (if $n = 2$). In other words, every permutation of the 1-spaces of $V_{n+1}(\mathbb{L})$ inducing a permutation of the 2-spaces of $V_{n+1}(\mathbb{L})$ is induced by a vector space

(anti)automorphism. In fact, all information of $\mathsf{PG}(n,\mathbb{L})$ is already contained in the graph with the lines of $\mathsf{PG}(n,\mathbb{L})$, or equivalently, the 2-spaces of $V_{n+1}(\mathbb{L})$, as vertices, where two vertices are adjacent if the corresponding subspaces intersect nontrivially. This is the collinearity graph of the line Grassmannian of the projective space. More generally, every automorphism of the collinearity graph of *any* Grassmannian of $\mathsf{PG}(n,\mathbb{L})$ is induced by a semi-linear permutation of the underlyin g vector space $V_{n+1}(\mathbb{L})$, or a duality thereof, by a fundamental result of Chow [6].

In the present paper, we generalize this by considering the subspaces of other arbitrary dimensions. Since we will only use subspaces of $V_{n+1}(\mathbb{L})$, and not the vectors themselves, we prefer to work in the projective setting and hence use projective dimensions of the subspaces (1-spaces of $V_{n+1}(\mathbb{L})$ are points or 0-spaces of $\mathsf{PG}(n,\mathbb{L})$, 2-spaces of $V_{n+1}(\mathbb{L})$ are lines or 1-spaces of $\mathsf{PG}(n,\mathbb{L})$, 3-spaces of $V_{n+1}(\mathbb{L})$ are planes or 2-spaces of $\mathsf{PG}(n,\mathbb{L})$, . . . , k-spaces of $V_{n+1}(\mathbb{L})$ are $(k-1)$-spaces of $\mathsf{PG}(n,\mathbb{L})$, $0 \leq k \leq n+1$; note that the 0-space of $V_{n+1}(\mathbb{L})$ is a -1-space of $\mathsf{PG}(n,\mathbb{L})$). So we consider the bipartite graph $\Gamma_{i,j;k}^{n}(\mathbb{L})$ of i- and j-spaces of $\mathsf{PG}(n,\mathbb{L})$, where an i-space is adjacent to a j-space if their intersection is a k-space, $-1 \leq k \leq i \leq j \leq n$. This is a metric generalization of the set-up in the previous paragraph, because "intersecting in a k-space" is a well-defined Weyl di stance (a double coset in the Weyl group, which is the symmetric group on $n+1$ letters) between i-spaces and j-spaces, when $\mathsf{PG}(n,\mathbb{L})$ be viewed as a building, and its subspaces as vertices. From the incidence geometric point of view, however, the general set-up is the graph $\Gamma_{i,j;\geq k}^{n}(\mathbb{L})$ of i-spaces and j-spaces of $\mathsf{PG}(n,\mathbb{L})$, where an i-space is adjacent to a j-space if they are both incident or equal to a common k-space. This means that their intersection has dimension *at least* k, whence the notation.

We show that, if $\Gamma_{i,j;k}^{n}(\mathbb{L})$ or $\Gamma_{i,j;\geq k}^{n}(\mathbb{L})$ are not trivial (meaning not complete bipartite, nonempty, not a matching and not the complement of a matching), then they completely determine the structure of $\mathsf{PG}(n,\mathbb{L})$. In particular, every automorphism is induced by a semi-linear permutation of the underlying vector space, in some cases possibly a duality. Moreover, we will see that, apart from obvious isomorphisms, all these graphs are pairwise non-isomorphic, i.e., the values n, i, j, k and the skew field $\mathbb{L}$ are essentially determined by the respective graphs ("essentially" here means "up to certain dualities" which will be explained below).

The metric point of view fits into the framework of the classical Theorem of Beckman and Quarles [3] which states that the preservation of one single distance guarantees an isometry of the Euclidean plane, or more generally, the n-dimensional Euclidean space. Here we show that the preservation of one single Weyl distance between vertices on a building of type A_n guarantees an automorphism of the building. A similar result exists for distances between chambers in any building [2]. For vertices, only partial results exist, mainly treating the special case of largest distance (see below), or buildings in low rank (see [7]).

The main consequence concerns the graphs $\Gamma_{j;k}^n(\mathbb{L})$ and $\Gamma_{j;\geq k}^n(\mathbb{L})$ of j-spaces of $\mathsf{PG}(n,\mathbb{L})$ where two vertices are adjacent if they intersect in a k-space and if their intersection contains a k-space, respectively. Indeed, since the (possibly extended) bipartite doubles of these graphs are $\Gamma_{j,j;k}^n(\mathbb{L})$ and $\Gamma_{j,j;\geq k}^n(\mathbb{L})$, respectively, every graph automorphism of $\Gamma_{j;k}^n(\mathbb{L})$ or of $\Gamma_{j;\geq k}^n(\mathbb{L})$ is induced by a semi-linear permutation or a duality of the underlying vector space, as soon as these graphs are not trivial (meaning not empty and not complete). For the graphs $\Gamma_{j;\geq k}^n(\mathbb{L})$, this has been shown by Lim [9] with a beautiful geometric argument. In fact, Lim considers adjacency preserving (in both directions) *surjections* of the graphs. This slightly weaker hypothesis also suffices in our setting, but we find it more convenient to work with bijections and afterwards d educe th is slightly more general result, see Remark 3.17. We will not need to use Lim's results and thus provide an alternative approach to Lim's theorem.

The proof in the case of $\Gamma_{i,j;\geq k}^n(\mathbb{L})$ uses the idea of a round-up triple introduced in [8], where opposition is handled (and "opposition" is just the maximal Weyl distance, using the "longest word"). In fact the idea of a round-up triple is a more conceptual way to formulate Lim's proof, and it allows to treat more general situations. But basically, Lim's proof and ours are very alike, when applied to $\Gamma_{j;\geq k}^n(\mathbb{L})$. Our results can also be seen as the completion of Lim's results in the most general case.

But the method of round-up triples does not work anymore for the graphs $\Gamma_{i,j;k}^n(\mathbb{L})$. This is rather surprising since a similar idea for opposite chambers, see [1] carried over to single Weyl distance between chambers, see [2]. For $\Gamma_{i,j;k}^n(\mathbb{L})$, we have to use "round-up quadruples", which considerably complicates things, and excludes $|\mathbb{L}| = 2$. Here, Lim's approach does not work anymore.

This brings us to the special case of finite $\mathbb{L}$, where our results can be proved using the classification of maximal subgroups of the symmetric and alternating groups in [10]. But that proof does not give much insight into the problem, of course. In the finite case, we will denote the graphs $\Gamma_{i,j;k}^n(\mathbb{L})$ and $\Gamma_{i,j;\geq k}^n(\mathbb{L})$ by $\Gamma_{i,j;k}^n(|\mathbb{L}|)$ and $\Gamma_{i,j;\geq k}^n(|\mathbb{L}|)$, respectively, since a finite field is determined by its order.

Apart from the results of Lim and Chow mentioned above, another special case of our results concerns the case $\Gamma_{i,j;-1}^n(\mathbb{L})$ (or the bipartite complement $\Gamma_{i,j;\geq 0}^n(\mathbb{L})$). This has been treated by Blunck and Havlicek [4]. We are not aware of other special cases in the literature. However, for polar spaces, a lot of similar problems have been solved, but thus far not in such a full generality as we do for projective spaces in the present paper (see the references in [9]). This is our main motivation: settle the problem for projective spaces and polar spaces in the most general way. In the present paper, we deal with the projective spaces, and the analogue for polar spaces is work in progress.

This research is part of a larger programme to determine all situations for spherical buildings [11] where the family of pairs of vertices at certain fixed Weyl distance, or connected by an incidence condition, uniquely defines the building in question. Note

that this is not always true (so the projective spaces are, in this respect, a nice class of spherical buildings), see for instance [8], where the maximal distance between maximal singular subspaces in certain parabolic quadrics is a counter example, or [7], where again the maximal distance between points of a generalized hexagon is a counter example. In the present paper we will find other such examples, be it in the thin case (but one of the examples is strongly related to the smallest thick generalized quadrangle).

Indeed, in the last section of the present paper, we also prove the analogue of our main results for the *thin* case, i.e., for vector spaces or projective spaces over the field of order 1, hence just sets. For finite sets, everything will follow, with some additional work, from a group-theoretic result of Liebeck, Praeger and Saxl [10], but we also consider infinite sets. As mentioned in the previous paragraph, it is interesting to note that for finite sets, there are counter examples, i.e., there are situations where not all graph automorphisms are induced by a permutation of the starting set. For more details, see Section 4.2

2 Statements of the results

We now provide the exact statements alluded to in the introduction. Let $V_{n+1}(\mathbb{L})$ be an $(n+1)$-dimensional right vector space over the skew field $\mathbb{L}$, and let $\mathsf{PG}(n,\mathbb{L})$ be the associated projective space, i.e., the points of $\mathsf{PG}(n,\mathbb{L})$ are the 1-spaces of $V_{n+1}(\mathbb{L})$, and a k-space of $\mathsf{PG}(n,\mathbb{L})$, $0 \leq k \leq n-1$, consists of the 1-spaces contained in subspace of dimension $k+1$ of $V_n(\mathbb{L})$. The *empty subspace* of $\mathsf{PG}(n,\mathbb{L})$ corresponds to the trivial subspace of $V_n(\mathbb{L})$ and has project dimension -1.

We define the graphs $\Gamma^n_{i,j;k}(\mathbb{L}) = \Gamma^n_{j,i;k}(\mathbb{L})$ and $\Gamma^n_{i,j;\geq k}(\mathbb{L}) = \Gamma^n_{j,i;\geq k}(\mathbb{L})$ as above. The *bipartite complement* of a bipartite graph Γ is the graph obtained from Γ by interchanging edges and non-edges between the biparts, while keeping no edges within the biparts.

Main Result 2.1 *Let $\mathbb{L}$ and $\mathbb{L}'$ be two skew fields, and let $-1 \leq k \leq i \leq j \leq n-1$, $-1 \leq k' \leq i' \leq j' \leq n'-1$ be integers, with $-1 \notin \{i, i'\}$ and $n \geq 2$.*

   (i) If $i = j = k \geq 0$, then $\Gamma^n_{i,j;k}(\mathbb{L})$ is a matching.

   (ii) If $i = j = n-1 = k+1$ or $i = j = 0 = k+1$, then $\Gamma^n_{i,j;k}(\mathbb{L})$ is the complement of a matching.

   (iii) If $n+k < i+j$, then $\Gamma^n_{i,j;k}(\mathbb{L})$ is an empty graph (a graph with vertices but no edges).

   (iv) If $\mathbb{L} \cong \mathbb{L}'$, $n = n'$, $i' = n-1-j$, $j' = n-1-i$ and $k' = n-1+k-i-j$, then $\Gamma^n_{i,j;k}(\mathbb{L}) \cong \Gamma^{n'}_{i',j';k'}(\mathbb{L}')$.

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(v) If $i + j \leq n - 1$, $(i,j) \neq (0,0)$, $k < j$ and $i' + j' \leq n' - 1$, then $\Gamma^n_{i,j;k}(\mathbb{L}) \cong \Gamma^{n'}_{i',j';k'}(\mathbb{L}')$ if and only if $\mathbb{L} \cong \mathbb{L}'$ and $(i,j,k,n) = (i',j',k',n')$. In this case every graph isomorphism is induced by a semi-linear bijection from $V_n(\mathbb{L})$ to $V_{n'}(\mathbb{L}')$, or possibly to $V^*_{n'}(\mathbb{L}')$ (the dual of $V_{n'}(\mathbb{L}')$) if $i + j = n - 1$ and $\mathbb{L}' \cong (\mathbb{L}')^*$ (the latter is the opposite skew field).

Note that the restrictions in (v) are justified by (iv), so that we really cover all possible cases. The same thing holds for the next result, where the restrictions in (iv) are justified by (iii).

Main Result 2.2 *Let $\mathbb{L}$ and $\mathbb{L}'$ be two skew fields, and let $-1 \leq k \leq i \leq j \leq n - 1$, $-1 \leq k' \leq i' \leq j' \leq n' - 1$ be integers, with $-1 \notin \{i, i'\}$ and $n \geq 2$.*

(i) *If $n + k \leq i + j$ or $k = -1$, then $\Gamma^n_{i,j;\geq k}(\mathbb{L})$ is a complete bipartite graph.*

(ii) *If $k = i + j + 1 - n$, then $\Gamma^n_{i,j;\geq k}(\mathbb{L})$ is the bipartite complement of $\Gamma^n_{i,j;k-1}(\mathbb{L})$; if $i = k$, then $\Gamma^n_{i,j;\geq k}(\mathbb{L}) \cong \Gamma^n_{i,j;k}(\mathbb{L})$.*

(iii) *If $\mathbb{L} \cong \mathbb{L}'$, $n = n'$, $i' = n - 1 - j$, $j' = n - 1 - i$ and $k' = n - 1 + k - i - j$, then $\Gamma^n_{i,j;\geq k}(\mathbb{L}) \cong \Gamma^{n'}_{i',j';\geq k'}(\mathbb{L}')$.*

(iv) *If $i + j \leq n - 1$, $-1 \neq k < j$ and $i' + j' \leq n' - 1$, then $\Gamma^n_{i,j;\geq k}(\mathbb{L}) \cong \Gamma^{n'}_{i',j';\geq k'}(\mathbb{L}')$ if and only if $\mathbb{L} \cong \mathbb{L}'$ and $(i,j,k,n) = (i',j',k',n')$. In this case every graph isomorphism is induced by a semi-linear bijection from $V_n(\mathbb{L})$ to $V_{n'}(\mathbb{L}')$, or possibly to $V^*_{n'}(\mathbb{L}')$ (the dual of $V_{n'}(\mathbb{L}')$) if $i + j = n - 1$ and $\mathbb{L}' \cong (\mathbb{L}')^*$ (the latter is the opposite skew field).*

We will also show that all graphs in (v) of Main Result 2.1 are distinct from those in (iv) of Main Result 2.2, except for $k = \min\{i,j\}$ in both.

A special case worth mentioning is the graph $\Gamma^n_{i,j;i}(\mathbb{L})$, with $0 \leq i < j \leq n - 1$. This is the bipartite graph consisting of i-subspaces and j-subspaces, where adjacency is just defined by containment. It leads to the most straightforward generalization of the Fundamental Theorem of Projective Geometry, and, as we will see, we will need to prove it separately in advance.

Another special case occurs when $i = j$; in this case one can define the non-bipartite graphs $\Gamma^n_{j;k}(\mathbb{L})$ and $\Gamma^n_{j;\geq k}(\mathbb{L})$ (see above), see Corollaries 3.12 and 3.16.

Notation. The incidence graph of $\mathsf{PG}(n, \mathbb{L})$ is an n-partite graph denoted by $\Gamma^n_{[0,n-1]}(\mathbb{L})$. If we restrict this graph to the subspaces of dimensions $i, i+1, \ldots, j-1, j$, then we denote the resulting $(j - i + 1)$-partite graph by $\Gamma^n_{[i,j]}(\mathbb{L})$. In such a graph, the k-neighbor of a

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vertex v, with $i \leq k \leq j$ and with $\dim(v) \neq k$, are the subspaces of dimension k incident with v. The k-neighborhood of v is the set of k-neighbors of v.

For $0 \leq j \leq n-1$, the j-Grassmann graph is the graph with vertices the j-spaces of $\mathsf{PG}(n, \mathbb{L})$, where two j-spaces are adjacent if they intersect in a $(j-1)$-space. This is the collinearity graph of the so-called j-Grassmannian geometry, which is defined as follows. The points are the j-spaces and the lines are the sets of j-spaces containing a fixed $(j-1)$-space J_- and being contained in a fixed $(j+1)$-space J_+, with $J_- \subseteq J_+$. If $j \notin \{0, n-1\}$, then the j-Grassmann graph uniquely determines the j-Grassmannian geometry. In any case, the j-Grassmannian geometry completely determines $\mathsf{PG}(n, \mathbb{L})$ in the sense that the automorphism groups of both structures coincide (possibly up to the dualities).

For a set S of subspaces (possibly just points), we define $\langle S \rangle$ to be the subspace generated by all members of S. If S consists of two distinct points p_1, p_2, then we also denote the unique line passing through these points by $p_1 p_2$. Finally, for a k-subspace K, we denote by $\mathrm{Res}(K)$ the projective space of dimension $n-k-1$ obtained from the underlying vector space by factoring out K, and we call it the *residue of K*. Hence the i-spaces of $\mathrm{Res}(K)$, $-1 \leq i \leq n-k-1$, are the quotients W/K, where W is an $(i+k+1)$-space of $\mathsf{PG}(n, \mathbb{L})$ containing K.

3 Proofs

3.1 Generalities and the case $k = 0$

The assertions (i) to (iv) of Main Result 2.1 and (i) to (iii) of Main Result 2.2 are easy to verify. Hence we concentrate on showing (v) of Main Result 2.1 and (iv) of Main Result 2.2.

So let there be given a graph $\Gamma \cong \Gamma_{i,j;k}^n(\mathbb{L})$ (with $n, i, j, k, \mathbb{L}$ as in (v) of Main Result 2.1) or $\Gamma \cong \Gamma_{i,j;\geq k}^n(\mathbb{L})$ (with $n, i, j, k, \mathbb{L}$ as in (iv) of Main Result 2.2), except that we do not assume that $i \leq j$. We provide an algorithmic proof, determining the parameters as we go along. An exception is the family of graphs $\Gamma \cong \Gamma_{i,j;k}^n(2)$, $k < \min\{i, j\}$, which we must handle separately. Hence we will assume that Γ is not isomorphic to such a graph.

In the course of the proof, we will have to pick at certain moments a subspace of certain dimension satisfying different incidence conditions. We first prove a lemma that will imply that we can do so in the most restrictive case (which will take care of all other cases, too, that we will encounter).

Lemma 3.1 *Let $a \geq 2$ and let $0 \leq b < a$ be natural numbers. Let B be b-space in $\mathsf{PG}(a, \mathbb{L})$, let B_1, B_2 be two subspaces of $\mathsf{PG}(a, \mathbb{L})$ of dimension at most $b-1$, and let B_3*

be a subspace of $\mathsf{PG}(a, \mathbb{L})$ *of dimension at most* $b - 2$ *(if* $b = 0$, *then* B_3 *is the empty space). Then there exists an* $(a - b - 1)$*-space* C *disjoint from* $B \cup B_1 \cup B_2 \cup B_3$.

Proof We assume $a \geq 3$, as the case $a = 2$ is easy (there is always a point not incident with a given line and distinct from two other given points, and there is always a line not incident with a given point). By possibly extending the spaces B_1, B_2, B_3, we may assume that they have precisely dimension $b - 1, b - 1, b - 2$, respectively.

We claim that $B \cup B_1 \cup B_2 \cup B_3$ is nonempty. Indeed, if $|\mathbb{L}|$ is infinite, then this follows from the fact that $\mathsf{PG}(a, \mathbb{L})$ is not the union of a finite number of (proper) subspaces. Now let $|\mathbb{L}| = q$ be finite. We may extend the subspaces B, B_1, B_2, B_3 in such a way that they have maximal dimension, i.e., we may assume that $b = a - 1$. Then, taking into account that $B \cap B_1$ is at least $(a - 2)$-dimensional, and similarly for $B \cap B_2$ and $B \cap B_3$, we count at most

$$\underbrace{\frac{q^a - 1}{q - 1} + \frac{q^{a-1} - 1}{q - 1} - \frac{q^{a-2} - 1}{q - 1}}_{B \cup B_1} + \underbrace{\frac{q^{a-1} - 1}{q - 1} - \frac{q^{a-2} - 1}{q - 1}}_{B_2 \backslash B} + \underbrace{\frac{q^{a-2} - 1}{q - 1} - \frac{q^{a-3} - 1}{q - 1}}_{B_3 \backslash B}$$

points in $B \cup B_1 \cup B_2 \cup B_3$. The claim follows if we show that this number is strictly less than $\frac{q^{a+1} - 1}{q - 1}$, which follows immediately from the obvious inequality

$$q^{a+1} > q^a + 2q^{a-1} - q^{a-2} - a^{q-3},$$

for all $a \geq 3$ and all $q \geq 2$.

Now we continue by induction on $a \geq b + 1$. The case $a = b + 1$ follows from the claim above (then C is just a point outside $B \cup B_1 \cup B_2 \cup B_3$). Let $a > b + 1$. Let x be a point outside $B \cup B_1 \cup B_2 \cup B_3$. Then by the induction hypothesis we obtain an $(a - b - 2)$-space C' in $\mathrm{Res}(x)$ disjoint from $\langle B, x \rangle / x, \langle B_1, x \rangle / x \langle B_2, x \rangle / x, \langle B_3, x \rangle / x$. The corresponding subspace C in $\mathsf{PG}(a, \mathbb{L})$ intersects each of B, B_1, B_2, B_3 exactly in the point x. Hence C is disjoint from $B \cup B_1 \cup B_2 \cup B_3$ and the lemma is proved. $\qquad\square$

Now we start by isolating the case $k = \min\{i, j\}$, $(i, j) \notin \{(0, n-1), (n-1, 0)\}$, $|j - i| > 1$. For a vertex v of Γ, we denote by $\Gamma(v)$ the set of neighbors of v. Also, $V(\Gamma)$ is the set of vertices of Γ, and $V^v(\Gamma)$ is the set of vertices in the bipart of v.

Proposition 3.2 *For the graph* Γ *the parameter* k *equals* $\min\{i, j\}$ *with* $(i, j) \notin \{(0, n-1), (n-1, 0)\}$ *and* $|j - i| > 1$, *if and only if* Γ *satisfies the following property* (min)

(min) *For some vertex* v, *the family* $\mathcal{F}(v) = \{\Gamma(v) \cap \Gamma(w) : w \in V^v(\Gamma)\}$ *forms a poset under inclusion with the property that, if two elements have a greatest common lower bound, then it is obtained by intersecting the two elements. Also, every two maximal elements have a greatest common lower bound and every element is contained in a maximal element.*

Proof If $k = \min\{i, j\}$, say $i = k$, and v is a j-space, then $\mathcal{F}(v)$ is the poset of all subspaces of dimension at least $\max\{i, 2j - n\}$, viewed as sets of the i-spaces they contain, of a projective space of dimension j. It is clear that this poset satisfies (min) as soon as $i < j - 1$ and $2j - n < j - 1$. The first condition is equivalent with $|j - i| > 1$ and the second with $j < n - 1$, so with $(i, j) \neq (0, n - 1)$ in view of $i + j \leq n - 1$.

Now suppose $0 \leq k < \min\{i, j\}$ (we cease to assume $i \leq j$ as we did in the previous paragraph). Let J_1 and J_2 be two j-spaces intersecting in a $(j - 1)$-space J'. Let J_3 be such that $\Gamma(J_1) \cap \Gamma(J_3)$ is maximal in $\mathcal{F}(J_1)$ and contains $\Gamma(J_1) \cap \Gamma(J_2)$. We claim first that $J_1 \cap J_2 \subseteq J_3$. Indeed, suppose not. Then we can select a k-subspace K contained in $J_1 \cap J_2$ but not in J_3. Applying Lemma 3.1 in $\mathrm{Res}(K)$, we see that there exists an i-space I intersecting both J_1 and J_2 in K, and intersecting J_3 in $J_1 \cap J_2 \cap K$, which has dimension strictly less than k. Hence $I \in \Gamma(J_1) \cap \Gamma(J_2) \setminus \Gamma(J_3)$, a contradiction. Now there are two possibilities.

Assume that $J_3 \notin \langle J_1, J_2 \rangle$. Then there is a $(k + 1)$-space K in $\langle J_1, J_2 \rangle$ intersecting $J_1 \cap J_2$ in a $(k - 1)$-space. We can now pick an i-space I through K not intersecting $J_3 \setminus K$. Hence I is adjacent to both J_1, J_2, but not to J_3, contradicting the maximality of $\Gamma(J_1) \cap \Gamma(J_3)$ in $\mathcal{F}(J_1)$.

Hence we may assume $J_3 \in \langle J_1, J_2 \rangle$. If $\Gamma(J_1) \cap \Gamma(J_2) = \Gamma(J_1) \cap \Gamma(J_3)$, then $\Gamma(J_1) \cap \Gamma(J_2)$ was already maximal in the first place. If $\Gamma(J_1) \cap \Gamma(J_2) \neq \Gamma(J_1) \cap \Gamma(J_3)$, then, since there exists a collineation fixing J_1 and interchanging J_2 and J_3, there is a vertex in $\Gamma(J_1) \cap \Gamma(J_2)$ that does not belong to $\Gamma(J_1) \cap \Gamma(J_3)$, again a contradiction.

We conclude that $\Gamma(J_1) \cap \Gamma(J_2)$ is maximal itself. Now consider any j-space J_4 not contained in $\langle J_1, J_2 \rangle$ and such that $J_1 \cap J_2 \subseteq J_4$. As above, we know that $\Gamma(J_1) \cap \Gamma(J_2) \neq \Gamma(J_1) \cap \Gamma(J_4)$. Hence, by (min), there exists a j-space J_5 with $\Gamma(J_1) \cap \Gamma(J_5) = \Gamma(J_1) \cap \Gamma(J_2) \cap \Gamma(J_4)$. But, also as above, if J_5 does not contain $J_1 \cap J_2$, then we can select a k-space K in $J_1 \cap J_2$ not contained in J_5, and an i-space I through K such that $I \cap J_5 = I \cap K$, $I \cap J_i = K$, for $i \in \{1, 2, 4\}$. Then I belongs to $\Gamma(J_1) \cap \Gamma(J_2) \cap \Gamma(J_4) \setminus \Gamma(J_5)$, a contradiction. Hence $J_1 \cap J_2 \subseteq J_5$. But then $\Gamma(J_1) \cap \Gamma(J_5)$ is maximal, the final contradiction. Hence (min) is not satisfied.

Now let $k = -1$. Let J_1, J_2, J_3 be three different j-spaces. It is easy to see that through any point of $J_3 \setminus (J_1 \cup J_2)$ one can find an i-space disjoint from both J_1 and J_2. Hence every element of $\mathcal{F}(J_1)$ is maximal and so (min) cannot be satisfied. $\square$

Proposition 3.3 *The parameters $i, j, n, \mathbb{L}$ are uniquely determined by the graph $\Gamma_{i,j;i}^n(\mathbb{L})$, for $i \leq j \leq n - i - 1$. Moreover, every graph automorphism is induced by a semi-linear permutation of the underlying vector space.*

Proof Fix any vertex v. Then we consider the poset $P_v = \{\Gamma(v) \cap \Gamma(w_1) \cap \ldots \Gamma(w_t) : w_1, \ldots, w_t \in V^v(\Gamma), t \in \mathbb{N}\}$. The length of a maximal chain in P_v is precisely $j - i$. We

define a new graph Γ' as follows. The vertices are the intersections of a finite number of neighborhoods of vertices of $V^v(\Gamma)$. Adjacency is containment made symmetric. It is clear that Γ' is isomorphic to $\Gamma^n_{[i,j]}(\mathbb{L})$. Now we can extend this graph "at both ends" as follows. We define a new graph Γ^i where the vertices are the i-spaces, adjacent when they are adjacent in Γ' to a common vertex representing an $(i+1)$-space. The graph Γ^i is the i-Grassmann graph and hence it has two kinds of maximal cliques: all i-spaces contained in an $(i+1)$-space (these maximal cliques are visible in Γ'), and all i-space contained in an $(i-1)$-space. We add the latter maximal cliques to the graph Γ' with natura l adjace ncy between the new vertices and the i-spaces, and a new vertex I' is adjacent to a vertex L representing an ℓ-space, $i+1 \le \ell \le j$, if I' and L are adjacent to a common i-space. We do the same "at the other end" (with j-spaces) and obtain the graph $\Gamma^n_{i-1,j+1}(\mathbb{L})$, we keep doing this until every pair of the new vertices is adjacent to a common old vertex introduces at the previous step "next to it". Then we have $\Gamma^n_{[0,j+i]}(\mathbb{L})$. This uniquely determines i (and hence j, since we already knew $j-i$). But now we can extend this graph "at the right" $n-i-j-1$ times to obtain $\Gamma^n_{[0,n-1]}(\mathbb{L})$, uncovering n. The Fundamental Theorem of Projective Geometry now applies, $\mathbb{L}$ follows and so does the proposition. $\qquad\square$

So we are left with the graphs $\Gamma^n_{i,j;i}(\mathbb{F})$, $i+1 = j \le n-i-1$ or $(i,j) = (0,n-1)$, which we call of *type* I; graphs $\Gamma^n_{i,j;\ge k}(\mathbb{F})$ with $0 \le k < \min\{i,j\}$ and $i+j \le n-1$, which we call of *type* II; graphs $\Gamma^n_{i,j;k}(\mathbb{F})$ with $-1 \le k < \min\{i,j\}$ and $i+j \le n-1$, which we call of *type* III.

Now we characterize the graphs $\Gamma^n_{i,j;i}(\mathbb{F})$ with $i+1 = j \le n-i-1$, among these.

Lemma 3.4 *Let Γ be a graph of type I,II or III. Then $\Gamma \cong \Gamma^n_{i,j;i}(\mathbb{F})$ with $i+1 = j \le n-i-1$ if and only if every vertex is the intersection $\Gamma(v) \cap \Gamma(w)$ of the neighborhoods of two vertices v, w.*

Proof   Clearly the property holds if $\Gamma \cong \Gamma^n_{i,j;i}(\mathbb{F})$, with $i+1 = j \le n-i-1$. Now suppose Γ satisfies the stated property. Then Γ is not of type II or III since through any given k-space K contained in two j-spaces J_1, J_2, one can select at least two i-spaces intersecting two given j-spaces only in K (indeed, projecting from K, this amounts to choose two $(i-k-1)$-spaces disjoint from two given $(j-k-1)$-spaces in $(n-k-1)$-dimensional space, which can clearly be done). $\qquad\square$

By Proposition 3.3, every automorphism of $\Gamma^n_{i,i+1;i}(\mathbb{F})$ is induced by a semi-linear mapping of the underlying vector space, or a duality (which happens if $2i+2 = n$).

¿From now on, we do not call a graph $\Gamma^n_{i,i+1;i}(\mathbb{F})$ of type I anymore, since we already characterized it.

Let Γ be a graph of type I, type II or type III. We introduce a property of triples and quadruples, respectively, of vertices in the same bipart.

Suppose one of the biparts of Γ are the j-spaces, and the other consists of the i-spaces. Let J_1, J_2, J_3 be three j-spaces. Then we say that $\{J_1, J_2, J_3\}$ is a Γ-*round-up triple* if no i-space is Γ-adjacent to exactly two of J_1, J_2, J_3 and some i-space is Γ-adjacent to all of J_1, J_2, J_3. Also, $\{J_1, J_2, J_3\}$ is a *regular round-up triple* if $J_1 \cap J_2 \cap J_3$ is a $(j-1)$-space and if $\langle J_1, J_2, J_3 \rangle$ is a $(j+1)$-space.

Now let J_1, J_2, J_3, J_4 be four j-spaces in $\mathsf{PG}(n, \mathbb{L})$. Then we say that $\{J_1, J_2, J_3, J_4\}$ is a Γ-*round-up quadruple* if every vertex that is Γ-adjacent to at least two among J_1, J_2, J_3, J_4 is adjacent to at least three of them, and some vertex is adjacent to at least three of them. Also, $\{J_1, J_2, J_3, J_4\}$ is called a *regular round-up quadruple* if the four j-spaces all contain a fixed $(j-1)$-space and are themselves contained in a fixed $(j+1)$-space.

We first investigate when Γ contains Γ-round-up triples, then prove that, when they do, these are precisely the regular round-up triples. Using these regular round-up triples, we determine the parameters of the graph in a canonical way and show that Γ determines $\mathsf{PG}(n, \mathbb{L})$.

Then we go on doing the same with Γ-round-up quadruples for the remaining graphs.

3.2 Γ-round-up triples

We start with a characterization of regular round-up triples and quadruples.

Lemma 3.5 *Let J_1, J_2, J_3, J_4 be (not necessarily different) j-spaces such that the intersection of the distinct pairs is a fixed subspace D, say of dimension $d \geq -1$. Then $d = j - 1$ and $\dim \langle J_1, J_2, J_3, J_4 \rangle = j+1$ if and only if every line intersecting two different members of $\{J_1, J_2, J_3, J_4\}$, say J_{ℓ_1}, J_{ℓ_2}, also intersects one of J_{ℓ_3}, J_{ℓ_4}, with $\{\ell_1, \ell_2, \ell_3, \ell_4\} = \{1, 2, 3, 4\}$.*

Proof   The lemma is trivial if $|\{J_1, J_2, J_3, J_4\}| \leq 2$. So we may assume that all of J_1, J_2, J_3 are distinct, and that either J_4 is distinct from all of J_1, J_2, J_3, or $J_3 = J_4$. Also, if $d = j - 1$ and $\dim \langle J_1, J_2, J_3, J_4 \rangle = j+1$, then it is easy to see that the stated condition is satisfied.

Suppose now the stated condition is satisfied. If $j = 0$, then the assertion is easy. Suppose now $j \geq 1$ and assume for a contradiction that $d < j - 1$. Then there is some line L contained in $J_2 \setminus D$. Consider any point p_1 in $J_1 \setminus D$ and let p_2, p_2', p_2'' be three points on L. Two of the lines $p_1 p_2, p_1 p_2', p_1 p_2''$ must then intersect either J_3 or J_4, say J_3, by the condition. Hence the plane $\langle p_1, L \rangle$ intersects J_3 in a line L'. But then the point $L \cap L'$ belongs to $J_2 \cap J_3$, hence to D, a contradiction. Hence $d = j - 1$.

The line joining a point of $J_1 \setminus D$ with a point of $J_2 \setminus D$ intersects $J_3 \cup J_4$, clearly in a point not belonging to D. Hence at most one of J_3, J_4, say J_4, does not belong to $\langle J_1, J_2 \rangle$.

Similarly, one of J_1, J_2 belongs to $\langle J_3, J_4 \rangle$, say J_1. But then J_4 and J_2 belong to $\langle J_1, J_3 \rangle$, which implies $\dim \langle J_1, J_2, J_3, J_4 \rangle = \dim \langle J_1, J_3 \rangle = j + 1$. $\qquad \square$

If Γ is of type I, then it is easy to verify that a Γ-round-up triple is either a set of three collinear points or a set of three hyperplanes containing the same $(n-2)$-space.

Now suppose that Γ is of type II, with $\Gamma \cong \Gamma^n_{i,j;\geq k}(\mathbb{L})$, $k > -1$ (we do not assume $i \leq j$) or Γ is of type III, with $\Gamma \cong \Gamma^n_{i,j;k}(\mathbb{L})$, $k \geq -1$. In these cases, we note the following equivalent more manageable condition for being a Γ-round-up triple. For type II, it follows immediately from the fact that, if U, V are two subspaces of $\mathsf{PG}(n, \mathbb{L})$, with $\dim V \leq i$, $\dim U = j$ and $\dim(U \cap V) < k$, then there is an i-space W containing V with $W \cap U = V \cap U$. Note that we will only state things for one choice of (i, j); interchanging i and j usually results in another property, which we assume tacitly.

Observation 3.6 *A triple $\{J_1, J_2, J_3\}$ of j-spaces is a Γ-round-up triple, with Γ of type II, if and only if every subspace of dimension at most i spanned by two k-spaces contained in two respective members of $\{J_1, J_2, J_3\}$ intersects the third member in a subspace of dimension at least k.*

Lemma 3.7 *A triple $\{J_1, J_2, J_3\}$ of j-spaces is a Γ-round-up triple, with Γ of type III, if and only if every subspace L of dimension at most i spanned by two k-spaces contained in two respective members of $\{J_1, J_2, J_3\}$ and intersecting these subspaces in the respective k-spaces, intersects the third member in a subspace of dimension precisely k.*

Proof   The proof is a simplified version of the proof of Lemma 3.13 (see below).

Assume for a contradiction that the assertion is false. By renumbering if necessary we may then assume that K_1 and K_2 are k-spaces contained in J_1, J_2, respectively, and that $\langle K_1, K_2 \rangle \cap J_3$ is not k-dimensional. Our goal is to construct an $(i-1)$-space I' containing $\langle K_1, K_2 \rangle$ and intersecting J_3 in a $(k-1)$-space.

We consider an arbitrary i-space I intersecting J_1 in K_1 and J_2 in K_2 (it is easy to see that such a subspace exists). Since $\{J_1, J_2, J_3\}$ is a Γ-round-up triple, we know that I intersects J_3 in a k-space K_3. Since $K_3 \not\subseteq \langle K_1, K_2 \rangle$ by our assumption, we can consider a hyperplane I' of I containing $\langle K_1, K_2 \rangle$ and not containing K_3.

In $\mathrm{Res}(I')$, the spaces $\langle J_1, I' \rangle / I'$, $\langle J_2, I' \rangle / I'$ and $\langle J_3, I' \rangle / I'$ have dimension $j - k - 1$, $j - k - 1$ and $j - k$, respectively. By Lemma 3.1, we can find a point in $\mathrm{Res}(I')$ which avoids these three subspaces. This point corresponds to an i-space I^* intersecting J_1 and J_2 in k-spaces and intersecting J_3 in a $(k-1)$-space. This clearly contradicts the fact that $\{J_1, J_2, J_3\}$ is a Γ-round-up triple. $\qquad \square$

Suppose now $\{J_1, J_2, J_3\}$ is a Γ-round-up triple. In the next statements, the arguments are usually independent of the type of Γ; however, if they differ, then we write the argument

for type II in the ordinary way, whereas the argument for type III will be written in square brackets. It will be clear which part of the argument for type II it replaces.

Lemma 3.8 *The Γ-round-up triple $\{J_1, J_2, J_3\}$ satisfies $J_1 \cap J_2 = J_3 \cap J_1 = J_2 \cap J_3$.*

Proof   Choose the subscripts such that $\dim(J_1 \cap J_2)$ be maximal among $\{\dim(J_\ell \cap J_{\ell'}) : \ell \in \{1,2,3\}, \ell' \in \{1,2,3\} \setminus \{\ell\}\}$. For a contradiction, we may hence suppose that there exists a point $p \in (J_1 \cap J_2) \setminus J_3$. There are two cases to distinguish.

1. Suppose $\dim(J_1 \cap J_2) \geq k$. Consider a k-space K in $J_1 \cap J_2$ containing p. Observation 3.6 implies that K intersects J_3 in a subspace of dimension at least [exactly] k, which is absurd.

2. Suppose $\dim(J_1 \cap J_2) = \ell < k$. Let K_a be any k-space in J_a containing $J_1 \cap J_2$, $a = 1, 2$. By Observation 3.6, the subspace $\langle K_1, K_2 \rangle$ intersects J_3 in a subspace K_3 of dimension at least [exactly] k. Since $\dim\langle K_1, K_2 \rangle = 2k - \ell$ and $\langle K_1, K_2 \rangle$ contains K_3, a dimension argument implies $\dim K_1 \cap K_3 \geq \ell$, By maximality of ℓ, we have equality. Moreover, this also implies $J_1 \cap J_3 = K_1 \cap K_3$, and so $K_1 \cap K_3 \neq J_1 \cap J_2$. Replacing K_1 by a k-space K_1' in J_1 containing $J_1 \cap J_2$ but not $K_1 \cap K_3$, the same argument implies $J_1 \cap J_3 \subseteq K_1'$, a contradiction.

The lemma is proved.   □

Lemma 3.9 *Every Γ-round-up triple $\{J_1, J_2, J_3\}$ is a regular round-up triple.*

Proof   By Lemma 3.8, we know $J_1 \cap J_2 = J_3 \cap J_1 = J_2 \cap J_3$. Define $\ell = \dim J_1 \cap J_2$. Again, we distinguish two cases.

1. Suppose $\ell \geq k$. Let $p_a \in J_a \setminus J_3$ be arbitrary, $a = 1, 2$. For $a = 1, 2$, choose a k-space K_a containing p_a and intersecting $J_1 \cap J_2$ in a $(k-1)$-space, with the latter independent of a. Observation 3.6 [Lemma 3.7] and the fact that $k < i$ imply that the $(k+1)$-space $\langle K_1, K_2 \rangle$ intersects J_3 in at least [precisely] a k-space K_3 intersecting $J_1 \cap J_2$ in $K_1 \cap K_2$, hence in precisely a k-space. Consequently, the line $p_1 p_2$ intersects K_3, and hence J_3, in a point. Lemma 3.5 shows the assertion.

2. Suppose $\ell < k$. We claim this situation cannot occur. Indeed, let K_a be a k-space through $J_1 \cap J_2$ inside J_a, $a = 1, 2$. Observation 3.6 [Lemma 3.1] and Lemma 3.8 together imply that $\langle K_1, K_2 \rangle \cap J_3 =: K_3$ is k-dimensional. Now let K_1' be a k-space in J_1 through $J_1 \cap J_2$ such that $\langle K_1, K_1' \rangle$ is $(k+1)$-dimensional (this is possible since $k < j$). Then $\langle K_1', K_2 \rangle \cap J_3 =: K_3'$ is again k-dimensional. Since $\langle K_2, K_3 \rangle = \langle K_1, K_2 \rangle$

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and $K_1' \nsubseteq \langle K_1, K_2 \rangle$, we conclude $K_3' \neq K_3$. Now $\dim \langle K_1, K_1', K_2 \rangle = 2k - \ell + 1$ and $\langle K_3, K_3' \rangle \subseteq \langle K_1, K_1', K_2 \rangle$. A dimension argument (the Grassmann identity) implies now that $\langle K_1, K_1' \rangle \cap \langle K_3, K_3' \rangle$ has dimension at least $(k+1) + (k+1) - (2k - \ell + 1) = \ell + 1$. Since $\langle K_1, K_1' \rangle \cap \langle K_3, K_3' \rangle$ is contained in $J_1 \cap J_3$, this is a contradiction. Our claim is proved. $\qquad\square$

Lemma 3.10 *A regular round-up triple is a Γ-round-up triple if and only if Γ has type* I *or* II.

Proof The assertion is clear for graphs of type I and II. Now let $\Gamma \cong \Gamma_{i,j;k}^n(\mathbb{L})$ be of type III. Let $\{J_1, J_2, J_3\}$ be a regular round-up triple of j-spaces. Since $k < j$, we can select a k-space K in $J_1 \cap J_2$. Since $k < i$, we can select an i-space I intersecting $J_1 \cap J_2$ in K, and intersecting $\langle J_1, J_2 \rangle$ in a $(k+1)$-space contained in J_1 (hence not in $J_1 \cap J_2$). Then I is adjacent with both J_2 and J_3, but not with J_1, a contradiction. Note that this argument also works for $k = -1$. $\qquad\square$

So we can distinguish graphs of type I and II from those of type III.

We now make two further simplifications.

First we split off the case $k = -1$ in type III graphs and forget about them, since their bipartite complements belong to type II.

Lemma 3.11 *Let $\Gamma = \Gamma_{i,j;k}^n(\mathbb{L})$ be a graph of type* III. *Then $k = -1$ if and only if the bipartite complement Γ^* of Γ admits Γ^*-round-up triples.*

Proof Clearly the bipartite complement of $\Gamma_{i,j;0}^n(\mathbb{L})$ is $\Gamma_{i,j;\geq 0}^n(\mathbb{L})$ and this admits $\Gamma_{i,j;\geq 0}^n(\mathbb{L})$-round-up triples, namely, regular round-up triples.

Now suppose $k \geq 0$. A Γ^*-round-up triple is, without loss, a triple $\{J_1, J_2, J_3\}$ of j-spaces such that no i-space intersects exactly one of J_1, J_2, J_3 in a k-space. We show that such a triple does not exist. Indeed, suppose for a contradiction that $\{J_1, J_2, J_3\}$ is such a triple. Then we select a point $p \in J_3 \setminus (J_1 \cup J_2)$ (this is possible since J_3 cannot be covered by the union of J_1 and J_2), and a k-space K in J_3 containing p. It is easy to see that we can find an i-space I intersecting J_3 in K and intersecting J_a, $a = 1, 2$, in $K \cap J_a$. Indeed, it suffices to find in $\mathrm{Res}(K)$ an $(i - k - 1)$ space avoiding the three subspaces $\langle J_1, K \rangle / K$, $\langle J_2, K \rangle / K$ and J_3 / K, which have dimension at most j, at most j, exactly $j - k - 1$, respectively. $\qquad\square$

Hence, from now on, we assume that type III graphs have $k \neq -1$.

Now we distinguish type I from type II graphs. So let Γ be a graph of type I or type II. Notice that it suffices to recognize when $i, j \in \{0, n-1\}$, since for type II graphs neither i nor j belongs to $\{0, n-1\}$ (since $0 \leq k < \min\{i, j\}$).

We define a new graph Γ_1 with vertex set, without loss, the set of j-spaces, with two j-spaces adjacent if they are contained in a Γ-round-up triple. This is the j-Grassmann graph of $\mathsf{PG}(n, \mathbb{L})$, and so two j-spaces are contained in a unique maximal clique if and only if $j \in \{0, n-1\}$. Hence, if in Γ_1 every pair of distinct vertices is contained in a unique maximal clique, then Γ has type I, and otherwise it has type II. If it has type I, then we can apply Proposition 3.3. Hence from now on, we only have to deal with graphs of type II and III.

We first handle the case of graphs of type II, thus finishing the proof of Main Result 2.2.

3.3 Proof of Main Result 2.2

The graph Γ_1 above determines $n, j, \mathbb{L}$, and likewise we can determine i. It does not directly determine k, although this can be proved indirectly. Nevertheless, we prefer to give a rather elegant algorithm to determine the parameters of Γ.

Remember that $\Gamma \cong \Gamma^n_{i,j;\geq k}(\mathbb{L})$, with $0 \leq k < \min\{i, j\} \leq n - \max\{i, j\} - 1$.

We define another graph Γ_2 with vertex set the set of j-spaces together with the set of maximal cliques of Γ_1. A j-space is adjacent to a maximal clique if it is contained in it, and two maximal cliques are adjacent if their intersection contains at least two Γ_1-adjacent j-spaces. This way, the tripartite graph isomorphic to $\Gamma^n_{[j-1,j+1]}(\mathbb{L})$ of $(j-1)$-, j- and $(j+1)$-spaces arises with natural adjacency (namely, incidence). But we do not know yet from our data which classes precisely correspond to $(j-1)$-spaces, and which to $(j+1)$-spaces.

ommon $(i+1)$-space. This way, we can construct the $(i-1)$-spaces as the maximal cliques which do not coincide with the i-neighborhood of an $(i+1)$-space in $\Gamma^n_{[i,j+1]}(\mathbb{L})$. Adding these maximal cliques with natural adjacency (containment) we obtain $\Gamma^n_{[i-1,j+1]}(\mathbb{L})$. If every pair of $(i-1)$-spaces have a common i-neighbor, then $i-1 = 0$. If not, then we continue until this happens. The number of steps we have to do this defines i uniquely. Similarly we can extend the graph $\Gamma^n_{[0,j+1]}(\mathbb{L})$ at the other side to gradually build $\Gamma^n_{[0,j+2]}(\mathbb{L})$, $\Gamma^n_{[0,j+3]}(\mathbb{L})$, etc., until we obtain $\Gamma^n_{[0,n-1]}(\mathbb{L})$. The number of steps this takes determines $(n-1) - (j+1)$ uniquely. Hence we have determined all parameters and reconstructed the entire projective space.

We note that, since $k < i$, every i-space is Γ-adjacent to all members of maximal cliques of both kinds. We define four different bipartite graphs Γ^ϵ_X, with $\epsilon \in \{-1, +1\}$ and $X \in \{\exists, \forall\}$, with one class of vertices each time the set of i-spaces. The other class of vertices of the two graphs Γ^{-1}_X, $X \in \{\exists, \forall\}$, is one class of Γ_2 distinct from the class corresponding to the j-spaces, and for the other two graphs we take the other class of Γ_2 distinct from the class of j-spaces. An i-space is $\Gamma^\epsilon_\exists$-adjacent to a clique of j-spaces if it is adjacent to some member of the clique, $\epsilon \in \{-1, +1\}$, and an i-space is $\Gamma^\epsilon_\forall$-adjacent to

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a clique of j-spaces if it is adjacent to all members of the clique, $\epsilon \in \{-1, +1\}$. To fix the ideas, we can choose the notation in such a way that the superscript -1 corresponds to cliques of j-spaces containing a fixed $(j-1)$-space, and the superscript $+1$ to cliques of j-spaces contained in a f ixed $(j+1)$-space (but this can for the moment not directly be derived from the graph Γ and the information we have up to now).

One easily checks the following isomorphisms:

$$\Gamma_\exists^- \cong \Gamma_{i,j-1;\geq k-1}^n(\mathbb{L}), \qquad \Gamma_\exists^+ \cong \Gamma_{i,j+1,\geq k}^n(\mathbb{L}),$$

$$\Gamma_\forall^- \cong \Gamma_{i,j-1,\geq k}^n(\mathbb{L}), \qquad \Gamma_\forall^+ \cong \Gamma_{i,j+1,\geq k+1}^n(\mathbb{L}).$$

Notice that we can repeat each of these four constructions for each of the four cases. But, starting from a minus graph, the appropriate plus graph brings us back to the original graph, and similarly for starting from a plus graph. Hence we can define with self-explaining notation the following graphs:

$$\Gamma_\exists^{-m} \cong \Gamma_{i,j-m;\geq k-m}^n(\mathbb{L}), \qquad \Gamma_\exists^{+m} \cong \Gamma_{i,j+m,\geq k}^n(\mathbb{L}),$$

$$\Gamma_\forall^{-m} \cong \Gamma_{i,j-m,\geq k}^n(\mathbb{L}), \qquad \Gamma_\forall^{+m} \cong \Gamma_{i,j+m,\geq k+m}^n(\mathbb{L}),$$

for all natural m for which the right hand side makes sense. It is clear that the smallest m for which $\Gamma_\exists^{-m}$ becomes complete bipartite is $m = k + 1$. Also, the smallest m for which $\Gamma_\exists^{+m}$ becomes complete bipartite is $m = n + k - i - j$. Since $k + 1 \leq n + k - i - j$, we can deduce k and $n - i - j$ from this information.

It is also clear that the smallest m for which $\Gamma_\forall^{-m}$ becomes an empty graph is $j - k + 1$, and the smallest m for which $\Gamma_\forall^{+m}$ becomes an empty graph is $i - k + 1$. Form this, we deduce j and i.

If we would have started with looking for Γ-round-up triples of i-spaces, then we would also have found first k, then $n - i - j - 1$, and then $i - k + 1$ and $j - k + 1$, which also determines all parameters uniquely.

Hence, in both cases, our method reveals i, j, k and n and Main Result 2.2 follows from this and the fact that $\Gamma_{[j-1,j+1]}^n(\mathbb{L})$ determines $\mathsf{PG}(n, \mathbb{L})$ completely.

We mention the following special case of Main Result 2.2. Suppose $-1 \leq k \leq j \leq n - 1$, $j \neq -1$, and let $\Gamma_{j;\geq k}^n(\mathbb{L})$ be the graph of j-spaces of $\mathsf{PG}(n, \mathbb{L})$ where two j-spaces are adjacent if their intersection contains a k-space, and let $\Gamma_{j;k}^n(\mathbb{L})$ be the graph of j-spaces of $\mathsf{PG}(n, \mathbb{L})$ where two j-spaces are adjacent if they intersect in a k-space.

Corollary 3.12 *Let $\mathbb{L}$ and $\mathbb{L}'$ be two skew fields, and let $-1 \leq k \leq j \leq n - 1$, $-1 \leq k' \leq j' \leq n' - 1$ be integers, with $-1 \notin \{j, j'\}$ and $n \geq 2$.*

(i) If $n + k \leq 2j$ or $k = -1$, then $\Gamma_{j;\geq k}^n(\mathbb{L})$ is a complete graph.

(ii) *If $j = k$, then $\Gamma^n_{j;\geq k}(\mathbb{L})$ is an empty graph.*

(iii) *If $\mathbb{L} \cong \mathbb{L}'$, $n = n'$, $j' = n - 1 - j$ and $k' = n - 1 + k - 2j$, then $\Gamma^n_{j;\geq k}(\mathbb{L}) \cong \Gamma^{n'}_{j';\geq k'}(\mathbb{L}')$.*

(iv) *If $k = 2j + 1 - n$, then $\Gamma^n_{j;\geq k}(\mathbb{L})$ is the complement of $\Gamma^n_{j;k-1}(\mathbb{L})$.*

(v) *If $2j \leq n - 1$, $-1 < k < j$ and $2j' \leq n' - 1$, then $\Gamma^n_{j;\geq k}(\mathbb{L}) \cong \Gamma^{n'}_{j';\geq k'}(\mathbb{L}')$ if and only if $\mathbb{L} \cong \mathbb{L}'$ and $(j, k, n) = (j', k', n')$. In this case every graph isomorphism is induced by a semi-linear bijection from $V_n(\mathbb{L})$ to $V_{n'}(\mathbb{L}')$, or possibly to $V_{n'}^*(\mathbb{L}')$ (the dual of $V_{n'}(\mathbb{L}')$) if $2j = n - 1$ and $\mathbb{L}' \cong (\mathbb{L}')^*$ (the latter is the opposite skew field).*

Proof   Recall that the (extended) bipartite double 2Γ ($\overline{2}\Gamma$) of a given graph Γ is obtained by taking two copies of the vertex set Γ (without the edges) and defining a vertex of one copy to be incident with a vertex of the other copy if the corresponding vertices of Γ are (equal or) adjacent in Γ.

Then the corollary follows easily from Main Result 2.2 and the observation that $\overline{2}\Gamma^n_{j;\geq k}(\mathbb{L}) \cong \Gamma^n_{j,j;\geq k}(\mathbb{L})$ and $2\Gamma^n_{j;k}(\mathbb{L}) \cong \Gamma^n_{j,j;k}(\mathbb{L})$ (the latter is needed for (iv)). $\qquad \square$

We now go on with graphs of type III.

3.4   Proof of Main Result 2.1

Now we prove Main Result 2.1. So $\Gamma \cong \Gamma^n_{i,j;k}(\mathbb{L})$, with $i + j \leq n - 1$ and $0 \leq k < \min\{i, j\}$. Remember that $k \neq -1$ because of Lemma 3.11.

Our proof is again symmetric in i and j. Hence throughout one can interchange i and j. The method used in the previous subsection with round-up triples does not work because of Lemmas 3.9 and 3.10. But we now use the idea of the Γ-round-up quadruples and show that every such quadruple is a regular round-up quadruple. This proof is considerably more involved, although the main structure is kept, and some results can be proved using a slight generalization.

It is easy to show that every regular round-up quadruple is a Γ-round-up quadruple. Hence we show the converse.

We start with a lemma which can be seen as the counterpart to Observation 3.6.

Lemma 3.13 *Let $\{J_1, J_2, J_3, J_4\}$ be a Γ-round-up quadruple. Then the following holds for arbitrary $\ell_1, \ell_2, \ell_3, \ell_4$ with $\{\ell_1, \ell_2, \ell_3, \ell_4\} = \{1, 2, 3, 4\}$. Whenever K_1 and K_2 are k-spaces in J_{ℓ_1} and J_{ℓ_2}, respectively, with $\langle K_1, K_2 \rangle \cap J_{\ell_a} = K_a$, for $a \in \{1, 2\}$, and with $\dim\langle K_1, K_2 \rangle \leq i$, then the subspace $\langle K_1, K_2 \rangle$ intersects at least one of J_{ℓ_3}, J_{ℓ_4} in precisely a k-space.*

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Proof Assume for a contradiction that the assertion is false. By renumbering if necessary we may then assume that K_1 and K_2 are k-spaces contained in J_1, J_2, respectively, and that none of $\langle K_1, K_2 \rangle \cap J_3$ and $\langle K_1, K_2 \rangle \cap J_4$ is k-dimensional.

Our main goal is to construct an $(i-1)$-space I' containing $\langle K_1, K_2 \rangle$ and intersecting, say, J_3 in a $(k-1)$-space, while $\dim(I' \cap J_4) \neq k$.

We consider an arbitrary i-space I intersecting J_1 in K_1 and J_2 in K_2 (which exists by applying Lemma 3.1 in $\mathrm{Res}(\langle K_1, K_2 \rangle)$). Since $\{J_1, J_2, J_3, J_4\}$ is a Γ-round-up quadruple, we know that I intersects, say, J_3 in a k-space K_3. We will construct I' as a hyperplane of I. Our construction depends on $\dim(I \cap J_4)$.

First case: $\dim(I \cap J_4) \notin \{k, k+1\}$.
This is an easy case. Indeed, first note that $K_3 \not\subseteq \langle K_1, K_2 \rangle$ by our main assumption. We can thus consider a hyperplane I' of I containing $\langle K_1, K_2 \rangle$ and not containing K_3.

Second case: $I \cap J_4 = K_4$ is k-**dimensional.**
In this case we know that none of K_3, K_4 is contained in $\langle K_1, K_2 \rangle$, and hence $\langle K_1, K_2 \rangle$ is not a hyperplane of I, as otherwise $I' = \langle K_1, K_2 \rangle$ satisfies our needs. So we can first choose an $(i-2)$-space I'' through $\langle K_1, K_2 \rangle$ neither containing K_3, nor K_4. If $\langle I'', K_3, K_4 \rangle$ has dimension $i-1$, then a hyperplane of I through I'' distinct from $\langle I'', K_3, K_4 \rangle$ does the job. If $\langle I'', K_3, K_4 \rangle = I$, then we can consider a line L intersecting K_3 and K_4 in respective distinct points and not intersecting I''. Now the space $\langle I'', p \rangle$, with p a point on L not in $K_3 \cup K_4$ is an $(i-1)$-space meeting our conditions.

Third case: $\dim(I \cap J_4) = k+1$.
Put $R_4 = I \cap J_4$. If K_3 is not contained in $\langle K_1, K_2, R_4 \rangle$, then we find a hyperplane I' of I through $\langle K_1, K_2, R_4 \rangle$ not containing K_3 and we are done. So we may assume that K_3 is contained in $\langle K_1, K_2, R_4 \rangle$. As in the previous case (for instance, considering a k-space in R_4), it is easy to see that we can find a hyperplane I_1 of I containing K_1, K_2, not containing K_3 and intersecting R_4 in a k-space K_4. Notice that K_4 is not contained in $\langle K_1, K_2 \rangle$ by assumption, so we can find an $(i-2)$-space I'_1 containing $\langle K_1, K_2 \rangle$ and intersecting K_4 in a $(k-1)$-space K'_4. Hence $I'_1 \cap J_1 = K_1$, $I'_1 \cap J_2 = K_2$, $\dim(I'_1 \cap J_3) \leq k-1$ and $\dim(I'_1 \cap J_4) = k-1$. In $\mathrm{Res}(I'_1)$, the spaces $\langle J_1, I'_1 \rangle / I'_1$, $\langle J_2, I'_1 \rangle / I'_1$ and $\langle J_4, I'_1 \rangle / I'_1$ have dimension $j-k-1$, $j-k-1$ and $j-k$, respectively. Since $j-k < n-i$, we can find a point in $\Re s(I'_1)$, which avoids these three subspaces, and also avoids $\langle J_3, I'_1 \rangle / I'_1$ in case $\dim(I'_1 \cap J_3) = k-1$ (the union of two hyperplanes and two subhyperplanes is never the entire space for $|\mathbb{L}| > 2$). That point corresponds to an $(i-1)$-space I', which does the trick, interchanging the roles of J_3 and J_4.

So in all three cases, we found an $(i-1)$-space I', and in $\mathrm{Res}(I')$, the spaces $\langle J_1, I' \rangle / I'$, $\langle J_2, I' \rangle / I'$ and $\langle J_3, I' \rangle / I'$ have dimension $j-k-1$, $j-k-1$ and $j-k$, respectively. As in the third case above, we can find a point in $\mathrm{Res}(I')$ which avoids these three subspaces, and which also avoids $\langle J_4, I' \rangle / I'$ in case $\dim(I' \cap J_4) = k-1$. This point

corresponds to an i-space I^* intersecting J_1 and J_2 in k-spaces, intersecting J_3 in a $(k-1)$-space, and intersecting J_4 in a space of dimension distinct from k. This contradiction to $\{J_1, J_2, J_3, J_4\}$ being a Γ-round-up quadruple concludes the proof of the lemma. $\qquad\square$

We now show that the J_a, $a \in \{1, 2, 3, 4\}$, pairwise intersect in a common space. This is the analogue of Lemma 3.8.

Lemma 3.14 *If $\{J_1, J_2, J_3, J_4\}$ is a Γ-round-up quadruple, then $J_1 \cap J_2 = J_2 \cap J_3 = J_3 \cap J_4 = J_1 \cap J_3 = J_2 \cap J_4 = J_1 \cap J_4$.*

Proof   Let the maximum dimension of $J_a \cap J_b$, $\{a, b\} \subseteq \{1, 2, 3, 4\}$, $a \neq b$, be ℓ. There are two possibilities.

1. Suppose $\ell \geq k$. We may assume $\dim(J_1 \cap J_2) = \ell$. By Lemma 3.13, every k-space contained in $J_1 \cap J_2$ is also contained in $J_3 \cup J_4$. Since $J_1 \cap J_2$ cannot be the union of $J_1 \cap J_2 \cap J_3$ and $J_1 \cap J_2 \cap J_4$ unless one of the latter two spaces coincides with $J_1 \cap J_2$, we may assume that $J_1 \cap J_2 \subseteq J_3$. By maximality of ℓ, we already have $J_1 \cap J_2 = J_3 \cap J_1 = J_2 \cap J_3$. Now we choose a k-space K in $J_1 \cap J_2$ and a point $p \in J_3 \setminus J_1$. Applying Lemma 3.1 in $\mathrm{Res}(\langle K, p \rangle)$, we find an i-space I_p containing $\langle K, p \rangle$, intersecting J_1 and J_2 precisely in K, intersecting J_3 in $\langle K, p \rangle$, and disjoint from $J_4 \setminus J_3$ (note that $\dim \mathrm{Res}(\langle K, p \rangle) = n - k - 2$, $\dim \langle J_1, p \rangle / \langle K, p \rangle = \dim \langle J_2, \rangle / \langle K, p \rangle = j - k - 1$, $\dim J_3 / \langle K, p \rangle = j - k - 2$, $\dim \langle K, J_4, p \rangle / \langle K, p \rangle \leq j$, and $\dim I_p / \langle K, p \rangle = i - k - 2$).

   It follows that $I_p \cap J_4$ is a k-space. But $I_p \cap J_4 \subseteq \langle K, p \rangle$, by construction. So $J_4 \cap J_3$ contains a k-space contained in $\langle K, p \rangle$. If $J_2 \cap J_3 \cap J_4$ has dimension $\leq \ell - 2$, then we can choose K such that $K \cap J_4$ is at most $(k-2)$-dimensional, a contradiction. Hence $J_2 \cap J_3 \cap J_4$ has dimension $\ell - 1$ (if it had dimension ℓ, the lemma was proved). So $\langle J_2 \cap J_3, J_3 \cap J_4 \rangle$ has dimension at most $\ell + 1$. If $\dim \langle J_2 \cap J_3, J_3 \cap J_4 \rangle < j$, then by picking p outside $\langle J_2 \cap J_3, J_3 \cap J_4 \rangle$, we again obtain a contradiction. Hence $\ell + 1 = j$ and so $\dim(J_3 \cap J_4) = \ell$. By maximality of ℓ we can repeat the argument of the previous paragraph, and conclude that either J_1 or J_2 must contain $J_3 \cap J_4$, clearly a contradiction. Hence $J_2 \cap J_3 \cap J_4$ is ℓ-dimensional after all, proving the lemma in this case.

2. Suppose $\ell < k$. Let $m \leq \ell$ be the maximal dimension of $J_1 \cap J_a$, $a = 2, 3, 4$ and suppose $\dim(J_1 \cap J_2) = m$. Let K_b be any k-space in J_b containing $J_1 \cap J_2$, $b = 1, 2$. By Lemma 3.13, the subspace $\langle K_1, K_2 \rangle$ intersects one of J_3, J_4, say J_3, in a k-subspace K_3. As in the second part of the proof of Lemma 3.8, we deduce $J_1 \cap J_3 = K_1 \cap K_3$. If this equals $J_1 \cap J_2$, then fine. Otherwise we choose another k-space K_1' in J_1 containing $J_1 \cap J_2$ and not containing $K_1 \cap K_3$. Then either the space $\langle K_1', K_2 \rangle$ intersects J_3 in a k-space (and then the same argument as in the second

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part of the proof of Lemma 3.8 leads to $J_1 \cap J_2 = J_2 \cap J_3 = J_3 \cap J_1$), or $\langle K_1', K_2 \rangle$ intersects J_4 in a k-space K_4. Then $J_1 \cap J_4 = K_1' \cap K_4$. If the latter equals $J_1 \cap J_2$, then fine again. Otherwise we choose a k-space K_1'' in J_1 containing $J_1 \cap J_2$ and neither containi ng $K_1 \cap K_3$ nor $K_1' \cap K_4$. Then $\langle K_1'', K_2 \rangle$ intersects one of J_3, J_4, say J_3, in a k-space, and the same argument as above leads to $J_1 \cap J_2 = J_2 \cap J_3 = J_3 \cap J_1$.

Hence in any case we may assume that $J_1 \cap J_2 = J_2 \cap J_3 = J_3 \cap J_1$ (possibly by interchanging the roles of J_3 and J_4). Now let f be the maximal dimension of $J_4 \cap J_c$, $c = 1, 2, 3$. If $f = -1$ and $m \geq 0$, then the second part of the proof of Lemma 3.9 can be copied and applied to J_1, J_2, J_3 and thus yields a contradiction. If $f = m = -1 = \ell$, then the lemma is proved. Hence we may assume that $f \geq 0$ and $m \geq 0$ by symmetry. By the arguments above (interchanging J_1 and J_4, and f and m), we may assume that $J_2 \cap J_3 = J_3 \cap J_4 = J_4 \cap J_2$. But then $m = f = \ell$ and by maximality of ℓ, $J_1 \cap J_4$ is not bigger than $J_1 \cap J_2$, hence coincides with it.

The lemma is proved. $\qquad\square$

We can modify the proof of Lemma 3.9 to prove the following lemma.

Lemma 3.15 *Every Γ-round-up quadruple $\{J_1, J_2, J_3, J_4\}$ is a regular round-up quadruple.*

Proof   By Lemma 3.14, we know $J_a \cap J_b = J_c \cap J_d$, for all $a, b, c, d \in \{1, 2, 3, 4\}$, $a \neq b$ and $c \neq d$. Put $\ell = \dim(J_1 \cap J_2)$. There again are two possibilities to consider separately.

1. Assume $\ell \geq k$. Let $p_a \in J_a \setminus J_3$ be arbitrary, $a = 1, 2$. For $a = 1, 2$, choose a k-space K_a containing p_a and intersecting $J_1 \cap J_2$ in a $(k-1)$-space, with the latter independent of a. Lemma 3.13 implies that the $(k+1)$-space $\langle K_1, K_2 \rangle$ intersects either J_3 or J_4 in k-space K intersecting $J_1 \cap J_2$ in $K_1 \cap K_2$. Hence the line $p_1 p_2$ intersects K, and hence either J_3 or J_4, in a point. Lemma 3.5 shows the assertion.

2. As in the proof of Lemma 3.9, we prove that the situation $\ell < k$ leads to a contradiction. Indeed, the argument is similar to the argument in the second part of the proof of Lemma 3.9, but we consider three k-subspaces K_1, K_1', K_1'' in J_1 through $J_1 \cap J_2$ contained in a common $(k+1)$-space, choose a k-space $K_2 \subseteq J_2$ with $J_1 \cap J_2 \subseteq K_2$ and note that at least two of $\langle K_1, K_2 \rangle$, $\langle K_1', K_2 \rangle$, $\langle K_1'', K_2 \rangle$ intersect the same subspace among J_3, J_4 in k-spaces. Then we argue as in the second part of the proof of Lemma 3.9 and derive a contradiction.

This completes the proof of the lemma. $\qquad\square$

As in the previous subsection, we can now again define Γ_1 with vertex set the set of j-spaces, and adjacency being contained in a common Γ-round-up quadruple. The graph Γ_1 has two distinguishable kinds of maximal cliques which we use to define the graph Γ_2, isomorphic to $\Gamma^n_{[j-1,j+1]}(\mathbb{L})$. Recall that Γ_2 is tripartite and that the triparts correspond to j-subspaces and the two kinds of maximal cliques in Γ_1. We define a j-*Grassmann line* as the set of j-spaces contained in a given Γ-round-up quadruple Q, or forming such a quadruple with any three members of Q. Alternatively, these are the intersections of a member of one class with a member of the other class of maximal cliques of Γ_1 if this intersection contains at least two elements.

Let I be an i-space intersecting some j-space J of a given maximal clique Q in a k-space K. There are four possibilities.

(1) The clique Q consists of all j-spaces through a given $(j-1)$-space J' and $K \subseteq J'$. Then I is Γ-adjacent to either none, all, or all but one members of an arbitrary j-Grassmann line in Q.

(2) The clique Q consists of all j-spaces through a given $(j-1)$-space J' and $K \cap J'$ has dimension $k-1$. Then I is Γ-adjacent to either none, exactly one, or all members of an arbitrary j-Grassmann line in Q.

(3) The clique Q consists of all j-spaces in a given $(j+1)$-space J'' and $\dim(I \cap J'') = k+1$. Then I is Γ-adjacent to either none, all, or all but one members of an arbitrary j-Grassmann line in Q.

(4) The clique Q consists of all j-spaces in a given $(j+1)$-space J'' and $I \cap J'' = K$. Then I is Γ-adjacent to either none, exactly one, or all members of an arbitrary j-Grassmann line in Q.

If we define the bipartite graph with vertices at the one hand the i-spaces and at the other hand the maximal cliques of fixed type, and we define an i-space I and a clique Q to be adjacent if I is Γ-adjacent to either none, all, or all but one members of an arbitrary j-Grassmann line in Q, then in Case (1) we denote the graph $\Gamma^{-1}_{\forall_{-1}}$ and it is isomorphic to $\Gamma^n_{i,j-1;k}(\mathbb{L})$; in Case (3) we denote it by $\Gamma^{+1}_{\forall_{-1}}$ and it is isomorphic to $\Gamma^n_{i,j+1;k+1}(\mathbb{L})$.

If we now define an i-space I and a clique Q to be adjacent if I is Γ-adjacent to either none, exactly one, or all members of an arbitrary j-Grassmann line in Q, then in Case (2) we denote the graph $\Gamma^{-1}_{\exists_1}$ and it is isomorphic to $\Gamma^n_{i,j-1;k-1}(\mathbb{L})$; in Case (4) we denote it by $\Gamma^{+1}_{\exists_1}$ and it is isomorphic to $\Gamma^n_{i,j+1;k}(\mathbb{L})$.

We can again repeat these four constructions to the appropriate graphs and obtain, with self-explaining notation, the following isomorphisms.

$$\Gamma^{-m}_{\exists_1} \cong \Gamma^n_{i,j-m;k-m}(\mathbb{L}), \qquad \Gamma^{+m}_{\exists_1} \cong \Gamma^n_{i,j+m,k}(\mathbb{L}),$$

$$\Gamma^{-m}_{\forall_{-1}} \cong \Gamma^n_{i,j-m,k}(\mathbb{L}), \qquad \Gamma^{+m}_{\forall_{-1}} \cong \Gamma^n_{i,j+m,k+m}(\mathbb{L}),$$

for all natural m for which these make sense.

The smallest m for which the next graph in the series $\Gamma_{\exists_1}^{-m'}$ cannot be defined (since every i-space adjacent to at least one member of the appropriate maximal clique, is always adjacent to all but one members of a $(j - m - 1)$-Grassmann line), is $m = k + 1$. Similarly, the smallest m for which the next graph in the series $\Gamma_{\exists_1}^{+m'}$ becomes void is $m = n + k - i - j$, since an i-space and an $(n + k - i + 1)$-space inside an n-space always meet in at least a $(k + 1)$-space (and this does not hold for the previous step!). As in the previous section, this determines k and $n - i - j$ uniquely.

Now completely similar to the previous section, the smallest value of m for which $\Gamma_{\forall_{-1}}^{-m}$ becomes an empty graph is $j - k + 1$, and the smallest value of m for which $\Gamma_{\forall_{-1}}^{+m}$ becomes empty is $i - k + 1$. This determines i and j, and so we determined all values i, j, k, n.

The proofs of the Main Results are complete now.

We mention the following special case of Main Result 2.1. Suppose $-1 \le k \le j \le n - 1$, $j \neq -1$ and recall that $\Gamma_{j;k}^n(\mathbb{L})$ denotes the graph of j-spaces of $\mathsf{PG}(n, \mathbb{L})$ where two j-spaces are adjacent if they intersect in a k-space.

Corollary 3.16 *Let $\mathbb{L}$ and $\mathbb{L}'$ be two skew fields, and let $-1 \le k \le j \le n - 1$, $-1 \le k' \le j' \le n' - 1$ be integers, with $-1 \notin \{j, j'\}$ and $n \ge 2$.*

   (i) If $j = n - 1$ and $k = n - 2$, or $j = 0$ and $k = -1$, then $\Gamma_{j;k}^n(\mathbb{L})$ is a complete graph.

   (ii) If $n + k < 2j$ or $j = k$, then $\Gamma_{j;k}^n(\mathbb{L})$ is an empty graph.

   (iii) If $\mathbb{L} \cong \mathbb{L}'$, $n = n'$, $j' = n - 1 - j$ and $k' = n - 1 + k - 2j$, then $\Gamma_{j;k}^n(\mathbb{L}) \cong \Gamma_{j';k'}^{n'}(\mathbb{L}')$.

   (iv) If $2j \le n - 1$, $-1 \le k < j$, $(k, j) \neq (-1, 0)$ and $2j' \le n' - 1$, then $\Gamma_{j;k}^n(\mathbb{L}) \cong \Gamma_{j';k'}^{n'}(\mathbb{L}')$ if and only if $\mathbb{L} \cong \mathbb{L}'$ and $(j, k, n) = (j', k', n')$. In this case every graph isomorphism is induced by a semi-linear bijection from $V_n(\mathbb{L})$ to $V_{n'}(\mathbb{L}')$, or possibly to $V_{n'}^*(\mathbb{L}')$ (the dual of $V_{n'}(\mathbb{L}')$) if $2j = n - 1$ and $\mathbb{L}' \cong (\mathbb{L}')^*$ (the latter is the opposite skew field).

Proof Again by taking bipartite double of the graphs in question and then applying Main Result 2.1. $\qquad\square$

Remark 3.17 In the statements of all our results, we may substitute "graph isomorphism" by "graph epimorphism", where *epimorphism* means a surjective mapping on the vertices such that two vertices in the source graph are adjacent if and only if their images are adjacent. Indeed, if there would exist a non-bijective graph epimorphism, then there are two vertices in the source graph with exactly the same neighborhood set. But it is trivial to see that this is impossible for all the graphs we defined, as soon as they are nontrivial.

4 The finite and the thin case

4.1 The finite case

In the finite case, there is a shortcut to prove our main results, though very un-algorithmic, but rather efficient.

It follows from the main result and Table VI in [10] that $\mathsf{P\Gamma L}(n+1, q)$, extended with a duality whenever possible, is a maximal subgroup of the symmetric or alternating group $\mathsf{Sym}(N), \mathsf{Alt}(N)$, where N is the number of j-spaces of $\mathsf{PG}(n, q)$, except possibly when $q \in \{2, 3\}$ and $N = q^{m-1}(q^m - 1)/(2, q - 1)$. But in this case, N can never be the number of j-spaces of $\mathsf{PG}(n, q)$, for any j, since this number is never divisible by q. Hence, if the automorphism group of the graph $\Gamma^n_{i,j;k}(q)$ or $\Gamma^n_{i,j;\geq k}(q)$, with the appropriate restrictions to make the graph nontrivial, were larger than the corresponding group induced by semi-linear permutations, then the full symmetric or alternating group would act on at least one of the bipartition classes. We now claim that this implies that the graph is complete bipartite, which is a contradiction. Indeed, we have the following lemma.

Lemma 4.1 *Let v_1, v_2 be two distinct vertices of one of the nontrivial graphs $\Gamma^n_{i,j;k}(q)$ or $\Gamma^n_{i,j;\geq k}(q)$ in the same bipart, then there are at least two neighbors of v_1 not adjacent to v_2.*

Proof For $\Gamma^n_{i,j;k}(q)$, $k > -1$, and $\Gamma^n_{i,j;\geq k}(q)$, with $i + j \leq n + 1$, this follows immediately from the easy fact that, for every pair v_1, v_2 of j-spaces (i-spaces) there exists a pair w_1, w_2 of k-spaces contained in v_1 but not in v_2. For $\Gamma^n_{i,j;-1}(q)$, this follows from the previous fact for $\Gamma^n_{i,j;0}(q)$ and with the roles of v_1 and v_2 interchanged. $\qquad\square$

Now let v be a vertex of $\Gamma^n_{i,j;k}(q)$ or $\Gamma^n_{i,j;\geq k}(q)$. Let w be a neighbor of v, and let w' be an arbitrary vertex in the same bipart as w, but not adjacent to v. Let θ be an automorphism of the graph stabilizing the set of neighbors of v distinct from w, and mapping w onto w' (which exists since the alternating group acts as an automorphism group on each bipart). By the previous lemma, v is fixed, and so v is adjacent to w'. Varying v and w', we see that the graph is complete bipartite, proving our claim.

This in particular proves Main Result 2.1 for $|\mathbb{L}| = q = 2$. The graphs $\Gamma^n_{i,j;k}(2)$ and $\Gamma^n_{i,j;\geq k}(2)$, with the appropriate restrictions, are not isomorphic to any other graph of type III because it has a different automorphism group. These graphs are mutually non-isomorphic because they pairwise have different orders (cardinalities of the biparts) and bi-valences (we leave the details to the reader).

4.2 The finite thin case

The same argument works in the *thin* case, except that $\mathsf{Sym}(2n-1)$ acting on all $(n-1)$-subsets of a $(2n-1)$-set is not maximal in $\mathsf{Sym}(\frac{1}{2}\binom{2n}{n})$ (indeed, it is contained in $\mathsf{Sym}(2n)$ acting on partitions of sizes n,n of a $2n$-set), giving rise to a counter example, and there are three other small exceptions, one of which also gives rise to a counter example. We now provide the details.

Let $\Omega^{(n)}$ be an n-set and let Ω_j be the set of all j-subsets of Ω, $0 \leq j \leq n$. Suppose i,j,k are integers with $0 \leq k \leq i \leq j \leq n-1$ and let $\Gamma^n_{i,j;\geq k}$ and $\Gamma^n_{i,j;k}$ be the bipartite graphs with biparts Ω_j and Ω_i, where a j-subset is adjacent to an i-subset if their intersection is of size at least k and precisely k, respectively. For convenience, we also denote $\Gamma^n_{i,j;k}$ by $\Gamma^n_{j,i;k}$ and $\Gamma^n_{i,j;\geq k}$ by $\Gamma^n_{j,i;\geq k}$, $0 \leq k \leq i \leq j \leq n-1$.

Theorem 4.2 *Let $0 \leq k \leq i \leq j \leq n-1$ and $0 \leq k' \leq i' \leq j' \leq n'-1$ be integers with $0 \notin \{i,i'\}$ and $n \geq 2$.*

(i) *If $i = j = k \geq 1$, then $\Gamma^n_{i,j;k}$ and $\Gamma^n_{i,j;\geq k}$ are matchings; if $i+j = n$ and $k = 0$, then $\Gamma^n_{i,j;k}$ is a matching.*

(ii) *If $i = j = n-1 = k+1$ or $i = j = 1 = k+1$, then $\Gamma^n_{i,j;k}$ is the complement of a matching.*

(iii) *If $n+k < i+j$, then $\Gamma^n_{i,j;k}$ is an empty graph; if $n+k \leq i+j$ or $k = 0$, then $\Gamma^n_{i,j;\geq k}$ is a complete bipartite graph.*

(iv) *If $k = i+j+1-n$, then $\Gamma^n_{i,j;\geq k}$ is the bipartite complement of $\Gamma^n_{i,j;k-1}$; if $i = k$, then $\Gamma^n_{i,j;\geq k} \cong \Gamma^n_{i,j;k}$.*

(v) *If $n = n'$, $j' = n-j$ and $k' = i-k$, then $\Gamma^n_{i,j;k} \cong \Gamma^{n'}_{i,j';k'}$ and $\Gamma^n_{i,j;\geq k}$ is the bipartite complement of $\Gamma^{n'}_{i,j';\geq k'+1}$.*

(vi) *If $i,j \leq n/2$, $k < j$, $k \leq i/2$ if $j = n/2$, $i',j' \leq n'/2$, $k \leq i'/2$ if $j' = n'/2$, then $\Gamma^n_{i,j;k} \cong \Gamma^{n'}_{i',j';k'}$ if and only if $(i,j,k,n) = (i',j',k',n')$. In this case every graph isomorphism is induced by a bijection from $\Omega^{(n)}$ to $\Omega^{(n')}$ (this is also trivially true for $(i,j,k) = (i',j',k') = (1,1,1)$), except if $(n,i,j,k) = (6,2,2,1)$, where one can compose with an arbitrary collineation belonging to $\mathsf{L}_4(2)$ acting on the points of $\mathsf{PG}(3,2)$, and except if $(n,i,j,k) = (4n^*-1, 2n^*-1, 2n^*-1, n^*-1)$, with $n^* \geq 2$ an integer, where the full automorphism group is $\mathsf{Sym}(4n^*)$ acting on the $(2n^*, 2n^*)$-partitions of a $4n^*$-set.*

(*vii*) *If $i, j \leq n/2$, $0 \neq k < j$, $k \leq (i+1)/2$ if $j = n/2$, $i', j' \leq n'/2$, and $k \leq (i'+1)/2$ if $j' = n'/2$, then $\Gamma^n_{i,j;\geq k} \cong \Gamma^{n'}_{i',j';\geq k'}$ if and only if $(i, j, k, n) = (i', j', k', n')$. In this case every graph isomorphism is induced by a bijection from $\Omega^{(n)}$ to $\Omega^{(n')}$ (the latter is also trivially true for $(i, j, k) = (i', j', k') = (1, 1, 1)$).*

Proof   The assertions (*i*) up to (*v*) are easy to check. We now prove (*vi*) and (*vii*). As before, it suffices, under the given conditions, to reconstruct (i, j, k, n) from the graphs $\Gamma^n_{i,j;k}$ and $\Gamma^n_{i,j;\geq k}$.

Suppose we know that the automorphism group of $\Gamma^n_{i,j;k}$ or $\Gamma^n_{i,j;\geq k}$ is $\mathsf{Sym}(n)$. Then we already know n. From the sizes of the biparts we deduce i and j, and then the bi-valence of the graph reveals k. Hence it suffices to show that the automorphism group of $\Gamma^n_{i,j;k}$ and $\Gamma^n_{i,j;\geq k}$ is $\mathsf{Sym}(n)$.

Noting that no two vertices of $\Gamma^n_{i,j;k}$ or $\Gamma^n_{i,j;\geq k}$ have exactly the same set of neighbors, it suffices to show that the automorphism group induced on one of the biparts is $\mathsf{Sym}(n)$. This follows immediately if $\mathsf{Sym}(n)$ is a maximal subgroup of the symmetric group acting on $\binom{n}{j}$ or $\binom{n}{i}$ letters. Now according to [10], this is the case, except possibly in the following cases (under the restrictions of (*vi*) and (*vii*) of Theorem 4.2):

$$(n, i, j) \in \{(6, 2, 2), (10, 3, 3), (12, 4, 4), (2\ell - 1, \ell - 1, \ell - 1) : \ell \in \mathbb{N}, \ell \geq 3\}.$$

- $(n, i, j) = (6, 2, 2)$.

  This is an easy case, as up to taking bipartite complement, we may suppose we have $\Gamma^6_{2,2;0}$ or $\Gamma^6_{2,2;1}$ (and we can distinguish these by their valences, which are 6 and 8, respectively). For $\Gamma^6_{2,2;0}$, two vertices of the same bipart correspond to disjoint pairs if they have exactly one neighbor, and to intersecting pairs if they have three common neighbors. If we define a graph on one bipart by declaring two vertices adjacent if they have three common neighbors in $\Gamma^6_{2,2;0}$, then we can recover the 3-subsets as the maximal cliques of size 3 of that graph. Hence we can derive $\Gamma^6_{2,3;2}$, which is not in our list of exceptions, and the result follows. For $\Gamma^6_{2,2;1}$, we note that the vertices of one bipart adjacent to a vertex of the other bipart can be considered as the points of the unique generalized quadrangle of order 2 opposite (non-collinear to) a given point. In $\mathsf{PG}(3, 2)$, these sets are just all complements of planes, hence $\mathsf{L}_4(2)$ acts on $\Gamma^6_{2,2;1}$, and $\mathsf{Sym}(6)$ is contained in it as $\mathsf{Sp}_4(2)$.

- $(n, i, j) = (10, 3, 3)$.

  In the case there are essentially four different cases: $\Gamma^{10}_{3,3;k}$, for $k = 0, 1, 2$, and $\Gamma^{10}_{3,3;\geq 2}$. These graphs can again be distinguished by their valences (these are 35, 63, 21 and 22, respectively). In the cases $k = 0, 2$, the number of common neighbors of two triples in the same bipart is equal to $4, 10$ or 20, and to $0, 4$ or 8, respectively,

according to whether the triples have exactly no, exactly one, or exactly two points in common. Hence on each bipart the structure of $\Gamma_{3;2}^{10}$ can be recovered. The maximal cliques of size 8 in that graph correspond with pairs of $\Omega(10)$, and so we can uniquely build $\Gamma_{3,2;2}^{10}$, which is not in our list of possible exceptions.

If $k = 1$, an interesting phenomenon occurs: the graph on any bipart where two vertices are adjacent when they have exactly 30 neighbors is a strongly regular graph $(120, 63, 30, 36)$. Hence we can only recover $\Gamma_{3;1}^{10}$. But the maximal cliques of size 7 of that graph clearly correspond to 7-sets of $\Omega(10)$ (where they induce a Fano plane), and there is one vertex not adjacent to any seven of these vertices. Such a vertex corresponds to a triple which is disjoint from all triples in the maximal clique. So we can recover when two triples are disjoint, after all, and hence when they share 2 elements, too. The argument of the previous paragraph now applies.

Concerning $\Gamma_{3,3;\geq 2}^{10}$, the number of common neighbors of two triples in the same bipart is equal to $0, 4$ or 10, respectively, according to whether the triples have exactly no, exactly one, or exactly two points in common. The same argument as above now applies.

- $(n, i, j) = (12, 4, 4)$.

  Here are essentially six possibilities, namely $\Gamma_{4,4;k}^{12}$, $k = 0, 1, 2, 3$, and $\Gamma_{4,4;\geq \ell}^{12}$, $\ell = 2, 3$. The valences again tell us these cases apart (the valences are 70, 224, 168, 32, 201 and 33, respectively). In the following table, where the rows and columns of the upper 4×4 square are numbered from 0 to 3, the (i, j)-entry is the number of common neighbors of two vertices of the same bipart in $\Gamma_{4,4;i}^{12}$ which correspond to 4-sets of $\Omega(12)$ intersecting in a j-set. The two bottom rows tell us in the jth position exactly the same for the graphs $\Gamma_{4,4;\geq \ell}^{12}$, $\ell = 2, 3$. The labels are written on the left and the top and are self-explaining.

	0	1	2	3
0	1	5	15	35
1	96	100	100	126
2	36	54	64	84
3	0	0	4	10
≥ 2	36	72	102	138
≥ 3	0	0	4	12

It follows from the last column of this table that in each bipart we can uniquely and unambiguously construct $\Gamma_{4;3}^{12}$ (the vertices are the 4-sets of $\Omega^{(12)}$ and two 4-sets are adjacent if they intersect in a 3-set). It is easy to see that we can identify the 3-subsets with the maximal cliques of $\Gamma_{4;3}^{12}$ consisting of 9 vertices. Hence we can build $\Gamma_{3,4;3}^{12}$, which is not in the list of possible exceptions.

- $(n, i, j) = (2\ell - 1, \ell - 1, \ell - 1)$, $\ell \in \mathbb{N}, \ell \geq 3$.

  In this case, it follows from Table III of [10] that the only way in which $\mathsf{Sym}(2\ell - 1)$ is not the full automorphism group of $\Gamma^n_{i,j;k}$ or $\Gamma^n_{i,j;\geq k}$ is when $\mathsf{Sym}(2\ell)$ is the full automorphism group and we can represent the vertices of our graph as the (ℓ, ℓ)-partitions of $\Omega^{(2\ell)}$, with "induced" adjacency. We now take a closer look at this induced adjacency. Suppose two $(\ell - 1)$-sets of $\Omega^{(2\ell-1)}$ intersect in a set of size m. Then the partition classes of the corresponding (ℓ, ℓ)-partitions of $\Omega^{(2\ell)}$ intersect in sets of sizes $m + 1$ and $\ell - m - 1$. Conversely, if the partition classes of two (ℓ, ℓ)-partitions of $\Omega^{(2\ell)}$ intersect in sets of sizes $m + 1$ and $\ell - m - 1$, then the corresponding $(\ell - 1)$-sets of $\Omega^{(2\ell-1)}$ intersect in a set of size either m or $\ell - m - 2$.

  From this we easily deduce that the automorphism group of a graph $\Gamma^n_{i,j;k}$ or $\Gamma^n_{i,j;\geq k}$ is $\mathsf{Sym}(2\ell)$ if and only if whenever two vertices are adjacent as soon as the corresponding subsets intersect in a set of size m, then the two vertices corresponding to subsets intersecting in a set of size $\ell - m - 2$ are also adjacent. This is never the case for $\Gamma^n_{i,j;\geq k}$ and only the case for $\Gamma^n_{i,j;k}$ if $k = \ell - k - 2$; hence if $k = (\ell - 2)/2$.

This completes the proof of the theorem. $\qquad\qquad\square$

Similarly to Corollaries 3.12 and 3.16, one has also a corollary to Theorem 4.2. We will not explicitly mention it, but the reader can easily state it for himself. We content ourselves by mentioning that all automorphisms of the graphs $\Gamma^n_{j;\geq k}$ where vertices are the j-subsets of $\Omega^{(n)}$, adjacent if they share at least k elements ($1 \leq k < j \leq n - 1$) and $\Gamma^n_{j;k}$ with same vertex set but adjacency now defined as the intersection exactly being a k-subset, are induced by ordinary permutations of $\Omega^{(n)}$, except that the automorphism group of $\Gamma^{(4n^*-1)}_{2n^*-1;n^*-1}$ is $\mathsf{Sym}(4n^*)$. The graph $\Gamma^6_{2;1}$ is *not* an exception here because we can consider the extended bipartite double, which is the bipartite complement of $\Gamma^6_{2,2;0}$, and the latter has no exceptional behavior. Note, however, that the ordinary bipartite double of $\Gamma^6_{2;1}$ is $\Gamma^6_{2,2;1}$, and this pai r consti tutes an example of a connected non-bipartite graph whose automorphism group is much smaller than that of its bipartite double (usually the size of the automorphism group of the bipartite double is just twice the size of the original graph).

4.3   The infinite thin case

Finally, we take a look at the analogue of the thin case for an infinite set. Let $0 \leq k \leq i \leq j$, with $k < j$ be integers and let Ω be any infinite set. Let the graph $\Gamma^\Omega_{i,j;k}$ be the bipartite graph with vertices the i-subsets and the j-subsets of Ω, where an i-set is adjacent to a j-set if they intersect in a k-set. Similarly we define $\Gamma^\Omega_{i,j;\geq k}$ in the obvious way, and also $\Gamma^\Omega_{j;k}$ and $\Gamma^\Omega_{j;\geq k}$.

We first show the following.

Proposition 4.3 *The elements of Ω are recoverable from $\Gamma^{\Omega}_{j;k}$, $0 < k < j$. In particular, every graph automorphism is induced by a permutation of Ω.*

Proof   We first reconstruct the k-subsets of Ω. We claim that an infinite clique in the graph corresponds to an infinite number of j-subsets containing a fixed k-subset.

Indeed, let C be an infinite clique and suppose by way of contradiction that there are three members of the clique, say J_1, J_2, J_3 with $J_1 \cap J_2 \neq J_2 \cap J_3$. Let J be an arbitrary member of the clique distinct from J_1, J_2, J_3. If $|J \cap (J_1 \cup J_2 \cup J_3)| = k$, then clearly this k-subset must be contained in $J_1 \cap J_2 \cap J_3$, a contradiction. Hence $|J \cap (J_1 \cup J_2 \cup J_3)| > k$. We now have infinitely many choices for J and only a finite number of choices (bounded by 2^{3j}) for the intersection $J \cap (J_1 \cup J_2 \cup J_3)$. It follows that there are two j-subsets J, J' in the clique intersecting $J_1 \cap J_2 \cap J_3$ in the same set. But then $|J \cap J'| > k$, a contradiction.

Now the transitive closure of the relation "shares at least two members" among the infinite cliques of the graph defines an equivalence relation whose classes are in natural bijective correspondence with the k-subsets. Hence we can reconstruct the bipartite graph $\Gamma^{\Omega}_{k,j}$ of k-sets and j-sets, where adjacency is given by symmetrized inclusion. Note that the valence of a vertex corresponding to a j-subset is $\binom{j}{k}$.

We can recognize the $(k+1)$-subsets as the sets with minimal > 1 number of common neighbors of two j-subsets (and then they have exactly $k+1$ common neighbors). This way we determine k, and since we know $\binom{j}{k}$ already, we also know j. But we also recognize $\Gamma^{\Omega}_{k+1,k}$. Now, two vertices representing $(k+1)$-subsets are at distance $2k$ from one another if and only if they intersect in a single element. This relation defines $\Gamma^{\Omega}_{k+1;1}$. As in the beginning of this proof, we can now reconstruct $\Gamma^{\Omega}_{k+1,1}$, telling us exactly which elements are contained in each $(k+1)$-set.

This proves the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now let $0 \leq k \leq i \leq j$, with $k < j$ and let Γ be either isomorphic to $\Gamma^{\Omega}_{i,j;k}$ or isomorphic to $\Gamma^{\Omega}_{i,j;\geq k}$ (where Ω is an infinite set of any cardinality). Let $\mathcal{A}$ be an infinite set of vertices belonging to the same bipart of Γ. Then we say that $\mathcal{A}$ is an ℓ-*star*, $\ell \in \mathbb{N} \setminus \{0\}$, if it satisfies the following properties.

(S1) If v is a vertex belonging to the other bipart, then v is either adjacent to exactly $\ell + 1$ members of $\mathcal{A}$, or not adjacent to exactly ℓ members of $\mathcal{A}$, or adjacent to no members of $\mathcal{A}$.

(S2) For every subset $A \subset \mathcal{A}$ containing exactly $\ell + 1$ elements, there exists a finite nonzero number of vertices v of Γ such that v is adjacent to every member of A and not adjacent to every member of $\mathcal{A} \setminus A$.

(S3) For every subset $B \subset \mathcal{A}$ containing exactly ℓ elements, there exists a finite nonzero number of vertices v of Γ such that v is not adjacent to every member of B and adjacent to every member of $\mathcal{A} \setminus B$.

Also, we say that $\mathcal{A}$ is an ℓ-*flower*, $\ell \in \mathbb{N} \setminus \{0\}$, if it satisfies the following properties.

(F1) If v is a vertex belonging to the other bipart, then v is either adjacent to exactly $\ell + 1$ members of $\mathcal{A}$, or adjacent to all members of $\mathcal{A}$, or adjacent to no members of $\mathcal{A}$.

(F2) For every subset $A \subset \mathcal{A}$ containing exactly $\ell + 1$ elements, there exists a finite nonzero number of vertices v of Γ such that v is adjacent to every member of A and not adjacent to every member of $\mathcal{A} \setminus A$.

(F3) There are infinitely many vertices adjacent to all members of $\mathcal{A}$.

Finally, $\mathcal{A}$ will be called an i-*regular star* if it consists of the vertices corresponding to all i-sets of Ω containing a fixed $(i-1)$-set. Similarly for a j-regular star. If $i = 1$, then an i-regular star coincides with Ω.

Basically, we want to show that the graphs $\Gamma_{i,j;k}^{\Omega}$ contain unique ℓ-stars, with $\ell \in \{j-k, i-k\}$, which are either i-regular or j-regular stars. Moreover these graphs do not contain any ℓ-flowers. Also, we want to show that the graphs $\Gamma_{i,j;\geq k}^{\Omega}$ contain unique ℓ-flowers, with $\ell \in \{j-k, i-k\}$, which are either i-regular or j-regular stars. Moreover these graphs do not contain any ℓ-stars. But first we isolate some more-or-less trivial cases.

Given Γ (with $k \leq i \leq j$, $k < j$), we can easily decide whether $k = i$ or not; indeed, $k = i$ if and only if the valence of the vertices of one bipart is finite. In this case, we make a new graph by throwing away the bipart with infinite valence and declare two remaining vertices adjacent when they have a unique common neighbor in Γ. The graph thus obtained is clearly isomorphic to $\Gamma_{j;i}^{\Omega}$, and we can apply Proposition 4.3. Hence every graph automorphism of $\Gamma_{i,j;\geq i}^{\Omega} \cong \Gamma_{i,j;i}^{\Omega}$ is induced by a permutation of Ω.

Also, one checks that the only cases in which there are vertices with finite covalence (meaning that there are only a finite number of vertices of the other bipart not adjacent to a vertex of one bipart) are the cases of the graphs $\Gamma_{1,j;0}^{\Omega}$, $j \geq 1$. But in these cases, clearly every graph automorphism of the bipartite complement is induced by a permutation of Ω.

So from now on, we may assume that no vertex has finite valence nor finite covalence.

Finally, we can isolate the graphs $\Gamma \cong \Gamma_{i,j;0}^{\Omega}$ (case $k = 0$) from the rest as follows.

Lemma 4.4 *In each of the graphs $\Gamma_{i,j;k}^{\Omega}$ and $\Gamma_{i,j;\geq k}^{\Omega}$ with $k > 0$, there exists in each bipart a finite set of vertices not adjacent with a common vertex. But in the graph $\Gamma_{i,j;0}^{\Omega}$ each finite set of vertices contained in one of the biparts is adjacent to some vertex.*

Proof   In the bipart of the i-subsets one can consider a set S of $j+1$ disjoint i-subsets. Then an arbitrary j-subset can intersect at most j members of S nontrivially. Hence, if $k > 0$, no vertex is adjacent to all members of S. The same thing is true for i and j interchanged. However, if we consider any finite number of i-subsets, then we can always find a j-subset disjoint from all these i-subsets. And the same thing holds for i and j interchanged. This proves the lemma. $\square$

If $k = 0$, then we consider the complement of the graph $\Gamma^\Omega_{i,j;0}$, which is $\Gamma^\Omega_{i,j;\geq 1}$. So we may assume $k > 0$ from now on.

Proposition 4.5 *If $\Gamma \cong \Gamma^\Omega_{i,j;k}$, with $1 \leq k < i \leq j$, then every ℓ-star, $\ell \geq 0$, is either a j-regular star, or an i-regular star (and both occur), and $\ell = i - k$ or $\ell = j - k$, respectively. Also, Γ does not contain ℓ-flowers, for any natural ℓ.*

Proof   To fix the ideas, we assume that $\mathcal{A}$ is an ℓ-star or ℓ-flower contained in the bipart corresponding to the j-sets. We will cease to assume $i \leq j$ in order to treat the case of i-sets at the same time (but we do assume $k < j$).

By assumptions (S3) and (F3), there is an i-set I adjacent to every member of an infinite subset $\mathcal{A}_0$ of $\mathcal{A}$. Since $k > 0$, since every member of $\mathcal{A}_0$ intersects I in exactly k elements of Ω, and since I contains a finite number of k-subsets, there must be some k-subset $K \subseteq I$ contained in every member of an infinite subset $\mathcal{A}_K \subseteq \mathcal{A}_0$. Let $X \supseteq K$ be a set with the property that the set $\mathcal{A}_X$ of elements of $\mathcal{A}_K$ that contain X has infinitely many elements. Clearly $|X| \leq j - 1$, and K has this property. Hence we can pick a maximal set X with that property. It follows that for any $p \in \Omega \setminus X$, the number of elements of $\mathcal{A}_X$ containing p is a finite number n_p. An easy consequence of that fact is that for every finite subset $C \subseteq \Omega \setminus X$, there exist infinitely many members of $\mathcal{A}_X$ missing C, and only a finite number of members of $\mathcal{A}_X$ meet C nontrivially. Hence a str aightfor ward inductive argument implies that for any positive integer m, we can find a subset $A_m \subseteq \mathcal{A}_X$ of cardinality m such that no member of $\mathcal{A}_X$ intersects two members of A_m nontrivially outside X, and such that A_m contains an arbitrary given member of $\mathcal{A}_X$.

fine n_a as n_x for all $x \in J_a$, $a = 1, 2, \ldots, i$. Now, since $i > 1$, we can remove from I the element in J_2 and add a second element of J_1. The obtained i-set I' is not adjacent to at most $\ell - n_2 + (n_1 - 1)$ members of $\mathcal{A}$; hence $n_2 \leq n_1 - 1$. Likewise $n_1 \leq n_2 - 1$, a contradiction. Hence there exists some element $x \in \Omega$ contained in infinitely many members of $\mathcal{A}$. A similar argument as in the previous paragraph leads to the existence of a finite set $X \subseteq \Omega$ contained in all members of an infinite subset $\mathcal{A}_X$ of $\mathcal{A}$ such that every point p outside X is contained in finitely many (say, n_p) members of $\mathcal{A}_X$.

implies that I is adjacent to exactly $i + 1$ members of $\mathcal{A}$. We then consider any $(i + 1)$-subset A of $\mathcal{A}_X$. By (S2) there is an i-set I' disjoint from the union of A and intersecting all other members of $\mathcal{A}_X$. Since $I' \cap X = \emptyset$, this clearly contradicts the fact that I' is

finite and $\mathcal{A}_X$ is infinite. Hence $\mathcal{A} = \mathcal{A}_X$. If $n_p = 0$ for some $p \in \Omega \setminus X$, then we consider an i-set I'' containing p and disjoint from X and obtain that I'' is not adjacent to exactly $|\{q \in I'' : n_q = 1\}| < i$ elements, a contradiction. Hence $\mathcal{A}$ is an i-star.

We consider A_{i-k+1} and consider an i-set I such that $|I \cap X| = k - 1$ and $|I \cap J| = k$, for every $J \in A$. Then clearly, I is not adjacent with infinitely many members of $\mathcal{A}_X$, and adjacent with at least $i - k + 1 \neq 0$ members of $\mathcal{A}_X$. It follows from (S1) and (F1) that I is adjacent with exactly $\ell + 1$ members of $\mathcal{A}$, hence with at most $\ell + 1$ members of $\mathcal{A}_X$. This implies $i - k \leq \ell$.

Now we consider A_{i-k} and consider an i-set I' such that $|I' \cap X| = k$ and $|I' \cap J| = k + 1$, for every $J \in A$. Then clearly, I' is adjacent with infinitely many members of $\mathcal{A}_X$, and not adjacent with at least $i - k \neq 0$ members of $\mathcal{A}_X$. This contradicts (F1), hence Γ cannot contain an ℓ-flower.

Consider A_ℓ. Then, by (S3), there is an i-set I'' not adjacent to all members of A_ℓ and adjacent to everything else. It is easy to see that the latter implies $|I'' \cap X| = k$. The former implies that we can distribute the $i - k$ elements of $I'' \setminus X$ among the members of A_ℓ such that every member gets at least one element. Hence $i - k \geq \ell$, which yields $i - k = \ell$. Also, it now follows that I'' contains exactly one element in $J \setminus X$, for each $J \in A_\ell$. Varying this element over $J \setminus X$ (leading to different i-sets not adjacent with exactly the same set of members of $\mathcal{A}$), we see that for each point $p \in J \setminus X$, we have $n_p = 1$. Since we could choose one of the elements of A_ℓ completely arbitrarily, we conclude that $n_p \in \{0, 1\}$, for all $p \in \Omega \setminus X$.

If $n_p = 1$, for all $p \in \Omega \setminus X$, then $\mathcal{A}_X$ is an $(i - k)$-star and a j-regular star, and this implies easily $\mathcal{A} = \mathcal{A}_X$, which proves the assertion.

So we may assume that there exists a point $p \in \Omega \setminus X$ with $n_p = 0$. Then clearly $\mathcal{A} \neq \mathcal{A}_X$ as we can easily produce an i-set intersecting X in a $(k-1)$-set, containing a point p with $n_p = 0$; and containing a point q with $n_q = 1$ (this is possible since $i \geq k + 1$); this i-set is adjacent to at least one and at most $i - k$ vertices of $\mathcal{A}_X$.

So let $J \in \mathcal{A} \setminus \mathcal{A}_X$. Then there is a point $x \in X \setminus J$. We choose a k-set K in X containing x. Then we choose $i - k$ members of $\mathcal{A}_X$ disjoint from J. In each of these members, we choose a point not in X. The union of these points with K is an i-set I^* which is not adjacent with exactly $i - k$ members of $\mathcal{A}_X$; but I^* is not adjacent with J, too, a contradiction.

The assertion is proved. $\qquad \square$

Proposition 4.6 *If* $\Gamma \cong \Gamma^{\Omega}_{i,j;\geq k}$, *with* $1 \leq k < i \leq j$, *then every ℓ-flower, $\ell \geq 1$, is either a j-regular star, or an i-regular star (and both occur), and $\ell = i - k$ or $\ell = j - k$, respectively. Also, Γ does not contain ℓ-stars, for any natural ℓ.*

Proof   Let $\mathcal{A}$ be an ℓ-flower or ℓ-star in Γ. As before we assume that it consists of j-sets, and we drop the assumption $i \leq j$. Similarly as in the proof of Proposition 4.5, there exists a set $X \subseteq \Omega$, with $k \leq |X| \leq j - 1$, such that the set $\mathcal{A}_X$ of members of $\mathcal{A}$ containing X is infinite, and every element $p \in \Omega \setminus X$ is contained in finitely many members of $\mathcal{A}$.

Select ℓ members of $\mathcal{A}_X$ arbitrarily. If $\mathcal{A}$ is an ℓ-star, then by (S3), there exists an i-set I adjacent to every member of $\mathcal{A}_X$ except for the selected ℓ members. It is easily seen that the fact that I is adjacent with infinitely members of $\mathcal{A}_X$ implies that $|I \cap X| \geq k$. But then I is adjacent with every member of $\mathcal{A}_X$, a contradiction to $\ell \geq 1$. Hence $\mathcal{A}$ is an ℓ-flower.

Suppose now that there exists $J \in \mathcal{A} \setminus \mathcal{A}_X$. Then we can close a k-subset in X intersecting J in less than k elements. We add $i - k$ elements of $\Omega \setminus (X \cup J)$ and obtain an i-set adjacent to every element of $\mathcal{A}_X$, but not to J, contradicting (F1). Hence $\mathcal{A} = \mathcal{A}_X$.

Now we again define n_p as the number of members of $\mathcal{A}$ containing p, $p \in \Omega \setminus X$. Similarly as in the proof of Proposition 4.5 (case $k \geq 1$), one shows $i - k \leq \ell$. Also, we can select a set A of $\ell + 1$ members of $\mathcal{A}$ pairwise intersecting in only X. Then (F2) implies that there is some i-set I' adjacent to all members of A, and to no member of $\mathcal{A} \setminus A$. The latter implies that $|I' \cap X| \leq k - 1$. Since every member of A must intersect I' in at least k elements, the maximum value for $\ell + 1$ is $i - (k - 1)$; hence $\ell = i - k$. Also, since we can choose one member of A completely arbitrarily in $\mathcal{A}$, we deduce that $n_p = 1$ as soon as $n_p \neq 0$, for every $p \in \Omega \setminus X$. By deleting from I' an element outside X and replacing it with some element q, also outside X, but with $n_q = 0$, we obtain a contradiction and have hence shown that $n_p = 1$, for all $p \in \Omega \setminus X$.

We now claim $|X| = j - 1$. Indeed, assume $|X| \leq j - 2$. Let $J \in \mathcal{A}$. We choose an i-set I'' with exactly $k - 1$ elements in X and $k + 1$ elements in J. Then I'' is adjacent to at least one, but at most $i - (k - 2)$ members of $\mathcal{A}$, a contradiction.

This completes the proof of the proposition. $\qquad\square$

We can now show the following theorems.

Theorem 4.7 *Let $0 \leq k \leq i \leq j$, with $k < j$ and let Γ be either isomorphic to $\Gamma^{\Omega}_{i,j;k}$ or isomorphic to $\Gamma^{\Omega}_{i,j;\geq k}$, $k > 0$ (where Ω is an infinite set of any cardinality). Then every graph automorphism of Γ is induced by a permutation of Ω. Also, all graphs $\Gamma^{\Omega}_{i,j;k}$ and $\Gamma^{\Omega}_{i,j;\geq k}$ (with the given restrictions on the parameters) are pairwise non-isomorphic.*

Proof   We already discussed the cases in which there are vertices with either finite valence or finite covalence. Hence we may assume that no vertex has finite valence or finite covalence. Also, if for every finite set of one of the biparts, there exists a vertex adjacent to all members of that set, then we know by Lemma 4.4 that $k = 0$, and we go on

with the bipartite complement. Then Propositions 4.5 and 4.6 imply that the only ℓ-stars and ℓ-flowers of Γ are j-regular stars and i-regular stars, which are $(i-k)$-stars or -flowers and $(j-k)$-stars or -flowers, respectively. This determines $i-k$ and $j-k$ (if we went on with the bipartite complement then this already determines all parameters of the original graph $\Gamma^{\Omega}_{i,j;0}$). We consider one bipart B, say containing the vertices of the $(i-k)$-flowers or -stars, and define a new bipartite graph Γ_i with B as one bipart, and the $(i-k)$-stars or -flowers as o ther bip art. Adjacency is containment made symmetric. Then $\Gamma' \cong \Gamma^{\Omega}_{j-1,j;j-1}$ and the finite valence reveals j. Also, the assertion about the automorphism group now follows from the discussion preceding Proposition 4.5. $\qquad\square$

Theorem 4.8 *Let $0 \leq k < i$ and let Γ be either isomorphic to $\Gamma^{\Omega}_{i;k}$ or isomorphic to $\Gamma^{\Omega}_{i;\geq k}$, $k > 0$ if $i > 1$ (where Ω is an infinite set of any cardinality). Then every graph automorphism of Γ is induced by a permutation of Ω. Also, all graphs $\Gamma^{\Omega}_{i;k}$ and $\Gamma^{\Omega}_{i;\geq k}$ (with the given restrictions on the parameters) are pairwise non-isomorphic.*

Proof   If $i = 1$, this is trivial (and we also have trivial graphs). In the other cases, the graphs are not trivial, and the theorem follows from the previous one by taking the bipartite double. $\qquad\square$

Again, in the above results, we may assume graph-epimorphisms instead of automorphisms, see also Remark 3.17.

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