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Inventiones mathematicae

Half Moufang implies Moufang for finite generalized quadrangles

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Summary. A finite generalized quadrangle has two types of panels. If each panel of one type is Moufang, then every panel is Moufang. Hence by a theorem of Fong and Seitz [1] the quadrangle is classical or dual classical.

1 Introduction

Throughout this note we let $\mathcal{S}=(P, B, I)$ be a finite generalized quadrangle (GQ) of order (s, t), $s>1$, $t>1$. (For definitions and results not given in full here we recommend the monograph [2], hereafter denoted by FGQ.) A panel of $\mathcal{S}$ is an ordered triple (x, L, y), where x and y are distinct points incident with the line L, or an ordered triple (L, p, M), where L and M are distinct lines incident with the point p. For a given panel (x, L, y), let $H(x, L, y)$, be the group of all collineations of $\mathcal{S}$ which fix x and y linewise and L pointwise. For each line M through x, $M \neq L$, $H(x, L, y)$ acts semiregularly on the points of M different from x. Hence $|H(x, L, y)|$ divides s. We say the panel (x, L, y) is Moufang provided $|H(x, L, y)|=s$. For a panel (L, p, M), the group $H(L, p, M)$ is defined in a dual manner. And (L, p, M) is Moufang provided $|H(L, p, M)|=t$. If every panel of $\mathcal{S}$ is Moufang, then by a theorem of Fong and Seitz [1] (cf. J. Tits [3], p. 240), $\mathcal{S}$ must be isomorphic to one of the so-called classical or dual classical examples (cf. FGQ).

The main result of this paper is that if every panel of one type is Moufang, then all panels are Moufang. By point-line duality we may state this as follows:

1.1 Theorem. *If every panel of the form (x, L, y) is Moufang, then also every panel of the form (L, p, M) is Moufang. Hence $\mathcal{S}$ is classical or dual classical.*

The starting point is the following immediate corollary of 9.1.1 and 5.6.2 of FGQ:

1.2 Theorem. *Suppose that every panel of $\mathcal{S}$ of the form (x, L, y) is Moufang. Then one of the following must hold:*

i. Each point of $\mathcal{S}$ *is regular* (*so* $t \leqq s$ *by* 1.3.6 *of FGQ*), *or*
ii. For each pair $\{x, y\}$ *of noncollinear points,* $|\{x, y\}^{\perp\perp}| = 2$, *or*
iii. $\mathcal{S} \cong H(4, s)$ (*so* $\mathcal{S}$ *is classical*; cf. 3.1.1 *of FGQ*).

The cases $s \leqq t$ and $s > t$ are somewhat different, but both utilize the theory of elation generalized quadrangles (EGQ). Recall that an elation about a point p is a collineation of $\mathcal{S}$ that fixes p linewise and acts semiregularly on the points of $P \setminus p^{\perp}$. And if G is a group of elations about p acting regularly on $P \setminus p^{\perp}$, we say $(\mathcal{S}^{(p)}, G)$ is an EGQ. See Chapters 8, 9 and 10 of FGQ for definitions and results about EGQ.

Note. For any line N, N^* denotes the set of points incident with N.

2 A local Moufang condition

Main Hypothesis: $\mathcal{S}$ has three distinct points x_1, p, y_1 and two distinct lines L, M with $x_1 \, I L I \, p I M I \, y_1$ for which both (x_1, L, p) and (y_1, M, p) are Moufang.

For $x \in L^* \setminus \{p\}$, put $A(x) = H(x, L, p)$ and $A = \langle A(x) \colon x \in L^* \setminus \{p\} \rangle$. Analogously, for $y \in M^* \setminus \{p\}$, put $B(y) = H(y, M, p)$ and $B = \langle B(y) \colon y \in M^* \setminus \{p\} \rangle$. Finally, put $G = \langle A, B \rangle$.

Since $H(x_1, L, p)$ is transitive on $M^* \setminus \{p\}$ and $H(y_1, M, p)$ is transitive on $L^* \setminus \{p\}$, the following result is an immediate consequence of the Main Hypothesis.

2.1 Lemma. $|A(x)| = |B(y)| = s$ *for all* $x \in L^* \setminus \{p\}$ *and all* $y \in M^* \setminus \{p\}$.

For $a \in A$, $b \in B$, the commutator $[a, b] = a^{-1} b^{-1} ab$ fixes L pointwise, p linewise, and M pointwise. By 2.4.1 of FGQ it follows that $[a, b]$ acts semiregularly on the points of $P \setminus p^{\perp}$. In particular:

2.2 Lemma. *For* $x \in L^* \setminus \{p\}$ *and* $y \in M^* \setminus \{p\}$, *and for* $a \in A$, $b \in B$, $[a, b]$ *acts semiregularly on the points of* $\{x, y\}^{\perp} \setminus \{p\}$. *Hence* $|[a, b]|$ *divides* t, *and* $ab = ba$ *if* $[a, b]$ *fixes some point of* $P \setminus p^{\perp}$.

2.3 Lemma. *Let* $z \in P \setminus p^{\perp}$ *with* $L I x I M_1 I z I L_1 I y I M$. *Let* $\mathrm{id} \neq a \in A(x)$, $\mathrm{id} \neq b \in B(y)$. *Then the following are equivalent*:
i. $z^{ab} = z^{ba}$
ii. $[a, b]$ *fixes* z^{ba}
iii. $[a, b] = \mathrm{id}$
iv. $M_1^{ba} = M_1^{b}$
v. $L_1^{ab} = L_1^{a}$
vi. $L_1^{a} \perp M_1^{b}$

Proof. Easy to check with the aid of Fig. 1. □

With the setup as in lemma 2.3, suppose $ab = ba$, so $z^{ab} = z^{ba}$, etc. Let N be any line through x, $L \neq N \neq M_1$. Define $z_1 \in P \setminus p^{\perp}$ by $N I z_1 \perp y$. Since $ab = ba$, clearly $z_1^{ab} = z_1^{ba}$, and the above argument shows that N^b is fixed by a. As b is a bijection from the set of lines through x to the set of lines through x^b, each line through x^b must be fixed by a. We have essentially proved the following:

2.4 Lemma. *For* $\mathrm{id} \neq a \in A(x)$ *and* $b \in B(y)$, *if* a *and* b *commute, then* $a \in A(x^b)$. *If* $ab \neq ba$, *then no line through* x^b *can be fixed by* a.

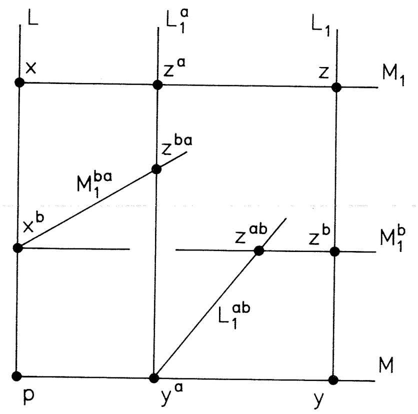

Fig. 1

2.5 Lemma. *If x, w are distinct points of $L^*\backslash\{p\}$ and $a \in A(x)$, then either a fixes all lines through w or only L through w.*

Proof. For any $y \in M^*\backslash\{p\}$, there is a unique $b \in B(y)$ with $w = x^b$. □

2.6 Lemma. *If $\theta \in A$ and θ fixes some line M_1 meeting L at a point x, $x \neq p$, then $\theta \in A(x)$. Hence whenever $\theta \in A$ fixes one line $(\neq L)$ through a point x $(\neq p)$ on L, then θ fixes x linewise.*

Proof. Let $\theta \in A$ satisfy the hypothesis. Let $z \in M_1^*\backslash\{x\}$ and suppose $z^\theta = z_1$. There must be a unique $a \in A(x)$ for which $z_1^a = z$. So $z^{\theta a} = z$. But θa fixes z, fixes L pointwise and p linewise. By 2.4.1 of FGQ, $\theta a = \mathrm{id}$, forcing $\theta \in A(x)$. □

The above proof actually shows the following:

2.7 Lemma. *$A(x)$ is the group of all collineations θ of $\mathcal{S}$ which fix L pointwise, p linewise, and fix some line $(\neq L)$ through x.*

For $c, d \in G$, and for $H \leq G$, write $c^d = d^{-1}cd$ and $H^c = c^{-1}Hc$. Then the following is immediate.

2.8 Lemma. *$A(x)^b = A(x^b)$ for all $x \in L^*\backslash\{p\}$ and all $b \in B$. Hence $B \leq N_G(A)$, forcing $A \lhd G$. Similarly, $B \lhd G$, and $G = AB$.*

For fixed $x \in L^*\backslash\{p\}$ and fixed $y \in M^*\backslash\{p\}$, $G = \langle A, B \rangle = \langle A(x), B(y) \rangle$. Hence $G = \langle A(x), B \rangle = \langle A, B(y) \rangle = A(x) \cdot B = A \cdot B(y)$. As $A(x) \cap B = \{\mathrm{id}\} = A \cap B(y)$, also $|G| = |A(x)| \cdot |B| = |A| \cdot |B(y)|$. But $|A(x)| = s = |B(y)|$, so $|A| = |B|$.

2.9 Lemma. *G acts semiregularly on $P\backslash p^\perp$.*

Proof. Write $G = A(x) \cdot B$ for some $x \in L^*\backslash\{p\}$. And suppose $\theta = ab$ fixes $z \in P\backslash p^\perp$, for some $a \in A(x)$, $b \in B$. Define y by $M I y \perp z$. Then $z = z^{ab} \perp y$ and $y^{ab} = y^a$ imply $y^a = y$. Hence $a = \mathrm{id}$. So $z^b = z$, forcing $b \in B(y)$ (by lemma 2.6). But then $b = \mathrm{id}$. □

3 EGQ and the local Moufang condition

Suppose that for some point p there is a group G for which $(\mathcal{S}^{(p)}, G)$ is an EGQ. Let $L_0, \ldots, L_t$ be the lines through p. Let e be a fixed but arbitrary point of $P\backslash p^\perp$. For $g \in G$ we can identify g with the point e^g. For $0 \leq i \leq t$, define x_i, M_i by $e I M_i I x_i I L_i$. Put $S_i = \{\theta \in G \colon M_i^\theta = M_i\}$, $S_i^* = \{\theta \in G \colon x_i^\theta = x_i\}$. Then $|S_i| = s$; $|S_i^*| = st$; $|G| = s^2 t$; $|S_i S_j \cap S_k| = 1 = |S_i^* \cap S_j|$ if i, j, k are distinct. And $\{S_j^*,$

$S_j S_0 \setminus S_j,\ S_j S_1 \setminus S_j, \ldots, S_j S_t \setminus S_j\}$ (omitting $S_j S_j \setminus S_j$) is a partition of G. (See Chapters 8, 9 and 10 of FGQ for these and many other results about EGQ).

We want to develop conditions that guarantee that panels of the form (L_j, p, L_k) are Moufang.

3.1 Lemma. *With the same notation as above, suppose* $0<j<k\leqq t$ *and suppose that both panels* (x_i, L_i, p), $i\in\{j, k\}$, *are Moufang, with* $H(x_i, L_i, p)=S_i\leqq G$ *and* $(\mathscr{S}^{(p)}, G)$ *an EGQ. Then the panel* (L_j, p, L_k) *is Moufang.*

Proof. Since S_j fixes all points of L_j and S_k acts regularly on the points of L_j different from p, it must be that no element of $S_j S_k \setminus S_j$ can fix any point of L_j different from p. Hence from the partition of G given above, S_j^* must contain all the elements of G that fix some point of L_j different from p. But each such point has a stabilizer of order st in G. Hence $\theta\in G$ fixes some point of L_j different from p if and only if $\theta\in S_j^*$ if and only if θ fixes all points of L_j. (Note: A corollary is that $S_j^*\lhd G$). Similarly, $\theta\in G$ fixes some point of L_k different from p if and only if $\theta\in S_k^*$ if and only if θ fixes all points of L_k. Then from $S_j^*\cap S_k^*\leqq H(L_j, p, L_k)$ and $t=|S_j^*\cap S_k^*|\leqq|H(L_j, p, L_k)|\leqq t$, it follows that (L_j, p, L_k) is Moufang. □

In the next lemma, the x_i, L_i, p, ... are as defined in the beginning of this section, but we do not require $\mathscr{S}$ to be an EGQ for some group.

3.2 Lemma. *Suppose* $s\leqq t$ *and* $|\{p, y\}^{\perp\perp}|=2$ *whenever* $y\in P\setminus p^{\perp}$. *Suppose* (x_i, L_i, p) *is Moufang for each* i, $0\leqq i\leqq t$. *Then each panel of the form* (L_i, p, L_j), $0\leqq i<j\leqq t$, *is Moufang.*

Proof. Put $G_p=\langle H(p, py, y)\colon y\in p^{\perp}\setminus\{p\}\rangle$. Then by 9.4.3 of FGQ, $(\mathscr{S}^{(p)}, G_p)$ is an EGQ. Now by lemma 3.1, the panel (L_i, p, L_j) is Moufang for $0\leqq i<j\leqq t$. □

4 Proof of Theorem 1.1

Suppose that every panel of $\mathscr{S}$ of the type (x, L, y) is Moufang, so the conclusion of theorem 1.2 holds.

If all points of $\mathscr{S}$ are regular and $s=t$, then $\mathscr{S}\cong W(s)$ (cf. 5.2.1 of FGQ).

Next assume that $s>t$. For some point p, let $L_0, \ldots, L_t$ be the lines through p and consider a panel of the form (L_j, p, L_k), $0\leqq j<k\leqq t$. For $i=j, k$ put $\hat{S}_i=\langle H(x, L_i, p)\colon L_i I x \neq p\rangle$, and let $G_{jk}=\langle\hat{S}_j, \hat{S}_k\rangle$. It is clear that G_{jk} fixes p linewise. By lemma 2.9 G_{jk} acts semiregularly on $P\setminus p^{\perp}$. Then by 9.4.2 of FGQ (with $r=1$), G_{jk} is transitive on $P\setminus p^{\perp}$, so $(\mathscr{S}^{(p)}, G_{jk})$ is an EGQ. Now by lemma 3.1, it follows that (L_j, p, L_k) is Moufang.

If $s\leqq t$ and $|\{x, y\}^{\perp\perp}|=2$ for each pair (x, y) of noncollinear points, then the desired result follows from lemma 3.2.

Finally, if $\mathscr{S}\cong H(4, s)$, then $\mathscr{S}$ is classical (and hence Moufang). □

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