Shult's Haircut Theorem revised*

Arjeh Cohen † Anneleen De Schepper ‡ Jeroen Schillewaert §

Hendrik Van Maldeghem ¶

Abstract

We discuss Shult's Haircut Theorem, which unifies several recognition theorems for Lie geometries. We point out why the conclusion of the original version (published posthumously without his final scrutiny) cannot be drawn from the hypotheses and state and prove a corrected version, obtained by restricting one condition and relaxing two other conditions.

1 Introduction

Polar spaces are point-line geometries associated with groups of classical Lie type. In 1974, together with Buekenhout, the late Ernest E. Shult introduced these spaces and extended earlier results of Veldkamp and Tits to a surprisingly simple and powerful characterisation of geometries of classical groups in terms of points and lines. Parapolar spaces form a wider class of point-line geometries than polar spaces. Since almost all point-line geometries of groups of Lie type are parapolar spaces, they provide a good starting point for recognitions of further geometries of groups of Lie type. This is well explained in Shult's celebrated book [8].

Shult's last work [9] focuses on a unification and combination of several characterisations of parapolar spaces into one big theorem, using one essential assumption that he called the "Haircut axiom". The "Haircut Theorem" appeared in a paper dedicated to Joseph A. Thas on the occasion of the latter's 70th birthday. After Shult submitted the manuscript, he did not have a chance to finalise the paper for publication because he became seriously ill before the refereeing process was terminated and passed away before the paper appeared.

The Haircut Theorem uses a gap in the spectrum of dimensions of the singular subspaces of symplecta obtained by intersecting them with point-perps. Recently, three of the current authors worked on another type of characterisation of parapolar spaces, mainly of exceptional type, see [6]. Shult's Haircut Theorem would induce certain shortcuts in their arguments. However, they stumbled upon some problems with the statement, and when discussing this with the first author of the current paper, also some problems with the proof emerged. In Subsection 3.2, we will carefully point out these problems and in Subsection 4.1, we state a corrected version of the Haircut Theorem. We were unable to keep exactly

*AMS Classification: 51E24, Key words: Parapolar spaces, spherical buildings, groups of Lie type

†Department of Mathematics and Computer Science, Eindhoven University of Technology, a.m.cohen@tue.nl

‡Department of Mathematics, Ghent University, Belgium, Anneleen.DeSchepper@ugent.be, supported by the Fund for Scientific Research Flanders—FWO Vlaanderen

§Department of Mathematics, University of Auckland (UoA), New Zealand, j.schillewaert@auckland.ac.nz, supported by UoA FDRF grant

¶Department of Mathematics, Ghent University, Belgium, Hendrik.VanMaldeghem@ugent.be

the same assumptions; we have to assume constant symplectic rank. Nevertheless, at the same time we relaxed two of the other conditions, one of which is the local connectivity (cf. Section 5).

The paper is organized as follows. In Section 2 we introduce notation. In Section 3 we state Shult's original version of the Haircut Theorem and discuss the statement, as well as certain parts of the proof. In Section 4 we state and prove the locally connected case of the *revised Haircut Theorem* and in the last section, we drop the assumption on local connectivity.

2 Notation

We refer the reader to [3, 8] for the definitions of polar spaces and parapolar spaces, with the basics about Tits buildings and their associated point-line geometries. A singular subspace of a parapolar space is defined as usual (it is a subspace, that is, a set of points containing each line that contains two collinear points of it, and each pair of its points is collinear). In this paper, polar spaces are assumed to be non-degenerate and have thick lines, that is, lines of size at least 3. We also assume that their rank is finite and at least 2, that is, for a given polar space ξ there is a natural number $r \geq 2$ such that the maximal singular subspaces of ξ are projective spaces of dimension $r-1$ (we view the empty set as a projective space of dimension -1, each singleton is a projective space of dimension 0 and every thick line is considered a projective space of dimension 1). We introduce some notation that we will use and at the same time recall some crucial notions.

A parapolar space will be denoted by Ω or by $(\mathscr{P},\mathscr{L})$, where $\mathscr{P}$ is the point set and $\mathscr{L}$ is the set of lines. We will view each line as a set of points. The axioms of a parapolar space imply that two points x,y are contained in at most one line. If they are, then we denote this line by xy. There is a family Ξ of convex subspaces of Ω, called *symplecta* (or *symps* for short), each of them isomorphic to a polar space with the property that each line of Ω is contained in at least one symp. Each polar space is a parapolar space with $\Xi = \{\mathscr{P}\}$.

We use standard notation for *collinearity*, that is, two points x,y in a common line are called *collinear*, in symbols $x \perp y$. The set of points collinear or equal to the point x is denoted by $x^\perp$. The *collinearity graph* is the graph whose vertex set is the point set and whose adjacency is collinearity. Its diameter is the *diameter* of the geometry.

We say that a parapolar space $\Omega = (\mathscr{P},\mathscr{L})$ has *constant* or *uniform symplectic rank r* when all symplecta have rank r. Some prominent examples of parapolar spaces do not have constant symplectic rank, see below. We will encounter parapolar spaces of constant symplectic rank 2 having singular subspaces which are not projective spaces. However, if all singular subspaces are projective spaces (which is always the case if there are no symps of rank 2), then we may define the point residue Ω_p, for $p \in \mathscr{P}$, as the point-line geometry with point set the set $\mathscr{L}_p$ of all lines on p, and with line set Π_p the set of singular planes on p, where each such singular plane is viewed as the set of lines in it on p (line pencils). If Ω_p is a connected geometry, then Ω is *locally connected at p*. If Ω is locally connected at each point, then Ω is *locally connected*.

If two points x,y of Ω do not lie in a common symp, but are collinear to a common point z, then z is unique with this property and $\{x,y\}$ is usually called a *special pair*. If Ω_z is defined and connected, then $\{x,y\}$ is special if and only if the distance between the lines xz and yz in the collinearity graph of Ω_z is at least 3. If this distance is at least 4, then $\{x,y\}$ is called an *extreme pair*, and if such an extreme pair exists, then Ω is called *extreme*. If no special pairs exist, then the parapolar space is called *strong*.

The motivating examples of parapolar spaces come from spherical buildings. For any building (spherical or not) over a type set I, the J-Grassmannian (referred to as the J-shadow space in [3]), with $J \subseteq I$, is the point-line geometry $(\mathscr{P},\mathscr{L})$ with $\mathscr{P}$ the set of simplices (or flags) of type J, and a typical line is obtained as the set of simplices of type J whose union with a fixed simplex of cotype $j \in J$ is a chamber. In all examples occurring in the present paper $|J| = 1$. If j is a node of a spherical diagram X_n with $n \geqslant 3$ which is neither an end node in case $\mathsf{X} = \mathsf{A}$ nor adjacent to a bond of valency 4, then the j-Grassmannian of any building of this type is a parapolar space. This follows from Proposition 10.6.8 of [2]. The analysis of points at mutual distance 2 in Theorem 10.2.6 of [2] also shows that the 1-Grassmannians of the buildings of type $\mathsf{Y}_{(1,l,m)}$ mentioned in §3 and depicted below are parapolar spaces if $l, m \geqslant 1$.

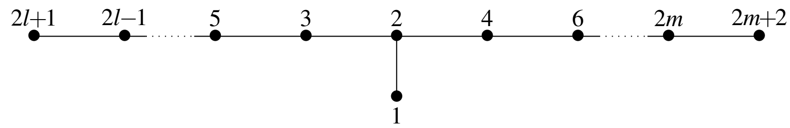

Figure 1: The Coxeter diagram of type $\mathsf{Y}_{(1,l,m)}$ with our labeling

If a building is canonically determined by its (connected and simply-laced) Coxeter diagram X_n and a (skew) field $\mathbb{L}$, then we denote it by $\mathsf{X}_n(\mathbb{L})$. If not, then we denote an arbitrary building of type X_n by $\mathsf{X}_n(*)$, thus making a distinction between the diagram and the building itself. The j-Grassmannian of the building $\mathsf{X}_n(\mathbb{L})$ is denoted $\mathsf{X}_{n,j}(\mathbb{L})$, and similarly for $\mathsf{X}_{n,j}(*)$. The 1-Grassmannians of buildings of type B_n and D_m, for instance, are polar spaces, denoted by $\mathsf{B}_{n,1}(*)$ and $\mathsf{D}_{m,1}(\mathbb{L})$, where $\mathbb{L}$ is a field if $m \geqslant 4$.

Such point-line geometries are often called *Lie incidence geometries*. For certain choices of j, the corresponding j-Grassmannian of a spherical building is called a *long root geometry*. Without going into detail we mention that those of exceptional type E are $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$ and $\mathsf{E}_{8,8}(\mathbb{K})$, for any field $\mathbb{K}$, using the Bourbaki labelling of nodes of connected Dynkin diagrams (which we will do throughout). The long root geometries $\mathsf{D}_{n,2}(\mathbb{K})$, $n \geq 5$, are examples of parapolar spaces without constant symplectic rank (rank 3 and rank $n-1$ symps occur). The 2- and 3-Grassmannians of buildings of type A_n, $n \geq 4$, are also called *line-* and *plane-Grassmannians*, respectively.

A *homomorphic image* of a parapolar space $\Omega = (\mathscr{P}, \mathscr{L})$ is the quotient geometry Ω/G, where G is a group of automorphisms of Ω, defined as follows. The points of Ω/G are the G-orbits on $\mathscr{P}$. Each line of Ω/G is the set of G-orbits on $\mathscr{P}$ of the set of points of a certain member of $\mathscr{L}$. If no non-trivial element of G maps a point onto a point at distance at most 4 in the collinearity graph of Ω, then Ω/G is a parapolar space and we call the homomorphic image *admissible*. An explicit example is contained in Section 3. Any admissible homomorphic image of a parapolar space Ω shall be denoted by Ω^h.

Other examples of parapolar spaces are the Cartesian product of any set of parapolar spaces, polar spaces and linear spaces. Here, we only need the Cartesian product of two linear spaces, where a linear space is a point-line geometry such that every pair of distinct points is contained in a unique line (and we assume lines to be thick). If $\Gamma_1 = (\mathscr{P}_1, \mathscr{L}_1)$ and $\Gamma_2 = (\mathscr{P}_2, \mathscr{L}_2)$ are linear spaces, then the Cartesian product geometry $\Gamma_1 \times \Gamma_2$ is the geometry with point set $\mathscr{P}_1 \times \mathscr{P}_2$ and set of lines $\{L_1 \times \{p_2\} \mid L_1 \in \mathscr{L}_1, p_2 \in \mathscr{P}_2\} \cup \{\{p_1\} \times L_2 \mid p_1 \in \mathscr{P}_1, L_2 \in \mathscr{L}_2\}$. The symps are the products $L_1 \times L_2$, with $L_i \in \mathscr{L}_i$, $i = 1,2$. The product of two linear spaces at least one of which is not a projective space is an example of a strong parapolar space of constant symplectic rank 2 having singular spaces which are not projective.

We are now armed with enough notation to state Shult's original version of the Haircut Theorem and our revision.

3 Original version of Shult's Haircut Theorem and some comments

3.1 The statement

We recall the main result from [9]. Below it is stated almost verbatim with the understanding that we explicitly exclude polar spaces in the hypotheses (since, in [9], they were excluded in the definition of parapolar spaces), corrected some typos, and often use the notation for buildings and Grassmannians introduced in Section 2.

Original version of Shult's Haircut Theorem. *Suppose Ω is a parapolar space of symplectic rank at least 3 that is not a polar space. Although we have not assumed constant symplectic rank, we shall assume the following:*

(i) *Each singular space possesses a finite projective dimension. Moreover, there exists an upper bound to the polar rank of a symplecton, and all symplecta have polar rank at least three.*

(ii) *We assume the "Haircut axiom":*

(H) *If ξ is a symplecton and x is a point not in ξ, then $x^\perp \cap \xi$ cannot be a hyperplane of a maximal singular subspace of ξ.*

(iii) *Ω is a locally connected space.*

Then Ω possesses a uniform symplectic rank k and one of the following occurs:

(1) *$k = 3$ and Ω is either a Grassmann space $\mathsf{A}_{n,k}(\mathbb{L})$, or a homomorphic image $\mathsf{A}_{2n-1,n}(\mathbb{L})/\langle\sigma\rangle$, where σ is a polarity of $\mathsf{A}_{2n-1}(\mathbb{L})$ of Witt index at most $n-5$, and $\mathbb{L}$ is any skew field.*

(2) *$k = 4$ and Ω is a Y_1-geometry or a twisted version thereof. In the latter case Ω is extreme. If the parapolar space Ω is neither extreme nor strong, then it is the long root geometry $\mathsf{E}_{6,2}(\mathbb{K})$, for some field $\mathbb{K}$. If Ω is strong, then it is the half-spin geometry $\mathsf{D}_{n,n}(\mathbb{K})$, $n \geq 5$, $\mathbb{K}$ any field.*

(3) *$k = 5$ and Ω is a homomorphic image of the building geometry $\mathsf{E}_{m+4,1}$, $m \geq 2$. If $m = 4$, then Ω is the Lie incidence geometry $\mathsf{E}_{8,1}(\mathbb{K})$, $\mathbb{K}$ any field. If $m = 3$, then Ω is the long root geometry $\mathsf{E}_{7,1}(\mathbb{K})$, $\mathbb{K}$ any field. If Ω is a strong parapolar space, then $m = 2$ and Ω is the Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathbb{K}$ any field.*

(4) *$k = 6$ and Ω is the Lie incidence geometry $\mathsf{E}_{7,7}(\mathbb{K})$, $\mathbb{K}$ any field, a strong parapolar space.*

(5) *$k = 7$ and Ω is the long root geometry $\mathsf{E}_{8,8}(\mathbb{K})$, $\mathbb{K}$ any field.*

Conversely, all of the listed geometries satisfy the hypotheses.

Clarifications for Case (2)**.** In Section 3.2.1 of [9], Case (2) is made more explicit. Summarized by use of our own terminology, it states that a Y_1-geometry or a twisted version thereof is either

(a) $\mathsf{Y}_{(1,l,m),1}(*)^h$, that is, an admissible homomorphic image of "the" 1-Grassmannian of a building of type $\mathsf{Y}_{(1,l,m)}$ (see Figure 1) for some $m \geqslant 2$ and $l \geqslant 1$ with $l \leq m$ (the remark that we will make in the next paragraph on the use of "the" also applies to this sentence), or

(b) a parapolar space all of whose point residues are isomorphic to either $\mathsf{A}_{2n-1,n}(\mathbb{L})$, or the homomorphic image $\mathsf{A}_{2n-1,n}(\mathbb{L})/\langle\sigma\rangle$ thereof, where σ is a polarity of $\mathsf{A}_{2n-1}(\mathbb{L})$ of Witt index at most $n-5$ (this obviously only occurs when $n \geq 5$), and $\mathbb{L}$ is any skew field (and both can occur within the same Ω).

About Case (a), in [9] it is stated that, if $l = 2$ and $m \in \{3,4\}$ then Ω is the Lie incidence geometry $\mathsf{E}_{7,2}(\mathbb{K})$ or $\mathsf{E}_{8,2}(\mathbb{K})$, for some field $\mathbb{K}$. About Case (b), in [9] it is stated that, if $l = 2$, and if one can recognize the different types of maximal singular subspaces globally, then Ω is the long root geometry $\mathsf{E}_{6,2}(\mathbb{K})$, for some field $\mathbb{K}$. It is not precisely defined what is meant in the conclusion of (2) with "a twisted version thereof". Most probably it means the spaces included in (b) but not in (a), although it is not clear at all that these can be obtained by some kind of "twist" of those in (a).

Clarifications for Case (3)**.** This case corresponds to (quotients of) buildings with Coxeter diagram E_n, defined as indicated in Figure 2. The sentence Ω *is a homomorphic image of the building geometry*

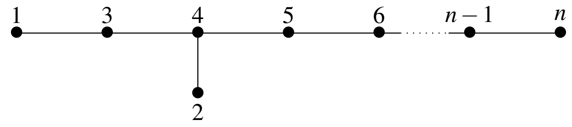

Figure 2: The Coxeter diagram of type E_n with extended Bourbaki labeling

$\mathsf{E}_{m+4,1}$, identifies the type (that is, the diagram) with the building. This makes the statement rather unclear: The use of "the" preceding "building" points to a unique building of this type; in the other statements, there is often is a field fixed in the background, and so one gets the impression that there is a unique such building for each field. This is supported by the following quote about that case on page 582 of [8]: *The buildings exist, but are they parameterized by the simply-laced diagram and an admissible division ring? One would suspect so.* But this is not true, because, for instance, there are at least two (affine) buildings of type $\mathsf{E}_9 = \widetilde{\mathsf{E}}_8$ over the field with 2 elements, namely a "2-adic" one and a "function field" one. We will introduce a notation below that takes this into account.

3.2 Some problems and counterexamples

On the geometries of the conclusion. Apart from the double use of the parameter k in the conclusion (1), an error in the original version of the Haircut Theorem is the absence of proper admissible homomorphic images of the Lie incidence geometries $\mathsf{D}_{n.n}(\mathbb{K})$, $n \geq 10$, $\mathsf{E}_{8,1}(\mathbb{K})$, $\mathsf{E}_{8,2}(\mathbb{K})$, $\mathsf{E}_{7,2}(\mathbb{K})$. Such images exist whenever we find an automorphism group of the corresponding building which maps every point of the Lie incidence geometry to a point at distance at least 5 of the original point. In particular this is satisfied whenever the diameter of the Lie incidence geometry is at least 5, the Coxeter diagram does not admit any non-trivial symmetry, and there is a non-trivial group of automorphisms such that every orbit consists of pairwise opposite elements of the building. Here is an explicit example of this situation for $\mathsf{D}_{n,n}(\mathbb{R})$, for even $n \geq 10$; see the subsequent paragraph for examples with exceptional types.

Counterexample. We start with a real Euclidean vector space $\mathbb{R}^{2n}$, $n \geq 10$ even, and the standard quadratic form $\kappa(x) = \sum_{i=1}^{2n} x_i^2$, with $[x_1, x_2, \ldots, x_{2n}]$ the coordinate-tuple of x with respect to a chosen basis $(e_1, e_2, \ldots, e_{2n})$. We write B_κ for the associated symmetric bilinear form. Then let α be the diagonal linear map $\alpha(x) = [x_1, \ldots, , x_n, -x_{n+1}, \ldots, -x_{2n}]$. Set $B(x,y) = B_\kappa(x, \alpha(y))$ and write $q(x) = \frac{1}{2}B(x,x)$. Note that $B(x,y) = B_\kappa(x, \alpha(y)) = B(\alpha(x), \alpha(y))$. Now let the point set $\mathscr{P}$ be the set of all maximal singular subspaces of $\mathbb{R}^{2n}$ with respect to q (which all have dimension n) intersecting $\langle e_1 - e_{n+1}, \ldots, e_n - e_{2n}\rangle$ in a subspace of even vector dimension and let $\mathscr{L}$ be the set of lines, where for any given subspace Y, singular with respect to q and of (affine) dimension $n-2$, the corresponding line is the set of all maximal singular subspaces in $\mathscr{P}$ containing Y. Then $\Omega = (\mathscr{P}, \mathscr{L})$ is a half spin geometry $\mathsf{D}_{n,n}(\mathbb{R})$.

Since $B_\kappa(x, \alpha(y)) = B(x,y) = 0$, for all 1-spaces $\langle x\rangle, \langle y\rangle$, singular with respect to q, and contained in a common member of $\mathscr{P}$, we see that $\alpha(X) = X^{\perp_\kappa}$, for all $X \in \mathscr{P}$. Moreover, $\alpha^2 = 1$ and $\alpha(X)$ belongs to $\mathscr{P}$. Since $\alpha(X) \cap X = X^{\perp_\kappa} \cap X = \{0\}$, the distance from X to $\alpha(X)$ equals $\frac{n}{2}$. Therefore, α induces an involution $\mathscr{P} \to \mathscr{P}$, which preserves collinearity in Ω. Hence we have found an order 2 automorphism of Ω sending each point to a point at (maximal) distance $\frac{n}{2}$. Write $A = \langle\alpha\rangle$. Since $n \geqslant 10$, the ball around a point $A(p)$ of Ω/A of radius 2 is equal to the ball around the original p of same radius in Ω. Hence Axiom (H) holds in Ω/A.

More about quotients of buildings. Besides the absence of proper admissible homomorphic images of the relevant point-line geometries, the paper [9] asserts that the corresponding spherical buildings do not admit quotient geometries of type M, as defined by Tits in [10]. We quote from [9] Remark 2.8 verbatim: *Geometries G with diagram* $\mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ *are already building geometries.* Whereas this is indeed true for the types D_n and E_6, as shown in [10] and [1], respectively, this is certainly false for both E_7 and E_8, as shown in [11]. Note that the above example does not produce a quotient geometry of type M; the rank 2 residues are destroyed, so the original building is not a 2-cover of its quotient with A. Conversely, not every "Grassmannian" of a geometry of type M arising from a quotient of a building with simply laced diagram defines a parapolar space. Indeed, a necessary condition for a spherical building to admit a quotient geometry is that the diameter of each k-Grassmannian is at least 3; so, loosely speaking, if the diameter of some k-Grassmannian is exactly 3 (and all others are at least 3), then a non-trivial quotient does not define a parapolar space "at node k". A quotient defines a parapolar space if and only if no point is mapped onto a point at distance ≤ 4.

For example, the compact forms of the complex Lie groups of exceptional types E_7 and E_8 are defined by Galois involutions σ_7 and σ_8 of the buildings $\mathsf{E}_7(\mathbb{C})$ and $\mathsf{E}_8(\mathbb{C})$, respectively, which map every flag to an opposite. This defines parapolar spaces $\mathsf{E}_{7,2}(\mathbb{C})/\langle\sigma_7\rangle$, $\mathsf{E}_{8,1}(\mathbb{C})/\langle\sigma_8\rangle$ and $\mathsf{E}_{8,2}(\mathbb{C})/\langle\sigma_8\rangle$. However, neither $\mathsf{E}_{7,7}(\mathbb{C})/\langle\sigma_7\rangle$, nor $\mathsf{E}_{7,1}(\mathbb{C})/\langle\sigma_7\rangle$, nor $\mathsf{E}_{8,8}(\mathbb{C})/\langle\sigma_8\rangle$ is a parapolar space since $\mathsf{E}_{7,1}$ and $\mathsf{E}_{8,8}$ have diameter 3.

About the Y_1-geometries. In Case (2) of the original version of the Haircut Theorem, it is stated (but not proved) that, if Ω is neither extreme nor strong, then it is isomorphic to $\mathsf{E}_{6,2}(\mathbb{K})$ for some field $\mathbb{K}$. In fact, it is wrong, as the Lie incidence geometries $\mathsf{E}_{7,2}(\mathbb{K})$ and $\mathsf{E}_{8,2}(\mathbb{K})$, for any field $\mathbb{K}$, are neither strong nor extreme. What we can prove is that, if the point residues are isomorphic to $\mathsf{A}_{5,3}(\mathbb{K})$, then Ω is the long root geometry $\mathsf{E}_{6,2}(\mathbb{K})$.

About the symplectic rank not required to be constant. The previous errors in [9] can easily be dealt with. We now point out two errors in the proof which do not seem to be so easily fixed.

The first one occurs in the proof of Lemma 4.1. Suppose we are given a parapolar space Σ satisfying the hypotheses of in the original version of Shult's Haircut Theorem. Let η be a symp of Σ of maximal polar rank n. This maximum exists and is at least 3 by (i) of the original version of the Haircut Theorem. Let A be a singular subspace of dimension $n-k-1 \leq n-3$ contained in η. The aim of Lemma 4.1 of [9] is to show that the residue Σ_A of Σ at A (which can be obtained by taking consecutive point residues) is connected.

Let $\mathscr{M}$ be the set of points of Σ_A corresponding to the singular subspaces of dimension $n-k$ lying in a fixed symplecton of polar rank $n-k+1$ and containing A. In the proof of Lemma 4.1, it seems to be assumed that each connected component of Σ_A contains either 0, 1, or all elements of $\mathscr{M}$. This however requires an argument, which is missing.

We will discuss this in greater detail. The proof of the lemma uses induction on $\dim(A)$. The case where $\dim(A) = -1$ is trivial; and since Σ is locally connected, Σ_A is by definition connected when $\dim(A) = 0$. So we may assume that $\dim(A) \geq 1$. We will even assume that $k = n-2$, so A is a line L (actually, using induction, we are allowed to do so, but even regardless of that, it serves as an enlightening example). Thus, $\Sigma_A = \Sigma_L$ and $n \geqslant 4$.

Suppose that Σ_L is not connected and let Γ be the connected component of Σ_L containing a plane of η. Then there are singular planes β and γ in Σ containing L, corresponding to respective points B and C in Σ_L, with $B \notin \Gamma$ and $C \in \Gamma$. In particular, the points B and C are not collinear, i.e., the planes β and γ are not contained in a singular subspace. It then follows from the parapolar space axioms that β and γ are jointly contained in a symp ζ of rank 3 (if the rank were higher, $B \in \Gamma$ after all). Now consider a point E in Γ which is not collinear to C (this exists since Γ contains the residue at L of the symplecton

η of rank at least 4), which corresponds to a plane ε containing L. The haircut axiom (H) implies that a point $p_E \in \varepsilon \setminus L$ is collinear to a plane μ in ζ containing L. The point M of Σ_L corresponding to μ then belongs to Γ and is distinct from C since E is collinear to M but not to C. In case ζ is non-thick, this implies that $B = M \in \Gamma$, a contradiction. In the thick case, let $\mathscr{M}$ be the set of points corresponding to the planes of ζ containing L. We now arrive at the point where the assumption is used that, if two distinct points (namely, C and M) of $\mathscr{M}$ belong to Γ, then all points of $\mathscr{M}$ do, in particular B. However, as mentioned above, we see no argument to support this conclusion.

The second error occurs in the arguments for the case $k = 4$, where symps of rank 2 have to be excluded. It suffices to show that the parapolar space under consideration is strong, and to do this, Lemma 2.3(3) is invoked, which however requires all symps to have rank at least 3. Again, we do not see how to mend this.

3.3 Conclusion

The paper [9] appeared without Shult being able to finalise the details of his important main theorem. The current statement and proof contain some inconsistencies. Taking the above observations into account, the validity of the theorem as published can be made to hold by simply requiring constant rank and incorporating the homomorphic images. However, as mentioned before, we prefer to state a slightly more general version, provided with a full proof.

4 The revised Haircut Theorem and its proof

4.1 Statement of the revised Haircut Theorem

In order to state our revision for the locally connected case, containing a slight extension, we need some additional notation concerning polar spaces of rank 2, that is, so-called *generalized quadrangles*. In our context, these are point-line geometries with thick lines so that each point is contained in at least two lines, each pair of points is contained in at most one line, and each point x is collinear to a unique point of any line not containing x (we refer to the latter as the *main axiom*). We call the generalized quadrangle *thick* if every point is contained in at least three lines; non-thick otherwise. In the latter case, every point is contained in exactly two lines. An *ideal subquadrangle* Γ' of a generalized quadrangle Γ is a subset of points of Γ together with a line set which is the set of all intersections of lines of Γ with Γ', which is itself a generalised quadrangle. Note that this implies that any line of Γ either is disjoint from Γ', or intersects Γ' in a line of Γ', in particular, never in a single point.

Theorem 4.1 (The revised Haircut Theorem—locally connected case)**.** *Suppose $\Omega = (\mathscr{P}, \mathscr{L})$ is a parapolar space of constant symplectic rank k at least 2 that is not a polar space. Assume the following three properties.*

- (i) *If $k \geq 3$, then each singular space possesses a finite projective dimension.*
- (ii) *The "Haircut axiom":*
  - (H) *If ξ is a symplecton and x is a point not in ξ, then $x^\perp \cap \xi$ cannot be a hyperplane of a maximal singular subspace of ξ.*
- (iii) *If $k \geq 3$, then Ω is a locally connected space; if $k = 2$, then Ω is strong.*

Then one of the following occurs:

- (0) *$k = 2$ and, if we denote the set of maximal singular subspaces of Ω by Σ and (as before) the set of symps of Ω by Ξ, then the point-block geometry $\Gamma = (\mathscr{P}, \Sigma)$ is a (thick or non-thick) generalized quadrangle with the property that Ξ is a family of proper ideal subquadrangles of Γ such that each pair of non-collinear points of Γ is contained in a unique member of Ξ and lines of different* *members of Ξ which intersect in at least two points, coincide. Moreover, $\mathscr{L}$ is the collection of lines of members of Ξ.*

(1) *$k = 3$ and Ω is either a Grassmann space $\mathsf{A}_{n,d}(\mathbb{L})$, $2 \leq d \leq \frac{n+1}{2}$, $n \geq 4$, or an admissible homomorphic image $\mathsf{A}_{2d-1,d}(\mathbb{L})/\langle\sigma\rangle$, where σ is a polarity of $\mathsf{A}_{2d-1}(\mathbb{L})$ of Witt index at most $d-5$, and $\mathbb{L}$ is any skew field.*

(2) *$k = 4$ and there are integers l and m with $1 \leq l \leq m$ and $m \geq 2$ and a field $\mathbb{K}$ such that each point residue Ω_p is isomorphic to either $\mathsf{A}_{m+l+1,l+1}(\mathbb{K})$ or $\mathsf{A}_{2l+1,l+1}(\mathbb{K})/\langle\sigma_p\rangle$, σ_p a polarity of Witt index at most $l-4$ (possibly) dependent on the choice of the point p. If some point residue is isomorphic to $\mathsf{A}_{m+l+1,l+1}(\mathbb{K})$ with $l < m$, then Ω is an admissible homomorphic image of the 1-Grassmannian of a building of type $\mathsf{Y}_{(1,l,m)}$. If some point residue is isomorphic to $\mathsf{A}_{5,3}(\mathbb{K})$, then Ω is the long root geometry $\mathsf{E}_{6,2}(\mathbb{K})$. If Ω is strong, then $l = 1$ and Ω is an admissible homomorphic image of the half spin geometry $\mathsf{D}_{m+3,m+3}(\mathbb{K})$, the homomorphisms being isomorphisms if $m \leq 6$. Also, Ω is extreme if and only if some point residue is neither a line Grassmannian (type $\mathsf{A}_{m+2,2}$) nor a plane Grassmannian (type $\mathsf{A}_{m+3,3}$).*

(3) *$k = 5$ and Ω is isomorphic to $\mathsf{E}_{m+4,1}(*)^h$, $m \geq 2$. If $m = 4$, then Ω is an admissible homomorphic image of the Lie incidence geometry $\mathsf{E}_{8,1}(\mathbb{K})$, $\mathbb{K}$ any field. If $m = 3$, then Ω is the long root geometry $\mathsf{E}_{7,1}(\mathbb{K})$, $\mathbb{K}$ any field. If Ω is a strong parapolar space, then $m = 2$ and Ω is the Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathbb{K}$ any field.*

(4) *$k = 6$ and Ω is the Lie incidence geometry $\mathsf{E}_{7,7}(\mathbb{K})$, $\mathbb{K}$ any field, a strong parapolar space.*

(5) *$k = 7$ and Ω is the long root geometry $\mathsf{E}_{8,8}(\mathbb{K})$, $\mathbb{K}$ any field.*

Conversely, all of the listed geometries satisfy the hypotheses.

If $k = 2$, then the geometry Γ of (0) is non-thick as soon as a maximal singular subspace is a projective space. This follows from noting that the parapolar spaces in (0) are so-called imbrex geometries (introduced and classified in [7]) and then using Theorem 3.9 of [7]. If Γ is non-thick, then Ω is the Cartesian product of two linear spaces, at least one of which does not consist of a unique line.

4.2 Strategy of the proof

We prove Theorem 4.1 in a sequence of lemmas, performing an induction on the symplectic rank k. We use ideas of [9] but avoid the use of sheaf theory, explained and introduced in [8], since some proofs using that theory in [8] are either only sketched, or omitted. In general, sheaf theory is used to recognize certain parapolar spaces from their common point residues. In [6] we have proved some local recognition theorems, and we will use those and one more, see Lemma 4.6. The proof of the latter resembles that of Lemma 5.1 of [6], but since there are some essential differences, we include it in full.

The following lemma will enable us to induct on the symplectic rank k.

Lemma 4.2. *Let k be a natural number greater than 1 and let Ω be a parapolar space of constant symplectic rank. Consider the following properties.*

(LC) *Ω is locally connected;*

(S) *Ω is strong;*

(FD) *Ω contains only finite-dimensional singular subspaces;*

(NP) *Ω is not a polar space;*

(SR)$_k$ *Ω has constant symplectic rank k.*

If $k \geqslant 3$, then Ω satisfies (LC), (H), (FD), (NP) *and* (SR)$_k$ *if and only if for each point p of Ω, the point residual Ω_p is a parapolar space satisfying* (H), (FD), (NP), (SR)$_{k-1}$ *and* (S).

Proof. Let $k \geq 3$. First suppose that Ω satisfies (LC), (H), (FD), (NP) and (SR)$_k$. The point residue Ω_p at each point p of Ω is a strong parapolar space of constant symplectic rank $k-1$, by Lemma 2.3 of [9], and satisfies (H) by Lemma 2.4 of [9]. Clearly, all singular subspaces of Ω_p are finite-dimensional. It remains to show that Ω_p is not a polar space. To see this, let ξ be a symp on p. Since Ω is connected and not a polar space, there is a point q outside ξ such that $q^\perp \cap \xi$ contains a point r. If r is not collinear with p, the local connectivity at r implies that we may assume that $q^\perp \cap \xi$ contains a line. Replacing r by the point on this line collinear with p (which exists since ξ is a polar space), we may assume that r is collinear with p. Again, by connectedness of Ω_r, we may assume, without loss of generality, that either $q \in p^\perp$, or $q^\perp \cap p^\perp$ contains a line through r. In the former case we have found a line through p not in ξ; in the latter case p and q are contained in a symp distinct from ξ and we reach the same conclusion. This shows that Ω_p is not polar.

Now suppose for each point p of Ω, the point residual Ω_p is a parapolar space satisfying (H), (FD), (NP), $(\mathrm{SR})_{k-1}$ (still $k \geq 3$), and (S). By definition Ω satisfies (LC). By Lemma 2.4 of [9], Ω satisfies (H). Clearly, Ω satisfies (FD), (NP) and $(\mathrm{SR})_k$. $\square$

Finally, we will use the following improvement of a result of Cooperstein [5], stated in the language and notation of the present paper, due to the first author.

Lemma 4.3 (Main Theorem of [4]). *Suppose Ω is a parapolar space of constant symplectic rank* 3 *that is not a polar space. Assume the following properties.*

(i) *Each singular space possesses a finite projective dimension.*

(ii) *The Haircut axiom:*

> (H) *If ξ is a symplecton and x is a point not in ξ, then $x^\perp \cap \xi$ cannot be a line.*

(iii) *Ω is strong (and hence in particular locally connected).*

Then the following occurs.

(1) *Ω is either a Grassmann space $\mathsf{A}_{n,d}(\mathbb{L})$, $2 \leq d \leq \frac{n+1}{2}$, $n \geq 4$, or a homomorphic image $\mathsf{A}_{2d-1,d}(\mathbb{L})/\langle\sigma\rangle$, where σ is a polarity of $\mathsf{A}_{2d-1}(\mathbb{L})$ of Witt index at most $d-5$, and $\mathbb{L}$ is any skew field.*

Conversely, the listed geometries satisfy the hypotheses.

We are now ready to embark on the proof of Theorem 4.1.

4.3 The cases $k = 2, 3$ of Theorem 4.1

The proof of the next lemma exploits ideas used in [4], by the first author, and ideas used in [7], by the third and fourth author. It completely settles the case $k = 2$ of Theorem 4.1.

Lemma 4.4 (Case $k = 2$ of Theorem 4.1). *Suppose Ω is a parapolar space of constant symplectic rank* 2 *that is not a polar space. Assume Ω satisfies the assumptions of Theorem 4.1, that is,*

(ii) *The Haircut axiom for rank 2:*

> (H_2) *If ξ is a symplecton and x is a point not in ξ, then $x^\perp \cap \xi$ cannot be a point.*

(iii) *Ω is strong.*

Then the conclusion of Case (0) *(for $k = 2$) of* Theorem 4.1 *holds.*

Conversely, the geometries Ω obtained from generalized quadrangles Γ as described in conclusion (0) *of* Theorem 4.1 *satisfy the hypotheses of the theorem.*

Proof. Suppose $\Omega = (\mathscr{P}, \mathscr{L})$ is a strong parapolar space of constant symplectic rank 2 satisfying (H_2) that is not a polar space. Since Ω is connected, each point lies on a line, and so each maximal singular subspace (whose existence is a direct consequence of Zorn's lemma) strictly contains a point. We first show that $\Gamma = (\mathscr{P}, \Sigma)$ is a generalized quadrangle satisfying the properties stated in seven steps.

Claim 4.4.1. *Two distinct maximal singular subspaces $M_1, M_2 \in \Sigma$ intersect in at most one point.*
Suppose for a contradiction that M_1, M_2 have a line L in common. Consider any point $p_i \in M_i \setminus L$, $i = 1, 2$. If $p_1 \notin p_2^{\perp}$, then they are together with L contained in a symp, contradicting the fact that symps have rank 2. Hence $p_1 \perp p_2$. Since p_i in $M_i \setminus L$ was arbitrary, M_1 and M_2 are together contained in a larger singular subspace, contradicting their maximality. ■

Claim 4.4.2. *A maximal singular subspace $M \in \Sigma$ and a symp $\xi \in \Xi$ either intersect in a line, or are disjoint.*
Suppose M and ξ have a point p in common. First note that $M \cap \xi$ is a singular subspace of ξ, and recall that $\dim(M) \geq 1$; in particular, if $M \subseteq \xi$, the statement is trivial. So we may consider a point $x \in M \setminus \xi$. Then by (H_2), x is collinear to a line L of ξ. Now, any singular subspace containing x and L shares the line xp with M, and so, by Claim 4.4.1, L is contained in M. ■

Claim 4.4.3. *For every $M \in \Sigma$ and every point $p \in \mathscr{P} \setminus M$, there exists a unique $N \in \Sigma$ containing p and intersecting M in a unique point.*
By Claim 4.4.1, N is unique if it exists: if $N' \in \Sigma$ enjoys the same properties, then a singular subspace containing $p, M \cap N$ and $M \cap N'$ shares with both N, N' a line. Hence it suffices to show that $p^{\perp} \cap M$ is non-empty. By connectivity we may assume that $p = p_0 \perp p_1 \perp p_2 \perp \ldots \perp p_\ell \in M$ is a shortest path connecting p with M. Assume for a contradiction that $\ell \geq 2$. Then, by strongness and minimality of the path, there exists a symp ξ containing $p_{\ell-2}$, $p_{\ell-1}$ and p_ℓ. By Claim 4.4.2, ξ and M have a line m in common, and so $p_{\ell-2}$ is collinear to some point of $m \subseteq M$, contradicting the minimality of ℓ. The claim is proved. ■

Claim 4.4.4. $\Gamma = (\mathscr{P}, \Sigma)$ *is a generalized quadrangle (with thick lines).*
By Claim 4.4.1. two points are contained in at most one line. The main axiom was proven in Claim 4.4.3. The lines of Γ are thick as $\dim(M) \geq 1$ for all $M \in \Sigma$. Since each point of $\mathscr{P}$ is contained in a member of Ξ, it belongs to at least two lines, each of which is contained in a different member of Σ. Therefore, the claim is proved. ■

Claim 4.4.5. *Each member of Ξ is a proper ideal subquadrangle of* Γ.
We already know that members of Ξ are (non-degenerate) polar spaces of rank 2 with thick lines, that is, generalized quadrangles. The fact that every non-empty intersection of a member of Σ with a member of Ξ is a line was proved in Claim 4.4.2. A member of Ξ must be proper because otherwise Ω would be a generalized quadrangle, which contradicts the assumption that it is not a polar space. ■

Claim 4.4.6. *Each pair of non-collinear points of Γ is contained in a unique member of Ξ.*
Since Ω is strong, this follows from the fact that Γ is a generalized quadrangle (and so has diameter 2) and the fact that the relation on $\mathscr{P}$ of collinearity in Γ coincides with collinearity in Ω, so the diameter of Ω is 2. ■

Claim 4.4.7. *Lines of different members of Ξ which intersect in at least two points, coincide.*
This is immediate from the fact that, in a parapolar space, two points are contained in at most one line. ■

It remains to verify that the geometries in the conclusion satisfy the hypotheses. To this end we suppose that $\Gamma = (\mathscr{P}, \Sigma)$ is a generalized quadrangle with the property that Ξ is a family of proper ideal subquadrangles of Γ such that each pair of non-collinear points of Γ is contained in a unique member of Ξ and lines of different members of Ξ which intersect in at least two points, coincide. We let $\mathscr{L}$ be the collection of lines of members of Ξ. We need to establish that $\Omega = (\mathscr{P}, \mathscr{L})$ is a strong parapolar space with Ξ as set of symps (and hence of constant symplectic rank 2) satisfying (H_2).

Since each line of Ω is contained in a member of Σ, collinearity in Ω implies collinearity in Γ. Conversely, if two points p, q are collinear in Γ, then by the properties of generalized quadrangles, there is a third point r collinear with q but not with p, so there is a member ξ of Ξ containing p and r. The line $M \in \Sigma$ on p and q contains a unique point collinear with r; this point must be q. The intersection $m := M \cap \xi$ contains p and so must be a line of ξ and hence a line of Ω. The line m of ξ contains a point collinear (in ξ) to r (and hence also collinear in Γ by the foregoing); this point must be q. We conclude that collinearity relation on $\mathscr{P}$ in Γ coincides with collinearity in Ω. We write $\perp$ for this collinearity. Since Γ is connected of diameter 2, so is Ω.

Suppose p is a point and l is a line of Ω not containing p, but having two points, say a and b, collinear to p. Then a and b are on a unique line $M \in \Sigma$ and both a and b belong to $p^\perp \cap M$. Since Γ is a generalized quadrangle, this forces p to be in M. In particular, p is collinear to all of l in Ω. This shows that Ω is a gamma space.

We claim that the members of Ξ are geodesically closed subspaces of Ω. To see this, assume x, y are non-collinear points of Ω belonging to a subquadrangle ξ in Ξ and z is a point collinear to both x and y. Let M be the unique line of Γ on x and z. Then y does not belong to M (since y and x are not collinear in Ω, they are not collinear in Γ). The intersection $M \cap \xi$ is a line of Ω contained in ξ, so there is a unique point in $y^\perp \cap M \cap \xi$. But z is the unique point of $y^\perp \cap M$, so z belongs to ξ. Since, by definition, ξ is a subspace of Ω, we find that ξ is a geodesically closed subspace of Ω.

By assumption, members of Ξ are generalized quadrangles and, for every pair of non-collinear points, there is a member of Ξ containing the pair. By definition, every line of Ω is contained in a member of Ξ. We conclude that Ω is a strong parapolar space with Ξ as set of symps.

As for (H_2), assume that ξ is a member of Ξ and p is a point of Ω such that $p^\perp \cap \xi$ contains a point q. The line M of Γ on p and q meets ξ in at least the point q, so, by properties of Ξ, the intersection $M \cap \xi$ is a line of Ω. Since this line belongs to M, it also lies in $p^\perp \cap \xi$, and we conclude that $p^\perp \cap \xi$ is not a point, thereby establishing (H_2).

Finally, since every polar space is the geodesic closure of any pair of non-collinear points contained in it, it follows from the fact that any $\xi \in \Xi$ is proper, that Ω is not polar. $\square$

We can now also finish the case $k = 3$ of Theorem 4.1. Due to Lemma 4.3 we only need to show that Ω is strong. This is equivalent to showing that the diameter of each point-residue equals 2. By Lemma 4.2, each such residual parapolar space is listed in the conclusion of Case (0) of Theorem 4.1, and hence has diameter 2.

Remark 4.5. Shult did not have Case (0) at his disposal. However, he nevertheless used Lemma 4.3, but proved directly that the point residues in Case (1) have diameter 2, see Lemma 2.5 of [9].

4.4 The case $k = 4$ of Theorem 4.1

So far, we have shown (0) and (1), that is, the cases $k = 2, 3$ of Theorem 4.1.

4.4.1 Proof of first statement of conclusion (2) of Theorem 4.1

In this case each point residue Ω_p is one of the parapolar spaces in conclusion (1) of Theorem 4.1, that is, $\mathsf{A}_{n,d}(\mathbb{L})$, $\mathsf{A}_{2d-1,d}(\mathbb{L})$ or $\mathsf{A}_{2d-1,d}(\mathbb{L})/\langle\sigma_p\rangle$, where σ_p is a polarity of $\mathsf{A}_{2d-1}(\mathbb{L})$ of Witt index at most $d-5$, (possibly) dependent on the point p; where n, d are integers with $2 \le d \le \frac{n+1}{2}$ and $\mathbb{L}$ a skew field. We claim that the integers n and d and the skew field $\mathbb{L}$ do not depend on the point p. Indeed, from the nature of Ω_p it follows that each singular plane through p is contained in exactly two maximal singular subspaces, say of dimensions d_1, d_2, which are defined over $\mathbb{L}$. If $d_1 < d_2$, then $d_1 = d + 1$

and $d_2 = n-d+2$ (in which case $\Omega_p = \mathsf{A}_{n,d}(\mathbb{L})$); if $d_1 = d_2$ then $d_1 = d_2 = d+1$ (in which case $\Omega_p \in \{\mathsf{A}_{2d-1,d}(\mathbb{L}), \mathsf{A}_{2d-1,d}(\mathbb{L})/\langle\sigma_p\rangle\}$). So for any point q collinear to p, the existence of a singular plane on the line pq shows that the point residue Ω_q comes with the same integers and skew field $\mathbb{L}$. By connectivity, the parameters coincide across Ω, showing the claim.

We conclude that either all point residues are isomorphic to $\mathsf{A}_{n,d}(\mathbb{L})$, for some integers n,d with $2 \le d < \frac{n+1}{2}$ and a skew field $\mathbb{L}$, or each point residue Ω_p is isomorphic to one of $\mathsf{A}_{2d-1,d}(\mathbb{L})$, $\mathsf{A}_{2d-1,d}(\mathbb{L})/\langle\sigma_p\rangle$, where $d \ge 2$ is an integer, $\mathbb{L}$ a skew field and σ_p is a polarity of $\mathsf{A}_{2d-1}(\mathbb{L})$ of Witt index at most $d-5$, possibly depending on the point p. In each of these cases, the symps are necessarily isomorphic to $\mathsf{D}_{4,1}(\mathbb{L})$, and so $\mathbb{L}$ is commutative. We put $l = d-1$ and $m = n-d$ and write $\mathbb{K}$ instead of $\mathbb{L}$. Then $(n,d) = (m+l+1, l+1)$, and the first statement of conclusion (2) of Theorem 4.1 has been established.

4.4.2 Proof of second statement of conclusion (2) of Theorem 4.1

As for the second statement, we assume $l < m$ (or, equivalently, $2d-1 < n$) and show that Ω is an admissible homomorphic image of a Grassmannian of a building. The proof roughly follows the strategy of the proof of Lemma 5.1 of [6]. It contains Lemma 5.2 of [6] (the proof of which was left to the reader) as a special case.

For convenience of the reader, we reproduce the Coxeter diagram $\mathsf{Y}_{(1,l,m)}$ of Figure 1 in Figure 3.

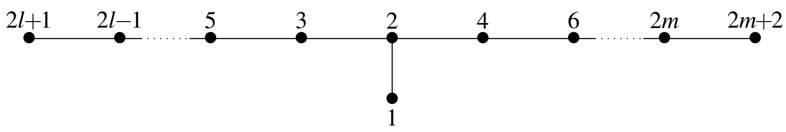

Figure 3: The Coxeter diagram of type $\mathsf{Y}_{(1,l,m)}$, reproduced

Lemma 4.6. *Let $\mathbb{K}$ be a division ring and suppose that $\Omega = (X, \mathscr{L})$ is a (locally connected) parapolar space with the property that each point residue is isomorphic to $\mathsf{A}_{m+l+1,l+1}(\mathbb{K})$, with $l < m$ and $l \ge 1$. Then $\mathbb{K}$ is a field and Ω is an admissible homomorphic image of the 1-Grassmannian of a building of type $\mathsf{Y}_{(1,l,m)}$ (see* Figure 3*).*

Proof. We will use the notion of a *chamber system (of type M)*. Since this is only needed in the current proof, we refer the reader to [10] for all definitions and background. For convenience we recall that the J-residue of a chamber c of a chamber system $\mathscr{C}$ over I, with $J \subseteq I$, is the connected component of c in the graph with vertex set $\mathscr{C}$ and edges the pairs of distinct j-adjacent chambers, with $j \in J$. A rank k residue is a J-residue with $|J| = k$. Each J-residue is a (connected) chamber system over the type set J, in the obvious way.

Let p be a point of Ω. We will use the isomorphism of Ω_p with $\mathsf{A}_{m+l+1,l+1}(\mathbb{K})$ in order to build a chamber system $\mathscr{C}$ of type $\mathsf{Y}_{(1,l,m)}$. Set $I = \{1,\dots,m+l+2\}$. The diagram A_{m+l+1} will be identified with the subdiagram of $\mathsf{Y}_{(1,l,m)}$ on $I \setminus \{1\}$. To avoid confusion about the labeling, we will only refer to the labeling of this diagram as given in Figure 3 (rather than the usual labeling of A_{m+l+1}), except in Claim 4.6.1 below. We will write $\mathscr{B}_p$ for the building of type $I \setminus \{1\}$ associated with Ω_p and $\mathscr{C}_p$ for the corresponding chamber system. It is well known that this building and chamber system are two representations of essentially the same object. More specifically, the chambers of $\mathscr{C}_p$ are viewed as maximal flags of objects of $\mathscr{B}_p$ and, for each $r \in I \setminus \{1\}$, the $(I \setminus \{1,r\})$-residues of $\mathscr{C}_p$ are identified with the objects of type r of $\mathscr{B}_p$.

For $r \in I \setminus \{1\}$, a subspace a of $p^\perp$ containing p is called a *local p-object of type r* if a_p, the point residue of a at p, is the 2-shadow of an object of $\mathscr{B}_p$ of type r. If $r = 2$, this means that a is a line on p, if $r = 3$

then a is a maximal singular subspace of Ω having dimension $m+2$ and containing p, and if $r=4$, it is a maximal singular subspace having dimension $l+2$ and containing p. Since the assignment of the 2-shadow to an object of $\mathscr{B}_p$ is a bijective map, there is a canonical bijective correspondence between the local p-objects of type r and the $(I\setminus\{1,r\})$-residues of $\mathscr{C}_p$ according to which a local p-object a of type r corresponds to the unique $(I\setminus\{1,r\})$-residue in $\mathscr{C}_p$ (this is an object of type r of $\mathscr{B}_p$) whose 2-shadow is a_p. If c is a chamber of $\mathscr{C}_p$ and $r\in I\setminus\{1\}$, we will write c_r for the local p-object of type r corresponding to the $(I\setminus\{1,r\})$-residue in $\mathscr{C}_p$ containing c (it corresponds to the object of type r in the maximal flag in $\mathscr{B}_p$ associated with c).

We will use the following two facts regarding local p-objects a and b of type r.

Claim 4.6.1.

(a) If L is a line containing p and M is a subspace of $L^\perp$ containing L and isomorphic to $L^\perp\cap a$ for a local p-object a containing L, then there is a unique local p-object b containing L such that $M=L^\perp\cap b$.

(b) If π is a singular plane containing p and N is a subspace of $\pi^\perp$ containing π and isomorphic to $\pi^\perp\cap a$ for a local p-object a containing π, then there is a unique local p-object b containing π such that $N=\pi^\perp\cap b$.

These are properties of the point residue Ω_p. Setting $k=l+1$ and $n=m+l+1$, we have $\Omega_p\cong \mathsf{A}_{n,k}(\mathbb{K})$, so we can interpret each object of the building $\mathsf{A}_{n,k}(\mathbb{K})$ via its k-shadow as a subspace of Ω_p. In this setting, we can rephrase the claim as follows.

(a′) If q is a point of $\mathsf{A}_{n,k}(\mathbb{K})$ and M is a subspace of $q^\perp$ containing q and isomorphic to $q^\perp\cap a$ for an object a containing q, then there is a unique object b containing q such that $M=q^\perp\cap b$.

(b′) If π is a line of $\mathsf{A}_{n,k}(\mathbb{K})$ and N is a subspace of $\pi^\perp$ containing π and isomorphic to $\pi^\perp\cap a$ for an object a containing π, then there is a unique object b containing π such that $N=\pi^\perp\cap b$.

If q and M satisfy the hypotheses of *(a′)*, the point residue of $\mathsf{A}_{n,k}(\mathbb{K})$ at q is the Cartesian product of two projective spaces over $\mathbb{K}$, one of dimension l and one of dimension m, and its subspace M/q is the Cartesian product of two singular subspaces of respective dimensions i and j, say, such that either $i<l$ and $j=m$ or $i=l$ and $j<m$. Since each point L/q, where L is a line of M/q, lies in exactly two maximal singular subspaces, one of dimension l and one of dimension m, the restriction $l<m$ enables us to determine which of the components of the Cartesian product decomposition of M/q belongs to which of the two maximal singular subspaces on L. As a consequence, we find that, in the former case, M is an object of type $1+l-i$ and, in the latter case, an object of type $1+l+j$.

The proof of *(b′)* is very similar. As in the argument for *(a′)*, the object b can still be recovered from the union of two singular subspaces on the line π (the union coincides with N). ∎

The chambers of $\mathscr{C}$ will be pairs (p,d), where $p\in\mathscr{P}$ and $d\in\mathscr{C}_p$. The r-adjacencies for $r\in I$ are defined as follows, where (p,c), $(q,d)\in\mathscr{C}$.

(i) If $r>1$ then $(p,c)\sim_r(q,d)$ if and only if $p=q$ and $c\sim_r d$.
(ii) if $r=1$, then $(p,c)\sim_r(q,d)$ if and only if, for each type $s\in I\setminus\{1\}$, we have $c_2^\perp\cap c_s=d_2^\perp\cap d_s$.

In (ii), the situation is that $L=c_2=d_2$ is a line containing both p and q and the objects of type $s\in I\setminus\{1,2\}$ corresponding to c in $\mathscr{B}_p$ and d in $\mathscr{B}_q$ coincide in the residue of L.

Since each r-adjacency is an equivalence relation, $\mathscr{C}$ is a chamber system over I. We now show that $\mathscr{C}$ is a chamber system of type $\mathsf{Y}_{(1,l,m)}$ (see [10]). According to Section 3.2 of [10] it suffices to show that, for pair $i,j\in I$, each $\{i,j\}$-residue is the chamber system of a (proper) projective plane if i and j are connected in $\mathsf{Y}_{(1,l,m)}$ and a generalized digon otherwise, and that $\mathscr{C}$ is connected, that is, the graph with vertex set $\mathscr{C}$ and adjacency relation the union over I of the i-adjacency relations, is connected.

Claim 4.6.2. *Let $i,j \in I$, $i \neq j$. Each $\{i,j\}$-residue of $\mathscr{C}$ is the chamber system of a (proper) projective plane, if i and j are connected by an edge in* Figure 3*, or a generalized digon otherwise.*

Suppose first that $1 \notin \{i,j\}$. Clearly, every $(I \setminus \{1\})$-residue of a chamber containing $p \in \mathscr{P}$ as element of type 1, is the chamber system $\mathscr{C}_p$ of type A_{m+l+1}, so the claim follows.

Now suppose $j = 1$ and $i \neq 2$. Then two chambers (p,c), (q,d) in the same $\{1,i\}$-residue in $\mathscr{C}$ share the same line $L \in \mathscr{L}$, so $L^{\perp} \cap c_s = L^{\perp} \cap d_s$ for each $s \in I \setminus \{1,i\}$. Moreover, by Claim 4.6.1*(a)*, applied to $L^{\perp} \cap d_i$ and to $L^{\perp} \cap c_i$, shows that there is a unique chamber c' in $\mathscr{C}_p$ containing L determined by $c'_s = c_s$ for all $s \in I \setminus \{1,i\}$, and $L^{\perp} \cap c'_i = L^{\perp} \cap d_i$. Similarly, there is a unique chamber d' in $\mathscr{C}_q$ containing L determined by $d'_s = d_s$ for all $s \in I \setminus \{1,i\}$, and $L^{\perp} \cap d'_i = L^{\perp} \cap c_i$. Now, (p,d') and (q,c') are chambers of $\mathscr{C}$ with $(p,c) \sim_i (p,c') \sim_1 (q,d) \sim_i (q,d') \sim_1 (p,c)$. This shows that the $\{1,i\}$-residue is the chamber system of a generalized digon.

Finally suppose $\{i,j\} = \{1,2\}$. Consider the $\{1,2\}$-residue R of a chamber (p,c) of $\mathscr{C}$. The local p-objects of type 3 and 4 are maximal singular subspaces of Ω whose intersection is a singular plane π of Ω containing p. Connectedness of R implies that each chamber (q,d) in R satisfies $d_3 \cap d_4 = \pi$ and $\pi^{\perp} \cap d_s = \pi^{\perp} \cap c_s$ for each $s \geqslant 3$. By Claim 4.6.1*(b)*, therefore, R consists of all chambers (q,d) where $q \in \pi$ and d is a chamber of $\mathscr{C}_q$ with $q \in d_2 \subseteq \pi$ such that d_s is the unique local s-object containing π determined by $d_s \cap \pi^{\perp} = c_s \cap \pi^{\perp}$ for all $i \geqslant 2$. This means that R is the chamber system of the projective plane π, and the claim is proved. ∎

Claim 4.6.3. *The chamber system $\mathscr{C}$ is connected.*

Indeed, let (p,c) and (q,d) be two chambers of $\mathscr{C}$. We argue that these two chambers are connected by a path in the adjacency graph. In view of connectivity of Ω, it suffices to show this if the points p and q are collinear. By connectivity of Ω_p we only need to find a path in case $c_2 = d_2$ is the same line L containing both p and q. We may now assume $p = q$ by replacing (q,d) by the 1-adjacent chamber (p,d') where, for each $s \in I \setminus \{1,2\}$, the object d'_s of $\mathscr{B}_p$ is uniquely determined by $L \subseteq d'_s$ and $L^{\perp} \cap d'_s = L^{\perp} \cap d'_s$. Now the claim follows from the observation that the chamber system naturally induced on all chambers containing p as type 1 element coincides with the $(I \setminus \{1\})$-residue $\mathscr{C}_p$. ∎

Next we want to show that $\mathscr{C}$ is a quotient of a building. According to Corollary 3 of [10], this amounts to show that every rank 3 residue is the chamber system of the quotient of a rank 3 building. In our case, the quotients of the rank 3 residues are trivial ones.

Claim 4.6.4. *Each rank* 3 *residue of $\mathscr{C}$ is the chamber system of a spherical building whose type is determined by the diagram $\mathsf{Y}_{(1,l,m)}$.*

As in the proof of Claim 4.6.1, this follows for $\{i,j,k\}$-residues if $1 \notin \{i,j,k\}$ using the chamber system $\mathscr{C}_p$, which is a spherical building. Also, Claim 4.6.4 follows from Claim 4.6.2 if $\{i,j,k\}$ is disconnected. Hence only the cases of residues of types $\{1,2,3\}$ and $\{1,2,4\}$ remain. These cases are of course similar, and we only consider the case of a $\{1,2,3\}$-residue, say R.

Let (p,c) be a chamber of $\mathscr{C}$ contained in R. The argument we give uses local p-objects of type 5. If $l = 1$, there are no such objects, but the arguments will still be valid if we define c_5 in this case as $p^{\perp}$. Then $S = c_4 \cap c_5$ is a singular 3-space of R. To see this, consider a chamber (q,d) of $\mathscr{C}$ which is i-adjacent to (p,c) for $i \in \{1,2,3\}$. If $i = 2$ or 3, then $p = q$ and $c_4 = d_4$ and $c_5 = d_5$ (with $d_5 = q^{\perp}$ if $l = 1$), so $c_4 \cap c_5 = d_4 \cap d_5$. If $i = 1$ and $p \neq q$, then $L = c_2 = d_2 = pq$ is a unique line contained in c_j and d_j satisfying $L^{\perp} \cap c_j = L^{\perp} \cap d_j$ for each $j \in \{4,5\}$ and so $L^{\perp} \cap S = L^{\perp} \cap d_4 \cap d_5$. But S is a singular subspace of $p^{\perp}$ of dimension 3 (since its point residue at p is a plane by the building properties of Ω_p) and contains L, so $S = d_4 \cap d_5$. Thus, for each chamber (q,d) adjacent to (p,c), we have $d_4 \cap d_5 = S$. By induction on the length of a path to (p,c) in R, we conclude that the intersection of the local q-objects of types 4 and 5 coincides with S for each point of R. This implies that R is the chamber system of the projective space S of dimension 3 (with types 1, 2, 3 for points, lines and planes). Hence the claim. ∎

As mentioned before, Corollary 3 of [10] implies that $\mathscr{C}$ is the quotient of the chamber system of a building of type $\mathsf{Y}_{(1,l,m)}$ by a group G. By construction, $\Omega = (\mathscr{P}, \mathscr{L})$ is the 1-Grassmannian of $\mathscr{C}$ and hence an admissible homomorphic image of the Grassmannian of the building arising from the quotient by G. $\square$

This proves the second statement of Case (2) of Theorem 4.1. If all point residues of Ω are isomorphic to $\mathsf{A}_{5,3}(\mathbb{K})$, then, by Lemma 5.8 of [6], Ω is isomorphic to $\mathsf{E}_{6,2}(\mathbb{K})$. This settles the third statement.

Since point residues of strong parapolar spaces have diameter 2, the Grassmannian $\mathsf{A}_{n,d}(\mathbb{K})$ has diameter d, and $\mathsf{A}_{2d-1,d}(\mathbb{K})/\langle\sigma\rangle$, which only exists for $d \geq 5$, has diameter at least 5, see Section 6.9 of [4]. As a consequence, the parapolar space Ω is strong if and only if $l = 1$, and it is extreme if and only if $l > 2$. If $l = 1$, then the diagram $\mathsf{Y}_{(1,l,m)}$ is equal to D_{m+3} up to the labeling of the indices and Ω is an admissible quotient of $\mathsf{D}_{m+3,m+3}(\mathbb{K})$. If, in addition, $m \leq 6$, then the diameter of Ω is at most 4, and so G must be trivial for the quotient to be admissible, so $\Omega \cong \mathsf{D}_{m+3,m+3}(\mathbb{K})$. This completes the proof of Case (2), where $k = 4$.

Conversely, the point-residues of all parapolar spaces listed in the conclusion of Case (2) of Theorem 4.1 appear in the conclusion of Case (1) of the same theorem. Lemma 4.2 now implies that all parapolar spaces listed in the conclusion of Case (2) of Theorem 4.1 satisfy the hypotheses of Theorem 4.1.

4.5 The cases $k \geq 5$ of Theorem 4.1

Suppose $k = 5$. By Lemma 4.2 and Case (2), Ω_p is isomorphic to $\mathsf{D}_{n,n}(\mathbb{K})^h$, for some field $\mathbb{K}$. Lemma 5.1 of [6] implies that Ω is isomorphic to $\mathsf{E}_{m+4,1}(*)^h$, as stated in (3) of the revised Haircut Theorem. The statements about the isomorphism type of Ω if $m = 2,3,4$, are immediate consequences of the classification of the buildings of the given type and knowledge of 1-Grassmannians of these buildings. Since the diameter of $\mathsf{D}_{n,n}(\mathbb{K})^h$ is at least 3 for $n \geq 6$, the parapolar space Ω is strong only if it is $\mathsf{E}_{6,1}(\mathbb{K})$ for some field $\mathbb{K}$. This establishes Case (3).

Suppose next that $k = 6$. By Lemma 4.2 and Case (3), Ω_p is isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$. It is easy to see that $\mathbb{K}$ does not depend on p. Hence Lemma 5.5 of [6] yields (4). Similarly, this yields (5), that is the case where $k = 7$, of the revised Haircut Theorem.

As in the last paragraph of the previous section, we use Lemma 4.2 to deduce that all listed geometries in the conclusion of Cases (3), (4) and (5) satisfy the hypotheses of the theorem.

Since $\mathsf{E}_{8,8}(\mathbb{K})$ is not strong, there are no locally connected parapolar spaces with constant symplectic rank $k \geq 8$ other than polar spaces admitting only finite-dimensional singular subspaces and satisfying the Haircut Axiom. This ends the proof of Theorem 4.1.

5 Locally disconnected case

In the case of constant symplectic rank at least 3, the local connectivity condition can be dispensed with by use of a technique described by Shult in Chapter 13 of [8], which we reviewed, used, and rephrased as unbuttoning in [6]. We recall the latter description.

Construction 5.1. Let I be a non-empty index set and let $\mathscr{F} = \{\Omega_i = (X_i, \mathscr{L}_i) \mid\in i \in I\}$ be a family of disjoint locally connected parapolar spaces of symplectic rank at least 3. Let $\mathscr{R}$ be an equivalence relation on the union $\widetilde{X} = \bigcup_{i\in I} X_i$ of the sets of points of all members of $\mathscr{F}$, satisfying the following two conditions (C1) and (C2), where δ is the distance function on the collinearity graph on $\widetilde{X}$ assuming the value ∞ at pairs from distinct point sets X_i.

(C1) Let $\widetilde{p},\widetilde{q},\widetilde{r},\widetilde{s}$ be four equivalence classes with respect to $\mathscr{R}$ (not necessarily distinct, but) satisfying $\widetilde{p} \notin \{\widetilde{q},\widetilde{r},\widetilde{s}\}$, and let $p_1, p_2 \in \widetilde{p}$, with $p_1 \neq p_2$. If $q_1,q_2 \in \widetilde{q}$, $r_1,r_2 \in \widetilde{r}$ and $s_1,s_2 \in \widetilde{s}$, then

$$\delta(p_2,q_1)+\delta(q_2,r_1)+\delta(r_2,s_1)+\delta(s_2,p_1) \geq 5.$$

(C2) The graph with vertex set $\mathscr{F}$, where two vertices Ω_i and Ω_j, $i,j \in I$, are adjacent if some point of Ω_i is contained in the same equivalence class as some point of Ω_j, is connected.

Set $X = \widetilde{X}/\mathscr{R}$. For each line L contained in some member of $\mathscr{F}$, we put $\widetilde{L} = \{\widetilde{p} \mid p \in L\}$ and define $\mathscr{L}$ as $\{\widetilde{L} \mid L \in \mathscr{L}_i$ for some $i \in I\}$. Then we denote the geometry $\Omega = (X,\mathscr{L})$ by $\Omega(\mathscr{F},\mathscr{R})$. If $\mathscr{R}$ is non-trivial, then we call Ω a *buttoned geometry*. The members of the family $\mathscr{F}$ are called *sheets*. If $\mathscr{C}$ is a class such that each parapolar space in $\mathscr{F}$ belongs to $\mathscr{C}$, then we say that Ω *is buttoned over* $\mathscr{C}$, and $\mathscr{F}$ is called the *sheet space* of Ω (i.e., not all members of $\mathscr{C}$ occur in $\mathscr{F}$ and those that do occur are allowed to occur multiple times). If the sheet space is a singleton, that is, $|I| = 1$, then we call Ω self-buttoned.

In the current terminology, the following theorem is a consequence of Corollary 13.5.3 of [8]; see also Theorem A.8 of [6].

Theorem 5.2. *Every parapolar space of symplectic rank at least* 3 *is either locally connected, or a buttoned parapolar space over the family of all locally connected parapolar spaces of symplectic rank at least* 3*. Also, the ranks of the symplecta of a buttoned parapolar space are precisely the ranks of the symplecta of the members of its sheet space, and similarly for the dimensions of maximal singular subspaces.*

We then have the following generalization of the revised Haircut Theorem for symplectic rank at least 3. For any integer $k \geqslant 2$, let $\mathscr{C}_k$ denote the family of all locally connected parapolar spaces of constant symplectic rank k satisfying (H) and only admitting finite-dimensional singular subspaces.

The revised Haircut Theorem—General case. *Suppose that Ω is a parapolar space of constant symplectic rank k at least* 3*. Assume the following properties.*

1. *Each singular space possesses a finite projective dimension.*
2. *The "Haircut axiom":*

    (H) *If ξ is a symplecton and x is a point not in ξ, then $x^\perp \cap \xi$ cannot be a hyperplane of a maximal singular subspace of ξ.*

If Ω is not locally connected, it is a buttoned parapolar space over $\mathscr{C}_k$. Conversely, every such parapolar space (which is not locally connected) satisfies the hypotheses.

Proof. Suppose that Ω satisfies the hypotheses of the first statement, so it is not locally connected. By Theorem 5.2, the space Ω is a buttoned parapolar space over the family of all locally connected parapolar spaces of constant symplectic rank k admitting only finite-dimensional singular subspaces. So all we need to show is that each sheet of Ω satisfies (H). Suppose, to the contrary, that Ω_* is a sheet of Ω having a point p_* that is collinear to a hyperplane H_* of a maximal singular subspace U_* of a symp ξ_* of Ω_*. Write $\widetilde{p}$ for the equivalence class of p_* with respect to the equivalence relation $\mathscr{R}$ pertaining to Ω, and write $\widetilde{q}$ for the equivalence class of a point q_* of $U_* \setminus H_*$. Then Condition (C1) implies that $\widetilde{p}$ and $\widetilde{q}$ are not collinear. Also, $\mathscr{R}$ restricted to $p_*^\perp \cup \xi_*$ is trivial, and so the image in Ω of $p_*^\perp \cup \xi_*$ is an isomorphic and isometric copy of it, violating Condition (H). Hence each sheet of Ω is a member of $\mathscr{C}_k$.

Let Ω be a buttoned parapolar space with sheet space $\mathscr{F}$ where $\mathscr{F}$ is a subset of $\mathscr{C}_k$. By Proposition A.9 of [6] it follows that all lines of an arbitrary symp ξ of Ω, and all lines of an arbitrary singular subspace U intersecting ξ in a hyperplane of a maximal singular subspace, are the unique images of lines contained in a common member of the sheet space. Hence it suffices to show that any self-buttoned parapolar space with sheet space $\{\Omega_*\}$ satisfies (H). This is the same reasoning as earlier in this proof used in the reverse direction. □

References

[1] A. E. Brouwer and A. M. Cohen, Some remarks on Tits geometries, *Indag. Math.* **45** (1983), 393–400.

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-Regular Graphs, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3. Folge **18** (1989).

[3] F. Buekenhout and A. M. Cohen, Diagram Geometry, Related to Classical Groups and Buildings, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3. Folge **57** (2013).

[4] A. M. Cohen, On a theorem of Cooperstein, *European J. Combin.* **4** (1983), 107–126.

[5] B. N. Cooperstein, A characterization of some Lie incidence structures, *Geometriae Dedicata* **6** (1977), 205–258.

[6] A. De Schepper, J. Schillewaert, H. Van Maldeghem and M. Victoor, On exceptional Lie geometries, Forum of Math., Sigma (2020), to appear.

[7] J. Schillewaert and H. Van Maldeghem, Imbrex geometries, *J. Combin. Theory Ser. A.* **127** (2014), 286–302.

[8] E. E. Shult, *Points and Lines, Characterizing the Classical Geometries*, Universitext, Springer-Verlag, Berlin Heidelberg (2011), xxii+676 pp.

[9] E. E. Shult, Parapolar spaces with the"Haircut" axiom, *Innov. Incid. Geom.* **15** (2017), 265–286.

[10] J. Tits, A local approach to buildings. in *The Geometric vein. The Coxeter Festschrift*, Coxeter symposium, University of Toronto, 21–25 May 1979, Springer-Verlag, New York 1982, 519–547.

[11] J. Tits, On the distance between opposite vertices in buildings of spherical type (appendix to 'A.E. Brouwer and A. M. Cohen: "Some remarks on Tits geometries"), *Indag. Math.* **45** (1983), 400–402.