LINEAR SPACES EMBEDDED INTO PROJECTIVE SPACES VIA BAER SUBLINES

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ABSTRACT. Every nontrivial linear space embedded in a pappian projective space such that the blocks of the linear space are projectively equivalent Baer sublines with respect to a separable quadratic field extension is either a Baer subspace, or a Hermitian unital.
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We consider linear spaces that are embedded into pappian projective spaces in such a way that each block of the linear space is a Baer subline of the projective space.

A *linear space* is an incidence structure $(Q, \mathscr{B})$ such that any two points of Q are on a unique block $B \in \mathscr{B}$, and every block has at least two points. A linear space is called nontrivial if it has more than one block.

A *Hermitian unital* in a pappian projective space consists of the absolute points of a unitary polarity of Witt index 1 of that space, with blocks induced by secant lines (see Section 1). The finite Hermitian unitals of order q are the classical examples of 2-$(q^3+1, q+1, 1)$-designs; they stem from unitary polarities of pappian planes of order q^2. In any case, the blocks of a hermitian unital are Baer sublines (in the sense of Definition 1.3 below) with respect to a separable quadratic field extension. Conversely, results by Lefèvre-Percsy [4] and by Faina and Korchmáros [1] state that in finite pappian planes the Hermitian unitals are characterized by that property.

In the present paper, we generalize those results in several directions: We consider pappian projective spaces of arbitrary dimension, drop the finiteness assumption, and then characterize

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the class of Hermitian unitals together with Baer subspaces as the linear spaces having blocks that are Baer sublines with respect to a separable quadratic field extension. (In planes, we do not even need the separability assumption.)

We state our result for the finite case first.

Theorem A. *Let V be a vector space over a finite field with $\dim V \geqslant 3$, and let $(Q, \mathscr{B})$ be a nontrivial linear space such that Q is a spanning set of points in the projective space $\mathrm{PG}(V)$, and every block $B \in \mathscr{B}$ is a Baer subline of $\mathrm{PG}(V)$.*

Then either $(Q, \mathscr{B})$ is a Baer subspace of $\mathrm{PG}(V)$, or $\dim V = 3$ and $(Q, \mathscr{B})$ is a Hermitian unital in the projective plane $\mathrm{PG}(V)$, in its standard embedding.

We obtain Theorem A as a special case of Theorem B below; the statement is simpler because each field of order q^2 has a unique subfield of order q. Also, the proof of the finite result is simpler; we complete it (in Proposition 4.6 below) before the proof of Theorem B is finished.

Theorem B. *Let $C|R$ be a separable quadratic extension of fields. Consider a vector space V over C, of dimension at least three. Let $(Q, \mathscr{B})$ be a nontrivial linear space such that Q is a subset spanning $\mathrm{PG}(V)$, and every block $B \in \mathscr{B}$ is a Baer subline of $\mathrm{PG}(V)$ with respect to $C|R$.*

Then $(Q, \mathscr{B})$ is either a Baer subspace (with respect to $C|R$) of $\mathrm{PG}(V)$ or the Hermitian unital $\mathscr{H}(C|R)$ in its standard embedding into $\mathrm{PG}(V)$.

In the planar case (viz., if $\dim V = 3$), Theorem B is an immediate consequence of the following result, which involves the mapping λ defined in 1.3 in Section 1 below. The non-planar case is covered by Theorem D at the end of the paper.

As in [2], we also consider, in the plane, generalized Hermitian unitals $\mathscr{H}(C|R)$ where $C|R$ is *any* (possibly inseparable) quadratic extension of fields; see Definition 1.1 below.

Theorem C. *Let $C|R$ be a quadratic extension of fields. Let $(Q, \mathscr{B})$ be a nontrivial linear space with $Q \subseteq \mathrm{PG}(2, C)$ such that every member of $\mathscr{B}$ is a Baer subline of $\mathrm{PG}(2, C)$ with respect to $C|R$. If the mapping $\lambda|_{\mathscr{B}} \colon \mathscr{B} \to \mathscr{L} \colon B \mapsto \lambda(B)$ is injective then one of the following holds:*

- **a.** *$(Q, \mathscr{B})$ contains O'Nan configurations, and is a Baer subplane with respect to $C|R$.*
- **b.** *$(Q, \mathscr{B})$ does not contain any O'Nan configuration, and $(Q, \mathscr{B})$ is the generalized Hermitian unital $\mathscr{H}(C|R)$, in its standard embedding.*

If $C|R$ is separable, then the mapping $\lambda|_{\mathscr{B}}$ is injective.

If one drops the assumption of injectivity for $\lambda|_{\mathscr{B}}$ then there are additional examples (naturally, with inseparable $C|R$), such as the projection into a line of an inseparable generalized Hermitian unital from its nucleus. If that projection is surjective, that is, if each element of R is a square in C, then one can endow every line with such a linear space; the union is again a linear space. This gives examples where $\lambda|_{\mathscr{B}}$ is neither injective nor constant.

1. Hermitian unitals, Baer subspaces, and Möbius geometry

Let $C|R$ be any quadratic (possibly inseparable) extension of fields; the classical example is $\mathbb{C}|\mathbb{R}$. The following is taken from [2]. We can write $C = R + \varepsilon R$, with $\varepsilon \in C \smallsetminus R$. There exist $t, d \in R$ such that $\varepsilon^2 - t\varepsilon + d = 0$, since $\varepsilon^2 \in R + \varepsilon R$. The mapping

$$\sigma\colon C \to C\colon x + \varepsilon y \mapsto (x + ty) - \varepsilon y \quad \text{for } x, y \in R$$

is a field automorphism which generates $\mathrm{Aut}_R C$: if $C|R$ is separable, then σ has order 2 and generates the Galois group of $C|R$; if $C|R$ is inseparable, then σ is the identity.

For any vector space V over C, we consider the pappian projective space $\mathrm{PG}(V)$: points are one-dimensional subspaces $[v] := vC$ of V; the line set $\mathscr{L}$ of $\mathrm{PG}(V)$ consists of all two-dimensional subspaces of V.

In particular, for any positive integer n, we consider the pappian projective space $\mathrm{PG}(n, C) := \mathrm{PG}(C^{n+1})$ of (projective) dimension n. We use homogeneous coordinates and write points as $[X_0, \dots, X_n] := (X_0, \dots, X_n)C$. Whenever we write $[X_0, \dots, X_n]$ or $[v]$, we tacitly assume that this *is* a point, i.e., that $(X_0, \dots, X_n)$ or v, respectively, is not trivial.

Assume that $C|R$ is separable, and let $h\colon V \times V \to C$ be a non-degenerate Hermitian or skew-Hermitian form on V of Witt index 1, with respect to σ. Mapping a point $[w]$ of $\mathrm{PG}(V)$ to the hyperplane $w^\perp := \{v \in V \mid h(v, w) = 0\}$ then gives a polarity of $\mathrm{PG}(V)$ (in the sense of Tits [5, 8.3.2, p. 128]). The *Hermitian unital* defined by that polarity has point set $U := \{[v] \mid v \leqslant v^\perp\}$; its blocks are induced by secant lines.

Definition 1.1 (see [2])**.** The *generalized Hermitian unital* $\mathscr{H}(C|R)$ is the incidence structure $(U, \mathscr{B})$ with the point set $U := \{[X_0, X_1, X_2] \mid X_0^\sigma X_1 + X_2^\sigma X_2 \in R\varepsilon\}$, and the set $\mathscr{B}$ of *blocks* consists of the intersections of U with secant lines, i.e. with lines of $\mathrm{PG}(2, C)$ containing more than one point of U.

In the next proposition, we identify $\mathscr{H}(C|R)$ in classical terms and motivate the name "generalized Hermitian unital". The nucleus of a quadric is the projective subspace corresponding to the radical of the associated polar form.

Proposition 1.2 (see [2, 2.2, 2.3])**.** *If $C|R$ is separable, then $\mathscr{H}(C|R) = (U, \mathscr{B})$ is the Hermitian unital arising from the skew-Hermitian form $h\colon C^3 \times C^3 \to C$ defined by*

$$h\left((X_0, X_1, X_2), (Y_0, Y_1, Y_2)\right) = \varepsilon^\sigma X_0^\sigma Y_1 - \varepsilon X_1^\sigma Y_0 + (\varepsilon^\sigma - \varepsilon) X_2^\sigma Y_2 \,.$$

If $C|R$ is inseparable, then $\mathscr{H}(C|R)$ is the projection of an ordinary quadric Q in a suitable projective space of dimension at least 3 from a subspace of codimension 1 in the nucleus of Q.

For every point p of U, there is a unique tangent to U in p; i.e., a unique line of $\mathrm{PG}(2, C)$ meeting U just in p.

Definition 1.3. Let E be any basis of V over C, and let $\langle E \rangle_R$ denote the R-span of E. A *Baer subspace* (*with respect to the extension* $C|R$) of $\mathrm{PG}(V)$ is the image $\gamma(P)$ of the point set $P := \{[X] \mid X \in \langle E \rangle_R \smallsetminus \{0\}\}$ under an element $\gamma \in \mathrm{PGL}(V)$. If $\dim V > 2$, we endow the point set $\gamma(P)$ with the set $\mathscr{L}_{\gamma(P)}$ of *blocks* that are obtained as intersections of $\gamma(P)$ with secant lines. Thus every Baer subspace of $\mathrm{PG}(n, C)$ is isomorphic to the projective space $\mathrm{PG}(n, R) \cong (P, \mathscr{L}_P)$ over R.

A Baer subplane of $\mathrm{PG}(V)$ (*with respect to* $C|R$) is a plane of a Baer subspace (*with respect to* $C|R$). A *Baer subline* (*with respect to* $C|R$) of $\mathrm{PG}(V)$ is a line of a Baer subspace (*with respect to* $C|R$).

In particular, for $n = 2$, Baer subspaces and Baer subplanes of $\mathrm{PG}(2, C)$ are the same. Using dimensions of subspaces over R, one sees immediately: Each Baer subplane in $\mathrm{PG}(2, C)$ has the property that every line of $\mathrm{PG}(2, C)$ intersects it in at least one point, and dually, every point of $\mathrm{PG}(2, C)$ is contained in at least one line intersecting the Baer subplane in more than one point (and that intersection is then a Baer subline).

For any Baer subline B, let $\lambda(B) := \langle B\rangle_C$ be the line of $\mathrm{PG}(n, C)$ containing B.

Möbius geometry

We will use various models for the classical Möbius plane related to the extension $C|R$, as follows. Let $\mathscr{M}$ be the geometry with point set $\mathrm{PG}(1, C)$ and blocks all Baer sublines with respect to $C|R$. Let $X^2 + \alpha X + \beta \in R[X]$ be an irreducible polynomial over R having roots in C. Let $\mathscr{O}$ be the quadric in $\mathrm{PG}(3, R)$ with equation $X_0^2 + \alpha X_0X_1 + \beta X_1^2 = X_2X_3$. Endowed with the set $\mathscr{C}$ of all nontrivial plane intersections (that is, plane intersections containing at least two points), this becomes a geometry isomorphic to $\mathscr{M}$. This is the classical Möbius plane related to the extension $C|R$. A planar model is obtained by ("stereographically") projecting $\mathscr{O}$ onto a plane from a point $p \in \mathscr{O}$: The points are then all points of the affine plane $\mathrm{AG}(2, R)$ plus a point ∞, and the blocks are some conics completely contained in $\mathrm{AG}(2, R)$, and all lines of $\mathrm{AG}(2, R)$ (with ∞ added to each line, but to no conic). We refer to that model as the *affine model related to* p.

Remark 1.4. The quadric $\mathscr{O}$ in three-space has a nucleus if, and only if, the extension $C|R$ is inseparable. In fact, the defining quadratic form $X_0^2 + \alpha X_0X_1 + \beta X_1^2 - X_2X_3$ has degenerate polar form if, and only if, the characteristic is 2 and $\alpha = 0$.

Proposition 1.5. *Let $(\mathscr{O}, \mathscr{C})$ be the Möbius plane related to the extension $C|R$, and assume that there exist a set $X \subseteq \mathscr{O}$ of points and a set $\mathscr{Y} \subseteq \mathscr{C}$ of circles in $\mathscr{M}$ such that $(X, \mathscr{Y})$ is a non-trivial linear space. Then $C|R$ is inseparable.*

Proof. Let $Y \in \mathscr{Y}$ be a block of that linear space and let $p \in X \setminus Y$ be a point of it outside Y. We take an affine view of the stereographic projection leading to the affine model related to p: the plane at infinity is the tangent plane to $\mathscr{O}$ in the point p, and the projection is a parallel projection. Without loss of generality, we assume that we project into the plane E containing Y.

The members of $\mathscr{Y}$ joining p with a point of Y are projected to affine lines in E. Since two points of X are on a unique member of $\mathscr{Y}$, these lines are parallel, and they are all tangent to Y. Hence the conic Y has a nucleus n in the projective completion of E, and that nucleus is a point at infinity of the affine plane. Now each of these affine lines is a tangent to $\mathscr{O}$ in three-space, and the line pn is also a tangent to $\mathscr{O}$. With respect to the polar form of the quadratic form defining $\mathscr{O}$, the point n is thus orthogonal to a non-planar set of points. This means that the polar form is degenerate, and $C|R$ is inseparable (see Remark 1.4). □

Applying Proposition 1.5 to a line L of $\mathrm{PG}(V)$ containing more than one member of $\mathscr{B}$, we obtain:

Corollary 1.6. *Let $C|R$ be a quadratic field extension, let V be a vector space of dimension at least 3 over C, and let $(Q,\mathscr{B})$ be a nontrivial linear space with $Q \subseteq \mathrm{PG}(V)$ such that every member of $\mathscr{B}$ is a Baer subline of $\mathrm{PG}(V)$ with respect to $C|R$. If $\lambda|_{\mathscr{B}}\colon \mathscr{B} \to \mathscr{L}\colon B \mapsto \lambda(B)$ is not an injective mapping, then the extension $C|R$ is inseparable.* □

2. Proof of Theorem C

In this section, let $C|R$ be a quadratic extension of fields, and let $(Q,\mathscr{B})$ be a nontrivial linear space with $Q \subseteq \mathrm{PG}(2,C)$ such that every member of $\mathscr{B}$ is a Baer subline of $\mathrm{PG}(2,C)$ with respect to $C|R$.

We assume in this section that the mapping $\lambda|_{\mathscr{B}}\colon \mathscr{B} \to \mathscr{L}\colon B \mapsto \lambda(B)$ is injective.

Theorem 2.1. *If $(Q,\mathscr{B})$ contains at least one O'Nan configuration, then $(Q,\mathscr{B})$ is a Baer subplane of $\mathrm{PG}(2,C)$ with respect to $C|R$.*

Proof. Let $\{p_1,p_2\}$, $\{p_3,p_4\}$ and $\{p_5,p_6\}$ be the two-element sets of points of the O'Nan configuration which are not joined by a block of that configuration. As $\mathrm{PGL}(3,C)$ acts transitively on quadrangles, there is a unique Baer subplane $(P,\mathscr{L}_P)$ of $\mathrm{PG}(2,C)$ (with respect to $C|R$) containing the four points p_1,p_2,p_3,p_4 and it obviously also contains p_5 and p_6. We may assume that p_1,p_3,p_5 are on a common block $B \in \mathscr{B}$. By our standing assumption of injectivity for λ, the block B is then the only block contained in $\lambda(B)$. Since Baer sublines with respect to $C|R$ are determined by any three of their points, the member of $\mathscr{L}_P$ containing p_1 and p_3 belongs to $\mathscr{B}$, just like the members of $\mathscr{L}_P$ that contain $\{p_2,p_3,p_6\}$, or $\{p_1,p_4,p_6\}$, or $\{p_2,p_4,p_5\}$, respectively. Every member of $\mathscr{L}_P$ distinct from those through p_1,p_2, or p_3,p_4, or p_5,p_6, intersects the union of foregoing four members of $\mathscr{L}_P$ in at least three points and hence also belongs to $\mathscr{B}$. It now easily follows that P is entirely contained in Q. Since every point outside P is contained in a line intersecting P in at least two points, our injectivity assumption for λ yields $P=Q$ and then $\mathscr{L}_P=\mathscr{B}$. □

The case with no O'Nan configurations

For the rest of this Section, we assume that $(Q,\mathscr{B})$ does not contain any O'Nan configuration.

Lemma 2.2. *Consider a triangle in $\mathrm{PG}(2,C)$ with vertices u,v,w, and let W and V be Baer sublines containing $\{u,v\}$, and $\{u,w\}$, respectively. Let c be a point of the line $uv=\lambda(W)$, but not contained in W. For each $p \in W$, let D_p be the Baer subline contained in the line pw and obtained by projecting V from c. Then there exists a unique Baer subline B of vw, with $v,w \in B$, such that the projection of $\bigcup\{D_p \mid p \in W \setminus \{u\}\}$ from u into vw coincides with $(vw \setminus B) \cup \{v,w\}$.*

Proof. In suitable homogeneous coordinates we have $u=[1,0,0]$, $v=[0,1,0]$, $w=[0,0,1]$, $W=\{[x,y,0] \mid x,y \in R\}$, and $V=\{[x,0,z] \mid x,z \in R\}$. Then $c=[k,1,0]$ with $k \in C \setminus R$. Let $p=[r,1,0]$ with $r \in R$. Then D_p is the set $\{[r,1,s(r-k)] \mid s \in R\} \cup \{w\}$. Its projection from u onto vw is the set $\{[0,1,s(r-k)] \mid s \in R\} \cup \{w\}$. Since $\{s(r-k) \mid r,s \in R\} = (C \setminus R) \cup \{0\}$, the points not in such a projection are exactly the points of the Baer subline $B=\{[0,y,z] \mid (y,z) \in R^2 \setminus \{(0,0)\}\}$, except for v and w. □

Lemma 2.3. *Let $W \in \mathscr{B}$ be a block, and let w be a point in $Q \setminus W$. For every point $p \in W$, let B_p be the block containing w and p.*

- **a.** *There exists a unique point $c \in \lambda(W)$ such that, for each $p \in W$, each $q \in B_p$, and each $t \in W$, the line cq contains a point of the block B_t.*
- **b.** *That unique point c does not belong to Q.*
- **c.** *The line joining that unique point c and w contains no point of Q apart from w.*

Proof. Let $p \in W$ be arbitrary, and let $q \in B_p \setminus \{w\}$ also be arbitrary. Select two points u and v in $W \setminus \{p\}$. There is a unique Baer subplane $(P, \mathscr{L}_P)$ containing B_u and B_v, and since $(Q, \mathscr{B})$ does not contain any O'Nan configuration, the point set P of that Baer subplane intersects W in just $\{u, v\}$. Let L be the unique line of $\mathrm{PG}(2, C)$ containing q and intersecting P in a Baer subline. By the previous sentence, $L \neq pw$. For $x \in \{u, v\}$, the line L intersects B_x in a point $q_x \neq w$. It follows that the unique Baer subline $B \subseteq L$ containing $\{q, q_u, q_v\}$ belongs to $\mathscr{B}$. Since the projection of a Baer subline is a Baer subline, $B \cap p'w$ is nontrivial for each $p' \in W$.

So we have shown that the set $A := \{q \in \mathrm{PG}(2, C) \mid q \in B_p \setminus \{w\}, p \in W\}$ has a partition into blocks. No other block is contained in A because we assume that there are no O'Nan configurations in $(Q, \mathscr{B})$. Hence that partition is unique. We now show that the lines containing the blocks in that partition all intersect in a common point. Let c be the intersection of two of them, say $\{c\} = L_1 \cap L_2$, with L_i containing a block $B_i \in \mathscr{B}$, with $B_i \subseteq A$. The line pencil in c has a unique Baer subpencil containing L_1, L_2 and cw. By uniqueness, this projects all blocks B_p with $p \in W$. Hence the intersection of any line through c and a point of A with A is a block of $(Q, \mathscr{B})$. Assertion a now follows. If c were in Q then it would belong to $W = \lambda(W) \cap Q$, and we would obtain an O'Nan configuration in $(Q, \mathscr{B})$, so Assertion b is true.

Aiming at a contradiction, assume that some point $a \in cw \setminus \{w\}$ belongs to Q. Then cw contains a block D of $(Q, \mathscr{B})$.

Suppose first that the line au intersects some block B_p, $p \in W \setminus \{u\}$, say in the point b. Let $(P, \mathscr{L}_P)$ be the Baer subplane containing $B_u \cup B_p$. Then P contains c and hence $a \in cw \cap bu$. It follows that each member of $\mathscr{L}_P$ containing a but not w belongs to $\mathscr{B}$, yielding many O'Nan configurations, a contradiction.

Now we apply Lemma 2.2. By the choice of c, the blocks D_p just coincide with B_p. Projecting the points of $\bigcup\{D_p \mid p \in W \setminus \{u\}\}$ from u into vw we obtain all points of vw except those in $B \setminus \{v, w\}$, for some Baer subline B with respect to $C|R$. For each $a \in D \setminus \{w\}$, the argument in the previous paragraph shows that ua meets vw in a point of B. So B is the unique Baer subline (with respect to $C|R$) of vw containing the image of D under the projection. But the projection does not contain v, as $c \notin D$, contradicting $v \in B$. This contradiction completes the proof of Assertion c. □

Lemma 2.4. *Let $C|R$ be a quadratic field extension, with $|R| > 4$. Then the point set H of the generalized Hermitian unital $\mathscr{H}(C|R)$ is determined by the union U of all blocks joining a fixed point w with each point of a block W not containing w, and a point a of $H \setminus U$, together with the property that $\mathscr{H}(C|R)$ is a linear space whose blocks are Baer sublines with respect to $C|R$ in $\mathrm{PG}(2, C)$.*

Proof. By Lemma 2.3, there is a unique point c with the property that each line of $\mathrm{PG}(2,C)$ joining c with any point of any block of $(Q,\mathscr{B})$ through w and a point on W induces a block of $(Q,\mathscr{B})$ intersecting every block through w and a point of W. We may assume that c lies on the line $\lambda(W)$ carrying the block W. By Lemma 2.3, no point of $cw \smallsetminus \{w\}$ belongs to Q. In the Möbius plane induced on $\lambda(W)$ by the Baer sublines with respect to $C|R$, we consider the derived affine plane $\mathscr{A}$ at a', where a' is the projection of a from w onto $\lambda(W)$. Then W is a conic in $\mathscr{A}$. Except for the possible nucleus n of W, every point b of $\mathscr{A} \smallsetminus W$ is on at most two tangents of W. By our assumption $|R| > 4$, the point lies on at least two secant lines M_1, M_2 of W in $\mathscr{A}$. Let M_i intersect W in the points a_{i1} and a_{i2}. Let B_{ij} be the block of $(Q,\mathscr{B})$ joining w with a_{ij}. Let P_i be the point set of the Baer subplane containing $B_{i1} \cup B_{i2}$, $i = 1, 2$. Obviously c belongs to P_i. By the choice of M_i, we see that a' is not contained in P_i, hence a is not contained in P_i and the unique line K_i of $\mathrm{PG}(2,C)$ through a intersecting P_i in a Baer subline does not contain w. Hence K_i intersects B_{ij} in some point $q_{ij} \neq w$, and $q_{i1} \neq q_{i2}$. Then, by injectivity of $\lambda_{\mathscr{B}}$, the block of $(Q,\mathscr{B})$ through q_{i1} and q_{i2} contains a, and contains a point b_i of bw. Obviously $b_1 \neq b_2$ and so the intersection $Q \cap bw$ is determined. If a nucleus n exists, it now also easily follows that $Q \cap nw$ is determined. □

We remark that Lemma 2.4 remains true if $|R| \leqslant 4$; the proof can be extended by considering the three small cases separately. As we do not need the more general result in the present paper, we omit that discussion.

Theorem 2.5. *If $(Q,\mathscr{B})$ does not contain any O'Nan configuration, then $(Q,\mathscr{B})$ is the generalized Hermitian unital $\mathscr{H}(C|R)$, and the embedding is standard.*

Proof. Assume first that Q is finite. Then $q := |R|$ is one less than the (then also finite) number of points on any block in $\mathscr{B}$, and $|C| = q^2$. Consider a triangle in $(Q,\mathscr{B})$ with vertices u, v, w. For every point p in the block $W \in \mathscr{B}$ containing u and v, let B_p be the block containing w and p. By Lemma 2.3, there is a unique point $c \in uv$ such that, for each $p \in W$, each $x \in B_p$, and each $t \in W$, the line cx intersects the block B_t nontrivially, and Q does not contain points of $cw \smallsetminus \{w\}$. We now show that every other line through w contains $q+1$ points of Q. Considering all blocks of $(Q,\mathscr{B})$ through u and a point of B_v, it follows from Lemma 2.2 (by projecting from w) that the only lines through w which possibly only contain one point of Q (namely, w itself), are projected from w onto a Baer subline B^* of uv containing u and v. By Lemma 2.3, B^* also contains c. Varying v over W, we see that the only line through w not containing any point of Q except for w is cw. This shows that Q contains $q^3 + 1$ points and $(Q,\mathscr{B})$ is a unital. This unital is the Hermitian unital $\mathscr{H}(\mathbb{F}_{q^2}|\mathbb{F}_q)$ (by [4] and [1], cp. [3]) since all blocks are Baer sublines, and the embedding is standard (cp. [2]).

Now assume that Q is infinite (in fact, we will only use $|R| > 4$). Since $(Q,\mathscr{B})$ does not contain O'Nan configurations, Lemma 2.3 holds. Consider a triangle in $(Q,\mathscr{B})$ with vertices u, v, w. For every point p in the block $W \in \mathscr{B}$ containing u and v, let B_p be the block containing w and p. Let $c \in uv$ be the unique point such that, for each $p \in W$, each point $q \in B_p$, and each $t \in W$, the line cq intersects the block B_t nontrivially.

Let U be the union of $\{c\}$ and all blocks of $(Q,\mathscr{B})$ joining w with a point of W, and let $A := \{t \in \mathrm{PG}(2,C) \mid t \in cp, p \in B_u\}$; the only points of Q in A are those on the blocks joining

w with a point of W. By Lemma 2.4, the theorem will be proved if we show that for every line $L \neq cw$ through w some subgroup of the stabilizer G of U in $\mathrm{PGL}_3(C)$ acts transitively on $L \smallsetminus A$. Clearly, G contains a group $G(w, cw)$ of elations with axis cw and center w such that $G(w, cw)$ acts transitively on $B_u \smallsetminus \{w\}$, and also a group $G(w, cu)$ of homologies with axis cu, center w, and acting transitively on $B_u \smallsetminus \{u, w\}$. Set $G^* = \langle G(w, cu), G(w, cw)\rangle$.

In suitable affine coordinates, the action of G^* on $uw \smallsetminus \{w\}$ is given by the affine maps $x \mapsto rx+t$, with $r, t \in R$ and $r \neq 0$. For any given $k \in C \smallsetminus R$, the set $\{rk + t \mid r, t \in R, r \neq 0\}$ coincides with $C \smallsetminus R$, and we obtain transitivity of G^* on $uw \smallsetminus B_u$.

Projecting from c (which is fixed by G^*) onto L (which is also stabilized by G^*) we obtain the transitivity we want.

So $(Q, \mathscr{B})$ is the generalized Hermitian unital $\mathscr{H}(C|R)$, in its standard embedding. □

Theorem C is a now a consequence of Theorem 2.1, Theorem 2.5, and Corollary 1.6. The planar case of Theorem B (where $\dim V = 3$) follows.

3. Generalized Hermitian unitals in the plane

The following observation shows that generalized Hermitian unitals have geometric dimension two. In Lemma 4.7 below, we generalize this to Hermitian linear spaces (with respect to *separable extensions*) in projective spaces of arbitrary dimension.

Recall that a *full* subspace of a linear space is a subset F of the point set such that, for any two points in F, every point on the block joining them belongs to F.

Lemma 3.1. *Every full subspace of a generalized Hermitian unital is either the unital itself, or a block, or has at most one point.*

Proof. A full subspace of a generalized Hermitian unital $\mathscr{U}(C|R)$ is also embedded in the projective plane $\mathrm{PG}(2, C)$ with all blocks being Baer sublines with respect to $C|R$. If the subspace is not contained in a block then it spans $\mathrm{PG}(2, C)$. Hence Theorem 2.5 says that the subspace is the generalized Hermitian unital itself (since it does not contain O'Nan configurations), in its standard embedding. □

Lemma 3.2. *Let $(Q, \mathscr{B})$ be a nontrivial linear space with $Q \subseteq \mathrm{PG}(2, C)$ such that each member of $\mathscr{B}$ is a Baer subline of $\mathrm{PG}(2, C)$ with respect to $C|R$, and that the mapping λ is injective. Then $(Q, \mathscr{B})$ is a Baer subplane if and only if it contains a triangle of blocks contained in a Baer subplane.*

Proof. Clearly every Baer subplane contains such a triangle of blocks. Conversely, let $\{u, v, w\}$ be the vertices of such a triangle, and choose two points p, q on different sides (and different from the vertices). The block joining p and q in the Baer subplane meets the third side in a point of the subplane, and that point belongs to Q. This gives an O'Nan configuration in $(Q, \mathscr{B})$, and Theorem 2.1 applies. □

We also have the following property of Hermitian unitals.

Lemma 3.3. *Let $\mathscr{H}(C|R) = (U, \mathscr{B})$ be a generalized Hermitian unital. Let $w \in U$ be any point and $W \in \mathscr{B}$ any block not containing w. Then every Baer subplane (of $\mathrm{PG}(2, C)$ with respect to $C|R$) containing w and W contains one, and only one, block of $\mathscr{H}(C|R)$ through w and some point of W.*

Proof. Let $(P, \mathscr{L}_P)$ be a Baer subplane (with respect to $C|R$) containing w and W. From Lemma 3.2 we infer that there is at most one block $B \in \mathscr{L}_P \cap \mathscr{B}$ through w and meeting W.

By Lemma 2.3.a there are a point c on $\lambda(W) \setminus W$ and a line L through c such that L intersects every block of $\mathscr{H}(C|R)$ through w and a point $t \in W$, and that intersection is outside $\{w, t\}$.

The lines L and $\lambda(W)$ meet in $c \notin W$, so $L \cap P$ is not a block of $(P, \mathscr{L}_P)$ but consists of just one point s. The block $B \in \mathscr{L}_P$ joining w and s meets $\lambda(W)$ in a point $t \in W$ because $(P, \mathscr{L}_P)$ is a projective plane and $W \in \mathscr{L}_P$. So s belongs to the block $B_t \in \mathscr{B}$ joining w and t in $\mathscr{H}(C|R)$. Now B_t and B are Baer sublines with respect to $C|R$, and $\{w, s, t\} \subseteq B_t \cap B$ yields $B_t = B$. □

4. Projective spaces of higher dimension

Let $C|R$ be a quadratic field extension, and let V be a vector space of (possibly infinite) dimension greater than three over C. Consider a nontrivial linear space $(Q, \mathscr{B})$ with $Q \subseteq \mathrm{PG}(V)$, such that Q spans $\mathrm{PG}(V)$, and such that each member of $\mathscr{B}$ is a Baer subline of $\mathrm{PG}(V)$ with respect to $C|R$. Assume that the mapping $\lambda|_{\mathscr{B}}$ is injective.

Definition 4.1. For any set X of points of $\mathrm{PG}(V)$, let $\mathscr{L}_X$ denote the set of intersections of X with secants; i.e. with lines of $\mathrm{PG}(V)$ that contain more than one point of X. A *flat* of X is the intersection of X with a plane of $\mathrm{PG}(V)$ containing at least one triangle of points in X.

Theorem 4.2. *Suppose there exists a plane π of $\mathrm{PG}(V)$ such that the flat $\pi_Q := Q \cap \pi$ is a Baer subplane of π. Then $(Q, \mathscr{B})$ is a Baer subspace of $\mathrm{PG}(V)$.*

Proof. It is obvious that the graph with vertices the flats, adjacent when they share a block, is connected. Hence, to show that every flat is a Baer subplane, it suffices to show this for flats sharing a block with π_Q. Let $\alpha_Q = Q \cap \alpha$ be such a flat, where α is a plane of $\mathrm{PG}(V)$. Select $p \in \alpha_Q \setminus \pi_Q$. For each block B in π_Q, we denote by β_B the flat obtained by intersecting Q with the plane spanned by $\{p\} \cup B$. Aiming at a contradiction, we assume that α_Q is a unital.

Claim. *For at most one block A of π_Q, the flat β_A is the point set of a Baer subplane.*

Suppose for a contradiction that β_{B_1} and β_{B_2} are Baer subplanes, with $B_1 \neq B_2$ two blocks of π_Q. As π_Q is a Baer subplane, the blocks B_1 and B_2 meet in a point $b \in \pi_Q$. Let Σ be the solid of $\mathrm{PG}(V)$ spanned by π and α, and let B be the block of $(Q, \mathscr{B})$ containing p and b. There is a unique Baer subspace Σ_B (with respect to $C|R$) of Σ containing all points of $\pi_Q \cup B$. For $i \in \{1, 2\}$, Σ_B clearly contains β_{B_i}, because a Baer subplane (with respect to $C|R$) is determined by any two of its blocks. Now pick $p_i \in B_i \setminus \{b\}$. For $i \in \{1, 2\}$, the block of Σ_B through p and p_i then belongs to $\mathscr{B}$. Lemma 3.2 implies that Q contains the intersection of Σ_B with the plane spanned by $\{p, p_1, p_2\}$. It now follows easily that $\Sigma_B \subseteq Q$. This implies that α_Q contains a Baer subplane, contradicting Lemma 3.3. Thus the Claim is established (under the assumption that the flat α_Q is not a Baer subplane).

Pick any point u of $\pi_q \setminus \alpha_Q$. By the Claim, we find two blocks C_1 and C_2 in π_q with $u \in C_1 \cap C_2$ such that the flats β_{C_1} and β_{C_2} are generalized Hermitian unitals. Let C be the block of $(Q, \mathscr{B})$ through p and u. Select any point $x \in \lambda(C) \setminus C$. For $i \in \{1, 2\}$, let P_i be the Baer subspace (with respect to $C|R$) of Σ_B defined by π_Q and the Baer subline (with respect to $C|R$) containing $\{p, x, u\}$.

Lemma 3.3 secures a point $u_i \in C_i$ such that P_i contains the block D_i of $(Q, \mathscr{B})$ through p and u_i. But then the flat determined by $\{p, u_1, u_2\}$ is a Baer subplane, by Lemma 3.2. Varying x in $\lambda(C) \setminus C$, we obtain at least $|R|$ such Baer subplanes, contradicting the Claim. This contradiction leaves only the possibility that each flat is a Baer subplane.

Now let $(M, \mathscr{L}_M)$ be a Baer subspace of $\mathrm{PG}(V)$ with respect to $C|R$, maximal among the Baer subspaces contained in Q. Aiming at a contradiction, we assume that there exists a point $x \in Q \setminus M$. Let K be the subspace of $\mathrm{PG}(V)$ spanned by $\{x\} \cup M$. For any point $y \in M$ and any block $B \in \mathscr{B}$ through y in M, the flat $\beta_B := Q \cap \langle x, B\rangle$ is a Baer subplane. Let A be the block joining x and y, and let K' be the Baer subspace of K generated by $M \cup A$. Then K' contains β_B, and K' is spanned (as a projective space over R) by $M \cup \{x\}$. Moreover, we have $A \subseteq Q$ because A is a block of β_B. Varying B through y, we see that $K' \cap Q$ contains every block joining x to a point in M. As every line in K' meets the hyperplane M, we obtain $K' \subseteq Q$, contradicting maximality of M. □

Hermitian linear spaces in projective space

Definition 4.3. A *Hermitian linear space in* $\mathrm{PG}(V)$ (*with respect to* $C|R$) is a set Q of points of $\mathrm{PG}(V)$ such that each member of $\mathscr{L}_Q$ is a Baer subline of $\mathrm{PG}(V)$ with respect to $C|R$, at least one flat of Q is not a Baer subplane, and $\mathrm{PG}(V)$ is generated by Q.

As there exists a flat, the dimension of V is at least 3 if there exists a Hermitian linear space in $\mathrm{PG}(V)$. The mapping $\lambda|_{\mathscr{L}_Q} : \mathscr{L}_Q \to \mathscr{L} : B \mapsto \lambda(B) = \langle B\rangle$ is injective by definition of $\mathscr{L}_Q$. From Theorem 4.2 we know that every flat is a Baer subplane if there exists at least one such flat. We conclude that every flat in a Hermitian linear space in $\mathrm{PG}(V)$ is isomorphic to the generalized Hermitian unital $\mathscr{U}(C|R)$, in its standard embedding into the plane inducing the flat.

Lemma 4.4. *Let Q be a Hermitian linear space in* $\mathrm{PG}(V)$. *If* $\dim V \geqslant 4$ *then no plane of* $\mathrm{PG}(V)$ *contains exactly one member of* $\mathscr{L}_Q$.

Proof. Aiming at a contradiction, suppose that some plane π contains exactly one member B of $\mathscr{B}$.

Assume $\dim V = 4$ first. Pick an arbitrary point $b \in B$. It is easy to find a plane α containing b, not containing B, and intersecting Q in a generalized Hermitian unital $\mathscr{U}$. Let $b' \in B \setminus \{b\}$. Let B' be a block of $\mathscr{U}$ containing b. The plane $\langle B', B\rangle$ intersects Q in a generalized Hermitian unital, the tangent line at b' intersects α in a point $x \in \lambda(B') \setminus Q$. Let $K \neq \alpha \cap \langle x, B\rangle$ be a line in α containing x and intersecting $\mathscr{U}$ in a block D. Then $B \not\subseteq \langle b', D\rangle$. Now the plane $\langle b', D\rangle$ contains two lines (namely, $\langle b', x\rangle$ and $\langle b', D\rangle \cap \pi$) through b' that intersect

Q in exactly one point, namely b'. But $\langle b', D\rangle \cap Q$ is a generalized Hermitian unital, which has only one tangent at b'. This contradiction shows that the lemma is true if $\dim V = 4$.

Now assume $\dim V > 4$. Pick a point $x \in \pi \setminus B$. Since Q generates $\mathrm{PG}(V)$, we find a finite set F of points of Q such that the subspace $\langle F\rangle$ contains x. Then $U := \langle F \cup B\rangle$ is a finite-dimensional subspace of $\mathrm{PG}(V)$ spanned by $U \cap Q$. Let W be a subspace of U maximal with respect to the properties $\pi \subseteq W$ and $W \neq \langle W \cap Q\rangle$ (such a subspace exists since π satisfies the given properties and $\dim U$ is finite). Since $W \neq U$, there exists a subspace $W' \supseteq W$ with $\dim W' = 1 + \dim W$. By maximality of W we have $\langle W' \cap Q\rangle = W'$. Hence we can pick points $p \in Q \setminus W$ and $q \in Q \setminus (\langle W \cap Q, p\rangle \cup W)$. Then $A := \langle p, q, B\rangle$ is a projective 3-space with $\langle A \cap Q\rangle = A$. The plane generated by B and $\langle p, q\rangle \cap W$ is contained in A but intersects Q in just B. This contradicts the previous paragraph. This contradiction completes the proof of the lemma. □

Lemma 4.5. *Let Q be a Hermitian linear space in $\mathrm{PG}(V)$. For each point $p \in Q$, the union T_p of the set of lines L with $L \cap Q = \{p\}$ is a hyperplane.*

Proof. We first show that T_p is a subspace. Let $u, v \in T_p \setminus \{p\}$ be two distinct points. Then $\langle p, u\rangle$ and $\langle p, v\rangle$ both intersect Q in only p. We may assume that $\langle p, u\rangle \neq \langle p, v\rangle$. Then the plane $\langle p, u, v\rangle$ does not intersect Q in a unital, since there would be two tangent lines at p to that unital. Using Lemma 4.4 we infer $\langle p, u, v\rangle \cap Q = \{p\}$. Thus $\langle p, u, v\rangle \subseteq T_p$, and T_p is a subspace.

It remains to show that each line intersects T_p nontrivially. Let L be an arbitrary line. We may assume $p \notin L$. Lemma 4.4 yields that the plane $\langle p, L\rangle$ intersects Q in either exactly $\{p\}$ (and then $L \subseteq T_p$), or a Hermitian unital (and then L intersects the tangent line $M \subseteq T_p$ at p to that unital nontrivially). □

The following Proposition establishes Theorem A.

Proposition 4.6. *If C is finite then either $(Q, \mathscr{B})$ is a Baer subspace (and $\mathrm{PG}(V)$ has arbitrary dimension), or a Hermitian unital (and $\mathrm{PG}(V)$ is a plane).*

Proof. The existence of tangent hyperplanes (as established by Lemma 4.5) rules out the finite case for $\dim V \geqslant 4$. Indeed, let $\mathscr{U}$ be any flat and $p \in Q \setminus \mathscr{U}$. A simple counting argument yields that every line of the plane $\langle \mathscr{U}\rangle$ meets $\mathscr{U}$ in at least one point. This leads to a contradiction with the definition of T_p and Lemma 4.5. □

Lemma 4.7. *Let Q be a Hermitian linear space in $\mathrm{PG}(V)$, with respect to a* (not necessarily separable) *quadratic field extension $C|R$. Then every full subspace S of $(Q, \mathscr{B})$ spanning $\mathrm{PG}(V)$ coincides with Q.*

Proof. Every point $p \in Q$ is contained in the span $\langle F\rangle$ of a finite subset F of S. Clearly $\langle F\rangle \cap Q$ is a Hermitian linear space in $\langle F\rangle$, and the full subspace $\langle F\rangle \cap S$ of $\langle F\rangle \cap Q$ spans $\langle F\rangle$ because it contains F. So it suffices to consider the case where $\mathrm{PG}(V) = \langle F\rangle$ (so $d := \dim V$ is finite). Without loss of generality, we may assume that F consists of exactly d points.

The existence of a Hermitian linear space in $\mathrm{PG}(V)$ implies $d \geqslant 3$. If $d = 3$ then $(Q, \mathscr{L}_Q)$ is a (generalized) Hermitian unital in the projective plane over C, and has been treated in Lemma 3.1.

Now we proceed by induction on d, starting with $d = 4$. Consider an arbitrary point $x \in Q$, and pick $p \in F$. The three points in $F \setminus \{p\}$ span a plane π meeting Q in a flat. Let L be the line joining x with p, and let z be the point where L meets π. We find a secant through z if either $C|R$ is separable or $C|R$ is inseparable but z is not the nucleus of the flat. In these cases, the plane generated by p and that secant defines a flat completely contained in S, and $x \in S$ follows.

There remains the case where $C|R$ is inseparable, and z is the nucleus of the flat in π. Then all points of $Q \setminus L$ belong to S, and $Q = \langle Q \setminus L\rangle = S$.

Now consider $d > 4$. By assumption F consists of d points spanning $\mathrm{PG}(V)$. Let p be one of them and let the projective subspace Π be spanned by the $d-1$ points in $F \setminus \{p\}$. Set $\mathscr{U} = \Pi \cap Q$, then $\mathscr{U} \subseteq \Pi \cap S \subseteq S$ by the induction hypothesis. Let $x \in Q \setminus \Pi$ be arbitrary, and let z be the point where the line $L = \langle p, x\rangle$ meets Π. If there exists a secant of $\mathscr{U}$ through z (in particular, if the extension $C|R$ is separable) then the plane spanned by p and that secant intersects Q in a flat and Lemma 3.1 implies $x \in S$.

It remains to consider the case where $C|R$ is inseparable, and every line through z and a point of $\mathscr{U}$ meets $\mathscr{U}$ in just that point. Take any block B of $\mathscr{U}$. Then $\langle B, z, p\rangle$ generates a projective subspace of projective dimension 3, and the induction hypothesis (for $d = 4$) yields $p \in S$. □

Constructing a polarity in the separable case

We now assume that Q is a Hermitian linear space in $\mathrm{PG}(V)$ with respect to a *separable* quadratic extension $C|R$.

Lemma 4.8. *Every point of* $\mathrm{PG}(V)$ *is contained in a secant.*

Proof. Let $x \in \mathrm{PG}(V)$ be arbitrary and select a block $B \in \mathscr{L}_Q$. If x is contained in the line $\lambda(B)$ spanned by B, then we are done. So suppose $\langle x, B\rangle$ is a plane, which then induces a Hermitian unital by Lemma 4.4. Clearly x is contained in a secant of the Hermitian unital since the extension $C|R$ is separable. □

Lemma 4.9. *Consider the Hermitian unital* $(U, \mathscr{B})$ *in the projective plane* $\mathrm{PG}(2, C)$*, and let* $x \mapsto x^\perp$ *be the (unique) polarity such that* U *is the set of absolute points of that polarity. On each secant* L *of the unital, the mapping* $x \mapsto x^\perp \cap L$ *coincides with the unique Baer involution (with respect to* $C|R$*) fixing each point of the block* $L \cap U$ *(and no other). If* L *is a tangent, the same geometric construction yields a constant map (mapping each point to the unique point in* $L \cap U$*).* □

Lemma 4.10. *Let* $x \in \mathrm{PG}(V) \setminus Q$ *be arbitrary. For each secant line* L *through* x *define the point* x_L *as the image of the unique Baer involution (with respect to* $C|R$*) on* L *fixing* $Q \cap L$ *pointwise. For each tangent line* L *through* x*, define* x_L *to be the unique point in* $L \cap Q$*. Then the set* S_x *of all such points* x_L *generates a hyperplane* T_x*.*

Proof. Let $S'_x \subseteq S_x$ be the set of points x_L with L a secant. We claim that $S'_x \cup \{x\}$ spans $\mathrm{PG}(V)$. Indeed, let K be any line through x and let L be a secant through x. By Lemma 4.4, the plane $\langle K, L\rangle$ induces a Hermitian unital in Q, and x is on a second secant M of that unital.

Now K intersects $\langle S'_x \rangle$ in a point of $\langle x_M, x_L \rangle$, and that point is distinct from x. Consequently, $K \subseteq \langle S'_x \cup \{x\} \rangle$ and the claim follows.

So there exists a set $E \subseteq S'_x$ such that $E \cup \{x\}$ is a basis of $\mathrm{PG}(V)$. Then E generates a hyperplane H. Let Q_H be the set of points of Q on the secants L through x such that $x_L \in H$. For each such secant L, the block $L \cap Q$ generates x and x_L. Hence $\langle Q_H \rangle = \langle H \cup \{x\} \rangle = \mathrm{PG}(V)$.

Now let $P \subseteq Q$ be the set of points $y \in Q$ such that $x_{\langle x,y \rangle} \in H$. Clearly $Q_H \subseteq P$. For any two points u, v of P, the plane $\pi := \langle x, u, v \rangle$ intersects Q in a Hermitian unital. If $L \subseteq \pi$ is a secant through x then the point x_L is contained in the line M of π spanned by $x_{\langle x,u \rangle}$ and $x_{\langle x,v \rangle}$ because that line is the polar of x with respect to the polarity corresponding to the Hermitian unital in π (see Lemma 4.9). If $L \subseteq \pi$ is a tangent through x then x_L lies on M, anyway.

We obtain that each point w on the block $\langle u, v \rangle \cap Q$ satisfies $x_{\langle x,w \rangle} \in H$. Hence P is a full subspace of Q. As x lies on at least two secants, we have $x \in \langle P \rangle$ and thus $\langle P \rangle = \langle S'_x \cup \{x\} \rangle = \mathrm{PG}(V)$. Lemma 4.7 implies $P = Q$. So $S_x \subseteq H$, and $T_x := H = \langle S_x \rangle$. The proof is complete. □

Theorem D. *Let Q be a Hermitian linear space in* $\mathrm{PG}(V)$, *with respect to a* separable *quadratic field extension $C|R$. Then there exists a (non-degenerate) polarity of* $\mathrm{PG}(V)$ *such that Q consists of the absolute points of that polarity. That polarity is represented by a non-degenerate Hermitian form with respect to the involution generating* $\mathrm{Gal}(C|R)$.

Proof. For $x \in Q$, we have constructed the tangent hyperplane T_x in Lemma 4.5. For $x \notin Q$, we use the hyperplane T_x as defined in Lemma 4.10. We claim that the mapping $x \mapsto T_x$ is a (non-degenerate) polarity (as defined by Tits [5, 8.3.2, p. 128]). It suffices to prove that the correspondence $x \in T_y$ is symmetric, that is, $x \in T_y$ if and only if $y \in T_x$. By considering a plane containing x and a secant through y, this reduces to the case of a plane, where the correspondence holds by Lemma 4.9.

The restriction of the polarity to any plane intersecting Q in a flat is represented by a Hermitian form (with respect to the involution generating $\mathrm{Gal}(C|R)$). Thus the polarity is represented by a Hermitian form with respect to the same involution (again, see [5, 8.3.2, p. 128]). □

Theorem D, together with the planar case of Theorem C, yields Theorem B.

Open Problems. Various questions suggest further research.

a. Do there exist linear spaces embedded in the projective plane $\mathrm{PG}(2, C)$ such that the point set generates the plane, every block is a Baer subline and Baer sublines with respect to distinct subfields of C occur as blocks?

b. Is there an analogue for Theorem B in the inseparable case?

c. If one omits the injectivity of λ in the conditions of Theorem C, are there additional examples besides the ones mentioned after the statement of Theorem C in the introduction?

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