The geometry of the Freudenthal-Tits Magic Square

Hendrik Van Maldeghem

Dedicated the memory of Jacques Tits

Abstract

We review some geometric properties of the Freudenthal-Tits Magic Square, considered as a square of buildings, incidence geometries and varieties, as partly originally defined by Jacques Tits. In particular we establish new links between the vertical and horizontal layers.

1 Introduction

On pages 141 and 142 of [28], Jacques Tits introduced two tables, each consisting of twelve geometries, more exactly varieties in a projective space, ordered in three rows of four entries (a 3×4 matrix). The last column of each table consists of geometries of exceptional type: The first table contains real forms of F_4, E_6 and E_7, respectively, while the second table is the complexification of the first one, which, in modern terms, just means the split form (in French "forme déployée", which rather means "unfolded form"). Each of the geometries in the last column is constructed using an octonion algebra; in the first table the algebra is a division algebra, in the second table it is the corresponding split algebra. The geometries in the preceding columns are the analogues of the ones in the last column, defined over a the real field, the compex field or its split variant $\mathbb{R} \times \mathbb{R}$, and the quaternion skew field or its split variant, the algebra of real 2×2 matrices. The goal of displaying these tables is to put forward the question whether there exists a fourth row completing the tables to squares, such that the last entry is a form of E_8 involving an octonion algebra, and the other entries are analogues of that over the real, complex and quaternion algebras, respectively. Jacques Tits then went on and predicted how these geometries (before complexification) should look like and which forms of E_8 they should represent, which dimensions they have, etc. Today we know exactly which geometries they are, they are so-called metasymplectic spaces. But still unsolved is a direct construction of the corresponding varieties. The ultimate goal of this geometric investigation of the Magic Square is to eventually fill that gap. However, the way to the solution of this problem is paved with diamonds and pearls of beautiful connections and geometrical delights. Some of these I want to share with the reader of this paper, which aims to tell a coherent story rather than collect in a formal way mathematical statements.

What one cannot find in this paper is the approach via the Lie algebras. Although also very interesting, we have to draw the line somewhere, and we prefer to do it between geometry and algebra. The "magic" in the Magic Square is the symmetry in the table despite the asymmetric construction of the Lie algebras. However, we will see that geometry provides enough magic in this sense.

2 Structure of the paper

This paper does not have the intention to prove a main result, but just to describe the beauty hidden in the Square. We will introduce quite some different shades of the Magic Square. Traditionally, there is the split form of the Magic Square, and there is the non-split form. Since we will introduce some variants of each of them, we will give them different names to distinguish them. Each variant takes into account in which form we consider a given geometric or combinatorial structure. Roughly, there are three levels, ranging from general to specific (in reverse chronological historical order of introduction in the literature). The first level is the level of the theory of buildings. At the time Jacques Tits wrote about the Magic Square, these were not explicitly around yet, but they provide a very suitable framework to start with. The non-split (called "Relative") and split form (called "Absolute"; terminology taken from algebraic groups) are defined in **Section 3** and **Section 4**, respectively. We assume the reader is somewhat familiar with building theory; we do not define buildings here but instead refer to the literature, in particular the seminal book of Jacques Tits [30]. It is best for the present paper to view buildings as simplicial complexes rather than chamber systems.

In **Section 5**, we put the two previous forms in one Square by considering the corresponding indices (terminology taken from Tits [29]), nowadays sometimes called Tits indices or Tits diagrams. We interpret these diagrams in the combinatorial way à la Mühlherr, Petersson & Weiss [17] as fixed point diagrams, briefly called *fix diagrams*. This will enable us to define the *Delayed Magic Square* later on in Subsection 9.2. The Fix Magic Square bears the ultimate magic in its twisted symmetry, which is explained in Section 5.

Geometries related to buildings existed before the buildings were defined. In fact, "Incidence Geometry" as a research field found its origin exactly in Tits' thesis [28] where he introduced his Magic Square. This geometric point of view is slightly less general than the building point of view in that it favours a certain type of vertices, called the "points". Incidence geometry detaches the points from the subspaces they belong to by introducing an incidence relation. As a consequence, geometries do not necessarily have to live inside some projective space as a kind of variety. We define geometries for the non-split form in **Section 6**, yielding the Relative Geometric Magic Square, and also for the split form in **Section 7**, yielding the Complexified Geometric Magic Square (terminology from Tits [28]). We mention some remarkable characterization results singling out exactly the complexified geometries and their residues.

In **Section 8** we return to the origin and define the Magic Square as a 4×4 table of varieties, that is, point-line geometries embedded in projective space, and defined with algebraic formulas. Nevertheless, we also provide some purely geometric constructions. In this section, we look at the Square row by row.

Finally, in **Section 9**, we point out some connections between the different rows and columns of the Magic Square. We show that one can walk from South-East to North-West by taking residuals and equator geometries (**Subsection 9.1**). We introduce the Delayed Magic Square (**Subsection 9.2**) as explained before, and point out two instances in which this Delayed Magic Square puts itself in the frontline—by studying the minimal number of quadrics intersecting in a variety of the second row (**Subsection 9.3**), and by studying so-called domestic automorphisms

(**Subsection 9.4**), giving rise to a second interpretation of the fix diagrams as opposition diagrams. In **Subsection 9.5** at last we bring together some conclusions about the global connectivity of the Square, made obvious in the present paper.

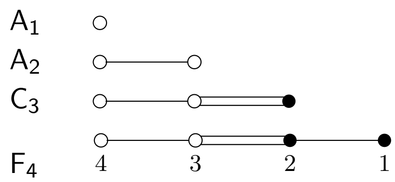
3 The Relative Magic Square

The Magic Square has many different forms, and each one usually has different interpretations. Let us review some of these (and remember we restrict ourselves to geometric approaches).

On the diagram level, the basic idea of the Magic Square is building up to the diagram of type F_4 via subdiagrams. This is established using the sequence $\mathsf{A}_1 - \mathsf{A}_2 - \mathsf{C}_3 - \mathsf{F}_4$. Notice that this is not in conformity with the Bourbaki labelling of the diagram of type F_4, as it starts with the vertex labelled 4 rather than with the vertex labelled 1.

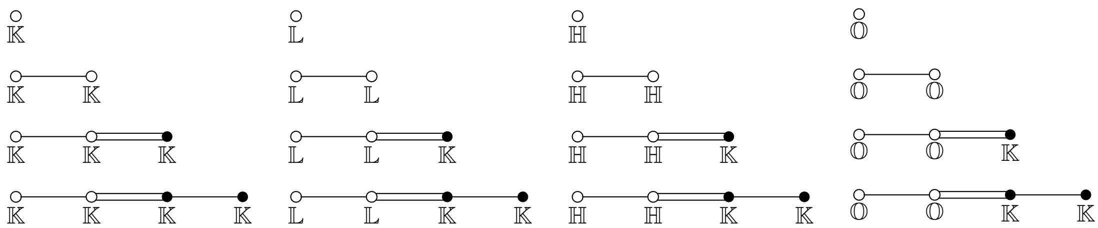
Before we draw any Dynkin diagram we would like to comment on the way these diagrams are usually drawn. Recall that each node of a Dynkin diagram represents a fundamental root of a crystallographic root system. Two nodes are not connected if the corresponding roots are perpendicular; otherwise we connect them with an edge, double edge, or triple edge according to whether the roots form an angle of 120, 135 or 150 degrees. Since Tits' thesis [28], one also furnishes every multiple edge with an arrow pointing to the smaller root. Dynkin diagrams have then be generalized to Coxeter diagrams, where nodes represent involutive generators, joined by an edge of weight n if the product of the generators has order n. In the diagram, one connects the nodes with an $(n-2)$-fold bond. Except for the case of $n = 6$, the Coxeter diagram of the Weyl group of a root system is exactly the Dynkin diagram where the arrows are removed. Now, Tits [27] already interpreted the Coxeter diagrams as diagrams belonging to the geometry of Lie groups corresponding to the given Dynkin diagrams. This idea was in full generality formalized by Buekenhout [3], and so we obtain Buekenhout diagrams, constructed as follows: each node represents a class of objects in an incidence geometry, two nodes being joined by an $(n-2)$-fold edge if the residue of each flag of corresponding cotype is a generalized n-gon. It turns out that the Buekenhout diagram of the natural geometry of a simple algebraic group is the same as the Coxeter diagram of its Weyl group. Hence, as one can see, in the geometry the arrows of the Dynkin diagrams are completely useless. However, there remains some geometric distinction between types C_n and B_n, which is a consequence of the commutation relations of the root groups. In the context of the Square, it is convenient to re-introduce this distinction, also for geometries. We do this by also considering the rank 1 residues, and distinguishing between "projective lines" and "polar lines". The idea is that a projective line will represent a line in a projective space, whereas a polar line will represent a polar space of rank 1 in a projective space. Examples of the latter are conics and Hermitian curves; hence point sets related to quadratic or Hermitian forms of Witt index 1. Heuristically, the points of a projective line are pairwise collinear, whereas every pair of points of a polar line is "symplectic" (at "distance 2" although there is no path of length 2 around; this "distance" must rather be thought of as a "grading", as for Lie algebras, and it does have a close connection with some 5-graded Lie algebras). The general rule is then that the rank 1 residues corresponding to short roots are polar lines, whereas the others are projective lines. Nodes corresponding to polar lines will be represented by white nodes, the other by black ones.

Viewed in this manner, we can build up to F_4 in the following way:



We have added the Bourbaki labelling for F_4 for clarity.

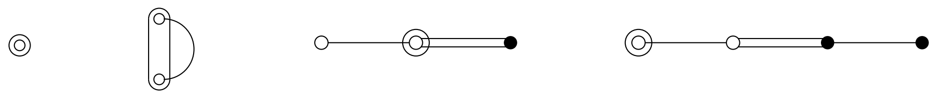
Now, these diagrams do not represent unique buildings, even if we would specify a field. Let us focus for a moment on the F_4 diagram. It is well known and follows from the classification of buildings of that type, that the projective plane corresponding to the residues of type $\{1, 2\}$ (the plane represented by the black nodes) is always a plane over a field $\mathbb{K}$, whereas the projective plane corresponding to the residues of type $\{3, 4\}$ is defined over a quadratic alternative division algebra over $\mathbb{K}$. Discarding the exceptional case of an inseparable field extension in characteristic 2, there are exactly four types of such algebras. And here is where the magic starts: F_4 has rank 4 and the corresponding buildings come in 4 shades. Equality of two numbers in geometry is never a coincidence, or at least has an exceptional consequence! With this information, we can already make a perfect Square. Let us mention the corresponding algebra below the nodes. We let $\mathbb{K}$ be the ground field, and then we denote by $\mathbb{L}$ an arbitrary quadratic Galois extension of $\mathbb{K}$, by $\mathbb{H}$ an arbitrary quaternion division algebra over $\mathbb{K}$ and by $\mathbb{O}$ an arbitrary octonion division algebra (Cayley-Dickson algebra) over $\mathbb{K}$. We order the algebras in increasing complexity (or increasing dimension over $\mathbb{K}$):



and call this table the *Relative Magic Square*. Its buildings are now all well defined and unique, given the algebraic structures, except that, in order to really have geometries and not "only" buildings, we need to decide which node we declare to correspond to points, and we also have to decide in the first row what we mean by a polar line over a quadratic alternative division algebra $\mathbb{A}$. Let us tackle the latter in the next section, the former being postponed to a later section.

4 The Absolute Magic Square

So what is a polar line over a quadratic alternative division algebra $\mathbb{A}$? In the spirit of our definition of "polar line", we should take a Hermitian form of Witt index 1 over $\mathbb{A}$. The corresponding field involution can be taken to be the natural involution in $\mathbb{A}$ defining the trace and norm (and given by the Cayley-Dickson process). Miraculously, if $\mathbb{A} = \mathbb{K}, \mathbb{L}, \mathbb{H}, \mathbb{O}$, these polar lines live naturally in split buildings of types $\ldots \mathsf{A}_1, \mathsf{A}_2, \mathsf{C}_3, \mathsf{F}_4$, respectively. In technical terms, using the vocabulary of Tits [29], these polar lines correspond to forms of algebraic groups of the respective types. Using the corresponding Tits diagrams (see [29]) together with our notation of black and white nodes to replace the arrows, the first row would become



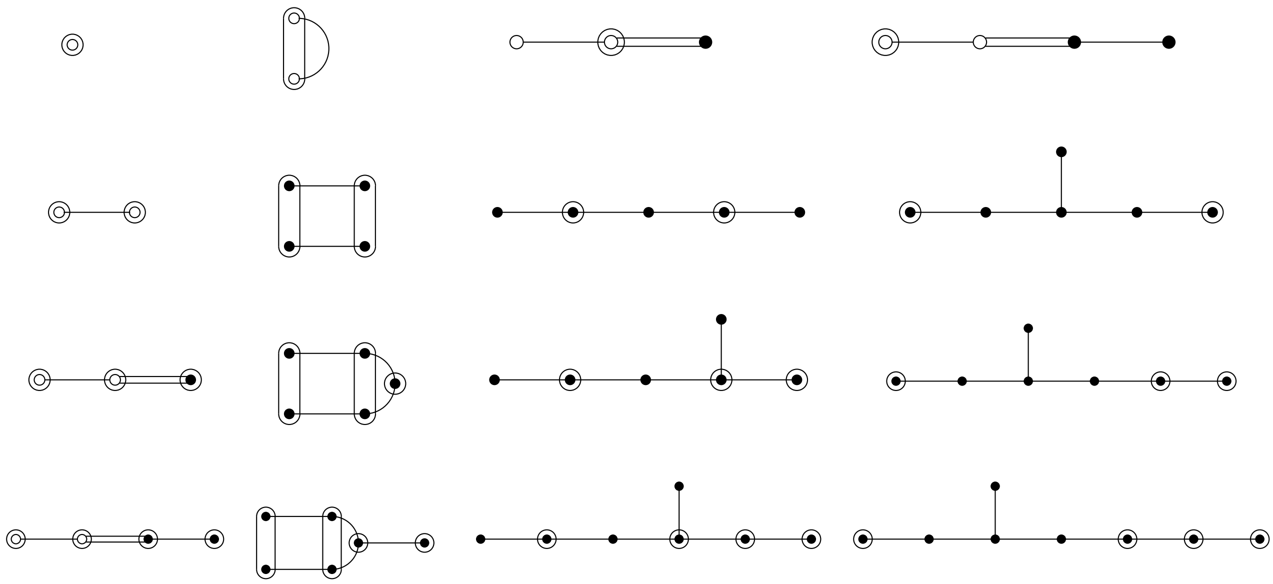
Consequently, in the above Square, both rows and columns are labelled by the sequence of diagrams $\mathsf{A}_1 - \mathsf{A}_2 - \mathsf{C}_3 - \mathsf{F}_4$. The latter are called the respective *absolute types* of the polar lines. Likewise, the projective planes over the quadratic alternative division rings as above have absolute types and corresponding Tits diagram, as also the other buildings appearing in the Relative Square. Just listing the absolute types, we obtain with the aid of [29]:

A_1	A_2	C_3	F_4
A_2	$\mathsf{A}_2 \times \mathsf{A}_2$	A_5	E_6
C_3	A_5	D_6	E_7
F_4	E_6	E_7	E_8

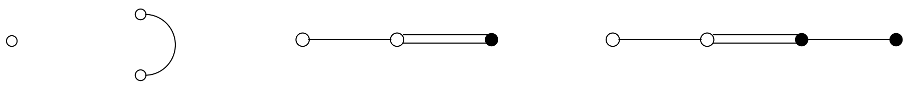
and call this the *Absolute Magic Square.* And here is again some magic: this table is symmetric! But this is not all. The symmetry goes further then simply the names of the types. It leads to another form of the Square.

5 The Fix Magic Square

Let us replace in the Absolute Square every absolute type with its Tits diagram, called *index* in [29]. Recall that a Tits diagram has the following geometric interpretation (see also [17]): The minimal flags fixed by the (Galois) descent group have the types given by the encircled nodes, and the diagram is drawn in a bent way if the descent group does not act type-preserving. In this context, and in the more general context of arbitrary automorphism groups fixing a flag opposite every fixed flag, this is sometimes referred to as the *fix diagram* of the automorphism group. Because of this, the following table will be called the *Fix Magic Square.*



Clearly the symmetry is kind of broken since the same absolute types do not produce the same Tits diagrams. To re-establish the symmetry, in an even more glorious form, we need to get ahead of the facts for a moment. Firstly, it will appear that an important ingredient in the algebraic construction of most varieties is that of a Hermitian matrix, that is, a matrix which is symmetric up to a "twist", in this case a field automorphism. It would be a betrayal to itself if the Magic Square would also be "just symmetric", without an extra twist. Secondly, it will also appear that one can make a case to replace the forms in the first row above by the anisotropic forms, that is, the empty Tits diagrams



(This schizophrenic behaviour of the first row can be compared with the behaviour of 'light' in physics—*particle* or *wave*. We will also always choose the interpretation that best fits our observations or our purposes.)

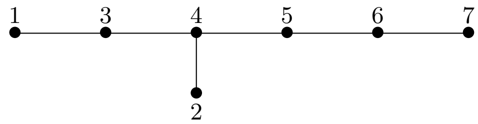
Now interpret the Tits diagrams in the Absolute Square (containing the row above as first row) as fix diagrams of certain involutions in Coxeter complexes. Let σ be an arbitrary involution of a Coxeter complex with a fix diagram in the Fix Square above. Let ρ be the opposition map, that is, ρ maps each vertex of the Coxeter complex to its (unique) opposite. Then $\sigma\rho = \rho\sigma$ is an involution of the Coxeter complex with fix diagram exactly the one lying symmetrically (symmetry with respect to the main diagonal) in the Square. This way, one sees that the Fix Square itself is a kind of Hermitian matrix. Isn't that real magic?

For example, the involution corresponding to the first column is just the identity; it follows that the involution in the first row is then the opposition map. Note that the North-West cell has a conflict with itself: the involution cannot be at the same time the identity and the opposition map. So this is a little bug which does not bother us; on the contrary, it will be helpful below when regarding this cell as the empty building.

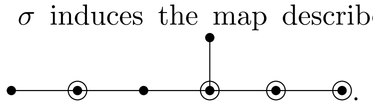
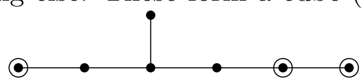
Let us give two additional examples of the above phenomenon.

Example 5.1 The Coxeter complex Σ of type D_6 can be defined using a set of twelve elements, say $\{-6, -5, \ldots, -1, 1, 2, \ldots, 6\}$. The vertices of Σ are all subsets of size distinct from 5 containing no pair of numbers with the same absolute value. The opposition map ρ is induced by the permutation of $\{-6, -5, \ldots, -1, 1, 2, \ldots, 6\}$ mapping each number x to its opposite $-x$. Let σ be induced by the permutation which interchanges the numbers i and j every time $|i+j| = 7$. Then σ stabilizes 2-cliques, 4-cliques and 6-cliques that contain an even number of positive numbers. These form a Coxeter complex of type C_3 on their own. The composition $\sigma\rho$ has exactly the same behaviour, although of course stabilizing different cliques. This explains why D_6 is on the diagonal of the Fix Square: It is self-conjugate.

Example 5.2 Recall the Bourbaki labelling of the nodes in the E_7 diagram:



The Coxeter complex Σ of type E_7 can be modelled on the Gosset graph Γ , which is defined as follows: The 56 vertices of Γ are the pairs from the respective 8-sets $\{1, 2, \ldots, 8\}$ and $\{1', 2', \ldots, 8'\}$. Two pairs from the same set are adjacent if they intersect in precisely one element; two pairs $\{a, b\}$ and $\{c', d'\}$ from different sets are adjacent if $\{a, b\}$ and $\{c, d\}$ are disjoint. The elements (vertices) of type $2, 3, 4, 5, 6, 7$ of Σ are the maximal 7-cliques, the maximal 6-cliques, the 4-cliques, the 3-cliques, the edges and the vertices, respectively. The elements of type 1 are the cross-polytopes of size 12 (so-called *hexacrosses* or 6-*orthoplexes*) contained in Γ (these are also the Coxeter subcomplexes of type D_6 as described in Example 5.1). There are 126 such, and 56 of these are determined by an ordered pair (i, j) with $i, j \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, $i \neq j$, and induced on the vertices $\{i, k\}$ and $\{j', k'\}$, $k \notin \{i, j\}$, whereas the other 70 are determined by a 4-set $\{i, j, k, \ell\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ and are induced on the vertices $\{s, t\} \subseteq \{i, j, k, \ell\}$, $s \neq t$, and $\{u', v'\} \subseteq \{1', 2', 3', 4', 5', 6', 7', 8'\} \setminus \{i', j', k', \ell'\}$, $u \neq v$.

The opposition map ρ is induced by interchanging a with a', for all $a \in \{1, 2, \ldots, 8\}$. Let σ be the involution induced by the map sending $x \in \{1, 2, \ldots, 8\}$ to $(x+4)'$, where $x+4$ has to be read mod 8. Let us briefly write $x \pm 4$ for $x+4$ mod 8. Clearly, σ does not fix any vertex of Γ, but it stabilizes precisely 24 edges, given by all pairs $\{\{a, b\}, \{(a \pm 4)', (b \pm 4)'\}\}$ with $a \neq b \pm 4$. It also stabilizes exactly 24 cross-polytopes of size 12, namely the eight determined by the ordered pairs $(i, i \pm 4)$, $i = 1, 2, \ldots, 8$, and the sixteen determined by all 4-subsets of $\{1, 2, \ldots, 8\}$ with the property that, for each $i \in \{1, 2, 3, 4\}$, it contains exactly one of i or $i+4$. In each such cross-polytope, σ induces the map described in Example 5.1. It now easily follows that σ has fix diagram . Now $\sigma\rho = \rho\sigma$ fixes the eight vertices $\{i, i+4\}$ and $\{i', (i+4)'\}$, $i = 1, 2, 3, 4$, the twelve edges $\{\{i, i+4\}, \{j', (j+4)'\}\}$, with $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3, 4\} \setminus \{i\}$, and the six cross-polytopes of size 12 determined by the 4-sets $\{i, j, i+4, j+4\}$, $i, j \in \{1, 2, 3, 4\}$, $i < j$, and nothing else. These form a cube (Coxeter complex of type C_3); the fix diagram of $\rho\sigma$ is now clearly  and we see the fix diagrams of σ and $\sigma\rho$ correspond under reflection about the main diagonal of the Fix Square.

The observation of this twisted symmetry will lead to rather interesting characterizations of certain automorphisms, see Section 9.4.

Digression—The Coxeter complexes in the second row of the Magic Square look very similar: Each can be modelled on the complement of the collinearity graph of a generalized quadrangle with three points per line (where we consider a single line as a degenerate generalized quadrangle; remember the rank 1 residues in the first cell are polar lines, so no collinear points exist, and hence the complement yields really a connected line with three points); they have order $(2, 0)$, $(2, 1)$, $(2, 2)$ and $(2, 4)$ and are uniquely determined by their order, for the columns 1, 2, 3, 4, respectively. One could ask whether something of that nature holds for the other rows. For the first row, we simply take away one line of the corresponding generalized quadrangle, and all edges joining vertices that correspond to points collinear to the same point of the line we took away; what remains is a graph that naturally models the Coxeter complex of the given geometry in the first row: The first cell becomes empty (compare with Subsection 9.5 where we argue why this cell could indeed also be seen as being empty; compare also with Subsection 8.2), the second cell becomes a hexagon, which models the Coxeter complex of type A_2 using the flags (chambers) as vertices; the third cell becomes a cuboctahedron, the fourth a 24-cell. Now what about the third row? We are looking for something that extends a generalized quadrangle. In the literature, an *extended generalized quadrangle of order* (s, t), $s, t \geq 1$, is a graph $\Gamma = (V(\Gamma), E(\Gamma))$ such that

(EGQ) For each vertex $v \in V(\Gamma)$, the point-line geometry with point set $\Gamma(v)$ (the set of vertices adjacent to v) and line set the maximal cliques of the graph induced on $\Gamma(v)$, is a generalized quadrangle of order (s, t).

It is shown in [7] that the diameter of an extended generalized quadrangle of order (s, t) is at most $s+1$. Moreover, it follows from Theorem 2 of [8] that there are exactly three extended generalized quadrangles of order $(2, t)$ with diameter 3, and these have respective values $t = 1, 2, 4$. It can easily be checked using the description in [7] that the Gosset graph is the "2-complement" of the unique extended generalized quadrangle of diameter 3 and order $(2, 4)$, where the 2-complement of a graph Γ is $\Gamma^2 - \Gamma$, that is, connect each pair of vertices at distance 2 and delete all other edges. The same is true for the ith cell of the third row: its Coxeter complex can be modelled on the 2-complement of the unique extended generalized quadrangle of diameter 3 and order $(2, 2^{i-2})$ in much the same way as the fourth cell and the Gosset graph.

In [8], a construction using quadrics is provided, making apparent the full automorphism group as a (subgroup of) an orthogonal classical group. But these groups must also be the Weyl groups of the corresponding Coxeter complex, hence Coxeter groups. This way, one sees that most Coxeter groups related to the Square are orthogonal classical groups. In fact, this even extends to the fourth row, but not entirely to the first one. We replace some cells of the Absolute Magic Square with their Coxeter group, using the notation of classical groups of the $\mathbb{ATLAS}$ [6], obtaining the following remarkable table.

$\mathsf{O}_2^+(2)$	$\mathsf{O}_3(2)$		
$\mathsf{O}_3(2)$	$\mathsf{O}_4^+(2)$	$\mathsf{O}_5(2)$	$\mathsf{SO}_6^-(2)$
	$\mathsf{O}_5(2)$	$2^5.\mathsf{O}_5(2)$	$2\times\mathsf{O}_7(2)$
	$\mathsf{SO}_6^-(2)$	$2\times\mathsf{O}_7(2)$	$2.\mathsf{O}_8^+(2).2$

6 The Relative Geometric Magic Square

We now turn to the geometries. We first briefly recall how to attach a geometry to a building by picking an arbitrary type of elements. We restrict ourselves to the irreducible thick case.

Everything that follows can be found at various places in [24], in particular in the chapters about parapolar spaces; see also [4]

Let Δ be an irreducible thick spherical building. Let n be its rank, let S be its type set and let $s\in S$. Then we define a point-line geometry $\Gamma=(X,\mathscr{L},*)$ as follows. The point set X is just the set of vertices of Δ of type s; the set $\mathscr{L}$ of lines are the flags of type $s^\sim$, where $s^\sim$ is the set of types adjacent to s in the Coxeter diagram of Δ. If x is a vertex of type s and F a flag of type $s^\sim$, then $x*F$ if $F\cup\{x\}$ is a flag. The geometry Γ is called a *Lie incidence geometry.* For instance, if Δ has type A_n, and $s=1$ (remember we use Bourbaki labelling), then Γ is the point-line geometry of a projective space. If X_n is the Coxeter type of Δ and Γ is defined using $s\in S$ as above, then we say that Γ has *type* $\mathsf{X}_{n,s}$. In the diagram, we replace the corresponding node by x.

Lie incidence geometries of type $\mathsf{B}_{n,1}$, $n\geq 2$, and $\mathsf{D}_{n,1}$, $n\geq 3$, are polar spaces of rank n, and those of the latter type are sometimes called *hyperbolic.* All Lie incidence geometries as defined above and which are not projective spaces or polar spaces, are *parapolar spaces.* Before recalling the definition of a parapolar space, we need to review some basics about point-line geometries.

Let $\Gamma=(X,\mathscr{L})$ be a point-line geometry (if the incidence relation is not mentioned, we assume it is induced by containment). Points $x,y\in X$ contained in a common line are called *collinear*, denoted as $x\perp y$; the set of all points collinear to x is denoted by $x^\perp$. We will always deal with situations where every point is contained in at least one line, so $x\in x^\perp$. Also, for $S\subseteq X$, we denote $S^\perp:=\{x\in X\mid x\perp s \text{ for all } s\in S\}$.

The *point graph* of Γ is the graph on X with collinearity as adjacency relation. The *distance* δ between two points $p,q\in X$ (denoted $\delta_\Gamma(p,q)$, or $\delta(p,q)$ if no confusion is possible) is the distance between p and q in the collinearity graph, where $\delta(p,q)=\infty$ if p and q are contained in distinct connected components of the point graph; If $\delta:=\delta(p,q)$ is finite, then a *geodesic path* or a *shortest path* between p and q is a path between them in the point graph of length δ. The *diameter* of Γ (denoted $\mathrm{Diam}\,\Gamma$) is the diameter of the point graph. We say that Γ is *connected* if every pair of vertices is at finite distance from one another. The point-line geometry Γ is called a *partial linear space* if each pair of distinct points is contained in at most one line.

A *subspace* of Γ is a subset A of X such that, if $x, y \in A$ are collinear and distinct, then all lines containing both x and y are contained in A. A subspace A is called *convex* if, for any pair of points $\{p, q\} \subseteq A$, every point occurring in a geodesic between p and q is contained in A; it is *singular* if $\delta(p, q) \leq 1$ for all $p, q \in A$. The intersection of all convex subspaces of Γ containing a given subset $B \subseteq X$ is called the *convex subspace closure* of B. A proper subspace H is called a *geometric hyperplane* if each line of Γ has either one or all its points contained in H.

Now a parapolar space is a point-line geometry satisfying the following three axioms:

(PPS1) There is line L and a point p such that no point of L is collinear to p.
(PPS2) The geometry is connected.
(PPS3) Let x, y be two points at distance 2. Then either there is a unique point collinear to both—and then the pair $\{x, y\}$ is called *special*—or the convex subspace closure of $\{x, y\}$ is a polar space—and then the pair $\{x, y\}$ is called a *symplectic pair*. Such polar spaces are called *symplecta*, or *symps* for short.
(PPS4) Each line is contained in a symplecton.

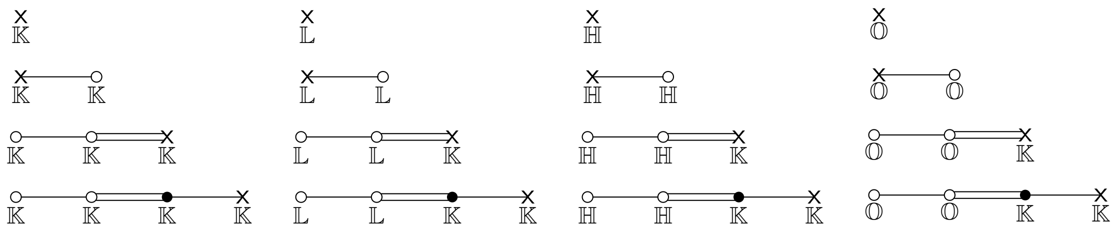
The parapolar spaces we will encounter all have the rather peculiar property that all symps have the same rank, which is then called the (uniform) *symplectic rank* of the parapolar space. In contrast, the maximal singular subspaces (which will be projective spaces) will not all have the same dimension. The *singular ranks* of a parapolar space with only projective spaces as singular subspaces (which is automatic if the symplectic rank is at least 3) are the dimensions of the maximal singular subspaces.

We need two more notions. A parapolar space without special pairs is called *strong*. And a parapolar space of uniform symplectic rank ≥ 3 is called *locally connected* if for every point p, the graph on the lines passing through p, adjacent when contained in a common symp (or, equivalently, a common singular plane), is connected. Equivalently, one can require that for every point p, the graph on the lines passing through p, adjacent when contained in a common singular plane, is connected.

A Lie incidence geometry is called a *long root geometry* if it has type $\mathsf{X}_{n,s}$, with s the so-called *polar node* of the diagram X_n, $\mathsf{X} \neq \mathsf{A}$ (the long root geometry for type A needs a more general definition of Lie incidence geometry using flags as points instead of vertices of the corresponding building, but we will not bother the reader with this). The polar node in case $\mathsf{B}, \mathsf{C}, \mathsf{E}$ and F (we will not need G_2) is given by the (unique) fundamental root not perpendicular to the longest (highest) root. The types are $\mathsf{B}_{n,2}$, $\mathsf{C}_{n,1}$, $\mathsf{D}_{n,2}$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, $\mathsf{E}_{8,8}$ and $\mathsf{F}_{4,1}$. Long root geometries share a lot of interesting (even characteristic) properties across all types; however, for most types, they are not the "simplest" Lie incidence geometry in the sense that other choices for points produce geometries with smaller diameter or simpler structure. For E_8, it is the simplest one in that sense. For F_4, it is debatable whether or not the Lie incidence geometry of type $\mathsf{F}_{4,4}$ is simpler than the one of type $\mathsf{F}_{4,1}$. They both have the same diameter and global structure;

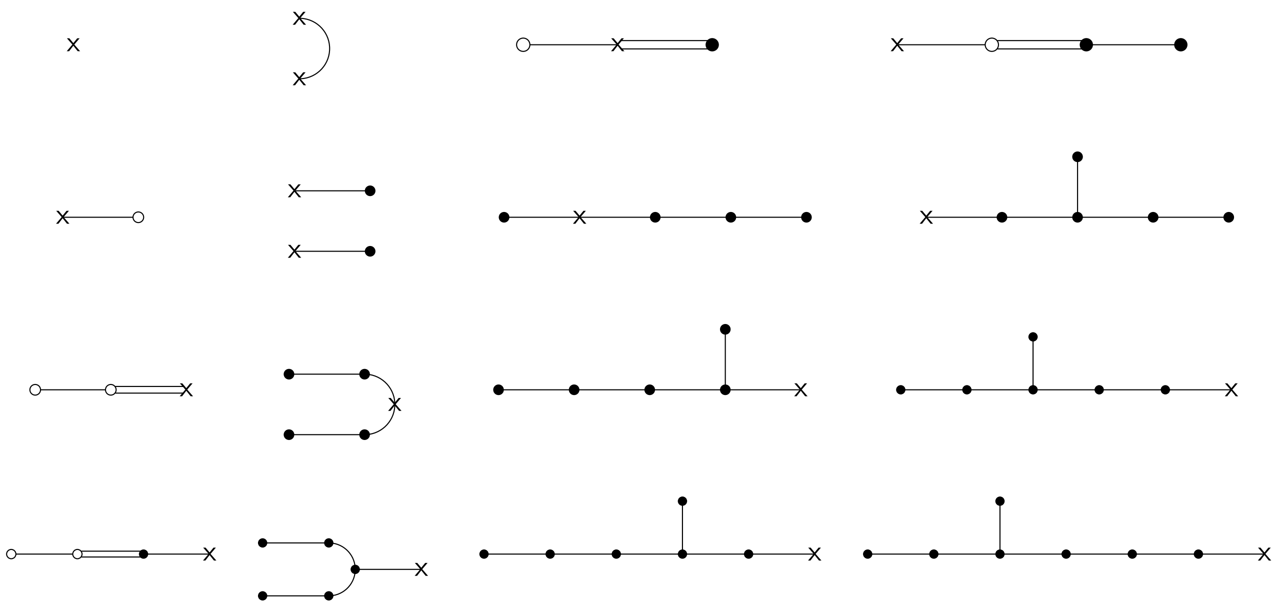
in fact every characterization theorem of parapolar spaces trying to single out the long root geometries also embraces the Lie incidence geometry of type $\mathsf{F}_{4,4}$. For an arbitrary building of type F_4, the Lie incidence geometry of type $\mathsf{F}_{4,1}$ or $\mathsf{F}_{4,4}$ is called a *metasymplectic space*, see [5] for an axiom system and more background information.

By the previous paragraph it is clear that the Square should contain long root geometries, and preferably in the fourth row, by the presence of type E_8 in the Absolute Square. So, carried over to the Relative Square, the geometries of the fourth row are the ones of type $\mathsf{F}_{4,1}$. So we choose the last node as the type of vertices to play the role of the points, and we replace the node with an "$\times$". We now do this for every row; however the diagrams in the second row are symmetric, so there we can choose the first vertex. Our Relative Geometric Square becomes:



7 The Complexified Geometric Magic Square

We will now attach to each cell of the Absolute Square a geometry. This is done simply by taking the Lie incidence geometry with respect to the node of the absolute diagram that corresponds to the x-node in the Relative Geometric Square. We get the following Complexified Geometric Square



Using diagram types, this Square looks as follows.

$\mathsf{A}_{1,1}$	$\mathsf{A}_{2,\{1,2\}}$	$\mathsf{C}_{3,2}$	$\mathsf{F}_{4,4}$
$\mathsf{A}_{2,1}$	$\mathsf{A}_{2,1}\times\mathsf{A}_{2,1}$	$\mathsf{A}_{5,2}$	$\mathsf{E}_{6,1}$
$\mathsf{C}_{3,3}$	$\mathsf{A}_{5,3}$	$\mathsf{D}_{6,6}$	$\mathsf{E}_{7,7}$
$\mathsf{F}_{4,1}$	$\mathsf{E}_{6,2}$	$\mathsf{E}_{7,1}$	$\mathsf{E}_{8,8}$

At this point the above squares represent 16 classes of Lie incidence geometries, with extraordinary properties and rather strong common characterizations. We mention two of them.

A parapolar space with uniform symplectic rank r possessing a singular subspace S of dimension $r-2$ with the property that $S^{\perp}$ is the union of two maximal singular subspaces of respective dimensions d_1, d_2 (possibly $d_1 = d_2$) is called $\{d_1, d_2\}$*-camel*. A parapolar space of uniform symplectic rank r with the property that no pair of symps intersects in a singular subspace of dimension k, $-1 \leq k \leq r-1$ is called k*-lacunary*. Table 1 shows some examples of $\{d_1, d_2\}$-camel and at the same time k-lacunary Lie incidence geometries with uniform symplectic rank r. The table shows the types; the fields or skew fields can be chosen arbitrarily (and in case of A_1 every set of at least three points qualifies; for A_2, every projective plane qualifies). All geometries have diameter 2, except the ones in the grey cells, which have diameter 3. All parapolar spaces are strong, except the ones written in white. All these examples have singular ranks d_1, d_2. Note that all geometries appearing in Table 1 either appear in the Complexified Magic Square, or are a residue of a geometry appearing in the second row of the Complexified Magic Square (and all geometries appearing in the 3×3 South-East corner of the Complexified Magic Square are in the table), with the understanding that types A_1 and A_2 stand for arbitrary projective lines and planes, respectively.

We now have the following two characterization theorems. The first one is taken from [13], the second one combines the main results of [11] and [12].

Theorem 7.1 *Let Δ be a parapolar space of uniform symplectic rank $r \geq 2$ all symps of which are hyperbolic and all singular subspaces of which are projective spaces (remember this is automatic when $r \geq 3$). Assume Δ is locally connected if $r \geq 3$ and strong if $r = 2$. If Δ is $\{d_1, d_2\}$-camel, with $d_1 + d_2 \leq 2r$, then Δ is one of the geometries in* Table 1.

Theorem 7.2 *The k-lacunary parapolar spaces with uniform symplectic rank $r \geq k+3$ are those in* Table 1 *for $r \in \{k+3, k+4, k+6\}$.*

Also the k-lacunary parapolar spaces with symplectic rank $k+2$ are classified. Most other long root geometries appear there, along with dual polar spaces of rank 3 and half spin geometries, and some others, see [11]. Note also that the conditions in [11] are slightly weaker in that the rank of the parapolar space is not assumed to be uniform.

r	d_1, d_2	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$k+2$	$k+1, k+2$		$\mathsf{A}_{1,1} \times \mathsf{A}_{2,1}$	$\mathsf{A}_{4,2}$	$\mathsf{D}_{5,5}$	$\mathsf{E}_{6,1}$	$\mathsf{E}_{7,7}$	$\mathsf{E}_{8,8}$
	$k+1, k+3$		$\mathsf{A}_{1,1} \times \mathsf{A}_{3,1}$	$\mathsf{A}_{5,2}$	$\mathsf{D}_{6,6}$	$\mathsf{E}_{7,1}$		
$k+3$	$k+2, k+3$	$\mathsf{A}_{1,1} \times \mathsf{A}_{2,1}$	$\mathsf{A}_{4,2}$	$\mathsf{D}_{5,5}$	$\mathsf{E}_{6,1}$	$\mathsf{E}_{7,7}$	$\mathsf{E}_{8,8}$	
	$k+3, k+3$	$\mathsf{A}_{2,1} \times \mathsf{A}_{2,1}$	$\mathsf{A}_{5,3}$	$\mathsf{E}_{6,2}$				
$k+4$	$k+3, k+4$	$\mathsf{A}_{4,2}$	$\mathsf{D}_{5,5}$	$\mathsf{E}_{6,1}$	$\mathsf{E}_{7,7}$	$\mathsf{E}_{8,8}$		
	$k+3, k+5$	$\mathsf{A}_{5,2}$	$\mathsf{D}_{6,6}$	$\mathsf{E}_{7,1}$				
$k+6$	$k+5, k+6$	$\mathsf{E}_{6,1}$	$\mathsf{E}_{7,7}$	$\mathsf{E}_{8,8}$				

Table 1: Types of some k-lacunary Lie incidence geometries with symplectic rank r and singular ranks d_1, d_2.

8 The original Tits Magic Square

The point-line geometries displayed in the Relative and Complexified Magic Squares have natural embeddings in projective spaces (as is the case for all Lie incidence geometries related to split algebraic groups, by [16]). For the second and the third rows, these natural embeddings are even the absolutely universal embeddings, discarding the first cells. This is conjectured to be true for the last row, too. The corresponding point sets of these embedded geometries are rational varieties, as they can be described with polynomial equations. The latter will also be a feature in the connectivity of the square.

In this section, we describe these varieties, row by row, except for the fourth row, where the problem of finding an elementary common description of these varieties is still open. Note that Jacques Tits encircles in [28] the points of the dual geometry in order to have the smallest dimensions for the point sets.

8.1 The second row

We begin with the second row, which lies central with respect to the rows we are going to handle. The main ingredient of every construction is a non-degenerate quadratic alternative algebra $\mathbb{A}$ over a field $\mathbb{K}$. In the context of the Magic Square, $\mathbb{A}$ is not an inseparable field extension of $\mathbb{K}$ in characteristic 2. However, when we will mention characterization results, these will nevertheless turn up as extra examples, so we might as well retain that possibility. There is a natural involution σ in $\mathbb{A}$, usually denoted by $\sigma : x \mapsto \overline{x}$, and a natural inclusion $\mathbb{K} \subseteq \mathbb{A}$ with the properties $x\overline{x} \in \mathbb{K}$ and $x + \overline{x} \in \mathbb{K}$.

The possibilities for $\mathbb{A}$ are

(i) the field $\mathbb{K}$ itself; σ is the identity,
(ii) a separable quadratic field extension; σ is the non-trivial Galois automorphism,
(iii) the direct product $\mathbb{K} \times \mathbb{K}$ (with componentwise addition and multiplication); σ interchanges the two components and the natural inclusion of $\mathbb{K}$ in $\mathbb{A}$ is diagonal,

(iv) a quaternion division algebra over $\mathbb{K}$; σ is the natural involution,
(v) a split quaternion algebra over $\mathbb{K}$, which is isomorphic to the algebra of 2×2 matrices over $\mathbb{K}$; σ is taking the adjugate matrix and the natural inclusion of $\mathbb{K}$ in $\mathbb{A}$ is via the scalar matrices,
(vi) an octonion division algebra over $\mathbb{K}$; σ is the natural involution,
(vii) a split octonion algebra over $\mathbb{K}$ with natural involution σ,
(viii) an inseparable field extension of $\mathbb{K}$ in characteristic 2; σ is the identity.

So let $\mathbb{A}$ be one of these algebras. Set $d := \dim_{\mathbb{K}} \mathbb{A}$. ($d$ might possibly be infinite.) Now consider the *Veronese map*

$$\rho_{\mathbb{A}} : \mathbb{A} \times \mathbb{A} \to \mathsf{PG}(3d+2, \mathbb{K}) : (a, b) \mapsto (a\overline{a}, b\overline{b}, 1; b, \overline{a}, a\overline{b}),$$

where we view $\mathsf{PG}(3d+2, \mathbb{K})$ as $\mathsf{PG}(V)$, with $V \cong \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{A} \times \mathbb{A} \times \mathbb{A}$. Then the Zariski closure of the full image of $\rho_{\mathbb{A}}$ is a variety which we denote by $\mathscr{V}(\mathbb{K}, \mathbb{A})$.

Perhaps one word about the Zariski closure. The best way to think about that here is as follows: if $\mathbb{A}$ is an associative division algebra, then one can make the image homogeneous, and so the Zariski closure coincides with the set of points

$$\{(a\overline{a}, b\overline{b}, c\overline{c}; b\overline{c}, c\overline{a}, a\overline{b}) \mid a, b, c \in \mathbb{A}\}.$$

In the non-associative case we can alternatively take the following union:

$$\{(1, b\overline{b}, c\overline{c}; b\overline{c}, c, \overline{b}) \mid b, c \in \mathbb{A}\} \cup \{(a\overline{a}, 1, c\overline{c}; \overline{c}, c\overline{a}, a) \mid a, c \in \mathbb{A}\} \cup \{(a\overline{a}, b\overline{b}, 1; b, \overline{a}, a\overline{b}) \mid a, b \in \mathbb{A}\}.$$

If $\mathbb{A}$ is not a division algebra, and $|\mathbb{K}| > 2$, then the Zariski closure of a point set P coincides with the *projective closure* of P, that is, the smallest set of points containing P and containing no affine line (an *affine line* is a projective line with one point deleted). If $|\mathbb{K}| = 2$ and $\mathbb{A}$ is not division, then first go to any extension field, take the projective closure, and then restrict the field again to $\mathbb{K}$.

If $\mathbb{A}$ is division, then we obtain a geometry of the Relative Geometric Magic Square; otherwise the Complexified Geometric Magic Square. We now briefly discuss the different possibilities and provide the possibly different classical definition of construction.

The quadric Veronese surface $\mathscr{V}_2(\mathbb{K})$—This is the image of the *Veronese map* $\mathsf{PG}(2, \mathbb{K}) \to \mathsf{PG}(5, \mathbb{K}) : (x, y, z) \mapsto (x^2, y^2, z^2, yz, zx, xy)$. Here $\mathbb{A} = \mathbb{K}$.

We see that, although in essence a projective plane, the quadric Veronese surface behaves like a strong parapolar space of diameter 2 without lines: Every pair of points determines a unique symp, which is here a polar space of rank 1, in particular a quadric of Witt index 1. This is made clear by the diagram in the first column and second row of any form of the Magic square, which shows that the rank 1 residues are polar lines rather than projective lines.

The Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$—This is the image of the *Segre map* $\mathsf{PG}(2, \mathbb{K}) \times \mathsf{PG}(2, \mathbb{K}) \to \mathsf{PG}(8, \mathbb{K}) : (x, y, z; u, v, w) \mapsto (xu, yu, zu, xv, yv, zv, xw, yw, zw)$. It is isomorphic to $\mathscr{V}(\mathbb{K}, \mathbb{K} \times \mathbb{K})$.

We may view the set of 3×3 matrices over $\mathbb{K}$ as a 9-dimensional vector space, and the set of symmetric 3×3 matrices as a 6-dimensional subspace. Then we may consider the corresponding projective spaces of (projective) dimension 8 and 5, respectively, in the classical way by considering the 1-spaces as the points. In this way, the Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$ corresponds exactly to the rank 1 3×3 matrices; explicitly

$$\mathbb{K}(xu, yu, zu, xv, yv, zv, xw, yw, zw) \leftrightarrow \mathbb{K}\begin{pmatrix} xu & yu & zu \\ xv & yv & zv \\ xw & yw & zw \end{pmatrix}.$$

Similarly, the quadric Veronese surface $\mathscr{V}_2(\mathbb{K})$ corresponds exactly with the rank 1 symmetric 3×3 matrices; explicitly

$$\mathbb{K}(x^2, y^2, z^2, yz, zx, xy) \leftrightarrow \mathbb{K}\begin{pmatrix} x^2 & yx & zx \\ xy & y^2 & zy \\ xz & yz & z^2 \end{pmatrix}.$$

In particular, $\mathscr{V}_2(\mathbb{K})$ is a subvariety of $\mathscr{S}_{2,2}(\mathbb{K})$ obtained by intersecting with a 5-dimensional subspace.

We also encounter other Segre varieties; in general $\mathscr{S}_{n,m}(\mathbb{K})$ is defined as the image in $\mathsf{PG}(nm-1, \mathbb{K})$ of the map $(x_i, y_j)_{1 \le i \le n, 1 \le j \le m} \mapsto (x_i y_j)_{1 \le i \le n, 1 \le j \le m}$. The images of the marginal maps defined by either fixing the x_i, $1 \le i \le n$, or the y_j, $1 \le j \le m$, are called the *generators* of the variety (in case of $\mathscr{S}_{2,2}(\mathbb{K})$ the generators are 2-dimensional projective subspaces, hence planes).

The line Grassmannian $\mathscr{G}_{2,6}(\mathbb{K})$—Denote the set of lines of $\mathsf{PG}(5, \mathbb{K})$, or equivalently, the set of 2-spaces of $\mathbb{K}^6$ by $\binom{\mathbb{K}^6}{\mathbb{K}^2}$. Then $\mathscr{G}_{2,6}(\mathbb{K})$ is the image of the Plücker map $\binom{\mathbb{K}^6}{\mathbb{K}^2} \to \mathsf{PG}(14, \mathbb{K})$: $\langle (x_1, x_2, \ldots, x_6), (y_1, y_2, \ldots, y_6) \rangle \mapsto (x_i y_j - x_j y_i)_{1 \le i < j \le 6}$. Denote the coordinate of $\mathsf{PG}(14, \mathbb{K})$ corresponding to the entry $x_i y_j - x_j y_i$ by p_{ij}, $1 \le i < j \le 6$. By restricting to $y_1 = y_2 = y_3 = x_4 = x_5 = x_6 = 0$, we see that $\mathscr{S}_{2,2}(\mathbb{K})$ is a subvariety of $\mathscr{G}_{2,6}(\mathbb{K})$ obtained by intersecting with an 8-dimensional projective subspace with equation $p_{12} = p_{13} = p_{23} = p_{45} = p_{46} = p_{56} = 0$.

The line Grassmannian $\mathscr{G}_{2,6}(\mathbb{K})$ is isomorphic to $\mathscr{V}(\mathbb{K}, \mathbb{A})$, with $\mathbb{A}$ a split quaternion algebra over $\mathbb{K}$. The isomorphism can be seen by taking $a = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ and $b = \begin{pmatrix} x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}$ in the definition of $\mathscr{V}(\mathbb{K}, \mathbb{A})$ above.

The Cartan variety $\mathscr{E}_{6,1}(\mathbb{K})$—This variety is traditionally defined using a trilinear or cubic form. It is an exceptional variety in the sense that it cannot be defined, using classical notions like Plücker or Grassmann coordinates, from a projective space. We introduce the cubic form in the geometric way explored in [33].

Let $\Gamma = (X, \mathscr{L})$ be the generalized quadrangle of order $(2, 4)$, that is, the points and lines on the quadric in $\mathsf{PG}(5, 2)$ with equation $X_0 X_1 + X_2 X_3 = X_4^2 + X_4 X_5 + X_5^2$. Let $\mathscr{S}$ be a *regular spread*, that is, a set of nine lines of Γ partitioning the point set and enjoying the property that, if $L, M \in \mathscr{L}$, $L \neq M$, then the set of points off $L \cup M$ but contained in a line intersecting both L and M are the points of a member of $\mathscr{S}$. Label the standard basis of a 27-dimensional vector space over $\mathbb{K}$ with the points of Γ, and use x_p, $p \in X$, as the corresponding coordinate. Then we define the cubic form

$$C\left((x_p)_{p \in X}\right) = \sum_{\{p,q,r\} \in \mathscr{S}} X_p X_q X_r - \sum_{\{p,q,r\} \in \mathscr{L} \setminus \mathscr{S}} X_p X_q X_r.$$

Then the Cartan variety $\mathscr{E}_{6,1}(\mathbb{K})$ consists of all projective points $\langle v\rangle$ such that $\nabla C(v) = \vec{o}$ (the gradient in the classical sense). We have $\mathscr{E}_{6,1}(\mathbb{K}) \cong \mathscr{V}(\mathbb{K}, \mathbb{A})$, with $\mathbb{A}$ a split octonion algebra over $\mathbb{K}$.

Veronesean representations of projective planes—This is the case where $\mathbb{A}$ is a division algebra. If it is also associative, then $\mathscr{V}(\mathbb{K}, \mathbb{A})$ is the image of the map

$$\mathbb{A} \times \mathbb{A} \times \mathbb{A} \to \mathsf{PG}(3d+2, \mathbb{K}) : (a, b, c) \mapsto (a\overline{a}, b\overline{b}, c\overline{c}, b\overline{c}, c\overline{a}, a\overline{b}).$$

In general, it is the union of the three image sets

$$\{(a\overline{a}, b\overline{b}, 1, b, \overline{a}, a\overline{b}) \mid a, b \in \mathbb{A}\} \cup \{(a\overline{a}, 1, c\overline{c}, \overline{c}, c\overline{a}, a) \mid a, c \in \mathbb{A}\} \cup \{(1, b\overline{b}, c\overline{c}, b\overline{c}, c, \overline{b}) \mid b, c \in \mathbb{A}\}.$$

The variety $\mathscr{V}(\mathbb{K}, \mathbb{A})$ is a projective plane with obvious point set, and with as set of lines the d-dimensional quadrics with Witt index 1 entirely contained in it (if $|\mathbb{K}| = |\mathbb{A}| = 2$, we have to make an appropriate selection of those since every set of three non-collinear points is a conic, hence a 1-dimensional quadric of Witt index 1).

All the preceding varieties $\mathscr{V} = \mathscr{V}(\mathbb{K}, \mathbb{A})$ share the following properties, see [23]. There exists a unique set $\mathscr{H}$ of $(d+1)$-dimensional subspaces, called the *host spaces*, satisfying

(1) every pair of points of $\mathscr{V}$ is contained in at least one host space;
(2) the intersection of $\mathscr{V}$ with any host space is a non-empty non-degenerate d-dimensional quadric and the intersection of two distinct host spaces is always contained in $\mathscr{V}$;
(3) for each point p of $\mathscr{V}$, the space generated by the tangent spaces at p to the quadrics $\mathscr{V} \cap H$, with $p \in H \in \mathscr{H}$ has dimension $2d$.

These are precisely the properties, found by Zak [34], of the complex Severi varieties, that is, complex smooth $2d$-dimensional varieties in $\mathsf{PG}(3d+2, \mathbb{C})$ whose secant variety is not trivial (the secant variety of every complex smooth $2d$-dimensional variety in $\mathsf{PG}(n, \mathbb{C})$, with $n < 3d+2$ is trivial, that is, the whole of $\mathsf{PG}(n, \mathbb{C})$). Zak classified varieties satisfying the above three properties (over the complex numbers) and only found the varieties $\mathscr{V}(\mathbb{C}, \mathbb{A})$, with $\mathbb{A}$ either $\mathbb{C}$, $\mathbb{C} \times \mathbb{C}$, the split quaternions over $\mathbb{C}$, or the split octonions over $\mathbb{C}$. One can ask whether that classification can be extended to general fields. In fact, we can do more. The following results come from [23], [15] and [10].

Theorem 8.1 *Assume we have a point set $\mathscr{V}$ in some (not necessarily finite-dimensional) projective space over some field $\mathbb{K}$, a set of $(d+1)$-dimensional host spaces $\mathscr{H}$ satisfying* (1), (2) *and* (3) *above, for some natural number $d \geq 1$. Then there exists a non-degenerate quadratic alternative algebra $\mathbb{A}$ over $\mathbb{K}$ such that $\mathscr{V} = \mathscr{V}(\mathbb{K}, \mathbb{A})$.*

We emphasize that the quadrics obtained as intersection of $\mathscr{V}$ with the members of $\mathscr{H}$ should not be assumed to be isomorphic, or necessarily have the same Witt index. No assumption whatsoever is made on these quadrics, besides the fact that they are d-dimensional, that is, they span their host space. Actually, the assumptions in the aforementioned papers are slightly weaker in that (3) is replaced with the following axiom.

(3′) For each point p of $\mathscr{V}$, the space generated by the tangent spaces at p to the quadrics $\mathscr{V} \cap H$, with $p \in H \in \mathscr{H}$ has dimension at most $2d$.

Under that weaker condition, also the universal natural embeddings of the geometries in the conclusion of Theorem 7.1 have to be added to the conclusion. So completely different properties lead to exactly the same geometries, in different forms (pure vs. embedded), restricting Theorem 8.1 to sets $\mathscr{V}$ containing lines. This is part of the Magic so characteristic for the geometries in the Magic Square.

Besides using the Veronese map, there is another way to construct the varieties related to the second row of the Magic Square. Let $\mathbb{K}$, $\mathbb{A}$, σ and V again be as above. Then we may view V as the set of all *Hermitian* 3×3 *matrices*, that is, 3×3 matrices M with on the diagonal scalars and off the diagonal members of $\mathbb{A}$, so that $M^{\mathsf{t}} = \overline{M}$ (with t the *transpose operator*). Explicitly,

$$M = \begin{pmatrix} x_1 & X_3 & \overline{X}_2 \\ \overline{X}_3 & x_2 & X_1 \\ X_2 & \overline{X}_1 & x_3 \end{pmatrix}, x_1, x_2, x_3 \in \mathbb{K}, \;\; X_1, X_2, X_3 \in \mathbb{A}.$$

These form in fact a Jordan algebra (under the Jordan multiplication $A * B = \frac{1}{2}(AB + BA)$, provided that $\operatorname{char} \mathbb{K} \neq 2$). The projective points corresponding to all rank 1 such matrices constitute the variety $\mathscr{V}(\mathbb{K}, \mathbb{A})$. Writing down this condition explicitly by expressing that the columns of M are mutually proportional by right factors in $\mathbb{A}$, one obtains $\mathscr{V}(\mathbb{K}, \mathbb{A})$ as an intersection of $3d + 3$ quadrics with equations (where the x_i and the X_i refer to the coordinates as above in the matrix M):

$$\left\{ \begin{array}{rcl} X_i \overline{X}_i & = & x_{i+1} x_{i+2}, \\ x_i \overline{X}_i & = & X_{i+1} X_{i+2}, \end{array} \right. \quad \text{for all } i \in \{1, 2, 3\} \pmod 3.$$

Now it is the right moment to look at the first row.

8.2 The first row

The varieties of the first row of the Magic Square are obtained from the ones in the second row by a suitable hyperplane section. In the complexified case, the standard suitable hyperplane H is, with the above notation, the one with equation $x_1 + x_2 + x_3 = 0$. Something peculiar happens now for the third and fourth cells. Indeed, the lines of the geometries are not all lines of the varieties. For instance, $\mathscr{E}_{6,1}(\mathbb{K})$ contains subspaces of dimension 5, so a hyperplane intersects this in a subspace of dimension 4 or 5; yet the geometry of type $\mathsf{F}_{4,4}$ does not have any singular subspaces of that dimension. It is even more extreme: the 5-spaces of $\mathscr{E}_{6,1}(\mathbb{K})$ that are contained in H become the symps of the new geometry. And a line of $\mathscr{E}_{6,1}(\mathbb{K})$ in H is a line of the new geometry if and only if it is contained in at least two 5-spaces that are contained in H and $\mathscr{E}_{6,1}(\mathbb{K})$. Similarly for $\mathscr{G}_{2,6}(\mathbb{K})$ and the variety corresponding to the geometry of type $\mathsf{C}_{3,2}$.

In general, we denote by $\mathscr{V}'(\mathbb{K}, \mathbb{A})$ the variety of the first row in $\mathsf{PG}(3d + 1, \mathbb{K})$ obtained from $\mathscr{V}(\mathbb{K}, \mathbb{A})$ by intersecting with a suitable hyperplane of $\mathsf{PG}(3d + 2, \mathbb{K})$.

So this is a neat correspondence between the first two rows of the Complexified Geometric Magic Square. What about the Relative Geometric Magic Square? If we take the same equation

$x_1 + x_2 + x_3 = 0$ for H, then it really depends on the field whether we get a non-empty variety. For instance, considering the real numbers, we see that we get a complete anisotropic variety, since the sum of squares is only zero if each square is zero. Hence here we see that it makes sense to take the empty Tits diagrams in the first row of the Relative Magic Square. But over a finite field the second cell is a true Hermitian curve (the third and fourth cells do not exist over a finite field in the Relative Square). In fact, for a finite field one cannot find a hyperplane with empty intersection. This mixed behaviour implies different view points and interpretations on the first row, and we already saw an example of how we can choose a certain interpretation to prove a point.

There is another, related correspondence between the first and second row of the Complexified Geometric Magic Square, recently proved by De Schepper & Victoor [14]. Before we can state it, we state a duality property of the varieties of the second row. This is kind of folklore. We use the terminology of the characterization in Theorem 8.1.

Theorem 8.2 *Let $\mathbb{A}$ be any non-degenerate quadratic alternative algebra over the field $\mathbb{K}$. Then each host space W of $\mathscr{V}(\mathbb{K}, \mathbb{A})$ is contained in a unique hyperplane H of $\mathsf{PG}(3d+2, \mathbb{K})$ which intersects $\mathscr{V}(\mathbb{K}, \mathbb{A})$ precisely in the points that belong to W and $\mathscr{V}(\mathbb{K}, \mathbb{A})$. The set of all such hyperplanes when varying the host space is in the dual projective space isomorphic to $\mathscr{V}(\mathbb{K}, \mathbb{A})$.*

We denote this dual variety by $\mathscr{V}^*(\mathbb{K}, \mathbb{A})$. Now we can state the beautiful result of De Schepper & Victoor.

Theorem 8.3 *Let Q be any non-degenerate quadric of $\mathsf{PG}(3d+2, \mathbb{K})$ containing all points of $\mathscr{V}(\mathbb{K}, \mathbb{A})$, with $\mathbb{A}$ not division. Then the image in $\mathscr{V}^*(\mathbb{K}, \mathbb{A})$ of the set of host spaces of $\mathscr{V}(\mathbb{K}, \mathbb{A})$ contained in a singular subspace of Q is isomorphic to a variety dual to $\mathscr{V}'(\mathbb{K}, \mathbb{A})$.*

One expects the same to be true for the case of a division algebra $\mathbb{A}$ if one assumes that Q has maximal Witt index (this is proved in [14] for the smallest case, namely $\mathbb{A} = \mathbb{K}$).

Let us now have a look at the third row.

8.3 The third row

Again, we can construct all varieties corresponding to the geometries of the third row of the Relative and Complexified Geometric Magic Square in a uniform way using a Veronesean map. In the general case this goes as follows. For a 3×6 matrix M we denote by $p_{ijk}(M)$, $1 \le i < j < k \le 6$ the determinant of the 3×3 matrix obtained from M by only keeping the columns labeled i, j and k. Now, for $\mathbb{K}$ and $\mathbb{A}$ as before, define the matrix

$$M(x_1, x_2, x_3, X_1, X_2, X_3) = \begin{pmatrix} 1 & 0 & 0 & x_1 & X_3 & \overline{X}_2 \\ 0 & 1 & 0 & \overline{X}_3 & x_2 & X_1 \\ 0 & 0 & 1 & X_2 & \overline{X}_1 & x_3 \end{pmatrix}, x_1, x_2, x_3 \in \mathbb{K}, \;\; X_1, X_2, X_3 \in \mathbb{A}.$$

Now let D be the set of triples $\{123,125,126,134,135,145,156,234,236,246,256,345,346,456\}$. We may restrict to those p_{ijk} with $ijk \in D$ since clearly $p_{136} = -\overline{p}_{125}$, $p_{124} = \overline{p}_{236}$, $p_{235} = -\overline{p}_{134}$, $p_{356} = \overline{p}_{145}$, $p_{146} = -\overline{p}_{256}$, $p_{245} = -\overline{p}_{346}$. Then we define the *Veronese map* $\rho^{\dagger}_{\mathbb{A}}$ as follows

$$\rho^{\dagger}_{\mathbb{A}} : \mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{A}\times\mathbb{A}\times\mathbb{A} \to \mathsf{PG}(6d+7,\mathbb{K}) : (x_1,x_2,x_3,X_1,X_2,X_3) \mapsto (p_{ijk})_{ijk\in D}.$$

Then we define the variety $\mathscr{V}^{\dagger}(\mathbb{K},\mathbb{A})$ as the Zariski closure of the image of $\rho^{\dagger}_{\mathbb{A}}$.

Remark 8.4 One could wonder how to exactly calculate the 3×3 determinants when some entries belong to non-commutative and even non-associative structures. The answer is that it does not matter so much as long as the calculations are mutually consistent with cyclic permutations of the indices. For instance, once p_{145} chosen as $x_2X_2 - \overline{X}_3\overline{X}_1$, we have to set p_{256} equal to $\overline{X}_1\overline{X}_2 - x_3X_3$. Concerning p_{456}, any way we place brackets to ensure the result belongs to $\mathbb{K}$, is fine. For instance

$$p_{456} = x_1x_2x_3 + X_1(X_2X_3) + (\overline{X}_3\overline{X}_2)\overline{X}_1 - x_1X_1\overline{X}_1 - x_2X_2\overline{X}_2 - x_3X_3\overline{X}_3.$$

Another choice boils down to a simple recoordinatization. This freedom can be seen as another magic feature of the Square.

Each variety $\mathscr{V}^{\dagger}(\mathbb{K},\mathbb{A})$ can also be described as the intersection of a number of quadrics (see [16]). Their equations in case of $\mathbb{A}$ being a split octonion algebra involves some nice combinatorics using the Gosset graph. However, the exact description would take us too far here.

Dual polar spaces—In the case of a division algebra $\mathbb{A}$, the geometry underlying $\mathscr{V}^{\dagger}(\mathbb{K},\mathbb{A})$ is a dual polar space of rank 3. In this case, there is an efficient way to write the equations of the above mentioned quadrics down. The following is Theorem 10.38 of [13].

Theorem 8.5 *Let $\mathbb{A}$ be a finite-dimensional alternative quadratic division algebra over $\mathbb{K}$ and set $d = \dim_{\mathbb{K}}\mathbb{A}$. Let V be the $(6d+8)$-dimensional vector space over $\mathbb{K}$ consisting of the direct sum $\mathbb{K}^4 \oplus \mathbb{A}^3 \oplus \mathbb{K}^3 \oplus \mathbb{A}^3 \oplus \mathbb{K}$. We label the coordinates according to the generic point $(x,\ell_1,\ell_2,\ell_3,X_1,X_2,X_3,k_1,k_2,k_3,Y_1,Y_2,Y_3,y)$. Then the intersection of the $12d+7$ quadrics in $\mathsf{PG}(V)$ with following equation is the point set of the variety $\mathscr{V}^{\dagger}(\mathbb{K},\mathbb{A})$:*

$$\begin{array}{lll}
0 = xk_1 + \ell_2\ell_3 - X_1\overline{X}_1, & 0 = xY_1 + X_2X_3 - \ell_1\overline{X}_1, & 0 = k_2\overline{X}_1 + \ell_3Y_1 + X_2\overline{Y}_3,\\
0 = xk_2 + \ell_3\ell_1 - X_2\overline{X}_2, & 0 = xY_2 + X_3X_1 - \ell_2\overline{X}_2, & 0 = k_3\overline{X}_1 + \ell_2Y_1 + \overline{Y}_2X_3,\\
0 = xk_3 + \ell_1\ell_2 - X_3\overline{X}_3, & 0 = xY_3 + X_1X_2 - \ell_3\overline{X}_3, & 0 = k_3\overline{X}_2 + \ell_1Y_2 + X_3\overline{Y}_1,\\
0 = y\ell_1 + k_2k_3 - Y_1\overline{Y}_1, & 0 = yX_1 + Y_3Y_2 - k_1\overline{Y}_1, & 0 = k_1\overline{X}_2 + \ell_3Y_2 + \overline{Y}_3X_1,\\
0 = y\ell_2 + k_3k_1 - Y_2\overline{Y}_2, & 0 = yX_2 + Y_1Y_3 - k_2\overline{Y}_2, & 0 = k_1\overline{X}_3 + \ell_2Y_3 + X_1\overline{Y}_2,\\
0 = y\ell_3 + k_1k_2 - Y_3\overline{Y}_3, & 0 = yX_3 + Y_2Y_1 - k_3\overline{Y}_3, & 0 = k_2\overline{X}_3 + \ell_1Y_3 + \overline{Y}_1X_2
\end{array}$$

and $0 = xy + \ell_1k_1 - \ell_2k_2 - \ell_3k_3 - X_1Y_1 - \overline{Y}_1\overline{X}_1$.
Also, no quadric can be ommited, that is, the intersection of each proper subset of this set of $12d+7$ quadrics contains points off $\mathscr{V}^{\dagger}(\mathbb{K},\mathbb{A})$.

Concerning the cases $\mathbb{A} = \mathbb{K} \times \mathbb{K}$ and $\mathbb{A}$ a quaternion algebra, their exist elegant geometric constructions of the varieties $\mathscr{V}^\dagger(\mathbb{K}, \mathbb{A})$ using varieties related to the North-East cell. In particular, we can construct the Grassmannian $\mathscr{G}_{3,6}(\mathbb{K})$ (the cell of type $\mathsf{A}_{5,3}$) using the smaller Grassmannian $\mathscr{G}_{2,5}(\mathbb{K})$ (related to the cell of type $\mathsf{A}_{5,2}$), and we can construct the half spin geometry $\mathscr{HS}_6(\mathbb{K})$ (the cell of type $\mathsf{D}_{6,6}$) using the half spin geometry $\mathscr{HS}_5(\mathbb{K})$ (related to the cell of type $\mathsf{E}_{6,1}$).

The plane Grassmannian $\mathscr{G}_{3,6}(\mathbb{K})$—Let us construct $\mathscr{G}_{3,6}(\mathbb{K})$ in a completely geometric way. We proceed in three steps. (Proofs are easy an left to the reader.)

(Step 1) Let π_1 and π_2 be two disjoint planes in $\mathsf{PG}(5, \mathbb{K})$. Let α be a linear duality between between π_1 and π_2, that is, an incidence preserving bijective map from the points and lines of π_1 to the lines and points, respectively, of π_2 preserving the cross ratio. Taking the union of all planes of $\mathsf{PG}(5, \mathbb{K})$ joining a point $p \in \pi_1$ to its image $p^\alpha \subseteq \pi_2$, we obtain $\mathscr{G}_{2,4}(\mathbb{K})$.

(Step 2) It is well known that the points of $\mathscr{G}_{2,4}$ can be identified with the lines of $\mathsf{PG}(3, \mathbb{K})$ in such a way that collinear points go to concurrent lines. Embed $\mathscr{G}_{2,4}(\mathbb{K})$ in a 5-space W of $\mathsf{PG}(9, \mathbb{K})$ and choose a 3-space U disjoint from W. Let β be a bijective map from the point set of $\mathscr{G}_{2,4}(\mathbb{K})$ to the set of lines of U so that collinear points get mapped to concurrent lines, and so that β preserves the cross ratio. Taking the union of all planes of $\mathsf{PG}(9, \mathbb{K})$ joining a point $p \in \mathscr{G}_{2,4}(\mathbb{K})$ to its image $p^\beta \subseteq U$, we obtain $\mathscr{G}_{2,5}(\mathbb{K})$.

(Step 3) Let Π_1 and Π_2 be two copies of $\mathscr{G}_{2,5}(\mathbb{K})$ in disjoint 9-spaces of $\mathsf{PG}(19, \mathbb{K})$. We can consider Π_2 as a copy of $\mathscr{G}_{3,5}(\mathbb{K})$ and hence there is a bijective map γ from the set of points of Π_1 to the set of planes of Π_2 which are not contained in a 3-space contained in Π_2 mapping collinear points to planes sharing a point. Again we can choose γ such that it preserves the cross ratio. Taking the union of all solids of $\mathsf{PG}(19, \mathbb{K})$ joining a point $p \in \Pi_1$ to its image $p^\gamma \subseteq \Pi_2$, we obtain $\mathscr{G}_{3,6}(\mathbb{K})$.

The half spin geometry $\mathscr{HS}_6(\mathbb{K})$—Here, we also proceed in three steps.

(Step 1) Let Σ_1 and Σ_2 be two disjoint solids—that is, subspaces of dimension 3— in $\mathsf{PG}(7, \mathbb{K})$. Let α be a linear duality between between Σ_1 and Σ_2, that is, an incidence preserving bijective map from the points, lines and planes of Σ_1 to the planes, lines and points, respectively, of Σ_2 preserving incidence and the cross ratio. Taking the union of all solids of $\mathsf{PG}(7, \mathbb{K})$ joining a point $p \in \Sigma_1$ to its image $p^\alpha \subseteq \Sigma_2$, we obtain $\mathscr{HS}_4(\mathbb{K})$, which is isomorphic to a hyperbolic quadric (the triality quadric, or also called the quadric of Study).

(Step 2) It is well known that the points of $\mathscr{HS}_4(\mathbb{K})$ can be identified with one class of solids of $\mathscr{HS}_4(\mathbb{K})$ in such a way that collinear points go to solids sharing a line (this is due to triality). Embed two copies, say Ω_1 and Ω_2 of $\mathscr{HS}_4(\mathbb{K})$ in disjoint 7-spaces of $\mathsf{PG}(15, \mathbb{K})$ and choose a bijective mapping β from the point set of Ω_1 to one class of solids of Ω_2 such that collinear points are mapped to solids sharing a line and so that the cross ration is preserved. Taking the union of all 4-spaces of $\mathsf{PG}(15, \mathbb{K})$ joining a point $p \in \Omega_1$ to its image $p^\beta \subseteq \Omega_2$, we obtain $\mathscr{HS}_5(\mathbb{K})$.

(Step 3) Let Π_1 and Π_2 be two copies of $\mathscr{HS}_5(\mathbb{K})$ in disjoint 15-spaces of $\mathsf{PG}(31, \mathbb{K})$. Considering the set of maximal 4-spaces of $\mathscr{HS}_5(\mathbb{K})$, there is a bijective map γ from the set of points of Π_1 to the set of maximal 4-spaces of Π_2 mapping collinear points to 4-spaces sharing a plane. Again we can choose γ such that it preserves the cross ratio. Taking the union of all 5-spaces of $\mathsf{PG}(31,\mathbb{K})$ joining a point $p \in \Pi_1$ to its image $p^\gamma \subseteq \Pi_2$, we obtain $\mathscr{HS}_6(\mathbb{K})$.

A similar construction of $\mathscr{V}^\dagger(\mathbb{K},\mathbb{A})$ for $\mathbb{A}$ the split octonion algebra over $\mathbb{K}$ is not available. There exists a more involved algebraic construction, but it falls beyond the scope of this paper.

9 Connectivity of the Square

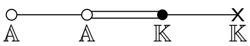
We now describe various results connecting in a systematic way cells of the Magic Square to other cells.

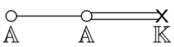
9.1 Global connectivity: Equator geometries and Residues

Diagrammatically, the first row and first column of the Absolute Magic Square seem to play an isolated role in that each cell is the type of a residue of a building of the type in any cell South, East and anywhere South-East of it, except if the original cell is in the first row or first column.

So we consider the South-East 3×3 corner of the Complexified Geometric Magic Square. The connectivity we would like to explain here is that of taking a point residual when going vertically North and taking an equator geometry when going horizontally West.

Going North: point residual—Let $(X,\mathscr{L})$ be the Lie incidence geometry over the field $\mathbb{K}$ of type $\mathsf{A}_{5,3}$, $\mathsf{D}_{6,6}$, $\mathsf{E}_{7,7}$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$ or $\mathsf{E}_{8,8}$ (for type A, this may even be a skew field). Let $p \in X$ be arbitrary. Then the *point residual at* p is the geometry $(\mathscr{L}_p,\mathscr{P}_p)$, where $\mathscr{L}_p$ is the set of lines that contain p, and $\mathscr{P}_p$ the set of planes that contain p (and natural inlcusion). This amounts to the same notion as the residue in the corresponding spherical building. Then $(\mathscr{L}_p,\mathscr{P}_p)$ is the Lie incidence geometry over $\mathbb{K}$ of respective type $\mathsf{A}_{2,1} \times \mathsf{A}_{2,1}$, $\mathsf{A}_{5,2}$, $\mathsf{E}_{6,1}$, $\mathsf{A}_{5,3}$, $\mathsf{D}_{6,6}$ and $\mathsf{E}_{7,7}$, that is, the type just North of the original type.

The same thing holds for the geometries in the fourth row of the Relative Geometric Square: the point residuals of the Lie incidence geometry 

for $\mathbb{A}$ a quadratic alternative division algebra, is 

.

Going West: Equator geometry—Let $\Delta = (X,\mathscr{L})$ be the (split) Lie incidence geometry over the (skew) field $\mathbb{K}$ of type $\mathsf{C}_{3,2}$, $\mathsf{A}_{5,2}$, $\mathsf{D}_{6,6}$, $\mathsf{E}_{7,1}$, $\mathsf{F}_{4,4}$, $\mathsf{E}_{6,1}$, $\mathsf{E}_{7,7}$ or $\mathsf{E}_{8,8}$. Let t be the type of the corresponding building corresponding to the objects whose residues are buildings of the type just West in the Absolute Square. We observe that t is always a self-opposite type. We choose two opposite objects Ω_1,Ω_2 of type t and denote by $E(\Omega_1,\Omega_2)$ the set of points in X exactly in the middle on a geodesic from Ω_1 to Ω_2 in the full incidence graph of the corresponding building. Endow $E(\Omega_1,\Omega_2)$ with the set of lines contained in it, then $E(\Omega_1,\Omega_2)$ is the Lie incidence geometry over $\mathbb{K}$ of respective type $\mathsf{A}_{2,\{1,2\}}$, $\mathsf{A}_{2,1} \times \mathsf{A}_{2,1}$, $\mathsf{A}_{5,3}$, $\mathsf{E}_{6,2}$, $\mathsf{C}_{3,2}$, $\mathsf{A}_{5,2}$, $\mathsf{D}_{6,6}$ or $\mathsf{E}_{7,1}$, that is, the type just West of the original type.

The above definition of $E(\Omega_1,\Omega_2)$ is perhaps not very intuitive, and not completely geometric, rather combinatorial. For each particular type, there is an individual more geometric description. Let us review these quickly.

(*i*) **Case** $\mathsf{C}_{3,2}$. Here, Ω_1 and Ω_2 are two opposite singular planes of Δ and $E(\Omega_1, \Omega_2)$ is the set of points of Δ collinear to a line of Ω_1 and one of Ω_2.

(*ii*) **Case** $\mathsf{A}_{5,2}$. Here, Ω_1 and Ω_2 are two opposite planes of Δ that are maximal singular subspaces and $E(\Omega_1, \Omega_2)$ is the set of points of Δ collinear to a line of Ω_1 and one of Ω_2.

(*iii*) **Case** $\mathsf{D}_{6,6}$. Here, Ω_1 and Ω_2 are two opposite singular 5-spaces of Δ and $E(\Omega_1, \Omega_2)$ is the set of points of Δ collinear to a plane of Ω_1 and one of Ω_2.

(*iv*) **Case** $\mathsf{E}_{7,1}$. Here, Ω_1 and Ω_2 are two opposite subgeometries (called *paras* in [9]) of type $\mathsf{E}_{6,1}$ of Δ and $E(\Omega_1, \Omega_2)$ is the set of points of Δ collinear to a 5-space of Ω_1 and one of Ω_2.

(*v*) **Case** $\mathsf{F}_{4,4}$. Here, Ω_1 and Ω_2 are two opposite symplecta of Δ and $E(\Omega_1, \Omega_2)$ is the set of points of Δ collinear to a line of Ω_1 and one of Ω_2.

(*vi*) **Case** $\mathsf{E}_{6,1}$. Here, Ω_1 and Ω_2 are two opposite singular 5-spaces of Δ and $E(\Omega_1, \Omega_2)$ is the set of points of Δ collinear to a solid of Ω_1 and one of Ω_2.

(*vii*) **Case** $\mathsf{E}_{7,7}$. Here, Ω_1 and Ω_2 are two opposite symplecta of Δ and $E(\Omega_1, \Omega_2)$ is the set of points of Δ collinear to a singular 5-space of Ω_1 and one of Ω_2.

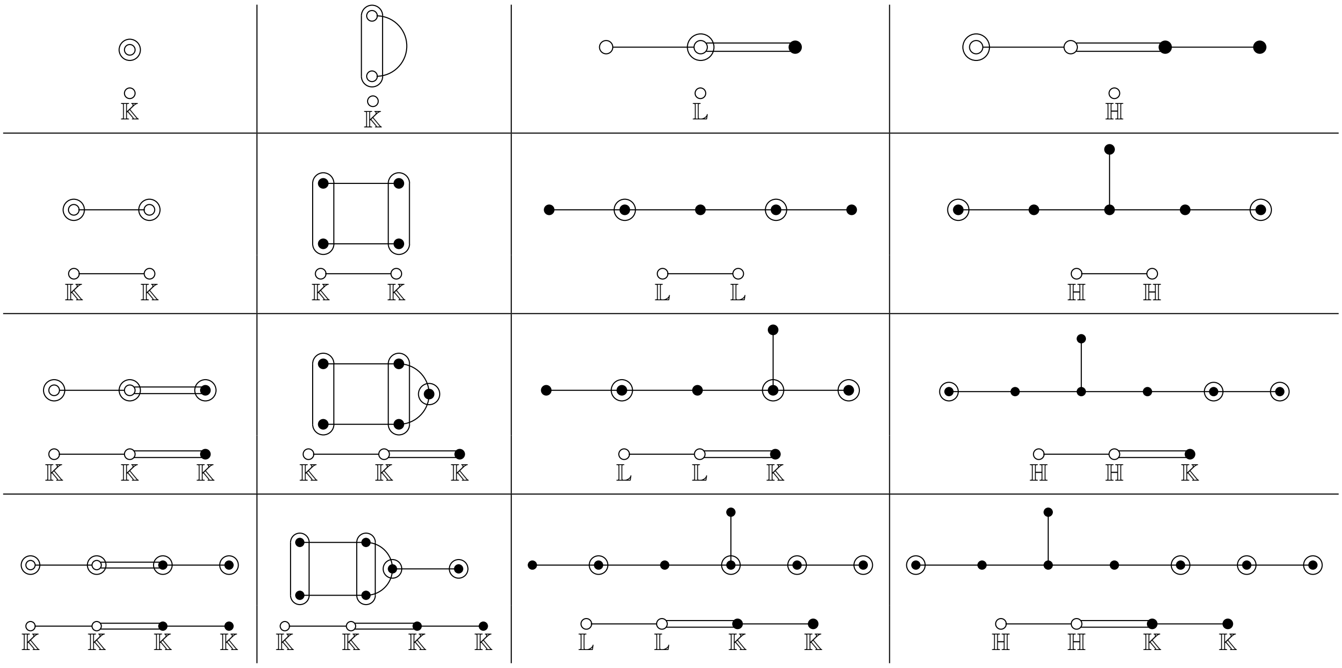
(*viii*) **Case** $\mathsf{E}_{8,8}$. Here, Ω_1 and Ω_2 are two opposite points of Δ and $E(\Omega_1, \Omega_2)$ is the set of points x of Δ such that $\{x, \Omega_1\}$ and $\{x, \Omega_2\}$ are both symplectic pairs.

The Lie incidence geometries of the first column are not equator geometries of those of the second column. However, the results of Section 6 in [9] show that they are the intersection of two equator geometries of the geometries of the third column. More exactly, and with the above notation, by considering a(n arbitrary) third object Ω_3 of type t opposite both Ω_1 and Ω_2 and then intersecting $E(\Omega_1, \Omega_2) \cap E(\Omega_1, \Omega_3)$.

9.2 The Delayed Magic Square

Recall that the Relative Magic Square originates from the Absolute Magic Square by Galois descent. A characteristic feature of this Galois descent is that in the cases of the Magic Square, the Galois group had order 2. Hence the geometries of the Relative Geometric Magic Square (and also the corresponding varieties) are fixed point sets of the geometries of the Complexified Geometric Magic Square (and of the corresponding varieties, respectively) under the action of an involution. That involution is semi-linear in the sense that there exists a (unique) involutive field automorphism σ of the underlying field such that, if c is the cross-ration of four points on a line, then c^σ is the cross-ration of the image of the four points (which are usually also four points on a line, but could also be points in the dual geometry). Now, in each case there also exists an involution that is linear (preserving cross-ratio rather than transforming it through a field automorphism) and has the same fix diagram as the Galois descent case. Magically, each fix geometry of such linear involution is isomorphic to the fix geometry of the Galois involution in the cell directly West to it, except for the ones in the first column, since there, the linear involution is just the identity.

In the next table we present the fix diagrams of the linear involutions together with the diagram of the fixed point set (which is a building, or a geometry depending on the point of view).



We call this the *Delayed Magic Square*. Some magic properties are mentioned in Paragraph 9.4 below. For now we content ourselves with another surprising connection. In order to explain and motivate it, we need some preparation.

9.3 Minimum number of quadrics for the varieties of the second row

Just like the Cartan variety was defined using a trilinear form, we can also define the other varieties $\mathscr{V}$ of the second row of the Complexified Geometric Magic Square using a trilinear form on a vector space V of dimension $n+1 \in \{6, 9, 15, 27\}$. For the second and third cell, the definition is just the same, that is

$$C\left((x_p)_{p\in X}\right) = \sum_{\{p,q,r\}\in\mathscr{S}} X_pX_qX_r - \sum_{\{p,q,r\}\in\mathscr{L}\setminus\mathscr{S}} X_pX_qX_r,$$

where now for the second cell (the variety $\mathscr{S}_{2,2}(\mathbb{K})$) $(X,\mathscr{L})$ is a generalized quadrangle $\mathsf{GQ}(2,1)$ of order $(2,1)$, that is, a 3×3 grid, and $\mathscr{S}$ is a spread of $\mathsf{GQ}(2,1)$; for the third cell (the variety $\mathscr{G}_{2,6}(\mathbb{K})$) $(X,\mathscr{L})$ is a generalized quadrangle $\mathsf{GQ}(2,2)$ of order $(2,2)$, and $\mathscr{S}$ is a spread of a subquadrangle of order $(2,1)$. For $\mathscr{S}_{2,2}(\mathbb{K})$ this amounts to the determinant of a 3×3 matrix with entries the nine coordinates; for $\mathscr{V}_2(\mathbb{K})$, it is the determinant of a 3×3 symmetric matrix.

Now let ϕ be the operator from V to V mapping a vector with coordinates $(x_p)_{p\in X}$ to $\nabla C(v)$. We then have $\phi(\phi(v)) = C(v)v$. The points $\langle v\rangle$ with $\phi(v) \neq \vec{o}$ but $C(v) = 0$ are called *grey points* [2] with respect to $\mathscr{V}$. Their image under ϕ belongs to $\mathscr{V}$. They are precisely the points lying on a tangent or secant line of $\mathscr{V}$.

Each coordinate of $\nabla C\left((x_p)_{p\in X}\right)$ defines a quadratic form, and so $\mathscr{V}$, being the set of all projective points $\langle v\rangle$ with $\phi(v) = \vec{o}$, is the intersection of $n+1$ quadrics. All these quadrics are linearly independent, and it is shown in [33] that (the equation of) every quadric containing $\mathscr{V}$ is a linear combination of (the equations of) these $n+1$ quadrics. However, it is nevertheless possible to produce a smaller number of quadrics with common intersection exactly $\mathscr{V}$. Indeed, the following theorem is proved in [33]:

Theorem 9.1 *Let $\mathscr{V}$ be either the quadratic Veronese surface $\mathscr{V}_2(\mathbb{K})$, the Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$, the line Grassmannian $\mathscr{G}_{2,6}(\mathbb{K})$, or the Cartan variety $\mathscr{E}_6(\mathbb{K})$, in n-dimensional projective space $\mathsf{PG}(n,\mathbb{K})$ over $\mathbb{K}$, with $n = 5, 8, 14, 26$, respectively. Then $\mathscr{V}$ is the intersection of $n-d$ quadrics and no less, where d is the dimension of a maximum dimensional projective subspace of $\mathsf{PG}(n,\mathbb{K})$ consisting solely of grey points. More precisely, the equivalence classes of the systems of $n-d$ linearly independent quadrics intersecting precisely in $\mathscr{V}$ are in natural bijective correspondence with the d-dimensional projective subspaces of $\mathsf{PG}(n,\mathbb{K})$ consisting solely of grey points.*

One could ask what the largest subspaces of grey points are. And here comes in some unexpected magic with the Delayed Magic Square. Indeed, as noted above, the image of each grey point under ϕ is a point of $\mathscr{V}$. Now it just happens so that, across all possible fields, the largest subspaces consisting entirely of grey points that are known have as image under ϕ precisely the subvariety given in the Delayed Magic Square obtained as fixed point set of a linear involution (this is also true for the first cell!).

As an example, taking as the cubic form for $\mathscr{S}_{2,2}(\mathbb{K})$ the determinant of the 3×3 matrix of coordinates x_{ij} (with self-explaining notation, $1\le i\le 3$, $1\le j\le 3$), a plane of grey points is determined by all anti-symmetric matrices. Noting that in this case $\nabla C\left((x_{ij})_{1\le i\le 3,\,1\le j\le 3}\right)$ is the adjugate matrix of $(x_{ij})_{1\le i\le 3,\,1\le j\le 3}$, one easily calculates that the global image of the set of anti-symmetric matrices under ϕ is exactly the set of symmetric matrices, that is, the image under ϕ of the corresponding plane of grey points is a subvariety of $\mathscr{S}_{2,2}(\mathbb{K})$ isomorphic to $\mathscr{V}_2(\mathbb{K})$.

9.4 Opposition diagrams

The magical twisted symmetry of the Fix Magic Square is the basis of the notion of an *opposition diagram* (called *domesticity diagram* in [31] when it was introduced in the literature for the first time).

Let θ be an automorphism of a spherical building Δ. Then the *opposition diagram* of θ is the Coxeter diagram of Δ where a set of nodes is encircled if there exists a flag of that type mapped under θ to an opposite flag, and no subflag of it has the same property. It is drawn in a bent way if the action of θ on the types does not coincide with the opposition relation. It is shown in [18] that, as soon as Δ has rank at least 3 and does not contain projective planes of order 2 as some residue, then there exists a flag of type the union of all encircled nodes mapped to an opposite flag. Hence, for such buildings, θ maps some chamber to an opposite if and only if in the opposition diagram of θ, every node is in an encircled set of nodes (such diagrams are called *full*). If the opposition diagram has nodes that are not encircled, then we call θ *domestic*. Equivalently, θ is domestic if it does not map any chamber to an opposite. For instance, if not all nodes of the diagram of a spherical building are polar (recall that the *polar nodes* are the nodes corresponding to the fundamental roots that are not perpendicular to the highest root), then every central collineation (in other words, every long root elation) is domestic and in the opposition diagram exactly the polar nodes are encircled. Note that all nodes are polar if and only if the diagram is A_1 or A_2, and indeed, in these cases central collineations are not domestic. (All (irreducible) Dynkin diagrams admit exactly one polar node except for A_n, $n \geq 2$, which admits two of them, namely, the two end nodes.)

Despite the immediate and easy example in the previous paragraph, domestic automorphisms seem to be rather rare, especially when we additionally assume that the automorphism also does not fix any chamber. In that case it is likely that also the fix diagram is not full. More background and details about domestic collineations, also when the building does contain planes of order 2 as residues, can be found in the papers [18], [19], [20], where buildings containing no planes of order 2 as residue are called *large*.

The main conjecture on domestic automorphisms and the Magic Square is a thickening of the twisted symmetry of the Fix Magic Square (with in the first row diagrams without encircled nodes to include the identity).

Conjecture 9.2 (i) *For each large spherical building with diagram in the Fix Magic Square, all domestic automorphisms with the given fix diagram are known and their opposition diagram is the fix diagram in the symmetric (with respect to the main diagonal) cell, except for the second half of the first column, where other possibilities for the opposition diagram then just those on the first row occur, if no subbuilding is fixed. Moreover, the fixed building in each case corresponds to the building of the Delayed Magic Square in the same cell.*

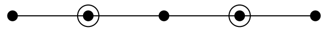
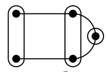
(ii) *Conversely, for each large spherical building with diagram in the Fix Magic Square, such that the corresponding fix diagram is not full, all automorphisms not fixing any chamber and with given opposition diagram are known and their fix diagram is the fix diagram in the symmetric cell. The fixed building is the corresponding building in the Delayed Magic Square.*

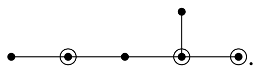
It is tacitly assumed that in the third column of the Delayed Magic Square the quadratic extension of the base field may also be inseparable in characteristic 2, and the quaternion division algebra in the fourth column may be a degree 4 inseparable extension in characteristic 2.

However, there is a counter example to (ii) of this conjecture for E_7. The renewed conjecture states that there are exactly two well known counter examples to part (ii) of the old conjecture, one for E_7 and one for E_8. We explain this now in some more detail, reviewing the different cells of the Square. The (i,j)-cell will mean the cell in row i and column j, $1 \leq i \leq 4$, $1 \leq j \leq 4$.

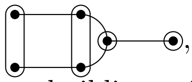
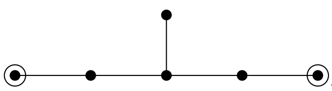
The first column and the first row. Here, the conjecture is trivial in view of the fact that the opposition diagram of an automorphism has no encircled nodes if and only if the automorphism is the identity (this follows from [1], where it is proved that every nontrivial automorphism of a spherical building maps some flag to an opposite) and the fact that, in large split buildings of type C_3 and F_4 only the identity is a domestic collineation fixing a subbuilding (the latter follows from [26] for type A_2, from the last section of [25] for type C_3, and from [20] for type F_4).

The cell with diagram $\mathsf{A}_2 \times \mathsf{A}_2$. This fix diagram is full and symmetric to itself, so nothing to prove here. The conjecture is trivially true.

The cells with diagram A_5. It is shown in [26] that every domestic collineation of a large projective space fixes at least one point. Hence the fix diagram  can never belong to a domestic collineation, showing in a trivial way part (i) of Conjecture 9.2 for the $(2,3)$-cell. Now let θ be an automorphism with fix diagram . Since the diagram is bent, θ is a duality. Domestic dualities in large projective spaces are classified in [26], and exactly the symplectic polarities qualify. For 5-dimensional spaces they have opposition diagram exactly . The fixed building is a symplectic polar space, which is exactly the $(3,2)$-cell in the Delayed Magic Square. Hence also part (i) of Conjecture 9.2 holds for the $(3,2)$-cell. Part (ii) of the conjecture follows immediately from part (i) and the uniqueness of domestic dualities in projective spaces.

The cell with diagram D_6. It is shown in [21] that a domestic automorphism of a large building of type D_6 either fixes at least one point, or has opposition diagram . Moreover it is shown in [21] that θ being domestic with this opposition diagram is equivalent to θ being linear, the fixed line set being a partition of the point set and arising from a Hermitian polar space of rank 3 in projective 5-space over a quadratic extension $\mathbb{L}$ of the base field $\mathbb{K}$ by so-called "field reduction" (considering $\mathbb{L}$ as a vector space of dimension 2 over $\mathbb{K}$, the point set of the Hermitian polar space in 5-dimensional projective space over $\mathbb{L}$ becomes a set (spread) of lines of the hyperbolic quadric in 11-dimensional projective space over $\mathbb{K}$). This Hermitian polar space corresponds exactly to the Delayed Magic Square at position $(3,3)$. Moreover, the fix diagram is the same as the opposition diagram. This shows the conjecture in this case. (But note that $\mathbb{L}$ may be inseparable, in which case "Hermitian" would rather be replaced with "mixed".)

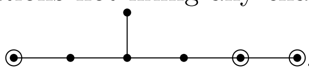
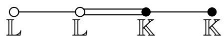
We now come to the most interesting cells, those containing the exceptional diagrams of any type E.

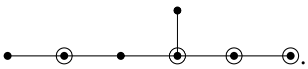
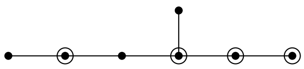
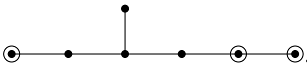
The cells with diagram E_6. The situation here is completely analogous to the situation of projective spaces (the cells with diagram A_5). Indeed, if the fix diagram of an automorphism θ is , then θ is a duality, and it is shown in [31] that the only domestic dualities in large buildings of type E_6 are the symplectic dualities, which have as fix point building a split building of type F_4 (the $(4,1)$-entry in the Delayed Magic Square). If θ is an automorphism having fix diagram , then θ is a collineation. But from the table of opposition diagrams in [18] follows that every domestic collineation, and hence also θ, fixes some object of type 2, a contradiction. Conjecture 9.2 follows for type E_6.

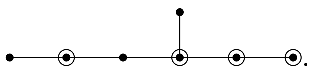
From now on, results are either not published yet, or even still in progress. So this serves rather as a survey of near-future results than of known results.

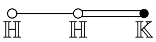
The cells with diagram E_7. This is the unique type for which the two distinct fix diagrams are both not full. So, Conjecture 9.2 hints at the existence of exactly two (classes of) domestic collineations fixing no chamber. But, according to the revised conjecture, there is a third. Here are the details (not published yet).

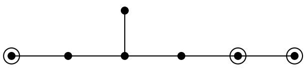
First we note that it is already shown in [20] that the two fix diagrams in the fix Magic Square of E_7 are the only two opposition diagrams for domestic collineations not fixing any chamber.

Now first suppose that some collineation θ has opposition diagram . This opposition diagram is realized for collineations which are the product of three perpendicular root elations, but if θ does not fix any chamber, then one can show that there is a quadratic extension $\mathbb{L}$ (may be inseparable in characteristic 2) such that the fixed building is 

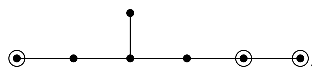
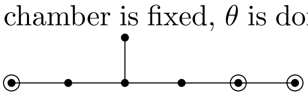
. The corresponding fix diagram is exactly . Conversely, if the latter is the fix diagram of a domestic collineation θ, then θ obviously does not fix any chamber. It can be shown that the only possible opposition diagrams are the two fix diagrams under consideration. But a collineation with opposition diagram  always fixes at least one object of type 7. Hence θ has opposition diagram , and we can apply the first arguments again.

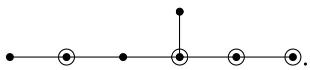
Now suppose that some collineation θ has opposition diagram . If θ does not fix any chamber, then one can show that there are two possibilities.

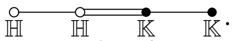
(1) Either there exists a quaternion division algebra $\mathbb{H}$ over the ground field $\mathbb{K}$ (possibly inseparable) such that the fixed building is 

, and the corresponding fix diagram is exactly  ;

(2) or there exist two opposite objects Ω_1, Ω_2 of type 1 such that the equator geometry $E(\Omega_1, \Omega_2)$ in the Lie incidence geometry of type $\mathsf{E}_{7,1}$ is pointwise fixed, and θ acts fixed point freely on the set $\mathscr{I}$ of all objects Ω of type 1 such that Ω is opposite at least one of Ω_1 and Ω_2, and $E(\Omega, \Omega_i) = E(\Omega_1, \Omega_2)$, for $i \in \{1, 2\}$ such that Ω is opposite Ω_i. The set $\mathscr{I}$ is a kind of imaginary line; its stabilizer (in the adjoint group) acts on it like $\mathsf{PSL}_2(\mathbb{K})$, and it takes a quadratic extension of $\mathbb{K}$ to have a fixed point free action on it (a 2×2 matrix with imaginary eigenvalues). This is the class of counter examples appearing in the revised conjecture.

Conversely, suppose θ is a domestic collineation with fixed diagram . Since no chamber is fixed, θ is domestic, and we already know that collineations with opposition diagram  have different fixed point structure, the opposition diagram of

θ is . This would prove (the revision of) Conjecture 9.2 for the case of E_7.

The cell with diagram E_8. It is already shown in [20] that the only opposition diagram of any domestic collineation of a building of type E_8 fixing no chamber is . Hence in order to complete the conjecture, it suffices to show that any domestic collineation with that opposition diagram has the same diagram as fix diagram, and that the corresponding fixed building is 

. However, the revised conjecture states that there is a second possibility, which has a full fix diagram. This collineation is very similar to the counter example in the case of E_7; it is also defined using the equator geometries of objects of type 8. Proofs are in progress here.

9.5 Facts and Figures

To conclude we emphasize the connectivity of the Square with some facts and figures, showing (1) increasing complexity going East and South, and (2) similarity between objects on the same row or column.

We start with a simple numerical observation. We saw that the first row of the Fix Magic Square leaves room for two interpretations. In the interpretation of involutions of Coxeter systems, the very first cell (at the utmost North-West position) is in contradiction with itself, and so for this moment we assume it corresponds to the empty cell, a Coxeter complex of rank 0. Writing down the ranks of the Coxeter systems for each cell, we obtain

0	2	3	4
2	4	5	6
3	5	6	7
4	6	7	8

and we observe that the rank in position (i, j) is the sum of the ranks in positions $(i, 1)$ and $(1, j)$, for all $\{i, j\} \subseteq \{1, 2, 3, 4\}$.

Common features of cells on the same columns are the following.

1. In the construction of the varieties in the Complexified Geometric Magic Square, and also in the definition of the Relative Magic Square, we saw that the first column uses a field, the second a field with a quadratic extension, the third a field with a quaternion algebra, and the fourth a field with an octonion algebra.

2. In the Fix Magic Square, interpreted as indices of semi-simple algebraic groups, the first column contains only split forms, the second quasi-split forms, the third non-split forms. with small disconnected anisotropic kernels (direct products of A_1), and the fourth non-split with a rather large connected anisotropic kernel (of type D_4).

3. Overall, the Galois groups of the indices aways have order 2, that is, the Galois descent is always with an involution. In the Delayed Magic Square, excluding the case of characteristic 2, the analogue of the Galois group is the group which pointwise fixes the subbuilding; in the first column it is the identity, in the second column it is always a group of order 2, in the third column it is the multiplicative group of norm 1 elements of a quadratic Galois extension, in the fourth column it is the multiplicative group of norm 1 elements of a quaternion algebra. This is a quite remarkable series, noting that the group of order 2 in the case of the second column can also be defined as the multiplicative group of elements of a field (in characteristic different from 2) with norm 1, where the norm here is just squaring (which is in conformity with the standard involution being the identity).

Common features of the cells on the same rows are the following.

1. The ranks of the buildings in the Relative Magic Square are the same as the row number. This is also equal to the number of encircled sets of vertices in the Fix Magic Square.

2. In the Relative Geometric Magic Square, the geometries in the first row are so-called Tits webs (coming from Moufang sets) and can be considered as unitals or, in Tits' language [28], σ-conics, in the second row projective planes, in the third row dual polar spaces (of diameter 3), and in the fourth row metasymplectic spaces.

3. In the Complexified Geometric Magic Square, the second row contains strong parapolar spaces of diameter 2 (except fo the first cell, where projective planes occur; although if we view each line of a projective plane as a polar line, as explained in Section 3, we obtain a parapolar space without lines. Compare with the one without symplecta we will meet in the symmetric cell below), the third row strong parapolar spaces of diameter 3 and the fourth row non-strong parapolar spaces of diameter 3 all of which are long root geometries. The first row one would ask? This contains what we could call "would-be" long root parapolar spaces: the building in the first cell is really too small to be a sensible geometry, the second cell is the long root geometry for A_2, but this does not contain symplecta, so it can hardly be seen as a parapolar space; the third cell is the line Grassmannian of a symplectic polar space, which is non-strong of diameter 3, but not a "pure" long root geometry since in the symplectic case, the first node is the polar one, and not the second, as is the case for B_3. Similarly for the fourth cell: Although there is seemingly symmetry here, it is the node with (Bourbaki) label 1 which is the polar node, and here we have the node with label 4 (but it also provides a metasymplectic space, a non-strong parapolar space of diameter 3 with thick symplecta of rank 3).

4. Going to the original Magic Square containing the varieties defined in Section 8, the first row contains a kind of Veronese representations of the corresponding unitals, the second

row contains the Veronese representations of certain projective planes (over alternative quadratic algebras), and over the complex numbers, these are exactly the Severi varieties. The third row constitutes the (analogues of the) Lagrangean manifolds, the fourth row are the varieties associated with the adjoint module.

This should convince the reader that the Geometric Magic Square is really an exceptionally beautiful object, well worth studying in some more detail.

References

[1] P. Abramenko & K. Brown, Automorphisms of non-spherical buildings have unbounded displacement, *Innov. Incid. Geom.* **10** (2009), 1–13.

[2] M. Aschbacher, The 27-dimensional module for E_6, I., *Invent. Math.* **89** (1987), 159–195.

[3] F. Buekenhout, Diagrams for geometries and groups, *J. Combin. Theory A* **27** (1979), 121–151.

[4] F. Buekenhout and A. M. Cohen, Diagram Geometry, Related to Classical Groups and Buildings, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3. Folge **57** (2013).

[5] A. M. Cohen, An axiom system for metasymplectic spaces, *Geom. Ded.* **12** (1982), 417–433.

[6] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker & R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[7] A Del Fra & D. Ghinelli, Classification of extended generalized quadrangles with maximum diameter, *Discrete Math.* **105** (1992), 13–23.

[8] A Del Fra, D. Ghinelli, T. Meixner & A. Pasini, Flag-transitive extensons of C_n geometries, *Geom. Dedicata* **37** (1991), 253–273.

[9] A. De Schepper, N. S. N. Sastry & H. Van Maldeghem, Buildings of exceptional type in buildings of type E_7, *Dissertationes Math.*, **573** (2022), 1–80. .

[10] A. De Schepper, J. Schillewaert and H. Van Maldeghem, A uniform characterisation of the varieties of the second row of the Freudenthal-Tits Magic Square over arbitrary fields, *submitted preprint*.

[11] A. De Schepper, J. Schillewaert, H. Van Maldeghem & M. Victoor, On exceptional Lie geometries, *Forum Math. Sigma* **9** (2021), paper No e2, 27pp.

[12] A. De Schepper, J. Schillewaert, H. Van Maldeghem & M. Victoor, A geometric characterization of Hjelmslev-Moufang planes, *Quart. J. Math.* (2021), haab043, https://doi.org/10.1093/qmath/haab043.

[13] A. De Schepper, J. Schillewaert, H. Van Maldeghem & M. Victoor, Construction and characterization of the varieties of the third row of the Freudenthal-Tits magic square, *submitted preprint*.

[14] A. De Schepper & M. Victoor, A geometric connection between the split first and second row of the Freudenthal-Tits magic square, *Innov. Incid. Geom.*, to appear.

[15] O. Krauss, J. Schillewaert and H. Van Maldeghem, Veronesean representations of Moufang planes. *Mich. Math. J.* **64** (2015), 819–847.

[16] W. Lichtenstein, A system of quadrics describing the orbit of the highest weight vector, *Proc. Amer. Math. Soc.* **84** (1982), 605–608.

[17] B. Mühlherr, H. Petersson & R. M. Weiss, *Descent in Buildings*, Annals of Mathematics Studies **190**, Princeton University Press, 2015.

[18] J. Parkinson & H. Van Maldeghem, Opposition diagrams for automorphisms of large spherical buildings, *J. Combin. Theory Ser.* A **162** (2019), 118–166.

[19] J. Parkinson & H. Van Maldeghem, Opposition diagrams for automorphisms of small spherical buildings, *Innov. Incid. Geom.* **17** (2019), 141–188.

[20] J. Parkinson & H. Van Maldeghem, Automorphisms and opposition in spherical buildings of exceptional type, I, *Canad. J. Math.*, 62pp, DOI: `https://doi.org/10.4153/S0008414X21000341`.

[21] J. Parkinson & H. Van Maldeghem, Automorphisms and opposition in spherical buildings of classical type, *submitted.*

[22] S. E. Payne & J. A. Thas, *Finite Generalized Quadrangles*, Research notes in Math. **110**, Pittman, 1984; second edition: Europ. Math. Soc. Series of Lectures in Mathematics, 2009.

[23] J. Schillewaert & H. Van Maldeghem, On the varieties of the second row of the split Freudenthal-Tits Magic Square, *Ann. Inst. Fourier* **67** (2017), 2265–2305.

[24] E. E. Shult, *Points and Lines, Characterizing the Classical Geometries*, Universitext, Springer, Heidelberg, 2011.

[25] B. Temmermans, J. A. Thas & H. Van Maldeghem, Collineations of polar spaces with restricted displacements, *Des. Codes Cryptogr.* **64** (2012), 61–80.

[26] B. Temmermans, J. A. Thas & H. Van Maldeghem, Domesticity in projective spaces, *Innov. Incid. Geom.* **12** (2011), 141–149.

[27] J. Tits, Espaces homogènes et groupes de Lie exceptionnels, in *Proc. Internat. Congr. Math.* 1954, vol. 1, Nordhoff, Groningen; North-Holland, Amsterdam (1954), 495–496.

[28] J. Tits, Sur certaines classes d'espaces homogènes de groupes de Lie, *Acad. Roy. Belg. Cl. Sci. Mém. Collect. 8^o (2)* **29** (1955), 5–268.

[29] J. Tits, Classification of simple algebraic groups, **in** *Algebraic groups and discontinuous subgroups*, Proc. Summer Mathematical Inst., Boulder, July 5–August 6, 1965, *Proc. Symp. Pure Math.* **9**, Amer. Math. Soc., Providence, RI (1966), 33–62.

[30] J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, Springer Lecture Notes Series **386**, Springer-Verlag, 1974.

[31] H. Van Maldeghem, Symplectic polarities in buildings of type E6, *Des. Codes Cryptogr.* **65** (2012), 115–125.

[32] H. Van Maldeghem & M. Victoor, Combinatorial and geometric constructions of spherical buildings, *Surveys in Combinatorics 2019*, Cambridge University Press (ed. A. Lo et al.), *London Math. Soc. Lect. Notes Ser.* **456** (2019), 237–265.

[33] H. Van Maldeghem & M. Victoor, On Severi varieties as intersections of a minimum number of quadrics, *Cubo* **24** (2022), 307–331..

[34] F. L. Zak, *Tangents and Secants of Algebraic Varieties*, Translation of Mathematical Monographs **127**, American Mathematical Society, 1983, vii+164 pages.