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Mixed relations for buildings of type F_4

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Dedicated to the memory of Jacques Tits

The main result of this paper is an explicit description of the representation of the metasymplectic space related to an arbitrary building of mixed type F_4 in 25-dimensional projective space. As an application, we study collineations of such spaces the fixed point structure of which is a Moufang quadrangle. We show that the exceptional Moufang quadrangles of type F_4 can be obtained as the intersection of the mixed metasymplectic space with a Baer subspace of the ambient projective space. We also determine the group of collineations fixing a mixed quadrangle and, more surprisingly, observe that it has infinite order, whereas it was generally believed to have just order 2. Finally, we classify collineations of the mixed metasymplectic space fixing mixed Moufang quadrangles arising from subspaces.

1. Introduction

The notion of a *group of mixed type* is due to Jacques Tits, see for instance [16]. In fact, Jacques seems to have cared a lot about mixed phenomena, and recognized them rather early. We can mention his general construction [15] of the Suzuki groups, of both classes of Ree groups, using in particular nonperfect fields, and his construction and classification of *Moufang quadrangles of mixed type* [18]. Moreover, in [16], Jacques describes an at that time new class of Moufang hexagons, namely precisely the ones of mixed type. In the same book he classifies the buildings of type F_4, and also here, one of the five classes is the class of mixed type. These were neglected by Freudenthal, who was well aware of all other classes. But, as well known, the only class of Moufang quadrangles which was overlooked by Jacques Tits in his original conjecture [17] were exactly those of mixed type F_4, which are constructed using a properly mixed building of type F_4. The quadrangles themselves were abstractly first constructed by Richard Weiss, and recognized as fixed point

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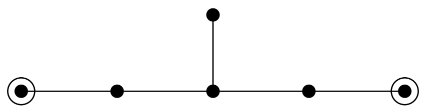
structures of a semilinear involution in buildings of mixed type F_4 (so to speak "mixed Galois descent") by Bernhard Mühlherr and Hendrik Van Maldeghem [7]. In the latter paper, it is also described how linear involutions give rise to Moufang quadrangles, which are of purely mixed type. In the present paper we construct the mixed buildings of type F_4 as varieties, or as metasymplectic spaces embedded in projective spaces of dimension 25 and show that these involutions can be extended to the projective spaces, implying that the associated Moufang quadrangles can be defined as the intersection of the varieties with either a Baer subspace, or a linear subspace (of dimension 15). In the case of mixed Galois descent, this really shows the similarity with ordinary Galois descent in linear algebraic groups, since we just take the rational points of a linear group.

However, our focus here is not on the mixed Galois descent (as there are other papers dealing with these in a much better algebraic way, see e.g., [3; 4]). Instead, we direct our attention to the linear case. "Linearization" of Galois descent groups have attracted some interest lately in connection with automorphisms of buildings with a restricted displacement spectrum. On top, it seems like the "linear descent groups" have a richer structure than their Galois counterparts. For instance, the Galois (descent) group related to the Tits index $\mathsf{E}_{6,2}^{28}$ (see [14])



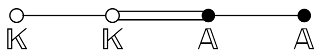
defining an octonion projective plane, has only order 2 whereas its linear analogue defining a quaternion projective plane consists of the multiplicative group of quaternion numbers with norm 1 (this is proved in ongoing work of Van Maldeghem with Yannick Neyt, James Parkinson and Magali Victoor in a paper in progress). For a long time, it was believed that the linear involution defining a mixed quadrangle in a mixed building of type F_4 is unique. In this paper we show that also that linear descent group has a richer structure.

The main achievement of this paper is an explicit description of the representation of the metasymplectic space related to an arbitrary building of mixed type F_4 in 25-dimensional projective space. It was also the main object of Roth's PhD thesis (under the guidance of Van Maldeghem). The construction and the proof that the construction works requires some long but rather elementary calculations, and we will often leave these to the reader. A lot of computations can be found in Roth's PhD thesis [10]. The usefulness of this construction shall be demonstrated by the application to the descent groups mentioned above.

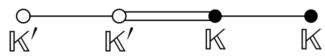
We now briefly comment on the construction itself. Buildings of type E_6 are generally best approached via the shadow geometry of type $\mathsf{E}_{6,1}$, meaning that we consider the geometry $\Delta = \mathsf{E}_{6,1}(\mathbb{K})$ of vertices of type 1 of a building of type E_6

over a field $\mathbb{K}$, where the lines are the panels of cotype 1. It is well known that Δ has a unique embedding $\mathscr{E}_{6,1}(\mathbb{K})$ in (and spanning) $\mathsf{PG}(26, \mathbb{K})$, the 26-dimensional projective space over $\mathbb{K}$. It is also folklore that the shadow geometry $\mathsf{F}_{4,4}(\mathbb{K})$ analogously defined from a (split) building of type F_4 over $\mathbb{K}$ has a representation $\mathscr{F}_{4,4}(\mathbb{K})$ in $\mathsf{PG}(25, \mathbb{K})$ arising from intersecting $\mathscr{E}_{6,1}(\mathbb{K})$ with an appropriate hyperplane. However, not all lines on $\mathscr{F}_{4,4}(\mathbb{K})$ are lines of $\mathsf{F}_{4,4}(\mathbb{K})$. Hence collinearity in $\mathsf{F}_{4,4}(\mathbb{K})$ does not coincide with collinearity on $\mathscr{F}_{4,4}(\mathbb{K})$. There is an algebraic description of $\mathscr{F}_{4,4}(\mathbb{K})$, as a point set; is there also an algebraic description of the collinearity on $\mathscr{F}_{4,4}(\mathbb{K})$? We will deduce one in case $\operatorname{char}\mathbb{K} = 2$.

Now, in general, for each quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$, there exists a building of type F_4 whose planes are coordinatized by $\mathbb{K}$ and $\mathbb{A}$ according to the following picture:

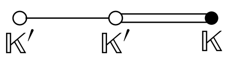


(black nodes represent short roots in the underlying root system). Now, if $\operatorname{char}\mathbb{K} = 2$, then we can consider a subfield $\mathbb{K}' \le \mathbb{K}$ such that $\mathbb{K}^2 \le \mathbb{K}'$ (the notation $\mathbb{K}^2$ means the set of all squares in $\mathbb{K}$). It follows that $\mathbb{K}$ can be seen in a natural way as a quadratic associative and commutative division algebra over $\mathbb{K}'$. Therefore there exists a well defined building $\mathsf{F}_4(\mathbb{K}', \mathbb{K})$ with diagram:



It is the building of mixed type F_4 associated to the pair of fields $(\mathbb{K}', \mathbb{K})$. It is for the corresponding shadow geometry $\mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$ of that building that we are going to provide algebraic equations and relations to fully describe the standard embedding in $\mathsf{PG}(25, \mathbb{K})$.

About the proof — Structure of the paper. The proof that the construction works will deviate slightly from the proof in [10] in that we avoid the study and use of the corresponding groups with a BN-pair, and the Lie-algebras. We emphasize though that this approach was very fruitful to discover the mixed relations (see below). However, once these relations are established, a different, more geometric, road can be taken to prove them. Large parts remain the same, though. The main difference is that we are going to recognize the geometry $\mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$ in a local fashion. This entails an adequate description of the local geometry, which is obtained by looking at the lines through a point; the type can be obtained by deleting in the diagram the vertex representing the points. We obtain:



This is a mixed dual polar space. Therefore, we start in Section 2 with describing an embedding of that geometry. We will derive it from a general description of the universal embeddings of dual polar spaces related to quadratic alternative division algebras as contained in [2].

Sections 2 and 3 perform preliminary work. In Section 3 we determine all collineations of certain mixed polar spaces of rank 3 with a given set of fixed points. This will enable us later to determine the linear descent group related to the linear "homogeneous involutions" constructed in [7].

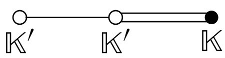
In Section 4 we show the main result: We construct the geometry $\mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$, which we will refer to as a *mixed metasymplectic space*, in $\mathsf{PG}(25, \mathbb{K})$, using quadratic algebraic expressions. This will be done by first describing $\mathscr{E}_{6,1}(\mathbb{K})$ (see above), intersecting with a suitable hyperplane, and then imposing a set of *mixed relations*, short ones and long one. We show that the stabilizer in $\mathsf{PG}(25, \mathbb{K})$ of the set of points satisfying all relations acts transitively on that set, and that it is locally a mixed dual polar space.

In Section 5 we discuss collineations with common fixed point structure (that is also a mixed Moufang quadrangle). In Section 6 we classify collineations fixing mixed Moufang quadrangles arising from subspaces. As we will explain, this has to do with our interest in automorphisms of spherical buildings having a restricted displacement spectrum.

Notation — Terminology. We will use standard notation as much as we can. The geometries under consideration are in fact *parapolar spaces*, but we try to avoid the theory of parapolar spaces. We will use some terminology, though, related to that theory. Points in projective space will usually be denoted by the span $\langle v \rangle$ of a vector v belonging to the associated underlying vector space; in coordinates we leave out the brackets since we conceive coordinate-tuples be determined up to a nonzero scalar.

2. The mixed dual polar space $\mathsf{B}_{3,3}(\mathbb{K}', \mathbb{K})$

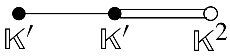
Let $\mathbb{K}$ be a field in characteristic 2 and let $\mathbb{K}' \leq \mathbb{K}$ be a subfield such that $\mathbb{K}^2 \leq \mathbb{K}'$. The mixed polar space $\mathsf{B}_{3,1}(\mathbb{K}', \mathbb{K})$ is the shadow geometry with diagram:



If $\mathbb{K}' = \mathbb{K}^2$, then it is the symplectic polar space in $\mathsf{PG}(5, \mathbb{K}')$; otherwise it can be explicitly defined as the points and lines contained in the set of points of $\mathsf{PG}(5, \mathbb{K}')$ with coordinate tuple $(x_1, x_2, x_3, x_4, x_5, x_6)$ satisfying the *mixed relation*

$$X_1X_2 + X_3X_4 + X_5X_6 \in \mathbb{K}^2.$$

The corresponding dual polar space $\mathsf{B}_{3,3}(\mathbb{K}', \mathbb{K})$ is the geometry of planes and plane pencils in $\mathsf{B}_{3,1}(\mathbb{K}', \mathbb{K})$ and has the following explicit description, providing right away an embedding in projective space. First note that in [2] the polar spaces are given in their *symplectic* disguise, meaning with respect to the diagram $\mathsf{C}_3(\mathbb{K}', \mathbb{K}^2)$:



Hence we can consider $\mathbb{K}'$ and $\mathbb{K}^2$ as vector spaces over $\mathbb{K}^2$ an define a vector space U as the direct sum

$$\mathbb{K}^2 \oplus \mathbb{K}^2 \oplus \mathbb{K}^2 \oplus \mathbb{K}^2 \oplus \mathbb{K}' \oplus \mathbb{K}' \oplus \mathbb{K}' \oplus \mathbb{K}^2 \oplus \mathbb{K}^2 \oplus \mathbb{K}^2 \oplus \mathbb{K}' \oplus \mathbb{K}' \oplus \mathbb{K}' \oplus \mathbb{K}^2,$$

and consider the corresponding projective space $\mathsf{PG}(U)$. In that space we gather the following set $\mathscr{S}$ of points, given by eight different types, according to the zeros appearing in the coordinates at certain places. We let k, ℓ, m run through $\mathbb{K}$, and x, x_1, x_2, x_3 through $\mathbb{K}'$

$$(1, k^2, \ell^2, m^2, x_1, x_2, x_3, \ell^2 m^2 + x_1^2, k^2 m^2 + x_2^2, k^2 \ell^2 + x_3^2, k^2 x_1 + x_2 x_3,$$
$$\ell^2 x_2 + x_1 x_3, m^2 x_3 + x_1 x_2, k^2 \ell^2 m^2 + k^2 x_1^2 + \ell^2 x_2^2 + m^2 x_3^2), \qquad \text{(VIII)}$$

$$(0, x_3^2, 1, x_1^2, x_1, x_1 x_3, x_3, k^2, \ell^2 x_1^2 + k^2 x_3^2, \ell^2,$$
$$x_2 x_3 + \ell^2 x_1, x_2, x_1 x_2 + k^2 x_3, k^2 \ell^2 + x_2^2), \qquad \text{(VII)}$$

$$(0, x_2^2, 0, 1, 0, x_2, 0, k^2, \ell^2, k^2 x_2^2, x_1 x_2, k^2 x_2, x_1, k^2 \ell^2 + x_1^2), \qquad \text{(VI)}$$

$$(0, 0, 0, 0, 0, 0, 0, 1, x_1^2, x_2^2, x_1 x_2, x_2, x_1, k^2), \qquad \text{(V)}$$

$$(0, 1, 0, 0, 0, 0, 0, 0, k^2, \ell^2, x, 0, 0, \ell^2 k^2 + x^2), \qquad \text{(IV)}$$

$$(0, 0, 0, 0, 0, 0, 0, 0, x^2, 1, x, 0, 0, k^2), \qquad \text{(III)}$$

$$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, k^2), \qquad \text{(II)}$$

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1). \qquad \text{(I)}$$

Suppose two points $p_1, p_2 \in \mathscr{S}$ are such that the line $\langle p_1, p_2 \rangle$ intersects the subspace

$$N = (0, 0, 0, 0, *, *, *, 0, 0, 0, *, *, *, 0)$$

nontrivially. It is clear that both points must have the same type. For instance, let their type be (VIII). Then, with self-explaining notation for the coordinates of p_1 and p_2, we have $k_1 = k_2$, $\ell_1 = \ell_2$, $m_1 = m_2$, $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$. So, N behaves like a nucleus subspace and we can project injectively the whole dual polar space $\mathscr{S}$ from it. Moreover, we can apply the isomorphism $\mathsf{PG}(7, \mathbb{K}^2) \to \mathsf{PG}(7, \mathbb{K})$ that takes the square root of each coordinate. This way we obtain the following representation $\mathscr{V}$ of $\mathsf{B}_{3,3}(\mathbb{K}', \mathbb{K}) \cong \mathsf{C}_{3,3}(\mathbb{K}', \mathbb{K}^2)$, it consists precisely of the following points (where l, ℓ, m run through $\mathbb{K}$, and x, x_1, x_2, x_3 through $\mathbb{K}'$):

(VIII) $(1, k, \ell, m, \ell m + x_1, km + x_2, k\ell + x_3, k\ell m + kx_1 + \ell x_2 + mx_3)$.

(VII) $(0, x_3, 1, x_1, k, \ell x_1 + kx_3, \ell, k\ell + x_2)$.

(VI) $(0, x_2, 0, 1, k, \ell, kx_2, k\ell + x_1)$.

(V) $(0, 0, 0, 0, 1, x_1, x_2, k)$.

(IV) $(0, 1, 0, 0, 0, k, \ell, \ell k + x)$.

(III) $(0, 0, 0, 0, 0, x, 1, k)$.

(II) $(0, 0, 0, 0, 0, 1, 0, k)$.

(I) $(0, 0, 0, 0, 0, 0, 0, 1)$.

We denote these coordinates by $(z_0, z_1, z_2, z_3, z_4, z_5, z_6, z_7)$ and prove:

Proposition 2.1. *The set* $\mathscr{V}$ *coincides exactly with* $(z_0, z_1, z_2, z_3, z_4, z_5, z_6, z_7)$, *the set of points satisfying*:

(a) $Z_0Z_7 + Z_1Z_4 + Z_2Z_5 + Z_3Z_6 = 0$.

(b) $Z_0Z_4 + Z_2Z_3 \in \mathbb{K}'$.

(c) $Z_0Z_5 + Z_1Z_3 \in \mathbb{K}'$.

(d) $Z_0Z_6 + Z_1Z_2 \in \mathbb{K}'$.

(e) $Z_7Z_1 + Z_5Z_6 \in \mathbb{K}'$.

(f) $Z_7Z_2 + Z_4Z_6 \in \mathbb{K}'$.

(g) $Z_7Z_3 + Z_4X_5 \in \mathbb{K}'$.

Proof. It is easy to verify that the coordinates of all points of $\mathscr{V}$ satisfy these relations. Now assume that a point with coordinates $(z_0, z_1, \dots, z_7)$ satisfies the above relations. Suppose first that $z_0 \neq 0$. Then we may assume $z_0 = 1$. Set $z_1 = k$, $z_2 = \ell$ and $z_3 = m$. Relation (b) yields $x_1 \in \mathbb{K}'$ such that $z_4 = x_1 + \ell m$. Likewise Relations (c) and (d) yield $x_2, x_3 \in \mathbb{K}'$ with $z_5 = x_2 + km$ and $z_6 = x_3 + k\ell$. Then Relation (a) yields $z_7 = k(\ell m + x_1) + \ell(km + x_2) + m(k\ell + x_3) = k\ell m + kx_1 + \ell x_2 + mx_3$, and we obtain a point of type (VIII).

Now suppose $z_0 = 0$. Then we first assume that $z_2 \neq 0$, so that we can set $z_2 = 1$. We set $z_1 = x_3$, $z_3 = x_1$, $z_4 = k$ and $z_6 = \ell$. Relation (b) implies $x_1 \in \mathbb{K}'$ ad Relation (d) yields $x_3 \in \mathbb{K}'$. Relation (f) yields $x_2 \in \mathbb{K}'$ such that $z_7 = x_2 + z_4z_6 = x_2 + k\ell$. Finally Relation (a) implies $z_5 = z_1z_4 + z_3z_6 = kx_3 + \ell x_1$. We obtain a point of type (VII).

So we may assume $z_0 = z_2 = 0$. Assume first that $z_3 \neq 0$, so that we may take $z_3 = 1$. We set $z_1 = x_2$, $z_4 = k$ and $z_5 = \ell$. Relation (c) implies $x_2 \in \mathbb{K}'$. Relation (g) yields $x_1 \in \mathbb{K}'$ such that $z_7 = x_2 + z_4z_5 = x_2 + k\ell$. Relation (a) then implies $z_6 = z_1z_4 = kx_2$. This yields a point of type (VI).

So we may assume $z_0 = z_2 = z_3 = 0$. Assume first $z_4 \neq 0$, so that we may take $z_4 = 1$. Relation (a) implies $z_1 = 0$. Relations (f) and (g) imply $z_5, z_6 \in \mathbb{K}'$. We obtain a point of type (V). If $z_1 \neq 0$, we may set $z_1 = 1$, and then $z_4 = 0$. Relation (e) yields $x \in \mathbb{K}'$ so that $z_7 = z_5 z_6 + x$. This gives rise to a point of type (IV). If also $z_1 = 0$, we can first assume $z_6 = 1$, so that Relation (e) implies $z_5 \in \mathbb{K}'$. This gives a type (III) point. If $z_6 = 0$, then types (II) and (I) follow. □

3. Quadrangles of mixed type

Let $\mathbb{K}$ be a field of characteristic 2 and let $\Omega := \mathsf{C}_{2,1}(\mathbb{K}) = \mathsf{C}_{2,1}(\mathbb{K}, \mathbb{K})$ be the corresponding symplectic quadrangle. That is, the point set of Ω is the point set of $\mathsf{PG}(3, \mathbb{K})$ and the lines are the lines of $\mathsf{PG}(3, \mathbb{K})$ that are totally isotropic with respect to the nondegenerate alternating form $X_1 Y_2 + X_2 Y_1 + X_3 Y_4 + X_4 Y_3$. Now let $\mathbb{K}'$ be a subfield of $\mathbb{K}$ containing all squares of $\mathbb{K}$; so we again have $\mathbb{K}^2 \leq \mathbb{K}' \leq \mathbb{K}$. Then the points of $\mathsf{PG}(3, \mathbb{K})$ whose coordinates (x_1, x_2, x_3, x_4) satisfy $X_1 X_2 + X_3 X_4 \in \mathbb{K}'$ induce a (mixed) subquadrangle Ω' of Ω; if $\mathbb{K}' \neq \mathbb{K}$, then the lines of Ω' are determined by the point set in that they are the only lines of $\mathsf{PG}(3, \mathbb{K})$ fully contained in Ω'. Indeed, suppose first two points $p = (x_1, x_2, x_3, x_4)$ and $q = (y_1, y_2, y_3, y_4)$ of Ω' are collinear in Ω; then for each $k \in \mathbb{K}$, we have

$$(x_1 + k y_1)(x_2 + k y_2) + (x_3 + k y_3)(x_4 + k y_4) \\ = (x_1 x_2 + x_3 x_4) + (y_1 y_2 + y_3 y_4) + k(x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3),$$

which implies that all points of the line $\langle p, q\rangle$ of $\mathsf{PG}(3, \mathbb{K})$ belong to Ω'. Suppose now that $p = (x_1, x_2, x_3, x_4)$ and $q = (y_1, y_2, y_3, y_4)$ are points of Ω' not collinear in Ω, but that every point of the line $\langle p, q\rangle$ lies in Ω'. Then the calculation above implies that $x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3 = 0$, contradicting the assumption.

We denote the quadrangle Ω' as $\mathsf{C}_{2,1}(\mathbb{K}, \mathbb{K}')$. The point rows of Ω are canonically projective lines over $\mathbb{K}$; the line pencils are canonically projective lines over $\mathbb{K}'$. Indeed, the isomorphism class of a line pencil is given by $(1, 0, 0, 0)^\perp \cap (0, 1, 0, 0)^\perp$, which is the set $\{(0, 0, x_3, x_4) \mid x_3 x_4 \in \mathbb{K}'\}$. This set equals $\{(0, 0, 0, 1)\} \cup \{(0, 0, 1, x) \mid x \in \mathbb{K}'\}$.

Now let $L \subseteq \mathbb{K}$ be a vector space over $\mathbb{K}'$ containing $\mathbb{K}'$ and generating $\mathbb{K}$ as a ring. Let $L' \subseteq \mathbb{K}'$ be a vector space over $\mathbb{K}^2$ containing $\mathbb{K}^2$ and generating $\mathbb{K}'$ as a ring. Then we restrict the coordinates to L and consider only those points (x_1, x_2, x_3, x_4) satisfying $X_1 X_2 + X_3 X_4 \in L'$. Again this defines a generalized quadrangle, denoted $\mathsf{C}_{2,1}(\mathbb{K}, \mathbb{K}'; L, L')$, also called a quadrangle of mixed type. If $L = \mathbb{K}$ and $L' = \mathbb{K}'$, then $\mathsf{C}_{2,1}(\mathbb{K}, \mathbb{K}'; L, L')$ coincides with $\mathsf{C}_{2,1}(\mathbb{K}, \mathbb{K}')$.

We now define a special and particular kind of quadrangles of mixed type. Again let $\mathbb{K}$ and $\mathbb{K}'$ be as above. Let $\alpha \in \mathbb{K}' \setminus \mathbb{K}^2$ and $\beta \in \mathbb{K} \setminus \mathbb{K}'$. Then $\mathbb{K}^2(\alpha)$ is a quadratic extension of $\mathbb{K}^2$ contained in $\mathbb{K}'$, hence also in the quadratic extension $\mathbb{K}'(\beta)$ of $\mathbb{K}'$.

Also $\beta\mathbb{K}'$ is contained in $\mathbb{K}'(\beta)$ and since $\mathbb{K}'(\beta)$ is, as a vector space over $\mathbb{K}'$, and hence also over $\mathbb{K}^2(\alpha)$, the direct sum of $\mathbb{K}'$ and $\beta\mathbb{K}'$, we see that $\mathbb{K}^2(\alpha)+\beta\mathbb{K}'$ is a vector space over $\mathbb{K}^2(\alpha)$. We set $L=\beta^{-1}\mathbb{K}^2(\alpha)+\mathbb{K}'$. It is a vector space over $\mathbb{K}^2(\alpha)$ containing $\mathbb{K}^2(\alpha)$ and being itself contained in $\mathbb{K}'(\beta)$. Clearly L generates $\mathbb{K}'(\beta)$ as a ring. Likewise, $L':=\alpha\mathbb{K}'^2(\beta^2)+\mathbb{K}^2$ is a vector space over $\mathbb{K}'^2(\beta^2)$ contained in, and generating as a ring, $\mathbb{K}^2(\alpha)$. Hence we have a mixed quadrangle $Q=\mathsf{C}_{2,1}(\mathbb{K}'(\beta),\mathbb{K}^2(\alpha);L,L')$. A point row of Q is the projective line $\mathsf{PG}(1,L)$ over L, which is itself a vector space over $\mathbb{K}^2(\alpha)$.

We now want to embed this point row in an algebraic way as an ovoid in a polar space of mixed type and rank 3.

Let $\Delta:=\mathsf{C}_{3,1}(\mathbb{K},\mathbb{K}')$ be the polar space of mixed type and rank 3 defined by the points of $\mathsf{PG}(5,\mathbb{K})$ whose coordinates $(x_1,x_2,\dots,x_6)$ satisfy $X_1X_4+X_2X_5+X_3X_6\in\mathbb{K}'$. Consider the following involution: $\theta:(x_1,x_2,\dots,x_6)\mapsto(x_1',x_2',\dots,x_6')$, with

$$x_1'=x_1,\quad x_4'=x_4,\quad x_2'=\alpha\beta^{-1}x_5,\quad x_5'=\alpha^{-1}\beta x_2,\quad x_3'=\beta^{-1}x_6,\quad x_6'=\beta x_3.$$

In the sequel we shall denote this briefly by

$$x_1\mapsto x_1,\quad x_4\mapsto x_4,\quad x_2\mapsto\alpha\beta^{-1}x_5,\quad x_5\mapsto\alpha^{-1}\beta x_2,\quad x_3\mapsto\beta^{-1}x_6,\quad x_6\mapsto\beta x_3.$$

It is easily checked that a generic fixed point, for $x_i\in\mathbb{K}$, $i=1,4,5,6$, has coordinates $(x_1,\alpha\beta^{-1}x_5,\beta^{-1}x_6,x_4,x_5,x_6)$. The condition to belong to Δ is

$$x_1x_4+\alpha\beta^{-1}x_5^2+\beta^{-1}x_6^2\in\mathbb{K}'.$$

Setting $(\infty)=(0,0,0,1,0,0)$ and $z=x_4+\beta^{-1}(x_6^2+\alpha x_5^2)$, this fixed point set can be identified with

$$F=\{(\infty)\}\cup\{(1,\alpha\beta^{-1}x_5,\beta^{-1}x_6,z+\beta^{-1}(x_6^2+\alpha x_5^2),x_5,x_6)\mid x_5,x_6\in\mathbb{K},z\in\mathbb{K}'\},$$

that is in other words, $\{(\infty)\}\cup L$, with $L=\beta^{-1}\mathbb{K}^2(\alpha)+\mathbb{K}'$, as above.

With action on the left, it is immediate that every linear collineation of $\mathsf{PG}(6,\mathbb{K})$ pointwise fixing F has matrix

$$\begin{pmatrix}1&\beta a&\beta b&0&\alpha a&b\\0&1+\beta c&\beta d&0&\alpha c&d\\0&\beta e&1+\beta f&0&\alpha e&f\\0&\beta g&\beta h&1&\alpha g&h\\0&\beta p&\beta q&0&1+\alpha p&q\\0&\beta r&\beta s&0&\alpha r&1+s\end{pmatrix},$$

with $a,b,c,d,e,f,g,h,p,q,r,s\in\mathbb{K}$. We now put some further restrictions on these coefficients by requiring that the collineation with above matrix preserves Δ and hence is a collineation of Δ pointwise fixing F.

The images of the base points belong to Δ; hence

$$\beta^2(ag+cp+er)+\beta p\in\mathbb{K}',\quad \beta^2(bh+dq+fs)+\beta s\in\mathbb{K}',$$
$$\alpha^2(ag+cp+er)+\alpha c\in\mathbb{K}',\qquad (bh+dq+fs)+f\in\mathbb{K}'.$$

Also, the image of (1, 0, 1, 0, 0, 0) lies in Δ, so combined with the second condition above, we find $\beta h\in\mathbb{K}'$. Likewise, now using the image of (1, 0, 0, 0, 0, 1), we obtain $h\in\mathbb{K}'$ and conclude $\beta\in\mathbb{K}'$ if $h\neq 0$. Since we assumed $\beta\in\mathbb{K}\setminus\mathbb{K}'$ above, this yields $h=0$. Similarly $a=b=g=0$. The matrix now looks like:

$$\begin{pmatrix}1&0&0&0&0&0\\0&1+\beta c&\beta d&0&\alpha c&d\\0&\beta e&1+\beta f&0&\alpha e&f\\0&0&0&1&0&0\\0&\beta p&\beta q&0&1+\alpha p&q\\0&\beta r&\beta s&0&\alpha r&1+s\end{pmatrix}$$

We now express that the image of a generic point $(x_1,\dots,x_6)$ belongs to Δ. This means that, for all $x_1,x_2,\dots,x_6\in\mathbb{K}$ such that $x_1x_2+x_3x_4+x_5x_6\in\mathbb{K}'$, this relation must hold:

$$x_1x_4+((1+\beta c)x_2+\beta dx_3+\alpha cx_5+dx_6)(\beta px_2+\beta qx_3+(1+\alpha p)x_5+qx_6)$$
$$+(\beta ex_2+(1+\beta f)x_3+\alpha ex_5+fx_6)(\beta rx_2+\beta sx_3+\alpha rx_5+(1+s)x_6)\in\mathbb{K}'.$$

This means that for all $x_2,x_3,x_5,x_6\in\mathbb{K}$,

$$x_2x_5+x_3x_6+((1+\beta c)x_2+\beta dx_3+\alpha cx_5+dx_6)(\beta px_2+\beta qx_3+(1+\alpha p)x_5+qx_6)$$
$$+(\beta ex_2+(1+\beta f)x_3+\alpha ex_5+fx_6)(\beta rx_2+\beta sx_3+\alpha rx_5+(1+s)x_6)\in\mathbb{K}'.$$

If we put two of x_2,x_3,x_5,x_6 equal to zero, then letting the others range through $\mathbb{K}$, we see that this implies that the coefficients of x_ix_j, $i,j\in\{2,3,5,6\}$, $i\neq j$, are all zero.

Hence:

The coefficient of x_2x_3 yields $\beta^2qc+\beta^2dp+\beta^2es+\beta^2rf=\beta(q+r)$.

The coefficient of x_5x_6 yields $\alpha qc+\alpha dp+\alpha es+\alpha rf=d+\alpha e$.

The coefficient of x_2x_6 yields $\beta qc+\beta pd+\beta es+\beta rf=q+\beta e$.

The coefficient of x_3x_5 yields $\alpha\beta qc+\alpha\beta pd+\alpha\beta es+\alpha\beta rf=\beta d+\alpha r$.

The first two equalities yield $r=\beta e$, the last two $q=\alpha^{-1}\beta d$. These four equalities now all reduce to

$$\alpha^{-1}\beta dc+dp+es+\beta ef=\alpha^{-1}d+e. \tag{3}$$

Expressing that the coefficient of x_2x_5 in the above expression is zero yields $\beta c + \alpha p = 0$. Expressing that the coefficient of x_3x_6 in the above expression is zero yields $\beta f + s = 0$. Substituting $p = \alpha^{-1}\beta c$ and $s = \beta f$ in (3), the left hand side becomes zero, hence $d = \alpha e$. Hence we have shown:

Proposition 3.1. *Let θ be a collineation of the mixed rank 3 polar space $\mathsf{C}_{3,1}(\mathbb{K}, \mathbb{K}')$ defined by the points of $\mathsf{PG}(5, \mathbb{K})$ whose coordinates $(x_1, x_2, \ldots, x_6)$ satisfy $X_1X_4 + X_2X_5 + X_3X_6 \in \mathbb{K}'$. Suppose that θ fixes the point $(x_1, \alpha\beta^{-1}x_5, \beta^{-1}x_6, x_4, x_5, x_6)$, for each $x_1, x_4, x_5, x_6 \in \mathbb{K}$. Then there exist $e, c, f \in \mathbb{K}$ such that θ is given by the following matrix (action on the left):*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+\beta c & \alpha\beta e & 0 & \alpha c & \alpha e \\ 0 & \beta e & 1+\beta f & 0 & \alpha e & f \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \beta^2\alpha^{-1}c & \beta^2 e & 0 & 1+\beta c & \beta e \\ 0 & \beta^2 e & \beta^2 f & 0 & \alpha\beta e & 1+\beta f \end{pmatrix},$$

with

$$\begin{cases} \beta(c^2 + \alpha e^2) + c \in \mathbb{K}', \\ \beta(f^2 + \alpha e^2) + f \in \mathbb{K}'. \end{cases}$$

It is easy to check that the set of all matrices as in the previous proposition defines a group of exponent 2. We will show later that these groups are large, although at present at first sight, it is not clear that there are plenty of $c, e, f \in \mathbb{K}$ that satisfy the given constraints.

We mention in passing that the above represents a central collineation if and only if the line determined by $(0, 1, 0, 0, 0, 0)$ and its image intersects F in the same point as the line determined by $(0, 0, 1, 0, 0, 0)$ and its image does. This is equivalent to $(\beta c, \beta e, \beta^2\alpha^{-1}c, \beta^2 e) = \lambda(\alpha\beta e, \beta f, \beta^2 e, \beta^2 f)$, for some $\lambda \in \mathbb{K}$, so to $(c, e, \beta\alpha^{-1}c, \beta e) = \lambda(\alpha e, f, \beta e, \beta f)$. This happens if and only if $(c, e) = \lambda(\alpha e, f)$, hence $c = \lambda^2\alpha f$ and $e = \lambda f$. This is so if and only if $cf = \alpha e^2$.

4. The embedding of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K}')$

Our approach is based on the description of the universal embedding of $\mathsf{E}_{6,1}(\mathbb{K})$ in $\mathsf{PG}(26, \mathbb{K})$ using the generalized quadrangle $\Gamma = (X, \mathscr{L})$ of order $(2, 4)$; see for instance [19]. This construction is particularly simple if the characteristic of the field is equal to 2, and runs as follows.

Let V be a 27-dimensional vector space with standard basis vectors e_p labeled by the points p of Γ. A generic vector of V can then be given by a coordinate tuple $(x_p)_{p\in X}$, $x_p \in \mathbb{K}$. For $p \in X$, let Q_p be the (degenerate hyperbolic) quadric in $\mathsf{PG}(V)$ with equation $\sum_{\{p,q,r\}\in\mathscr{L}} X_qX_r = 0$. The intersection of all Q_p, $p \in X$, together with all lines contained in it, constitute the universal embedding of the Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$, which is a 16-dimensional variety that we shall denote by $\mathscr{E}_{6,1}(\mathbb{K})$. The secant variety $\mathscr{E}_{6,1}^2(\mathbb{K})$ has equation $\sum_{\{p,q,r\}\in\mathscr{L}} X_p X_q X_r = 0$.

The variety $\mathscr{E}_{6,1}(\mathbb{K})$ has the following properties. The relation $\perp$ stands for the collinearity relation within $\mathscr{E}_{6,1}(\mathbb{K})$:

(1) Each pair of points x, y, which are not collinear on $\mathscr{E}_{6,1}(\mathbb{K})$, is contained in a unique nondegenerate hyperbolic quadric Q of rank 5 in some 9-dimensional subspace U_Q of $\mathsf{PG}(26, \mathbb{K})$. Moreover, $U_Q \cap \mathscr{E}_{6,1}(\mathbb{K}) = Q$. We also have $U_Q = \langle x, y, x^\perp \cap y^\perp\rangle$. Each such a quadric Q will be called a *symp*, referring to the theory of parapolar spaces whose roots lie in Freudenthal's exploration of the metasymplectic spaces.

(2) Symps intersect in either a unique point or a 4-space.

Now let $r_1, r_2, r_3 \in X$ be three fixed points with $\{r_1, r_2, r_3\} \in \mathscr{L}$. Then the hyperplane H with equation $H \leftrightarrow X_{r_1} + X_{r_2} + X_{r_3} = 0$ intersects $\mathscr{E}_{6,1}(\mathbb{K})$ in a geometric hyperplane, denoted by $\mathscr{H}$, enjoying the following properties:

(i) No symp is entirely contained in H.

(ii) Each point x of $\mathscr{H}$ is contained in a unique symp Q_x so that $x^\perp \cap Q_x \subseteq H$.

(iii) For a line $L \subseteq \mathscr{H}$, the following are equivalent:
  - (a) There exists a point $x \in L$ such that $L \subseteq Q_x$ (with Q_x as above).
  - (b) For every point $x \in L$, we have $L \subseteq Q_x$.
  - (c) There exist at least two 5-space containing L and themselves entirely contained in $\mathscr{H}$.

(iv) For each 5-space W entirely contained in $\mathscr{H}$, the mapping $\rho : W \to W^* : x \mapsto W \cap Q_x$ is well defined and defines a symplectic polarity in W (here, W^* denotes the dual of W, that is, the set of hyperplanes of W). The set of absolute lines with respect to ρ is precisely the set of lines of W belonging to $\mathfrak{L}$.

(v) The pair $\Delta = (\mathscr{H}, \mathfrak{L})$ is isomorphic to the metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K})$; each pair of points that is collinear in $\mathscr{E}_{6,1}(\mathbb{K})$ is contained in a *symplecton*, that is, a symplectic polar space of rank 3 corresponding to a 5-space of $\mathscr{E}_{6,1}(\mathbb{K})$ entirely contained in H, hence in $\mathscr{H}$ as in (iv) above.

Now let $\mathbb{K}'$ be a subfield of $\mathbb{K}$, as before, with $\mathbb{K}^2 \leq \mathbb{K}'$. Assume $\mathbb{K}' \neq \mathbb{K}$. Define the set of points with the suggestive name $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$ as the subset of points of $\mathscr{H}$ satisfying the twelve long mixed relations below and the twenty four short mixed relations below. In short, a generic short mixed relation looks like $X_{p_1}X_{q_1} + X_{p_2}X_{q_2} + X_{p_3}X_{q_3} \in \mathbb{K}'$, where $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ is a set of pairwise noncollinear points of Γ disjoint from $\{r_1, r_2, r_3\}$, and p_i, q_i are collinear to r_i,

$i = 1, 2, 3$. We call such a set a *special* 6-*coclique*; there are indeed twenty four such. Each such 6-set has a unique *opposite* by considering the points $p_i', q_i' \in X$ so that $\{p_i, p_i', r_i\}$ and $\{q_i, q_i', r_i\}$ belong to $\mathscr{L}$. Then a generic long mixed relation looks like

$$\sum_{i=1}^{3} X_{p_i} X_{p_i'} + X_{q_i} X_{q_i'} \in \mathbb{K}'.$$

Although it would be possible to show many results just using this rather abstract form, there is no way to efficiently write such arguments down. Hence we are going to use the concrete model given in 6.1.3 of [9] for Γ and then write out all relations explicitly.

We take as point set

$$X = \{1, 2, \ldots, 6\} \cup \{1', 2', \ldots, 6'\} \cup \{12, 13, \ldots, 56\}.$$

The lines of Γ are the 3-subsets $\{i, ij, j'\}$ and $\{i', ij, j\}$, $i, j \in \{1, 2, \ldots, 6\}$, $i < j$, and $\{ij, k\ell, mn\}$, with $\{i, j, k, \ell, m, n\} = \{1, 2, \ldots, 6\}$, $i < j$, $k < \ell$, $m < n$. We set $r_i = i(i+3)$.

Then here are the long mixed relations:

(I) $X_1X_{4'} + X_4X_{1'} + X_2X_{5'} + X_5X_{2'} + X_3X_{6'} + X_6X_{3'} \in \mathbb{K}'$.

(II) $X_1X_{4'} + X_{1'}X_4 + X_{24}X_{15} + X_{45}X_{12} + X_{34}X_{16} + X_{46}X_{13} \in \mathbb{K}'$.

(III) $X_1X_{4'} + X_{23}X_{56} + X_{45}X_{12} + X_3X_{6'} + X_{46}X_{13} + X_2X_{5'} \in \mathbb{K}'$.

(IV) $X_1X_{4'} + X_{56}X_{23} + X_{24}X_{15} + X_6X_{3'} + X_{34}X_{16} + X_5X_{2'} \in \mathbb{K}'$.

(V) $X_1X_{4'} + X_{35}X_{26} + X_{24}X_{15} + X_3X_{6'} + X_{46}X_{13} + X_5X_{2'} \in \mathbb{K}'$.

(VI) $X_1X_{4'} + X_{26}X_{35} + X_{45}X_{12} + X_6X_{3'} + X_{34}X_{16} + X_2X_{5'} \in \mathbb{K}'$.

(VII) $X_{1'}X_4 + X_{23}X_{56} + X_{45}X_{12} + X_{3'}X_6 + X_{46}X_{13} + X_{2'}X_5 \in \mathbb{K}'$.

(VIII) $X_{1'}X_4 + X_{56}X_{23} + X_{24}X_{15} + X_{6'}X_3 + X_{34}X_{16} + X_{5'}X_2 \in \mathbb{K}'$.

(IX) $X_{1'}X_4 + X_{35}X_{26} + X_{24}X_{15} + X_{3'}X_6 + X_{46}X_{13} + X_{5'}X_2 \in \mathbb{K}'$.

(X) $X_{1'}X_4 + X_{26}X_{35} + X_{45}X_{12} + X_{6'}X_3 + X_{34}X_{16} + X_{2'}X_5 \in \mathbb{K}'$.

(XI) $X_{23}X_{56} + X_{26}X_{35} + X_{12}X_{45} + X_{24}X_{15} + X_2X_{5'} + X_{2'}X_5 \in \mathbb{K}'$.

(XII) $X_{23}X_{56} + X_{35}X_{26} + X_{13}X_{46} + X_{34}X_{16} + X_3X_{6'} + X_{3'}X_6 \in \mathbb{K}'$.

And here are the short mixed relations written out:

(1) $X_1X_4+X_2X_5+X_3X_6\in\mathbb{K}'$. (2) $X_{1'}X_{4'}+X_{2'}X_{5'}+X_{6'}X_{3'}\in\mathbb{K}'$.
(3) $X_{1'}X_1+X_{24}X_{45}+X_{46}X_{34}\in\mathbb{K}'$. (4) $X_4X_{4'}+X_{15}X_{12}+X_{16}X_{13}\in\mathbb{K}'$.
(5) $X_1X_{56}+X_{45}X_3+X_2X_{46}\in\mathbb{K}'$. (6) $X_{4'}X_{23}+X_{12}X_{6'}+X_{5'}X_{13}\in\mathbb{K}'$.
(7) $X_1X_{23}+X_{24}X_6+X_5X_{34}\in\mathbb{K}'$. (8) $X_{4'}X_{56}+X_{15}X_{3'}+X_{2'}X_{16}\in\mathbb{K}'$.
(9) $X_1X_{26}+X_{24}X_3+X_5X_{46}\in\mathbb{K}'$. (10) $X_{4'}X_{35}+X_{15}X_{6'}+X_{2'}X_{13}\in\mathbb{K}'$.
(11) $X_1X_{35}+X_{45}X_6+X_2X_{34}\in\mathbb{K}'$. (12) $X_{4'}X_{26}+X_{12}X_{3'}+X_{5'}X_{16}\in\mathbb{K}'$.
(13) $X_{1'}X_{56}+X_{45}X_{3'}+X_{2'}X_{46}\in\mathbb{K}'$. (14) $X_4X_{23}+X_{12}X_6+X_5X_{13}\in\mathbb{K}'$.
(15) $X_{1'}X_{23}+X_{24}X_{6'}+X_{5'}X_{34}\in\mathbb{K}'$. (16) $X_4X_{56}+X_{15}X_3+X_2X_{16}\in\mathbb{K}'$.
(17) $X_{1'}X_{26}+X_{24}X_{3'}+X_{5'}X_{46}\in\mathbb{K}'$. (18) $X_4X_{35}+X_{15}X_6+X_2X_{13}\in\mathbb{K}'$.
(19) $X_{1'}X_{35}+X_{45}X_{6'}+X_{2'}X_{34}\in\mathbb{K}'$. (20) $X_4X_{26}+X_{12}X_3+X_5X_{16}\in\mathbb{K}'$.
(21) $X_{23}X_{26}+X_{12}X_{24}+X_{5'}X_5\in\mathbb{K}'$. (22) $X_{56}X_{35}+X_{45}X_{15}+X_{2'}X_2\in\mathbb{K}'$.
(23) $X_{23}X_{35}+X_{34}X_{13}+X_{6'}X_6\in\mathbb{K}'$. (24) $X_{56}X_{26}+X_{16}X_{46}+X_{3'}X_3\in\mathbb{K}'$.

The equation stemming from the degenerate quadric Q_p, $p\in X$, will be referred to as the *p-symp equation*, or briefly *symp equation* if we do not want to mention p.

Now we introduce two types of certain transformations of $\mathsf{PG}(V)$ directly with coordinates. The first type are so-called *short root elations*. There are twenty four basic ones, each related to a point p of Γ not on $\{r_1,r_2,r_3\}$. The short root elation $\sigma_p(k)$, $k\in\mathbb{K}$, *with center p* fixes all points of the subspace of $\mathsf{PG}(26,\mathbb{K})$ generated by all e_q, with q not Γ-collinear to p. In order to describe it abstractly, we call two points of $X\setminus\{r_1,r_2,r_3\}$ *sibling* if they are both Γ-collinear to the same member of $\{r_1,r_2,r_3\}$. If that member is r_i, $i\in\{1,2,3\}$, then we also talk about *r_i-sibling*. For each point $q\in X\setminus\{r_1,r_2,r_3\}$, there is a unique point, denoted q_*, that is collinear and sibling. Then the action of $\sigma_p(k)$ on the coordinates of a random point $(X_q)_{q\in X}$ is defined as follows:

$$X_q\mapsto\begin{cases}X_q & \text{if } p,q \text{ are sibling or collinear,}\\ X_p+kX_{r_i}+k^2X_{p_*},\ \{p,p_*,r_i\}\in\mathscr{L} & \text{if } q=p,\\ X_{r_j}+kX_{p_*},\ \{p,p_*,r_i\}\in\mathscr{L} & \text{if } p=r_j,\ j\neq i,\\ X_q+kX_{q'},\ \{p,q_*,q'\}\in\mathscr{L} & \text{otherwise.}\end{cases}$$

For example, if $p=1$, we have: $X_a\mapsto X_a$, for $a=1',2',3',4',5',6'$; 14; 4; 12, 13, 15, 16, 23, 26, 35, 56, and

$$X_1\mapsto X_1+kX_{14}+k^2X_{4'},$$
$$X_{25}\mapsto X_{25}+kX_{4'},\qquad X_{36}\mapsto X_{36}+kX_{4'},$$
$$X_2\mapsto X_2+kX_{15},\qquad X_3\mapsto X_3+kX_{16},$$
$$X_5\mapsto X_5+kX_{12},\qquad X_6\mapsto X_6+kX_{13},$$
$$X_{24}\mapsto X_{24}+kX_{5'},\qquad X_{34}\mapsto X_{34}+kX_{6'},$$
$$X_{45}\mapsto X_{45}+kX_{2'},\qquad X_{46}\mapsto X_{46}+kX_{3'}.$$

Lemma 4.1. *Each short root elation $\sigma_p(k)$ is a permutation of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$.*

Proof. This is a simple calculation. First we notice that all short root elations play the same role, so we only need to prove the assertion for $\sigma_1(k)$. Also, since $\sigma_1(k)$ is clearly an involution, we only need to show that $\sigma_1(k)$ preserves, in one direction, the symp equations, the long mixed relations and the short mixed relations. We provide an example or two of each.

The 16-symp equation $X_1X_{6'} + X_6X_{1'} + X_{23}X_{45} + X_{24}X_{35} + X_{25}X_{34} = 0$ is transformed under the action of $\sigma_1(k)$ onto

$$X_1X_{6'}+X_6X_{1'}+X_{23}X_{45}+X_{24}X_{35}+X_{25}X_{34}+k\big(X_{14}X_{6'}+X_{13}X_{1'}+X_{23}X_{2'}\\ +X_{5'}X_{35}+X_{25}X_{6'}+X_{4'}X_{34}\big)+k^2(X_{4'}X_{6'}+X_{4'}X_{6'})=0,$$

which holds by the 16-symp equation and the $3'$-symp equation in combination with the equation $X_{14}+X_{25}+X_{36}=0$.

The 56-symp equation $X_5X_{6'} + X_6X_{5'} + X_{12}X_{34} + X_{13}X_{24} + X_{14}X_{23} = 0$ is transformed under the action of $\sigma_1(k)$ onto $X_5X_{6'} + X_6X_{5'} + X_{12}X_{34} + X_{13}X_{24} + X_{14}X_{23} + k(X_{12}X_{6'} + X_{13}X_{5'} + X_{12}X_{6'} + X_{13}X_{5'})$, hence Q_{56} is fixed.

The long mixed relation (V) is transformed under the action of $\sigma_1(k)$ onto

$$X_1X_{4'}+X_{35}X_{26}+X_{24}X_{15}+X_3X_{6'}+X_{46}X_{13}+X_5X_{2'}\\ +k(X_{14}X_{4'}+X_{5'}X_{15}+X_{16}X_{6'}+X_{3'}X_{13}+X_{12}X_{2'})+k^2X_{4'}^2\in\mathbb{K}',$$

which holds by the 1-symp equation and the fact that $\mathbb{K}^2 \le \mathbb{K}'$.

Finally, the short mixed relation (5) is transformed under the action of $\sigma_1(k)$ onto

$$X_1X_{56}+X_{45}X_3+X_2X_{46}+k(X_{14}X_{56}+X_{2'}X_3+X_{45}X_{16}+X_{15}X_{46}+X_2X_{3'})\\ +k^2(X_{4'}X_{56}+X_2X_{16}+X_{3'}X_{15})\in\mathbb{K}',$$

which holds by the 23-symp equation and the short mixed relation (8), noting $k^2 \in \mathbb{K}^2 \le \mathbb{K}'$. □

Remark 4.2. In the given form, a short root elation does not preserve $\mathscr{E}_{6,1}(\mathbb{K})$. However, if we substitute in the second line of the definition of $\sigma_p(k)$ the coordinate X_{r_i} with $X_{r_k} + X_{r_\ell}$, with $\{i, j, \ell\} = \{1, 2, 3\}$, then we obtain the same action on $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$, and we do preserve $\mathscr{E}_{6,1}(\mathbb{K})$. The reason not to introduce it like that is that the given form is more convenient to calculate with.

The second type of transformations consists of so-called *long root elations*. There are again twenty four basic ones, each related to a special 6-coclique of Γ. The long root elation $\lambda_C(k)$, $k \in \mathbb{K}'$, *with axis* C, where C is a special 6-coclique, fixes all points of the subspace of $\mathsf{PG}(26, \mathbb{K})$ generated by all e_q, with q not contained in the opposite special 6-coclique. More precisely, the action of $\sigma_C(k)$ on the coordinates of a random point $(X_q)_{q\in X}$ is defined as follows, where for each point $p \in C$, the unique point that belongs to special 6-coclique opposite C and not collinear to p in Γ, is denoted by p_C.

$$X_q \mapsto \begin{cases} X_q & \text{if } q \notin C, \\ X_q + kX_{q_C} & \text{if } q \in C. \end{cases}$$

For example, if $C = \{1, 2, 3, 4, 5, 6\}$, we have

$$X_1 \mapsto X_1 + kX_{1'}, \quad X_2 \mapsto X_2 + kX_{2'}, \quad X_3 \mapsto X_3 + kX_{3'},$$
$$X_4 \mapsto X_4 + kX_{4'}, \quad X_5 \mapsto X_5 + kX_{5'}, \quad X_6 \mapsto X_6 + kX_{6'},$$

and all other coordinates are fixed.

Lemma 4.3. *Each long root elation $\sigma_C(k)$ is a permutation of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$.*

Proof. Again, this is an elementary calculation., and we may consider only $\lambda_C(k)$, $k \in \mathbb{K}'$, for $C = \{1, 2, 3, 4, 5, 6\}$. Then it is clear that ij-symp equations are fixed, for all $i, j \in \{1, 2, \ldots, 6\}$, $i \neq j$, and the same holds for all i-symp equations, $i \in \{1, 2', \ldots, 6\}$. The $1'$-symp equation is mapped onto the equality

$$X_2X_{12}+X_3X_{13}+X_4X_{14}+X_5X_{15}+X_6X_{16} \\ +k(X_{2'}X_{12}+X_{3'}X_{13}+X_{4'}X_{14}+X_{5'}X_{15}+X_{6'}X_{16}) = 0,$$

which is zero by the 1-symp equation itself and the $1'$-symp equation.

The long mixed relations (I), (II), (XI) and (XII) are preserved, whereas the other ones are mapped onto a combination with a short mixed relation. For instance, the long mixed relation (V) is mapped onto

$$X_1X_{4'}+X_{35}X_{26}+X_{24}X_{15}+X_3X_{6'}+X_{46}X_{13}+X_5X_{2'} \\ +k(X_{1'}X_{4'}+X_{3'}X_{6'}+X_{5'}X_{2'}) \in \mathbb{K}',$$

which holds by (V) itself, and by the short mixed relation (2) and the fact that $k \in \mathbb{K}'$.

Finally, the left hand sides of the short mixed relations with a term $X_iX_{i'}$, $i \in \{1, 2, \ldots, 6\}$, are mapped onto the sum of themselves with $kX_{i'}^2$ and hence that expression again belongs to $\mathbb{K}'$. The ones with a term $X_{i'}X_{j\ell}$ are preserved, whereas those with a term $X_iX_{j\ell}$ are mapped onto a combination of themselves with the one containing the term $X_{i'}X_{j\ell}$. Finally, short mixed relation (2) is fixed, and short mixed relation (1) is transformed under the action of $\lambda_C(k)$ onto

$$X_1X_4+X_2X_5+X_3X_6+k(X_{1'}X_4+X_1X_{4'}+X_{2'}X_5+X_2X_{5'}+X_{3'}X_6+X_3X_{6'}) \\ +k^2(X_{1'}X_{4'}+X_{2'}X_{5'}+X_{3'}X_{6'}) \in \mathbb{K}',$$

which holds because of short mixed relations (1) and (2), and long mixed relation (I), recalling that $k \in \mathbb{K}'$. □

Now it is clear that each collineation τ of Γ preserving $\{r_1, r_2, r_3\} = \{14, 25, 36\}$ defines a permutation of the coordinates that induces a collineation g_τ of $\mathscr{E}_{6,1}(\mathbb{K})$ preserving $\mathscr{H}$ and preserving the mixed relations. We denote the group of all such collineations by T. Hence the group of linear maps generated by T and all short and long root elations defined above, acts on $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$, and shall be denoted by G.

Proposition 4.4. *The group G acts transitively on the point set of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$.*

Proof. We will show that for each point $a \in \mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$, there exists $g \in G$ such that a^g is a base point. Since base points are clearly in one orbit of T, and hence also of G, this will suffice.

It is easily derived from the 14-, 25- and 36-symp equations that at least one coordinate x_p, $p \in X \setminus \{14, 25, 36\}$, is distinct from zero and hence can be chosen to equal 1 (indeed, these symp relations imply $x_{14}x_{25} = x_{25}x_{36} = x_{36}x_{14} = 0$, which, combined with $x_{14} + x_{25} + x_{36} = 0$, implies $x_{14} = x_{25} = x_{36} = 0$). Without loss, we may assume $x_{4'} = 1$. For ease of verification by the reader, we will explicitly write down the actions of the root elations that we use.

We first subsequently apply the following short root elations to a:

Short root elation $\sigma_{45}(x_{2'})$:

$$
\begin{aligned}
X_1 &\mapsto X_1 + x_{2'}X_5, & X_{1'} &\mapsto X_{1'} + x_{2'}X_{5'},\\
X_2 &\mapsto X_2 + x_{2'}X_4, & X_{2'} &\mapsto X_{2'} + x_{2'}X_{4'},\\
X_{56} &\mapsto X_{56} + x_{2'}X_{16}, & X_{35} &\mapsto X_{35} + x_{2'}X_{13},\\
X_{34} &\mapsto X_{34} + x_{2'}X_{23}, & X_{46} &\mapsto X_{46} + x_{2'}X_{26},\\
X_{25} &\mapsto X_{25} + x_{2'}X_{12}, & X_{14} &\mapsto X_{14} + x_{2'}X_{12},\\
X_{45} &\mapsto X_{45} + x_{2'}X_{36} + x_{2'}^2X_{12}.
\end{aligned}
$$

Short root elation $\sigma_{46}(x_{3'})$:

$$
\begin{aligned}
X_1 &\mapsto X_1 + x_{3'}X_6, & X_{1'} &\mapsto X_{1'} + x_{3'}X_{6'},\\
X_3 &\mapsto X_3 + x_{3'}X_4, & X_{3'} &\mapsto X_{3'} + x_{3'}X_{4'},\\
X_{56} &\mapsto X_{56} + x_{3'}X_{15}, & X_{26} &\mapsto X_{26} + x_{3'}X_{12},\\
X_{24} &\mapsto X_{24} + x_{3'}X_{23}, & X_{45} &\mapsto X_{45} + x_{3'}X_{35},\\
X_{36} &\mapsto X_{36} + x_{3'}X_{13}, & X_{14} &\mapsto X_{14} + x_{3'}X_{13},\\
X_{46} &\mapsto X_{46} + x_{3'}X_{25} + x_{3'}^2X_{13}.
\end{aligned}
$$

Short root elation $\sigma_{24}(x_{5'})$:

$$\begin{aligned} X_1 &\mapsto X_1+x_{5'}X_2, & X_{1'} &\mapsto X_{1'}+x_{5'}X_{2'},\\ X_5 &\mapsto X_5+x_{5'}X_4, & X_{5'} &\mapsto X_{5'}+x_{5'}X_{4'},\\ X_{26} &\mapsto X_{26}+x_{5'}X_{16}, & X_{23} &\mapsto X_{23}+x_{5'}X_{13},\\ X_{34} &\mapsto X_{34}+x_{5'}X_{35}, & X_{46} &\mapsto X_{46}+x_{5'}X_{56},\\ X_{25} &\mapsto X_{25}+x_{5'}X_{15}, & X_{14} &\mapsto X_{14}+x_{5'}X_{15},\\ X_{24} &\mapsto X_{24}+x_{5'}X_{36}+x_{5'}^2X_{15}. \end{aligned}$$

Short root elation $\sigma_{34}(x_{6'})$:

$$\begin{aligned} X_1 &\mapsto X_1+x_{6'}X_3, & X_{1'} &\mapsto X_{1'}+x_{6'}X_{3'},\\ X_6 &\mapsto X_6+x_{6'}X_4, & X_{6'} &\mapsto X_{6'}+x_{6'}X_{4'},\\ X_{35} &\mapsto X_{35}+x_{6'}X_{15}, & X_{23} &\mapsto X_{23}+x_{6'}X_{12},\\ X_{24} &\mapsto X_{24}+x_{6'}X_{26}, & X_{45} &\mapsto X_{45}+x_{6'}X_{56},\\ X_{36} &\mapsto X_{36}+x_{6'}X_{16}, & X_{14} &\mapsto X_{14}+x_{6'}X_{16},\\ X_{34} &\mapsto X_{34}+x_{6'}X_{25}+x_{6'}^2X_{16}. \end{aligned}$$

Short root elation $\sigma_5(x_{12})$:

$$\begin{aligned} X_1 &\mapsto X_1+x_{12}X_{45}, & X_3 &\mapsto X_3+x_{12}X_{56},\\ X_4 &\mapsto X_4+x_{12}X_{15}, & X_6 &\mapsto X_6+x_{12}X_{35},\\ X_{12} &\mapsto X_{12}+x_{12}X_{4'}, & X_{23} &\mapsto X_{23}+x_{12}X_{6'},\\ X_{24} &\mapsto X_{24}+x_{12}X_{1'}, & X_{26} &\mapsto X_{26}+x_{12}X_{3'},\\ X_{14} &\mapsto X_{14}+x_{12}X_{2'}, & X_{36} &\mapsto X_{36}+x_{12}X_{2'},\\ X_5 &\mapsto X_5+x_{12}X_{25}+x_{12}^2X_{2'}. \end{aligned}$$

Short root elation $\sigma_6(x_{13})$:

$$\begin{aligned} X_2 &\mapsto X_2+x_{13}X_{56}, & X_1 &\mapsto X_1+x_{13}X_{46},\\ X_5 &\mapsto X_5+x_{13}X_{26}, & X_4 &\mapsto X_4+x_{13}X_{16},\\ X_{23} &\mapsto X_{23}+x_{13}X_{5'}, & X_{13} &\mapsto X_{13}+x_{13}X_{4'},\\ X_{35} &\mapsto X_{35}+x_{13}X_{2'}, & X_{34} &\mapsto X_{34}+x_{13}X_{1'},\\ X_{25} &\mapsto X_{25}+x_{13}X_{3'}, & X_{14} &\mapsto X_{14}+x_{13}X_{3'},\\ X_6 &\mapsto X_6+x_{13}X_{36}+x_{13}^2X_{3'}. \end{aligned}$$

Short root elation $\sigma_2(x_{15})$:

$$X_1 \mapsto X_1 + x_{15}X_{24}, \qquad X_3 \mapsto X_3 + x_{15}X_{26},$$
$$X_4 \mapsto X_4 + x_{15}X_{12}, \qquad X_6 \mapsto X_6 + x_{15}X_{23},$$
$$X_{15} \mapsto X_{15} + x_{15}X_{4'}, \qquad X_{35} \mapsto X_{35} + x_{15}X_{6'},$$
$$X_{45} \mapsto X_{45} + x_{15}X_{1'}, \qquad X_{56} \mapsto X_{56} + x_{15}X_{3'},$$
$$X_{14} \mapsto X_{14} + x_{15}X_{5'}, \qquad X_{36} \mapsto X_{36} + x_{15}X_{5'},$$
$$X_2 \mapsto X_2 + x_{15}X_{25} + x_{15}^2X_{5'}.$$

Short root elation $\sigma_3(x_{16})$:

$$X_2 \mapsto X_2 + x_{16}X_{35}, \qquad X_1 \mapsto X_1 + x_{16}X_{34},$$
$$X_5 \mapsto X_5 + x_{16}X_{23}, \qquad X_4 \mapsto X_4 + x_{16}X_{13},$$
$$X_{26} \mapsto X_{26} + x_{16}X_{5'}, \qquad X_{16} \mapsto X_{16} + x_{16}X_{4'},$$
$$X_{56} \mapsto X_{56} + x_{16}X_{2'}, \qquad X_{46} \mapsto X_{46} + x_{16}X_{1'},$$
$$X_{25} \mapsto X_{25} + x_{16}X_{6'}, \qquad X_{14} \mapsto X_{14} + x_{16}X_{6'},$$
$$X_3 \mapsto X_3 + x_{16}X_{36} + x_{16}^2X_{6'}.$$

After applying these short root elations (in arbitrary order, call the composition g_1), the coordinates of the image $a^{g_1} = b$, denoted by $(y_p)_{p\in X}$, satisfy

$$y_{2'} = y_{3'} = y_{5'} = y_{6'} = y_{12} = y_{13} = y_{15} = y_{16} = 0 \quad \text{and} \quad y_{4'} = 1.$$

Then the following indicated short mixed relations reduce to and imply

$$(2)\ X_{1'}X_{4'} \in \mathbb{K}' \Longrightarrow y_{1'} \in \mathbb{K}', \qquad (4)\ X_4X_{4'} \in \mathbb{K}' \Longrightarrow y_4 \in \mathbb{K}',$$
$$(6)\ X_{4'}X_{23} \in \mathbb{K}' \Longrightarrow y_{23} \in \mathbb{K}', \qquad (8)\ X_{4'}X_{56} \in \mathbb{K}' \Longrightarrow y_{56} \in \mathbb{K}',$$
$$(10)\ X_{4'}X_{35} \in \mathbb{K}' \Longrightarrow y_{35} \in \mathbb{K}', \qquad (12)\ X_{4'}X_{26} \in \mathbb{K}' \Longrightarrow y_{26} \in \mathbb{K}'.$$

Moreover, the symp equation $X_{2'}X_{12} + X_{3'}X_{13} + X_{4'}X_{14} + X_{5'}X_{15} + X_{6'}X_{16} = 0$ implies $y_{14} = 0$.

Consider the following long root elations:

- $\lambda_{\{1,1',24,34,45,46\}}(y_{1'})$:

$$X_1 \mapsto X_1 + y_{1'}X_4, \qquad X_{1'} \mapsto X_{1'} + y_{1'}X_{4'}, \qquad X_{24} \mapsto X_{24} + y_{1'}X_{12},$$
$$X_{45} \mapsto X_{45} + y_{1'}X_{15}, \qquad X_{46} \mapsto X_{46} + y_{1'}X_{16}, \qquad X_{34} \mapsto X_{34} + y_{1'}X_{13}.$$

- $\lambda_{\{1,2,3,4,5,6\}}(y_4)$:

$$X_1 \mapsto X_1 + y_4X_{1'}, \qquad X_2 \mapsto X_2 + y_4X_{2'}, \qquad X_3 \mapsto X_3 + y_4X_{3'},$$
$$X_4 \mapsto X_4 + y_4X_{4'}, \qquad X_5 \mapsto X_5 + y_4X_{5'}, \qquad X_6 \mapsto X_6 + y_4X_{6'}.$$

- $\lambda_{\{1.5.6.23.24.34\}}(y_{23})$:

$$X_1 \mapsto X_1 + y_{23}X_{56}, \quad X_5 \mapsto X_5 + y_{23}X_{16}, \quad X_6 \mapsto X_6 + y_{23}X_{15},$$
$$X_{24} \mapsto X_{24} + y_{23}X_{3'}, \quad X_{34} \mapsto X_{34} + y_{23}X_{2'}, \quad X_{23} \mapsto X_{23} + y_{23}X_{4'}.$$

- $\lambda_{\{1,2,3,45,46,56\}}(y_{56})$:

$$X_1 \mapsto X_1 + y_{56}X_{23}, \quad X_2 \mapsto X_2 + y_{56}X_{13}, \quad X_3 \mapsto X_3 + y_{56}X_{12},$$
$$X_{45} \mapsto X_{45} + y_{56}X_{6'}, \quad X_{46} \mapsto X_{46} + y_{56}X_{5'}, \quad X_{56} \mapsto X_{56} + y_{56}X_{4'}.$$

- $\lambda_{\{1,2,6,34,35,45\}}(y_{35})$:

$$X_1 \mapsto X_1 + y_{35}X_{26}, \quad X_2 \mapsto X_2 + y_{35}X_{16}, \quad X_6 \mapsto X_6 + y_{35}X_{12},$$
$$X_{45} \mapsto X_{45} + y_{35}X_{3'}, \quad X_{34} \mapsto X_{34} + y_{35}X_{5'}, \quad X_{35} \mapsto X_{35} + y_{35}X_{4'}.$$

- $\lambda_{\{1,3,5,24,26,46\}}(y_{26})$:

$$X_1 \mapsto X_1 + y_{26}X_{35}, \quad X_3 \mapsto X_3 + y_{26}X_{15}, \quad X_5 \mapsto X_5 + y_{26}X_{13},$$
$$X_{24} \mapsto X_{24} + y_{26}X_{6'}, \quad X_{46} \mapsto X_{46} + y_{26}X_{2'}, \quad X_{26} \mapsto cr\,X_{26} + y_{26}X_{4'}.$$

If we apply these elations in arbitrary order (say the composition is g_2) to b, then the coordinates of $b^{g_2} = c$, which we denote by $(z_p)_{p\in X}$, satisfy $z_{1'} = z_4 = z_{23} = z_{26} = z_{35} = z_{56} = 0$, whereas we did not touch the coordinates that were already zero by virtue of applying g_1 to a. That is, $z_{2'} = z_{3'} = z_{5'} = z_{6'} = z_{12} = z_{13} = z_{15} = z_{16} = 0$.

The symp equation $X_{2'}X_{12} + X_{3'}X_{13} + X_{4'}X_{14} + X_{5'}X_{15} + X_{6'}X_{16} = 0$ implies $z_{14} = 0$. This implies $z_{25} = z_{36}$. Now, the symp equation $X_{1'}X_4 + X_1X_{4'} + X_{23}X_{56} + X_{26}X_{35} + X_{25}X_{36} = 0$ implies $z_1 = z_{25}^2 = z_{36}^2$. Now the short root elation $g_3 := \sigma_1(z_{25})$ turns the coordinates z_1, z_{25} and z_{36} equal to zero (and so we do assume $z_1 = z_{25} = z_{36} = 0$) and only alters the coordinates $z_2, z_3, z_5, z_6, z_{24}, z_{34}, z_{45}, z_{46}$.

But now the given symp equations reduce to and imply

$$\begin{aligned}
\text{24-symp equation}: &\ X_2X_{4'} = 0 \Longrightarrow z_2 = 0,\\
\text{34-symp equation}: &\ X_3X_{4'} = 0 \Longrightarrow z_3 = 0,\\
\text{45-symp equation}: &\ X_5X_{4'} = 0 \Longrightarrow z_5 = 0,\\
\text{46-symp equation}: &\ X_6X_{4'} = 0 \Longrightarrow z_6 = 0,\\
\text{2-symp equation}: &\ X_{24}X_{4'} = 0 \Longrightarrow z_{24} = 0,\\
\text{3-symp equation}: &\ X_{34}X_{4'} = 0 \Longrightarrow z_{34} = 0,\\
\text{5-symp equation}: &\ X_{45}X_{4'} = 0 \Longrightarrow z_{45} = 0,\\
\text{6-symp equation}: &\ X_{46}X_{4'} = 0 \Longrightarrow z_{46} = 0.
\end{aligned}$$

In conclusion, we have mapped an arbitrary point of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$ with $x_{4'}$ coordinate nonzero to the base point $\langle e_{4'}\rangle$. □

Now we denote the set of points of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$ briefly by $\mathscr{H}'$. Let $\mathfrak{L}'$ be the set of lines contained in $\mathscr{H}'$. Set $\Delta' = (\mathscr{H}', \mathfrak{L}')$. We have just shown that the geometry Δ' admits a transitive collineation group.

We now determine the *residual geometry* at a base point $b \in \mathscr{H}'$, that is, the geometry with point set the set of all members of $\mathfrak{L}'$ through b, and with for each plane π contained in $\mathscr{H}'$ and containing b, the set of lines through b and contained in π is a (typical) line of that residual geometry (this definition agrees with the one given in 13.4.1 of [11]). Let us take $b = \langle e_1 \rangle$.

Proposition 4.5. *The point residual of Δ' at $\langle e_1 \rangle$ is isomorphic to the dual polar space* $\mathsf{B}_{3,3}(\mathbb{K}', \mathbb{K})$. *Also, the point residual of Δ' at $\langle e_1 \rangle$ is contained in the tangent space T at $\langle e_1 \rangle$ of a unique symp ξ of $\mathscr{E}_{6,1}(\mathbb{K})$; we have $T = \langle \xi \rangle \cap H$.*

Proof. The symp equations imply that a point a of $\mathscr{H}'$ is collinear to $\langle e_1 \rangle$ in $\mathscr{E}_{6,1}(\mathbb{K})$ if and only if its coordinate x_q is zero as soon as $1 \perp q$ in Γ. Hence every line in $\mathscr{H}'$ intersects the subspace $W' \subset H$ defined as the intersection with H of the subspace of $\mathsf{PG}(26, \mathbb{K})$ generated by all $\langle e_r \rangle$ with r not collinear to 1. Consequently, if we are looking for the lines $\langle e_1, a \rangle \subseteq \mathscr{H}'$, we may assume $a \in W'$. Set $a = (x_q)_{q \in X}$.

Consider the short mixed relation (1): $X_1 X_4 + X_2 X_5 + X_3 X_6 \in \mathbb{K}'$. If $x_4 \neq 0$, then we can find $t \in \mathbb{K}$ such that (1) does not hold for the point $\langle t e_1 + a \rangle$. In other words, we can find a point on $\langle e_1, p \rangle$ not contained in $\mathscr{H}'$. We conclude that $x_4 = 0$. Likewise $x_{1'} = x_{56} = x_{23} = x_{26} = x_{35} = 0$. Also since $x_{14} = 0$, we have $x_{25} = x_{36}$ and the 14-symp equation $X_{25} X_{36} + X_1 X_{4'} + X_{1'} X_4 + X_{23} X_{56} + X_{26} X_{35} = 0$ yields $x_{25} = x_{36} = 0$.

Hence each line $\langle \langle e_1 \rangle, a \rangle$ of $\mathscr{H}'$ intersects $W'' = \langle e_2, e_3, e_5, e_6, e_{24}, e_{34}, e_{45}, e_{46} \rangle$ in a unique point. Conversely, each point $\langle f \rangle$ of $\mathscr{H}'$ in W' is Δ'-collinear to $\langle e_1 \rangle$ (as the point $\langle \lambda e_1 + f \rangle$ satisfies all mixed relations). This already shows the second assertion, with ξ the symp of $\mathscr{E}_{6,1}(\mathbb{K})$ corresponding to Q_1. So we can identify each member of $\mathfrak{L}'$ with a unique point having (truncated) coordinates $(x_2, x_3, x_5, x_6, x_{24}, x_{34}, x_{45}, x_{46})$. If we run these coordinates through all symp equations and mixed relations, then the only nonvanishing relations are the following:

$$X_2 X_{24} X_3 + X_{34} + X_5 X_{45} + X_6 X_{46} = 0,$$

$$X_2 X_5 + X_3 X_6 \in \mathbb{K}', \quad X_{24} X_{45} + X_{46} X_{34} \in \mathbb{K}',$$

$$X_2 X_{46} + X_3 X_{45} \in \mathbb{K}', \quad X_5 X_{46} + X_6 X_{24} \in \mathbb{K}',$$

$$X_3 X_{24} + X_5 X_{46} \in \mathbb{K}', \quad X_2 X_{34} + X_6 X_{45} \in \mathbb{K}'.$$

By Proposition 2.1 in Section 2, this is exactly a geometry isomorphic to $\mathsf{B}_{3,3}(\mathbb{K}', \mathbb{K})$. □

If we knew that Δ' were a parapolar space of symplectic rank 3, then the local recognition result Lemma 5.7 of [6] would imply that Δ' is the metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$. This will be taken care of in the next lemma. A *convex subspace* of Δ' is a set of points closed under joining collinear points with all points of the joining line and taking shortest paths between points in the graph with vertices the points of Δ', adjacent when collinear.

Lemma 4.6. (a) *Each pair of points of Δ' that is collinear in $\mathscr{E}_{6,1}(\mathbb{K})$ is contained in a convex subgeometry of Δ' isomorphic to $\mathsf{C}_{3,1}(\mathbb{K}, \mathbb{K}')$.*

(b) *Each pair of points of Δ' that is not collinear in $\mathscr{E}_{6,1}(\mathbb{K})$ is Δ'-collinear to at most one common point.*

Proof. First notice that by Properties (ii) and (iii) in the beginning of this section, Proposition 4.5 implies that $\mathfrak{L}' \subseteq \mathfrak{L}$. This implies that, if two points of Δ' are Δ-collinear to at most one common point, then they are also Δ'-collinear to at most one common point. Property (v) of the beginning of this section now says that the pairs of points of Δ contained in a symp of Δ are themselves contained in a 5-space of $\mathscr{E}_{6,1}(\mathbb{K})$. This yields (b).

So now let a, b be a pair of points of Δ' that is collinear in $\mathscr{E}_{6,1}(\mathbb{K})$. By Proposition 4.4, we may assume that $a = \langle e_1 \rangle$. Let $(x_p)_{p\in X}$ be a coordinate tuple for b. Expressing that also $e_1 + (x_p)_{p\in X}$ satisfies all symp equations, we deduce $x_{2'} = x_{3'} = x_{4'} = x_{5'} = x_{6'} = x_{12} = x_{13} = x_{14} = x_{15} = x_{16} = 0$.

Our first main step is to show that we may assume that $b \in \langle e_1, e_2, \ldots, e_6 \rangle$.

Suppose first that $x_4 = x_{1'} = x_{23} = x_{26} = x_{35} = x_{56} = 0$. Then the 14-symp equation implies $x_{25}x_{36} = 0$, yielding $x_{25} = x_{36} = 0$ (since $x_{14} = 0$ and $x_{14} + x_{25} + x_{36} = 0$). Hence the only nonzero coordinates of b are among $x_2, x_3, x_5, x_6, x_{24}, x_{34}, x_{45}$ and x_{46}. In the last paragraph of the proof of Proposition 4.5 we deduced that b is Δ'-collinear to a. We now claim that we can assume that $x_{24} = x_{34} = x_{45} = x_{46} = 0$. Indeed, suppose not. Without loss we may assume $x_{46} \neq 0$. Applying $\lambda_{\{4,5,6,12,13,23\}}(1)$

$$X_4 \mapsto X_4 + X_{56}, \quad X_5 \mapsto X_5 + X_{46}, \quad X_6 \mapsto X_6 + X_{45},$$
$$X_{12} \mapsto X_{12} + X_{3'}, \quad X_{13} \mapsto X_{13} + X_{2'}, \quad X_{23} \mapsto X_{23} + X_{1'},$$

if need be, we may also assume $x_5 \neq 0$. Applying $\lambda_{\{1',3',6',24,25,45\}}(x_{24}x_5^{-1})$, $\lambda_{\{1',2',6',34,35,45\}}(x_{34}x_5^{-1})$ and $\lambda_{\{1',2',3',45,46,56\}}(x_{46}x_5^{-1})$ (which all fix a) in a row, we see that we may assume $x_{24} = x_{34} = x_{46} = 0$ and $x_5 \neq 0$. The $4'$-symp equation now implies $x_5x_{45} = 0$, hence $x_{45} = 0$, proving our claim. Then $b \in \langle e_1, e_2, \ldots, e_6 \rangle$.

Now suppose at least one of $x_4, x_{1'}, x_{23}, x_{26}, x_{35}$ or x_{56} is not equal to zero. If one of x_{23}, x_{26}, x_{35} or x_{56} is nonzero, say $x_{23} \neq 0$, then by applying $\lambda_{\{2,3,4,15,16,56\}}(1)$ if need be, we may assume $x_4 \neq 0$. But then we apply $\lambda_{\{1',5',6',23,24,34\}}(x_{23}x_4^{-1})$, $\lambda_{\{1',3',5',24,26,46\}}(x_{26}x_4^{-1})$, $\lambda_{\{1',2',6',34,35,45\}}(x_{35}x_4^{-1})$ and $\lambda_{\{1',2',3',45,46,56\}}(x_{56}x_4^{-1})$, all in a row, and we see that we may assume $x_{23} = x_{26} = x_{35} = x_{56} = 0$, $x_4 \neq 0$. The 14-symp equation implies $x_4x_{1'} = x_{25}x_{36} = k^2$, with $k = x_{25} = x_{36}$. Applying

$\sigma_{1'}(kx_4^{-1})$ we see that we may assume $x_{1'} = 0$. Then $x_{25} = x_{36} = 0$ and the i'-symp equations for $i = 2, 3, 5, 6$ yield $x_{24} = x_{34} = x_{45} = x_{46}$. This again implies $b \in \langle e_1, e_2, \dots, e_6\rangle$.

Now note that $\langle e_1, e_2, \dots, e_6\rangle$ is a 5-space contained in $\mathscr{H}$, and hence by (iv) it intersects Δ in a symplectic polar space Ω of rank 3. Now all symp equations and all long mixed relations vanish on $\langle e_1, e_2, \dots, e_6\rangle$ and the only surviving short mixed relation is (1) $X_1X_4 + X_2X_5 + X_3X_6 \in \mathbb{K}'$, which defines a polar subspace Ω' of Ω isomorphic to $\mathsf{C}_{3,1}(\mathbb{K}, \mathbb{K}')$. Since Ω is convex in Δ, we conclude that Ω' is convex in Δ' and (a) is proved. □

We can now state the main result of this section.

Theorem 4.7. *Δ' is isomorphic to the metasymplectic space* $\mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$.

Proof. Lemma 4.6 implies that Δ' is a parapolar space and then Lemma 5.7 of [6] and Proposition 4.5 imply that Δ' is isomorphic to the metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$. □

In the proof of Lemma 4.6, we observed that the subspace $\langle e_1, e_2, \dots, e_6\rangle$ hosts a symplecton of Δ', that is, a convex subspace isomorphic to the polar space $\mathsf{C}_{3,1}(\mathbb{K}, \mathbb{K}')$. In fact, it is easily checked that this is true for each subspace $\langle e_{p_1}, e_{p_2}, \dots, e_{p_6}\rangle$, where $\{p_1, p_, \dots, p_6\}$ is a special 6-coclique. Hence each of the short mixed relations defines such a *host space* and such a symplecton, which we will refer to as a symplecton of the same type as the short mixed relation. For instance, the symplecton in $\langle e_1, e_2, \dots, e_6\rangle$ has type (1).

The next lemma will be needed in Section 5. An alternative proof consists of using the "principle of triality"; Theorem 3.2.1 of [12].

Lemma 4.8. *A collineation of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$ pointwise fixing the parabolic intersection of a symp of $\mathscr{E}_{6,1}(\mathbb{K})$ with $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$ is the identity.*

Proof. Without loss we may assume that the subspace

$$W = \langle e_1, e_4, e_{1'}, e_{4'}, e_{23}, e_{26}, e_{35}, e_{56}, e_{25} + e_{36}\rangle$$

is fixed pointwise. It can easily be verified that the only type-preserving collineation of the corresponding symp fixing W pointwise is the identity (we need type-preserving otherwise it is even not a collineation of $\mathscr{E}_{6,1}(\mathbb{K})$). Hence also $\langle e_{25}, e_{36}\rangle$ is fixed pointwise. Consider an arbitrary base point outside W. Without loss we can take $\langle e_2\rangle$. The short mixed relations (1), (11), (16) and (18) imply that $\langle e_2\rangle$ is fixed. Hence we see that all base points distinct from $\langle e_{14}\rangle$ are fixed. Any symp equation containing X_{14} then yields that also $\langle e_{14}\rangle$ is fixed. So the corresponding matrix is diagonal. Denote the (J, J)-entry by k_J, $J \in \{1, 2, \dots, 6, 1', 2', \dots, 6', 12, 13, \dots, 56\}$, and note that by assumption we may assume $k_1 = k_4 = k_{1'} = k_{4'} = k_{23} = k_{56} = k_{26} = k_{35} = k_{25} = k_{36} = 1$. Then the short mixed relations (1), (5) and (9) imply $k_2k_5 = k_2k_{46} = k_5k_{46} = 1$, yielding $k_2^2 = 1$ and so $k_2 = k_5 = k_{46} = 1$. Similarly all other k_J, except possibly k_{14}, are equal to 1. This again implies $k_{14} = 1$ and the assertion is proved. □

Remark 4.9. The parabolic intersection mentioned in Lemma 4.8 is called an *extended equator geometry* in [5].

Digression — Collinearity relations for $\mathscr{F}_{4,4}(\mathbb{K}, \mathbb{K})$. Since we now have a complete algebraic description of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$, the point set of $\mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$, and since, if $\mathbb{K}' \neq \mathbb{K}$, the lines of $\mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$ are precisely the lines of $\mathsf{PG}(V)$ fully contained in $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$, we can deduce an algebraic collinearity relation by expressing that, if $\langle v\rangle$ and $\langle w\rangle$, $v, w \in V$, are projective points that belong to $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$, then $\langle v\rangle$ and $\langle w\rangle$ are collinear in $\mathsf{F}_{4,4}(\mathbb{K}', \mathbb{K})$ if and only if, for every $k \in \mathbb{K}$, the point $\langle v + kw\rangle$ belongs to $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$. The result of this little and straightforward calculation is the following bilinearization of the symp equations and the mixed relations.

Theorem 4.10. *Two points with coordinates $(x_p)_{p\in X}$ and $(y_p)_{p\in X}$ contained in Δ' are collinear in Δ' if and only if the coordinates satisfy the following identities*:

(1) *For each point $p \in X$ of Γ, we have the identity*

$$\sum_{\{p,q,r\}\in\mathscr{L}} X_q Y_r + X_r Y_q = 0.$$

(2) *For ever special 6-coclique $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ of Γ (relative to $\{r_1, r_2, r_3\}$, that is, disjoint from this set), where p_i and q_i are collinear to r_i in Γ, we have the identity*

$$\sum_{i=1}^{3} X_{p_i} Y_{q_i} + X_{q_i} Y_{p_i} = 0.$$

(3) *For every pair of opposite special 6-cocliques $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ and $\{p'_1, p'_2, p'_3, q'_1, q'_2, q'_3\}$, with $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ as above and $p'_i, q'_i \in X$ so that $\{p_i, p'_i, r_i\}$ and $\{q_i, q'_i, r_i\}$ belong to $\mathscr{L}$, we have the identity*

$$\sum_{i=1}^{3} X_{p_i} Y_{p'_i} + X_{p'_i} Y_{p_i} + X_{q_i} Y_{q'_i} + X_{q'_i} Y_{q_i} = 0.$$

An adopted but rather simplified version of the arguments above can be given to complete the description of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$ with the collinearity relation exactly as given in the previous theorem. Hence Theorem 4.10 also holds for Δ! We conjecture that a similar description holds for the case where the characteristic of the field is not 2.

Note that this implies that two points of Δ' are Δ'-collinear if and only if they are Δ-collinear. This can also be deduced directly from Proposition 4.5 by noting that this proposition also holds for $\mathbb{K}' = \mathbb{K}$, where ξ is the image of $\langle e_1\rangle$ under the so-called symplectic polarity of $\mathscr{E}_{6,1}(\mathbb{K})$ defining Δ; see Lemma 4.19 of [5].

5. Involutions that produce generalized quadrangles

In this section, we provide a first application of the construction in the previous section. Briefly, we show that the involutions exhibited in [7] and that produce exceptional Moufang quadrangles of type F_4, and also quadrangles of mixed type, extend to the ambient projective space $\mathsf{PG}(25, \mathbb{K})$. As a corollary, the exceptional Moufang quadrangle of type F_4 related to the quadratic Galois extension $\mathbb{K}/\mathbb{F}$ of base fields, is shown to live in $\mathsf{PG}(25, \mathbb{F})$. Secondly, we show that the involution producing a quadrangle of mixed type is not unique, and that each nontrivial element of a whole group of exponent 2 fixes the given quadrangle and nothing more.

5A. ***The linear case.*** We start by defining an involution θ_0 of $\mathsf{PG}(V)$. Here, V is still our 27-dimensional vector space over the imperfect field $\mathbb{K}$ of characteristic 2, and $\mathbb{K}'$ is a subfield satisfying $\mathbb{K}^2 \leq \mathbb{K}' \leq \mathbb{K}$. We select arbitrary $\alpha \in \mathbb{K}' \setminus \mathbb{K}^2$ and $\beta \in \mathbb{K} \setminus \mathbb{K}'$. By definition, θ_0 depends on α and β and acts as follows on the coordinates $(x_p)_{p\in X}$ of an arbitrary point of $\mathsf{PG}(V)$:

$$
\begin{array}{lll}
x_1 \mapsto x_1, & x_2 \mapsto \alpha\beta^{-1}x_5, & x_3 \mapsto \beta^{-1}x_6, \\
x_4 \mapsto x_4, & x_5 \mapsto \alpha^{-1}\beta x_2, & x_6 \mapsto \beta x_3, \\
x_{1'} \mapsto x_{1'}, & x_{2'} \mapsto \alpha\beta^{-1}x_{5'}, & x_{3'} \mapsto \beta^{-1}x_{6'}, \\
x_{4'} \mapsto x_{4'}, & x_{5'} \mapsto \alpha^{-1}\beta x_{2'}, & x_{6'} \mapsto \beta x_{3'}, \\
x_{23} \mapsto \alpha^{-1}\beta^2 x_{56}, & x_{34} \mapsto \beta x_{46}, & x_{24} \mapsto \alpha^{-1}\beta x_{45}, \\
x_{56} \mapsto \alpha\beta^{-2}x_{23}, & x_{46} \mapsto \beta^{-1}x_{34}, & x_{45} \mapsto \alpha\beta^{-1}x_{24}, \\
x_{26} \mapsto \alpha^{-1}x_{35}, & x_{13} \mapsto \beta x_{16}, & x_{12} \mapsto \alpha^{-1}\beta x_{15}, \\
x_{35} \mapsto \alpha x_{26}, & x_{16} \mapsto \beta^{-1}x_{13}, & x_{15} \mapsto \alpha\beta^{-1}x_{12}, \\
x_{14} \mapsto x_{14}, & x_{25} \mapsto x_{25}, & x_{36} \mapsto x_{36}.
\end{array}
$$

Proposition 5.1. *The involution θ_0 is a collineation of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$. Also, the geometry consisting of the fixed points of θ_0 in $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$, where the lines are the fixed points contained in a common fixed symplecton, is a mixed quadrangle isomorphic to* $\mathsf{C}_{2,1}(\mathbb{K}'(\beta), \mathbb{K}^2(\alpha); \beta^{-1}\mathbb{K}^2(\alpha) + \mathbb{K}', \alpha\mathbb{K}'^2(\beta^2) + \mathbb{K}^2)$.

Proof. One easily calculates that all symp equations, the equation $X_{14} + X_{25} + X_{36} = 0$, and all mixed relations are satisfied by the image under θ_0 of a point of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$. The determination of the fixed points in $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$ requires a tedious calculation that is contained in [10]. We content ourselves with identifying the point rows and the line pencils of the corresponding generalized quadrangle (and then, in fact, the assertion also follows from [7] and Theorem 6.3). First the point rows. Clearly, the symplecton of type (1) is fixed and the involution induced in it by θ_0 is exactly the one considered in Section 3. The point rows follow. For a line pencil, we consider the two fixed points $\langle e_{1'}\rangle$ and $\langle e_4\rangle$. We identify each fixed symplecton ξ containing $\langle e_{1'}\rangle$ with its intersection with the fixed subspace $W := \langle e_1, e_{4'}, e_{23}, e_{25}, e_{26}, e_{35}, e_{36}, e_{56}\rangle$, that is, the line pencil can be identified with the fixed point set in W. We first determine the structure of $W \cap \mathscr{H}'$. The only nonvanishing symp equation yields $X_1X_{4'} + X_{23}X_{56} + X_{26}X_{35} = X_{25}^2$ (since $X_{25} + X_{36} = 0$). The nonvanishing long mixed relations are equivalent to the relations $X_1X_{4'} \in \mathbb{K}'$, $X_{23}X_{46} \in \mathbb{K}'$, $X_{26}X_{35} \in \mathbb{K}'$; the nonvanishing short mixed relations are equivalent to the relations $X_pX_q \in \mathbb{K}'$, for all pairs p, q in $\{1, 4', 23, 26, 35, 56\}$ except $\{1, 4'\}$, $\{23, 56\}$ and $\{26, 35\}$. Hence the union of all these conditions is simply that $W \cap H$ consists precisely of the points in $\langle e_1, e_{4'}, e_{23}, e_{26}, e_{56}, e_{35}, e_{25}+e_{36}\rangle$ satisfying $X_1X_{4'}+X_{23}X_{56}+X_{26}X_{35} = X_{25}^2$ and with all coordinates $x_1, x_{4'}, x_{23}, x_{26}, x_{35}$ and x_{56} belonging to $\mathbb{K}'$. We can project from the nucleus $\langle e_{25}+e_{36}\rangle$ onto $\langle e_1, e_{4'}, e_{23}, e_{26}, e_{56}, e_{35}\rangle$ and obtain a polar space isomorphic to $\mathsf{C}_{3,1}(\mathbb{K}', \mathbb{K}^2)$. The involution θ_0 induces the following involution in that space:

$$x_{1'} \mapsto x_{1'}, \quad x_{23} \mapsto \alpha^{-1}\beta^2 x_{56}, \quad x_{26} \mapsto \alpha^{-1}x_{35},$$
$$x_{4'} \mapsto x_{4'}, \quad x_{56} \mapsto \alpha\beta^{-2}x_{23}, \quad x_{35} \mapsto \alpha x_{26}.$$

Now if we substitute the coordinate tuple $(x_1, x_2, x_3, x_4, x_5, x_6)$ in the calculation of Section 3 with $(x_1, x_{23}, x_{26}, x_{4'}, x_{56}, x_{35})$ and (α, β) with (β^2, α^{-1}), then the result follows. □

Now let $\mathscr{F}_0 \subseteq \mathscr{H}'$ be the fixed point set of θ_0. it is clear that $\mathscr{F}_0$ is the intersection of $\mathscr{H}'$ and the 15-dimensional subspace of $\mathsf{PG}(V)$ given by the equations $X_{14} + X_{25} + X_{36} = 0$ and

$$X_2 = \alpha\beta^{-1}X_5,\; X_6 = \beta X_3,\; X_{56} = \alpha\beta^{-2}X_{23},\; X_{34} = \beta X_{46},\; X_{45} = \alpha\beta^{-1}X_{24},$$
$$X_{2'} = \alpha\beta^{-1}X_{5'},\; X_{6'} = \beta X_{3'},\; X_{35} = \alpha X_{26},\qquad X_{13} = \beta X_{16},\; X_{15} = \alpha\beta^{-1}X_{14}.$$

Here is the main result of this section.

Theorem 5.2. *The group of automorphisms of Δ' pointwise fixing $\mathscr{F}_0$ is a group of exponent 2 isomorphic to the additive group $\{(a, b) \in \mathbb{K}\times\mathbb{K} \mid a+\beta(a^2+\alpha b^2) \in \mathbb{K}'\}$, with standard addition. The value $(a, b) = (\beta^{-1}, 0)$ corresponds to θ_0.*

Proof. Let θ be an arbitrary collineation of Δ fixing $\mathscr{F}_0$ pointwise. In the host space of type (1), the points of $\mathscr{F}_0$ have generic coordinates $(x_1, \alpha\beta^{-1}x_5, \beta^{-1}x_6, x_4, x_5, x_6)$ (in natural ordering). By Proposition 3.1, the restriction of θ to $\langle e_1, e_2, \dots, e_6\rangle$ has the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+\beta a & \alpha\beta b & 0 & \alpha a & \alpha b \\ 0 & \beta b & 1+\beta c & 0 & \alpha b & c \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \beta^2\alpha^{-1}a & \beta^2 b & 0 & 1+\beta a & \beta b \\ 0 & \beta^2 b & \beta^2 c & 0 & \alpha\beta b & 1+\beta c \end{pmatrix}$$

with

$$\begin{cases} \beta(a^2+\alpha b^2)+a \in \mathbb{K}', \\ \beta(c^2+\alpha b^2)+c \in \mathbb{K}'. \end{cases}$$

We claim that the points $(x_{1'}, x_{2'}, \ldots, x_{6'}) \in \langle e_{1'}, e_{2'}, \ldots, e_{6'} \rangle$ and $(x_1, x_2, \ldots, x_6) \in \langle e_1, e_2, \ldots, e_6 \rangle$ are collinear in $\mathscr{E}_{6,1}(\mathbb{K})$ if and only if

$$(x_{1'}, x_{2'}, \ldots, x_{6'}) = \lambda(x_1, x_2, \ldots, x_6),$$

for some $\lambda \in \mathbb{K}^\times$. Indeed, expressing that the point with coordinates $(x_p)_{p\in X}$, with $x_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, 6\}$, $i < j$, belongs to $\mathscr{E}_{6,1}(\mathbb{K})$ implies by the ij-symp equation that $x_i x_{j'} = x_j x_{i'}$, for all distinct $i, j \in \{1, 2, \ldots, 6\}$. This easily implies the claim. It follows that the above matrix is also the matrix of θ restricted to $\langle e_{1'}, e_{2'}, \ldots, e_{6'} \rangle$.

Now, in a completely similar way, the restriction of θ to the host spaces of type (3) and (4), with the coordinates ordered as $(x_1, x_{45}, x_{46}, x_{1'}, x_{24}, x_{34})$ and $(x_{4'}, x_{15}, x_{16}, x_4, x_{12}, x_{13})$, has the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+\beta d & \alpha\beta e & 0 & \alpha d & \alpha e \\ 0 & \beta e & 1+\beta f & 0 & \alpha e & f \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \beta^2\alpha^{-1}d & \beta^2 e & 0 & 1+\beta d & \beta e \\ 0 & \beta^2 e & \beta^2 f & 0 & \alpha\beta e & 1+\beta f \end{pmatrix}$$

with

$$\begin{cases} \beta(d^2+\alpha e^2)+d \in \mathbb{K}', \\ \beta(f^2+\alpha e^2)+f \in \mathbb{K}'. \end{cases}$$

It follows that θ is an involution on the subspaces

$$\langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle, \qquad \langle e_{1'}, e_{2'}, e_{3'}, e_{4'}, e_{5'}, e_{6'} \rangle,$$
$$\langle e_1, e_{45}, e_{46}, e_{1'}, e_{24}, e_{34} \rangle, \quad \langle e_{4'}, e_{15}, e_{16}, e_4, e_{12}, e_{13} \rangle,$$

and hence on the subspace generated by these.

Note that, since $\langle e_1\rangle$ and $\langle e_{4'}\rangle$ are fixed, θ stabilizes the symp ξ_{14} defined by Q_{14}. Now a completely similar calculation as performed in the proof of Proposition 3.1 shows that the restriction of θ to $\langle \xi_{14}\rangle$, where coordinates are ordered as $(x_{25}, x_{36}, x_1, x_4, x_{1'}, x_{4'}, x_{23}, x_{26}, x_{35}, x_{56})$, has the following matrix M, where $\bar{0}_4$ is short hand notation for a subrow of four zeros, likewise $0|_4$ stands for a subcolumn of four zeros.

$$M := \begin{pmatrix} \epsilon+\alpha(r+q) & \epsilon'+\alpha(r+q) & \bar{0}_4 & \alpha g & \alpha h & h & \beta^2 g \\ \epsilon'+\alpha(r+q) & \epsilon+\alpha(r+q) & \bar{0}_4 & \alpha g & \alpha h & h & \beta^2 g \\ 0|_4 & 0|_4 & I_{4\times 4} & 0|_4 & 0|_4 & 0|_4 & 0|_4 \\ \beta^2 g & \beta^2 g & \bar{0}_4 & 1+\alpha r & \alpha\beta^2 p & \beta^2 p & \beta^2 r \\ h & h & \bar{0}_4 & \alpha p & 1+\alpha q & q & \beta^2 p \\ \alpha h & \alpha h & \bar{0}_4 & \alpha^2 p & \alpha^2 q & 1+\alpha q & \alpha\beta^2 p \\ \alpha g & \alpha g & \bar{0}_4 & \alpha^2\beta^{-2} r & \alpha^2 p & \alpha p & 1+\alpha r \end{pmatrix}$$

with $g, h \in \mathbb{K}$ and $r, p, q \in \mathbb{K}'$ and

$$\begin{cases} \alpha(r^2+\beta^2 p^2)+r = \beta^2 g^2, \\ \alpha(q^2+\beta^2 p^2)+q = h^2, \end{cases}$$

and with $\{\epsilon, \epsilon'\} = \{0, 1\}$. The latter choice for ϵ and ϵ' must be made to ensure that the map is type preserving. However, we will decide this later.

Independent of that is the fact that it now follows that θ is an involution on the subspace $\langle e_{25}, e_{36}, e_{23}, e_{26}, e_{35}, e_{56}, e_1, e_4, e_{1'}, e_{4'}\rangle$. Hence, by Lemma 4.8, θ^2 is the identity and so θ is globally an involution.

Now note that $V = \langle e_1, e_4, e_{1'}, e_{4'}\rangle \oplus \langle e_2, e_3, e_5, e_6\rangle \oplus \langle e_{2'}, e_{3'}, e_{5'}, e_{6'}\rangle \oplus \langle e_{24}, e_{34}, e_{45}, e_{46}\rangle \oplus \langle e_{12}, e_{13}, e_{15}, e_{16}\rangle \oplus \langle e_{25}, e_{36}, e_{23}, e_{26}, e_{35}, e_{56}\rangle \oplus \langle e_{14}\rangle$ and each of the direct factors is preserved under the involution θ. Hence the above matrices completely describe the involution θ, if we add $x_{14} \mapsto x_{14}$.

We now derive some relations between the parameters $a, b, c, d, e, f, p, q, r$. The short mixed relation (13) $X_1X_{56} + X_3X_{45} + X_2X_{46} \in \mathbb{K}'$ must hold for the image. We deduce $b = e$, $bd = ae$ and $bf = ce$ by considering the coefficients of X_2X_{45}, X_3X_{46}, X_5X_{24} and X_6X_{34} in the expression of the image.

From the mixed relation (7) $X_1X_{23} + X_{24}X_6 + X_{34}X_5 \in \mathbb{K}'$ and the coefficients of the corresponding monomials in the expression of the image, we deduce similarly $\alpha^2\beta^{-2}r = \alpha dc + \alpha^2 eb = \alpha^2 be + \alpha af$, hence $dc = af$.

From the mixed relation (9) $X_1X_{26} + X_{24}X_3 + X_5X_{46} \in \mathbb{K}'$ and the coefficients of the corresponding monomials, we deduce $\alpha^2 p = \alpha d(1+\beta c) + \alpha^2\beta be = \alpha^2\beta be + \alpha a(1+\beta f)$, which implies $a = d$. Note that a and b cannot be zero together, hence $c = f$.

From the mixed relation (11) $X_1X_{35}+X_{45}X_6+X_2X_{34}\in\mathbb{K}'$ and the coefficients of the corresponding monomials, we deduce $\alpha p=c(1+\beta a)+\alpha\beta b^2=\alpha\beta b^2+c(1+\beta a)$, hence $a(1+\beta c)=c(1+\beta a)$, so $a=c$.

We have

$$\begin{cases}\alpha r=\beta^2(a^2+\alpha b^2),\\ \alpha p=a+\beta(a^2+\alpha b^2).\end{cases}$$

Plugging this into the expression for g^2 one obtains $\alpha^{-1}(\beta^4a^4+\beta^4\alpha^2b^4+\beta^2a^2+\beta^4a^4+\beta^4\alpha^2b^4+\beta^2a^2+\beta^2\alpha b^2)=\beta^2g^2$. Hence $b=g$

Analogously, one shows that $h=\beta b$ and $q=r$. It can now be easily checked that $\epsilon=1$ leads to a type preserving map, whereas $\epsilon'=1$ leads to a type interchanging one.

The matrix M becomes, setting $u:=a^2+\alpha b^2$,

$$\begin{pmatrix}
1 & 0 & \bar{0}_4 & \alpha b & \alpha\beta b & \beta b & \beta^2 b\\
0 & 1 & \bar{0}_4 & \alpha b & \alpha\beta b & \beta b & \beta^2 b\\
0|_4 & 0|_4 & I_{4\times4} & 0|_4 & 0|_4 & 0|_4 & 0|_4\\
\beta^2 b & \beta^2 b & \bar{0}_4 & 1+\beta^2u & \beta^2(a+\beta u) & \alpha^{-1}\beta^2(a+\beta u) & \alpha^{-1}\beta^4 u\\
\beta b & \beta b & \bar{0}_4 & a+\beta u & 1+\beta^2 u & \alpha^{-1}\beta^2 u & \alpha^{-1}\beta^2(a+\beta u)\\
\alpha\beta b & \alpha\beta b & \bar{0}_4 & \alpha(a+\beta u) & \alpha\beta^2 u & 1+\beta^2 u & \beta^2(a+\beta u)\\
\alpha b & \alpha b & \bar{0}_4 & \alpha u & \alpha(a+\beta u) & a+\beta u & 1+\beta^2 u
\end{pmatrix}$$

with $a+\beta u\in\mathbb{K}'$. Conversely, it is easy to check that the map defined by the above matrices, with $c=d=f=a$, preserves all symp equations, and also all mixed relations. Hence it defines a collineation of Δ' fixing each point of $\mathscr{F}_0$ (and nothing more).

Clearly, $(a,b)=(\beta^{-1},0)$ yields the original involution θ_0. □

Standard group theoretic arguments show that the said group is equal to $C_G(\theta_0)$, the centralizer in G of θ_0 (indeed, the automorphism group of the fixed point quadrangle induced by G acts transitively on the fixed point set by Theorem 24.31 of [8]).

At first sight, it is not even clear that $a,b\in\mathbb{K}$ satisfying the constraint $a+\beta(a^2+\alpha b^2)\in\mathbb{K}'$ exist, besides the trivial case $a=b=0$ and the case $(a,b)=(\beta^{-1},0)$. However, an infinite number of appropriate a,b is given by

$$a=\frac{\alpha^{2t+1}\beta^{2s+1}}{1+\alpha^{2t+1}\beta^{2s+2}},\quad b=\frac{\alpha^t\beta^s}{1+\alpha^{2t+1}\beta^{2s+2}},$$

for arbitrary integers s,t. For all these values, $a+\beta(a^2+\alpha b^2)=0$. An example where this is equal to a nonzero member of $\mathbb{K}'$ is given in the next paragraph.

The group $C_G(\theta_0)$ can be seen as having cardinality at least $|\mathbb{K}\times\mathbb{K}'|$ by assigning, for every $k\in\mathbb{K}$ and $k'\in\mathbb{K}'$, the pair (k,k') to the pair (a,b) with

$$a=\beta^{-1}(1+\alpha k^2+\beta k')^{-1},\quad b=k\beta^{-1}(1+\alpha k^2+\beta k')^{-1}.$$

Here, $a+\beta(a^2+\alpha b^2)=k'(1+\alpha k^2+\beta k')^{-2}$.

5B. ***The semilinear case.*** We now present an explicit description of the involution related to an arbitrary Moufang quadrangle of mixed type F_4. As above, we will not show in detail that the stated involution is the right one — again a detailed proof can be found in the thesis [10] of Roth — it will however be clear from the data that we obtain a generic Moufang quadrangle of type F_4.

These data are the following. Let $\mathbb{K}$, $\mathbb{K}'$, α and β be as above. Let $\mathbb{E}/\mathbb{K}$ be a separable quadratic extension with Galois involution $x\mapsto\bar{x}$. Let $\mathbb{E}'=\mathbb{E}^2\mathbb{K}'$ (the composite field). Now suppose that the forms

$$\begin{aligned}\mathbb{E}\times\mathbb{E}\times\mathbb{K}'\to\mathbb{K}:(x,y,k)&\mapsto\beta^{-1}(x\bar{x}+\alpha y\bar{y})+k,\\ \mathbb{E}'\times\mathbb{E}'\times\mathbb{K}\to\mathbb{K}':(x,y,k)&\mapsto\alpha(x\bar{x}+\beta^2 y\bar{y})+k^2\end{aligned}\qquad(*)$$

are anisotropic, that is, they only become 0 for the zero-entry $(0,0,0)$.

It is shown in [7, Lemma 6.3] that the two conditions in $(*)$ are equivalent. However, it is convenient to have them both explicitly stated. We note another consequence of these data (not needed in [7] because of the slightly different form of the involution).

Lemma 5.3. *If $x\bar{x}\in\mathbb{E}'$, $x\in\mathbb{E}$, then $x=kx'$, with $k\in\mathbb{K}$ and $x'\in\mathbb{E}'$.*

Proof. Let $\mathbb{E}=\mathbb{K}(\delta)$, then $\mathbb{E}=\mathbb{K}(\delta^2)$ as $\mathbb{E}/\mathbb{K}$ is separable and char $\mathbb{K}=2$. Moreover, we can choose δ so that $\delta+\delta^2+t=0$, for some $t\in\mathbb{K}$. Then $\mathbb{E}^2=\mathbb{K}^2(\delta^2)$ and $\mathbb{E}'=\mathbb{K}'(\delta^2)$. Write $x=x_0+x_1\delta^2$. If $x_0=0$ or $x_1=0$, then the assertion is obvious, so assume $x_0\neq 0\neq x_1$. Now note that $\delta^4=\delta^2+t^2$ and one calculates that $x\bar{x}=x_0^2+x_0x_1+x_1^2t^2$. If $x\bar{x}\in\mathbb{E}'$, then, since $\mathbb{K}^2\le\mathbb{K}'\le\mathbb{E}'$, we have $x_0x_1\in\mathbb{E}'$ and hence in $\mathbb{K}'$ (since one easily sees $\mathbb{E}'\cap\mathbb{K}=\mathbb{K}'$). Consequently

$$x=x_0+x_1\delta^2=x_0^{-1}(x_0^2+x_0x_1\delta^2),$$

which shows the assertion. □

Extend the vector space V to V^* by tensoring with $\mathbb{E}$ and consider $\mathscr{F}_{4,4}(\mathbb{E}',\mathbb{E})$. Consider the following involution θ^* of $\mathsf{PG}(V^*)$:

$$
\begin{array}{lll}
x_1 \mapsto \bar{x}_1, & x_2 \mapsto \alpha\beta^{-1}\bar{x}_5, & x_3 \mapsto \beta^{-1}\bar{x}_6,\\
x_4 \mapsto \bar{x}_4, & x_5 \mapsto \alpha^{-1}\beta\bar{x}_2, & x_6 \mapsto \beta\bar{x}_3,\\
x_{1'} \mapsto \bar{x}_{1'}, & x_{2'} \mapsto \alpha\beta^{-1}\bar{x}_{5'}, & x_{3'} \mapsto \beta^{-1}\bar{x}_{6'},\\
x_{4'} \mapsto \bar{x}_{4'}, & x_{5'} \mapsto \alpha^{-1}\beta\bar{x}_{2'}, & x_{6'} \mapsto \beta\bar{x}_{3'},\\
x_{23} \mapsto \alpha^{-1}\beta^2\bar{x}_{56}, & x_{34} \mapsto \beta\bar{x}_{46}, & x_{24} \mapsto \alpha^{-1}\beta\bar{x}_{45},\\
x_{56} \mapsto \alpha\beta^{-2}\bar{x}_{23}, & x_{46} \mapsto \beta^{-1}\bar{x}_{34}, & x_{45} \mapsto \alpha\beta^{-1}\bar{x}_{24},\\
x_{26} \mapsto \alpha^{-1}\bar{x}_{35}, & x_{13} \mapsto \beta\bar{x}_{16}, & x_{12} \mapsto \alpha^{-1}\beta\bar{x}_{15},\\
x_{35} \mapsto \alpha\bar{x}_{26}, & x_{16} \mapsto \beta^{-1}\bar{x}_{13}, & x_{15} \mapsto \alpha\beta^{-1}\bar{x}_{12},\\
x_{14} \mapsto \bar{x}_{14}, & x_{25} \mapsto \bar{x}_{25}, & x_{36} \mapsto \bar{x}_{36}.
\end{array}
$$

Again, it is easy to verify that θ^* is a collineation of $\mathscr{F}_{4,4}(\mathbb{E}',\mathbb{E})$. For a generic fixed point $\langle v\rangle$ for θ^* we may assume that v itself is fixed; otherwise we consider $v+v^{\theta^*}$. So the set of fixed points in $\mathsf{PG}(V^*)$ for θ^* consists of the points $(x_p)_{p\in X}$ satisfying $x_1, x_4, x_{1'}, x_{4'}, x_{14}, x_{25}, x_{36} \in \mathbb{K}$ and

$$
\begin{array}{lllll}
x_2 = \alpha\beta^{-1}\bar{x}_5, & x_6 = \beta\bar{x}_3, & x_{56} = \alpha\beta^{-2}\bar{x}_{23}, & x_{34} = \beta\bar{x}_{46}, & x_{45} = \alpha\beta^{-1}\bar{x}_{24},\\
x_{2'} = \alpha\beta^{-1}\bar{x}_{5'}, & x_{6'} = \beta\bar{x}_{3'}, & x_{35} = \alpha\bar{x}_{26}, & x_{13} = \beta\bar{x}_{16}, & x_{15} = \alpha\beta^{-1}\bar{x}_{14}.
\end{array}
$$

We now show that two fixed points that belong to $\mathscr{F}_{4,4}(\mathbb{E}',\mathbb{E})$ are never collinear in $\mathsf{F}_{4,4}(\mathbb{E}',\mathbb{E})$. Indeed, if $a=(x_p)_{p\in X}$ and $b=(y_p)_{p\in X}$ were collinear fixed points, then Theorem 4.10 would imply via the bilinearization of the short mixed relation (1)

$$x_1y_4 + x_4y_1 + \alpha\beta^{-1}\bar{x}_5y_5 + \alpha\beta^{-1}x_5\bar{y}_5 + \beta x_3\bar{y}_3 + \beta\bar{x}_3y_3 = 0. \tag{4}$$

Since we assume that a and b belong to $\mathscr{F}_{4,4}(\mathbb{E}',\mathbb{E})$, we have short mixed relation (1)

$$x_1x_4 + \alpha\beta^{-1}x_5\bar{x}_5 + \beta\bar{x}_3x_3 \in \mathbb{E}' \quad\text{and}\quad y_1y_4 + \alpha\beta^{-1}y_5\bar{y}_5 + \beta\bar{y}_3y_3 \in \mathbb{E}'.$$

Suppose for a moment that $x_1=0$. Then by $(*)$, we deduce $x_3=x_5=0$, and so (4) becomes $x_4y_1=0$. Hence $y_1=0$ or $x_4=0$. Suppose $y_1=0$. Then likewise $y_3=y_5=0$ and y_4 is arbitrary. Similar conclusions hold if we assume $y_1=0$ instead of $x_1=0$.

Suppose now that $x_1\neq 0\neq y_1$. Note both belong to $\mathbb{K}$. Then multiplying relation (4) by x_1y_1 and substituting x_1x_4 by $k'+\alpha\beta^{-1}x_5\bar{x}_5+\beta\bar{x}_3x_3$, $k'\in\mathbb{K}'$ and y_1y_4 by $\ell'+\alpha\beta^{-1}y_5\bar{y}_5+\beta\bar{y}_3y_3$, $\ell'\in\mathbb{K}'$, we obtain, after combining terms,

$$\alpha\beta^{-1}(y_1x_5+x_1y_5)(\overline{y_1x_5+x_1y_5}) + \beta(\overline{y_1x_3+x_1y_3})(y_1x_3+x_1y_3) = k'+\ell' \in \mathbb{K}',$$

implying by (∗) that the tuples $(x_1, x_2, \ldots, x_6)$ and $(y_1, y_2, \ldots, y_6)$ are proportional with factor in $\mathbb{K}$.

We conclude that either the tuples $(x_1, x_2, \ldots, x_6)$ and $(y_1, y_2, \ldots, y_6)$ are proportional with factor in $\mathbb{K}$, or one of them is the 0-tuple. Set

$$P = \{1, 2, \ldots, 6, 1', 2', \ldots, 6', 12, 13, 15, 16, 24, 34, 45, 46\}.$$

Short mixed relations (2), (3) and (4) imply that, if $x_1 = x_2 = \cdots = x_6 = 0$, then $x_p = 0$, for all $p \in P$. Playing the same game as above, but now with the respective short mixed relations (2), (3) and (4), we can conclude that either the tuples $(x_p)_{p\in P}$ and $(y_p)_{p\in P}$ are proportional with factor in $\mathbb{K}$, or one of them is the 0-tuple. Suppose now that $(x_p)_{p\in P}$ is the 0-tuple. The 25-symp equation and the 36-symp equation imply that either $x_{14} = 0$, or $x_{25} = x_{36} = 0$. In any case $x_{25} = x_{36}$ and so $x_{14} = 0$. If $x_{23} = x_{26} = x_{35} = x_{56} = 0$, then the 14-symp equation implies $x_{25} = 0$ and then $(x_p)_{p\in X}$ is the 0-tuple, a contradiction. Without loss we may assume that $x_{35} \neq 0$. As in the proof of Proposition 5.1, the mixed relations imply that the 4-tuple $(x_{23}, x_{26}, x_{35}, x_{56})$ is proportional to a 4-tuple in $\mathbb{E}'$, say $(x_{23}, x_{26}, x_{35}, x_{56}) = \lambda(x'_{23}, x'_{26}, x'_{35}, x'_{56})$, with $x'_{23}, x'_{26}, x'_{35}, x'_{56} \in \mathbb{E}'$, $\lambda \in \mathbb{E}$. Since a is fixed by θ^*, $\lambda x'_{35} = \alpha\bar{\lambda}\bar{x}'_{26}$. Consequently, since $x_{35} \neq 0$,

$$\lambda\bar{\lambda} = \alpha^{-1}\lambda^2 x'_{34}\bar{x}'^{-1}_{26} \in \mathbb{E}'.$$

Lemma 5.3 implies that we may assume $\lambda \in \mathbb{K}$. Multiplying the coordinate tuple of a with λ^{-1}, and noting that we assumed $x_{25} \in \mathbb{K}$, we may additionally assume that x_{23}, x_{26}, x_{35} and x_{56} belong to $\mathbb{E}'$. Then the 14-symp equation yields

$$\alpha\beta^{-2}x_{23}\bar{x}_{23} + \alpha x_{26}\bar{x}_{26} + x_{25}^2 = 0.$$

Hence, by (∗), $x_p = 0$, for all $p \in X$, a contradiction again. So we may assume that $(x_p)_{p\in P}$ and $(y_p)_{p\in P}$ are proportional with factor in $\mathbb{K}$, say $(x_p)_{p\in P} = k(y_p)_{p\in P}$, $k \in \mathbb{K}$. Since a and b are collinear fixed points, the point $\langle c\rangle$, with $c = (x_p - ky_p)_{p\in X}$ is also a fixed point. Since $x_p - ky_p = 0$, for all $p \in P$, the foregoing implies $x_p - ky_p = 0$, for all $p \in X$ and consequently $a = b$.

Hence it now follows that also no plane is fixed by θ^*, and so θ^* only fixes points and symplecta. Since the symplecta of type (1), (2), (3) and (4) are fixed, the fixed point structure is a thick generalized quadrangle (where the lines of this quadrangle correspond to the fixed symps). Hence θ^* is a homogeneous involution in the sense of [7] (see 2.1 of that paper). Now, since θ^* is clearly semilinear (it involves the field automorphism $\mathbb{K} \to \mathbb{K} : x \mapsto \bar{x}$) the main result stated in Section 7 of [7] implies that the fixed point structure of θ^* is a Moufang quadrangle of type F_4. It follows that the latter can be seen as the intersection of $\mathscr{F}_{4,4}(\mathbb{E}', \mathbb{E})$ with a Baer subspace of $\mathsf{PG}(V^*)$ isomorphic to $\mathsf{PG}(V)$.

6. A classification

The above involutions have as fixed point sets Moufang quadrangles whose points are points of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$, whose lines are ovoids in symplecta, and whose line pencils with vertex p correspond to ovoids in the residue of p. One could wonder whether there are other collineations of $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$ with the same property. We will answer negatively to this question, at least in the linear case, that is, not only the case where the collineation comes from a linear map in V, but also the ovoids in the symplecta arise from intersections of the symplecta with subspaces of the ambient projective subspaces. The reason to restrict to this linear case is purely technical; the proof in the nonlinear case requires further technical results that would make the paper needlessly longer. Also, we are motivated by the following remark.

Remark 6.1. The interest in linear collineations (in the above sense) with the above properties stems from the fact that, besides being related to "mixed Galois descent", such collineations are also characterized by the property that their displacement spectrum avoids the extreme distances for chambers; in other words, no chamber is fixed and no chamber is mapped onto an opposite. This equivalence shall be proved in another paper where we hopefully classify such collineations for all metasymplectic spaces. Here we content ourselves with carrying out the classification for $\mathscr{F}_{4,4}(\mathbb{K}', \mathbb{K})$.

Remark 6.2. Before stating and proving the main theorem of this section, we note that the short and long root elations defined in Section 4 are actually a complete set of root elations for the standard apartment with point set $\{\langle e_p\rangle \mid p \in X \setminus \{r_1, r_2, r_3\}\}$. Then Theorem 7.10 of [1] shows that the group G (see Section 4) acts strongly transitively on Δ', which implies for instance that it acts transitively on pairs of opposite symps.

Theorem 6.3. *Let θ be a collineation of Δ' whose fixed points and fixed symplecta form a generalized quadrangle such that the set of fixed points in a fixed symplecton ξ is an ovoid of ξ obtained by intersecting the symplecton with a subspace of* $\mathsf{PG}(V)$, *and dually the set of fixed symplecta through a fixed point p is an ovoid in the polar space corresponding to the point residual of p. Then θ is conjugate to the involution θ_0 above for some appropriate $\alpha \in \mathbb{K}' \setminus \mathbb{K}^2$ and $\beta \in \mathbb{K} \setminus \mathbb{K}'$.*

Proof. First assume that θ is an involution. By Remark 6.2, we may assume that the symplecta corresponding to the short mixed relations (1) and (2) are fixed, and we can choose coordinates so that $\langle e_1\rangle$, $\langle e_4\rangle$, $\langle e_{1'}\rangle$ and $\langle e_{4'}\rangle$ are fixed, and $\langle e_2\rangle$ is interchanged with $\langle e_5\rangle$, $\langle x_3\rangle$ with $\langle e_6\rangle$, and hence $\langle e_{2'}\rangle$ with $\langle e_{5'}\rangle$ and $\langle e_{3'}\rangle$ with $\langle e_{6'}\rangle$. Now the symps (21) and (22) are interchanged, and since (3) is fixed, it follows that $\langle e_{24}\rangle$ and $\langle e_{45}\rangle$ are interchanged. Likewise, θ interchanges $\langle e_{12}\rangle$ and $\langle e_{15}\rangle$, also $\langle e_{34}\rangle$ and $\langle e_{46}\rangle$, and $\langle e_{13}\rangle$ and $\langle e_{16}\rangle$. The same technique implies that $\langle e_{23}\rangle$ and

$\langle e_{56}\rangle$ are interchanged as well as $\langle e_{26}\rangle$ and $\langle e_{35}\rangle$. It also follows that $\langle e_{14}\rangle$, $\langle e_{25}\rangle$ and $\langle e_{36}\rangle$ are fixed.

Since θ fixes $\langle e_1\rangle$ and $\langle e_4\rangle$, and a coordinate description of θ is determined up to a scalar factor, we may assume that θ fixes the coordinates X_1 and X_4 (note that θ^2 is the identity, and so if θ fixes X_1, it fixes X_4). Also, since $\langle e_3\rangle$ and $\langle e_5\rangle$ are interchanged, there is a constant $\beta\in\mathbb{K}$ such that θ maps the coordinate X_6 to βX_3, and hence it maps the coordinate X_3 to $\beta^{-1}X_6$ (again taking into account that θ^2 is the identity). Likewise there exists $\gamma\in\mathbb{K}$ such that θ maps X_2 to γX_5 and X_5 to $\gamma^{-1}X_2$. Setting $\alpha:=\beta\gamma$, we can hence assume that θ acts on the coordinates in $\langle e_1,e_2,\ldots,e_6\rangle$ as

$$X_1\mapsto X_1,\quad X_2\mapsto \alpha\beta^{-1}X_5,\quad X_3\mapsto\beta^{-1}X_6,$$
$$X_4\mapsto X_4,\quad X_5\mapsto \alpha^{-1}\beta X_2,\quad X_6\mapsto\beta X_3,$$

and consequently on $\langle e_{1'},e_{2'},\ldots,e_{6'}\rangle$ as (since $\langle e_i\mapsto\langle e_{i'}\rangle$ induces an isomorphism defined by collinearity in $\mathscr{E}_{6,1}(\mathbb{K})$)

$$X_{1'}\mapsto X_{1'},\quad X_{2'}\mapsto \alpha\beta^{-1}X_{5'},\quad X_{3'}\mapsto\beta^{-1}X_{6'},$$
$$X_{4'}\mapsto X_{4'},\quad X_{5'}\mapsto \alpha^{-1}\beta X_{2'},\quad X_{6'}\mapsto\beta X_{3'}.$$

Note that, in order for the fixed point sets in these spaces be ovoids of the symplecta, we need $\alpha\beta$ and β to belong to $\mathbb{K}\setminus\mathbb{K}'$. Now define $\lambda\in\mathbb{K}$ such that the coordinate X_{35} is changed to λX_{26}. Then the short mixed relation (11) implies $X_{45}\mapsto\lambda\beta^{-1}X_{24}$ and $X_{34}\mapsto\lambda\alpha^{-1}\beta X_{46}$. Then the short mixed relation (15) implies $X_{23}\mapsto\lambda\alpha^{-2}\beta^2X_{56}$. Now the symp equation $X_1X_{6'}+X_{1'}X_6+X_{23}X_{45}+X_{24}X_{35}+X_{25}X_{34}=0$ implies $\beta=(\lambda\alpha^{-2}\beta^2)(\lambda\beta^{-1})$; consequently $\lambda^2=\alpha^2$, or $\lambda=\alpha$. Similarly one now calculates all other coordinate images and exactly obtains the description of θ_0. Note that the short mixed relation (11) implies $\alpha\in\mathbb{K}'$. Now suppose for a contradiction that $\alpha\in\mathbb{K}^2$, say $\alpha=\gamma^2$. We argue inside the $\mathscr{E}_{6,1}(\mathbb{K})$-symp ξ_{14}. The mixed relations imply that the points of $\mathscr{F}_{4,4}(\mathbb{K}',\mathbb{K})$ in ξ_{14} are given by coordinates $(x_{25},x_{36};x_1,x_4,x_{1'},x_{4'};x_{23},x_{26},x_{35},x_{56})$, which may all be assumed to belong to $\mathbb{K}'$ and satisfy

$$X_{25}^2=X_{36}^2=X_1X_{4'}+X_{1'}X_4+X_{23}X_{56}+X_{26}X_{35}.$$

Then the points $e_{1'}=(0,0;0,0,1,0;0,0,0,0)$ and $e_\gamma:=(\gamma,\gamma;0,0,0,0;0,\gamma^2,1,0)$ are both symplectic to $e_1=(0,0;1,0,0,0;0,0,0,0)$ and these three points are fixed. But $\langle e_\gamma\rangle$ and $e_{1'}$ are symplectic, which implies that the fixed symplecta of $\mathscr{F}_{4,4}(\mathbb{K}',\mathbb{K})$ determined by $\langle e_1\rangle$ an $\langle e_{1'}\rangle$, and $\langle e_1\rangle$ and $\langle e_\gamma\rangle$ share a plane, a contradiction to our assumptions. Hence $\alpha\in\mathbb{K}'\setminus\mathbb{K}^2$ as required.

So we may from now on assume that θ is not an involution. We must show nonexistence. We can provide a synthetic argument using the notion of extended equator geometry and its properties proved in [5]; see Remark 4.9. Consider two opposite fixed points p, q (one can think of $\langle e_1 \rangle$ and $\langle e_{4'} \rangle$). The associated extended equator geometry $\widehat{E} := \widehat{E}(p, q)$ is also fixed (one can think of the intersection with the symp ξ_{14}). The argument in the previous paragraph implies that θ does not fix any plane in $\widehat{E}$, whereas the assumption that the fixed points in a symp form a subspace of the ambient projective space yields fixed lines of $\widehat{E}$. It also implies that the fixed point set in $\widehat{E}$, as a polar space, is a subspace S. The properties of S just mentioned imply that S has codimension 2 in $\Sigma := \langle \widehat{E} \rangle$ and contains the nucleus $\langle e_{25} + e_{36} \rangle$ of the corresponding parabolic quadric. Hence S can be thought of as the perp of a line $L \subseteq \Sigma$ in the symplectic representation of the parabolic quadric $\widehat{E}$. We now argue over the quadratic closure of $\mathbb{K}$, say $\mathbb{L}$. Then L is either an isotropic line, or not. In the first case, it is easily checked that θ is involutive on $\widehat{E}$, implying it is an involution by Lemma 4.8. Hence L is not isotropic. Note that no point of L is fixed, as otherwise S has codimension at most 1 in $\langle \widehat{E} \rangle$. Select two points $a, b \in L$ arbitrarily. Then θ pointwise fixes exactly $a^\perp \cap b^\perp$. Hence θ does not fix any singular 3-space of $\widehat{E}$ containing a fixed plane. It follows that θ does not fix any singular 3-space as, by a dimension argument, each such intersects $a^\perp \cap b^\perp$ in at least a line, and considering the projection of a fixed point collinear to that line onto the fixed 3-space, one obtains a fixed 3-space through a fixed plane, a contradiction. It now follows from Lemma 5.37 of [5] that no point of $\mathscr{F}_{4,4}(\mathbb{L}, \mathbb{L})$ collinear to some point of $\widehat{E}$ is fixed by θ. But then Corollary 5.38 of [5] implies that each point of $\mathscr{F}_{4,4}(\mathbb{L}, \mathbb{L})$ is special to some fixed point in $\widehat{E}$. Hence θ fixes no point outside $\widehat{E}$. This now clearly contradicts the assumption that θ pointwise fixes ovoids in fixed symps. □

Remark 6.4. A further weakening of the hypothesis could consist in only assuming a set of points with the following properties:

- The points and the symplecta containing at least two points of the set form a generalized quadrangle.
- Every symp intersecting the set in at least two points intersects it in an ovoid of the symp.
- Dually, the symps through a point of the set containing at least one further point of the set, form an ovoid in he polar space defined by the residue of the point.

By a result of Struyve [13], the generalized quadrangle is Moufang, and then, using the groups of perspectivities, one obtains ovoids with high transitivity properties (in fact, Moufang sets). However, ovoids like that, only pointwise fixed by the identity, in (different) polar spaces are known to exist. This makes this question harder, but more interesting, though. We do not have a conjecture, although it is tempting to believe that the only such sets are the Moufang quadrangles fixed by an involution, as in [7] or the present paper.

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