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Notes on Cooperstein Ovoids in Finite Geometries of Type $\mathsf{E}_{6,1}$

Hendrik Van Maldeghem

Department of Mathematics, Computer Science and Statistics, Ghent University, Krijgslaan 281, S9, 9000 Gent, Belgium; hendrik.vanmaldeghem@ugent.be

Abstract: A Cooperstein ovoid is a set of $q^8 + q^4 + 1$ pairwise non-collinear points in the Lie incidence geometry $\mathsf{E}_{6,1}(q)$. They were introduced by Cooperstein twenty-six years ago, motivated by the fact that possible non-existence of them would imply non-existence of ovoids in hyperbolic quadrics of rank 5. Since then, no progress has been made on their existence question. We prove that Cooperstein ovoids do not exist under some natural additional conditions. In particular, Cooperstein ovoids intersecting every symplecton of $\mathsf{E}_{6,1}(q)$ do not exist, Cooperstein ovoids which are the fixed points of a collineation do not exist, and Cooperstein ovoids which are the absolute points of a polarity of $\mathsf{E}_{6,1}(q)$ do not exist.

Keywords: blocking set; exceptional geometry; maximal cocliques; ovoids

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1. Introduction

Special substructures of finite geometries have proved their importance in many instances. They are used in graph theory (for instance to construct strongly regular graphs), coding theory (to construct codes that perform well), cryptography, in finite geometry itself (to construct other geometries, in proofs, in classification results), etc. A prominent substructure is an ovoid of a polar space. There has been a lot of work put into existence and non-existence results for these structures in the finite case, but we are still far from a complete answer. Many examples of interesting substructures arise as the set of fixed points of a collineation of the ambient geometry. For instance, with the notation introduced below, the (so-called) classical ovoids of the hyperbolic quadrics $\mathsf{Q}^+(3,q)$, $\mathsf{Q}^+(5,q)$, and $\mathsf{Q}^+(7,q)$ (the latter for q a power of 3) arise as fixpoint structure of a collineation. The next case for hyperbolic quadrics, $\mathsf{Q}^+(9,q)$, is still wide open for q a power of a prime distinct from 2 or 3 (for the latter, non-existence follows from a result of Blokhuis and Moorhouse [1], see [2], Proposition 2.6.17). In order to make some progress on the (remaining) existence question, Cooperstein [3] proved a connection between ovoids of $\mathsf{Q}^+(9,q)$ and maximal cocliques of size $q^8 + q^4 + 1$ in the Lie incidence geometry $\mathsf{E}_{6,1}(q)$, which we will refer to as *Cooperstein ovoids*, hoping that the non-existence of the latter would be easier to prove. To date, no one has been able to further explore this connection. The main goal of the present paper is to prove that Cooperstein ovoids cannot arise as fixpoint structure of a collineation of $\mathsf{E}_{6,1}(q)$. We also show the non-existence of Cooperstein ovoids under a natural geometric condition. In fact, the latter result follows from a more general observation connecting cocliques of $\mathsf{E}_{6,1}(q)$ with blocking sets of $\mathsf{E}_{6,1}(q)$ with respect to symplecta. More precisely, and with the notation introduced in Section 2, we prove the following two results.

Main Result 1. *Each coclique of points of $\mathsf{E}_{6,1}(q)$ is disjoint from at least one symplecton. In particular, there does not exist a Cooperstein ovoid intersecting each symplecton in at least one point.*

Main Result 2. *No Cooperstein ovoid arises as the set of fixed points of a collineation of $\mathsf{E}_{6,1}(q)$.*

Some ovoids, like the so-called Ree-Tits ovoids in $\mathsf{Q}^+(7,q)$, arise from polarities of an appropriate geometry (here, a generalized hexagon embedded in the quadric—this generalized hexagon itself arises from a triality of the oriflamme geometry defined by $\mathsf{Q}^+(7,q)$). We will also prove that a search along these lines for Cooperstein ovoids leads nowhere.

Main Result 3. *No Cooperstein ovoid arises as the set of absolute points of a polarity of* $\mathsf{E}_{6,1}(q)$.

Note that Main Result 1 is not true in the infinite case. Indeed, the naturally embedded quaternion and octonion Veroneseans (see [4]) are counterexamples: they are cocliques intersecting every symplecton non-trivially (see [4], Lemma 4.3). Moreover, they form the fixed point set of a collineation, which morally contadicts Main Result 2 (although in the infinite case one cannot speak of Cooperstein ovoids, these examples in the infinite case are as close as one can get, see Remark 3 below).

In the next section, we introduce the main objects of this paper and set notation. In the rest of the paper, we prove our main results.

2. Notation

Our main players are the so-called (finite) minuscule geometries of type E_6, denoted by $\mathsf{E}_{6,1}(q)$ when the underlying field is the finite field $\mathbb{F}_q$ of order q. We introduce these as a class of point-line geometries satisfying certain axioms, and then we list some well known properties that should sharpen the intuition for such geometries. Let us note that these were called *Hjelmslev–Moufang planes* by Springer and Veldkamp [5], who also collected a lot of properties of these geometries, which were first defined by Tits in [6].

It is convenient to formulate the properties and axioms in the language of parapolar spaces; these are geometries designed to be related to spherical buildings (and our main players are related to buildings of type E_6 as the notation already suggested). However, we will not need any theory of (spherical Tits) buildings. We begin with some basic definitions.

Throughout, we will work with incidence structures called *partial linear spaces*. In this subsection, we introduce the general definitions we will need.

2.1. Point-Line Geometries

Definition 1. *A point-line geometry is a pair $\Delta = (\mathcal{P}, \mathcal{L})$ with $\mathcal{P}$ a set and $\mathcal{L}$ a set of subsets of $\mathcal{P}$. The elements of $\mathcal{P}$ are called* points *and the members of $\mathcal{L}$ are called* lines*. If $p \in \mathcal{P}$ and $L \in \mathcal{L}$ with $p \in L$, we say that the point p* lies on *the line L, and the line L* contains *the point p, or* goes through *p. If two (not necessarily distinct) points p and q are contained in a common line, they are called* collinear*, denoted $p \perp q$. If they are not contained in a common line, we say that they are* non-collinear*. For any point p we denote $p^\perp := \{q \in \mathcal{P} \mid q \perp p\}$.*

A partial linear space *is a point-line geometry in which every line contains at least three points, and where there is a unique line through every pair of distinct collinear points p and q. That line is then denoted with pq. A* linear space *is a partial linear space in which every pair of points is collinear.*

Example 1. *Let V be a vector space of dimension at least 3. Let $\mathcal{P}$ be the set of 1-spaces of V, and let $\mathcal{L}$ be the set of 2-spaces of V, each of them regarded as the set of 1-spaces it contains. Then, $(\mathcal{P}, \mathcal{L})$ is called a* projective space (of dimension $\dim V - 1$) *and denoted by $\mathsf{PG}(V)$, or $\mathsf{PG}(n, \mathbb{K})$ if V is defined over the field $\mathbb{K}$ and has dimension $n+1$. If $\mathbb{K}$ is finite, say $|\mathbb{K}| = q$, then $\mathsf{PG}(n, \mathbb{K})$ is denoted by $\mathsf{PG}(n, q)$. Projective spaces are linear spaces.* Projective planes *are linear spaces in which every pair of lines intersect in a point; examples are $\mathsf{PG}(2, \mathbb{K})$, but there are also other examples. The finite planes $\mathsf{PG}(2, q)$ are characterized by the* Moufang property*, which states that every line L is a translation line, that is, the pointwise stabilizer of L contains a subgroup acting sharply transitively on the points off L.*

Definition 2. *Let $\Delta = (\mathcal{P}, \mathcal{L})$ be a partial linear space.*

(i) *The* point graph Γ *of* Δ *is the graph with vertices the points of* Δ*, adjacent when collinear. A* geodesic *in* Δ *between the points* x *and* y *is a minimal path in* Γ *from* x *to* y*. The* distance *between* x *and* y *in* Δ *is the graph theoretic distance in* Γ*.*

(ii) *The partial linear space* Δ *is called* connected *when the point graph is connected. The* diameter *of* Δ *is by definition the diameter of* Γ *(if it exists).*

(iii) *A subset* S *of* $\mathcal{P}$ *is called a* subspace *of* Δ *when every line* L *of* $\mathcal{L}$ *that contains at least two points of* S *is contained in* S*. A subspace that intersects every line in at least a point is called a* hyperplane*; it is* proper *if it does not coincide with* $\mathcal{P}$*. A subspace is called* convex *if it contains all points on every geodesic that connects any two points in* S*. We usually regard subspaces of* Δ *in the obvious way as subgeometries of* Δ*.*

(iv) *A subspace* S *in which all points are pairwise collinear is called a* singular *subspace. If* S *is, moreover, not contained in any other singular subspace, it is called a* maximal *singular subspace. If it is contained in at least one other singular subspace, but all such singular subspaces are maximal, then we call it* submaximal*. A singular subspace is called* projective *if, as a subgeometry, it is a projective space (cf. Example 1). Note that every singular subspace is trivially convex.*

(v) *For a subset* P *of* $\mathcal{P}$*, the* subspace generated by P *is denoted* $\langle P \rangle$ *and is defined to be the intersection of all subspaces containing* P*. The* convex hull of P *is defined to be the intersection of all convex subspaces that contain* P*. A subspace generated by three mutually collinear points, not on a common line, is called a* plane*. Note that, in general, this is not necessarily a singular subspace; however, we will only deal with geometries satisfying Axiom (GS) (see below), which implies that subspaces generated by pairwise collinear points are singular; in particular, planes will be singular subspaces.*

2.2. Polar and Parapolar Spaces

We recall the definition of a polar space, and gather some basic properties. We take the viewpoint of Buekenhout and Shult [7]. All results in this section are well known, the standard reference being [8]. Since we are only interested in *non-degenerate* polar spaces of *finite rank*, we include this in our definition.

Definition 3. *A* polar space *is a point-line geometry* Γ *in which for every point* p*, the set* $p^{\perp}$ *is a proper hyperplane, and each maximal nested family of singular subspaces is finite and has size* $r+1$ *at least 3. The integer* r *is the* rank *of the polar space.*

One shows that a polar space Γ is partial linear, and that each singular subspace is a projective space, see [7]. The maximal singular subspaces of a polar space of rank r have dimension $r-1$. Two singular subspaces are called Γ-*opposite* if no point of either of them is collinear to all points of the other.

Example 2. *Let* $\mathbb{K}$ *be a field,* n *an integer at least* 3*, and* $\mathcal{H}$ *a hyperbolic quadric in* $\mathsf{PG}(2n-1,\mathbb{K})$*, that is, a quadric with standard equation* $X_{-1}X_1 + X_{-2}X_2 + \cdots + X_{-n}X_n = 0$*. Then, the points and lines on* $\mathcal{H}$ *define a point-line geometry that is a polar space of rank* n *and that we will denote by* $\mathsf{D}_{n,1}(\mathbb{K})$*. We call it a* hyperbolic *polar space. Maximal singular subspaces of hyperbolic quadrics are often called* generators*. A hyperbolic polar space has the peculiar property that every submaximal singular subspace is contained in exactly two generators. Also, intersecting in a subspace of even codimension defines an equivalence relation in the set of generators and we call the two thus-obtained equivalence classes the* natural classes of generators*.*

We also recall the definition of a parapolar space and introduce the examples that we are concerned with in this paper.

Definition 4. *A* parapolar space Δ *is a connected point-line geometry, which is not a polar space, and for which every pair* $\{p,q\}$ *of points with* $|p^{\perp} \cap q^{\perp}| \geq 2$ *is contained in a convex subspace isomorphic to a polar space. Any such convex subspace is called a* symplecton *of* Δ*.*

A pair of points p and q is called special *if $|p^\perp \cap q^\perp| = 1$. A pair of non-collinear points p and q is called symplectic if $|p^\perp \cap q^\perp| \geq 2$. In this case, the convex hull of $\{p, q\}$ is a polar space, which we denote by $\xi(p, q)$. A parapolar space is called* strong *when it contains no pair of special points.*

Remark 1. *The definition of a parapolar space immediately implies that it is a partial linear space. Also, parapolar spaces are so-called* gamma spaces*, that is, they satisfy the following axiom, which is sometimes superfluously added in the definition.*

(GS) Every point is collinear to zero, one, or all points of any line.

In the present paper, we will only be concerned with parapolar spaces all symplecta of which have the same rank $r \geq 3$. We say that the parapolar space has (*constant* or *uniform symplectic*) rank r. If $r \geq 3$, then all singular subspaces are projective.

Example 3. *Let $\mathcal{H}$ be a hyperbolic quadric in $\mathsf{PG}(2n-1, \mathbb{K})$ as in Example 2, with $n \geq 5$. Let Υ_1 and Υ_2 be the two natural classes of generators. Let Ξ be the set of singular subspaces of dimension $r-3$ and set $L(W) = \{U \in \Upsilon_1 \mid W \subseteq U\}$ for each $W \in \Xi$. Then, the point-line geometry with point set Υ_1 and line set $\{L(W) \mid W \in \Xi\}$ is a strong parapolar space with diameter $\lfloor \frac{n}{2} \rfloor$ and rank 4. We denote it by $\mathsf{D}_{n,n}(\mathbb{K})$.*

Definition 5. *Let Γ be a non-degenerate polar or parapolar space of rank r and let U be a singular subspace of Γ of dimension at most $r-3$. We define $\mathsf{Res}_\Gamma(U)$ to be the point-line geometry $(\mathcal{P}, \mathcal{L})$ with:*

$$\mathcal{P} := \{\text{singular subspaces } K \text{ of } \Gamma \text{ with } U \subset K \text{ and } \mathsf{codim}_K U = 1\},$$
$$\mathcal{L} := \{\text{singular subspaces } L \text{ of } \Gamma \text{ with } U \subset L \text{ and } \mathsf{codim}_L U = 2\},$$

where any element of $\mathcal{L}$ is identified with the set of elements of $\mathcal{P}$ contained in it.

If U is a point, then we say that $\mathsf{Res}_\Gamma(U)$ is a point residual*.*

A lot of background information about parapolar spaces is provided in the standard reference [9].

2.3. Parapolar Spaces of Type $\mathsf{E}_{6,1}$

Let $\mathbb{K}$ be a field. We now introduce the parapolar space $\mathsf{E}_{6,1}(\mathbb{K})$. It is related to the building of type E_6 over the field $\mathbb{K}$, but we will not need that relationship; instead, we define this geometry by one of its characterizations in the literature, see ([9], Theorem 15.4.3). However, this relation to buildings, and hence, to groups of Lie type, explains why this geometry is often called a *Lie incidence geometry*, as we also did in the abstract.

Definition 6. *A parapolar space $\Delta = (X, \mathcal{L})$ is denoted by $\mathsf{E}_{6,1}(\mathbb{K})$ and called* of type $\mathsf{E}_{6,1}$ *if it satisfies the following axioms:*

- (i) *Two different points are either collinear or symplectic. In other words, Δ is strong and has diameter 2.*
- (ii) *The symplecta are hyperbolic polar spaces of rank 5 isomorphic to $\mathsf{D}_{5,1}(\mathbb{K})$.*
- (iii) *If a point p is collinear with at least one point of a symplecton ξ not containing p, then $p^\perp \cap \xi$ is a generator of ξ.*
- (iv) *Two different symplecta, with at least two common points, have a generator of both in common.*

Note that these axioms are not entirely independent, but we are not concerned about that.

2.4. Known Properties of $\mathsf{E}_{6,1}(\mathbb{K})$

In this section, we collect a number of well-known properties of the geometry $\Delta = (\mathcal{P}, \mathcal{L})$ of type $\mathsf{E}_{6,1}$. To have everything at one place, we also include the axioms. Most results are proved in [6], the others follow directly from (ii) below and a straightforward argument in the associated polar space $\mathsf{D}_{5,1}(\mathbb{K})$.

Lemma 1.

(i) *Δ is a strong parapolar space with diameter* 2.
(ii) *The point residual at any point is isomorphic to* $\mathsf{D}_{5,5}(\mathbb{K})$.
(iii) *All symplecta of Δ are isomorphic to* $\mathsf{D}_{5,1}(\mathbb{K})$.
(iv) *All singular subspaces of dimension $d \geq 2$ of Δ are isomorphic to* $\mathsf{PG}(d, \mathbb{K})$.
(v) *The maximal singular subspaces of Δ have dimension* 4 *and* 5. *They are referred to as the* 4*-spaces and* 5*-spaces, respectively.*
(vi) *Each singular subspace U of dimension* 4 *is contained in a unique maximal singular subspace. If the latter is a* 5*-space, then U is referred to as a $4'$-space.*
(vii) *If a point p is not contained in a symplecton ξ, but is collinear to at least one point of ξ, then it is collinear to all points of a $4'$-space U of ξ. The space $\langle p, U\rangle$ spanned by p and U is a* 5*-space.*
$(viii)$ *The maximal singular subspaces of a given symplecton ξ also contained in some other symplecton are* 4*-spaces and form one natural class of generators. The maximal singular subspaces of ξ contained in a* 5*-space of Δ are $4'$-spaces and form the other class.*
(ix) *Two distinct symplecta intersect in either a point or a* 4*-space.*
(x) *Each singular* 3*-space is contained in a unique maximal singular* 4*-space and a unique singular* 5*-space.*
(xi) *For a point p and* 5*-space W of Δ with $p \notin W$, either $p^\perp \cap W$ is a* 3*-space or $p^\perp \cap U$ is a point.*

Terminology 1.

- A point p not contained in a symplecton ξ is called *neighboring to* ξ if $x^\perp \cap \xi$ is a $4'$-space. It is called *opposite* ξ if $p^\perp \cap \xi = \varnothing$.
- In (xi) above, we say that p and W are *close* if $p^\perp \cap W$ is a 3-space, and they are *far* if $p^\perp \cap W$ is a single point.
- Two symplecta are called *adjacent* if they intersect in a 4-space.

Principle of Duality

Finally, we note that there is a *principle of duality* in $\mathsf{E}_{6,1}(\mathbb{K})$, for every field $\mathbb{K}$. This goes as follows. Let Ξ be the set of all symplecta of $\mathsf{E}_{6,1}(\mathbb{K})$. For each 4-space U, define L_U to be the set of all symplecta containing U. Set $\mathcal{U}$ be the set of all L_U, for U ranging over the set of all 4-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$. Then, the pair $(\Xi, \mathcal{U})$ is a point-line geometry isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$ again. Hence, points and symplecta play a similar role in $\mathsf{E}_{6,1}(\mathbb{K})$. We denote this dual geometry by $\mathsf{E}_{6,6}(\mathbb{K})$. We note that the planes and the 5-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$ are self dual notions, that is, each plane in $\mathsf{E}_{6,6}(\mathbb{K})$ consists of the symplecta of $\mathsf{E}_{6,1}(\mathbb{K})$ containing a given plane, and each 5-space in $\mathsf{E}_{6,6}(\mathbb{K})$ consists of the symplecta of $\mathsf{E}_{6,1}(\mathbb{K})$ intersecting a given 5-space in a $4'$-space.

2.5. The Finite Case

When the field $\mathbb{K}$ is finite, say isomorphic to $\mathbb{F}_q$ for some prime power q, then we denote the corresponding geometry $\mathsf{E}_{6,1}(\mathbb{K}) = (\mathcal{P}, \mathcal{L})$ by $\mathsf{E}_{6,1}(q)$. Its point graph (see Definition 2) is strongly regular with parameters (v, k, λ, μ), where k is the valency of Γ, λ is the number of vertices adjacent to two arbitrary adjacent vertices, and μ is the number of vertices adjacent to two arbitrary non-adjacent vertices. We have the following values (see also [2], Section 4.9):

$$\begin{cases} v = |\mathcal{P}| = |\Xi| = \frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)}, \\ k = q\frac{(q^8-1)(q^6-1)}{(q^3-1)(q-1)}, \\ \lambda = q - 1 + q^2\frac{(q^5-1)(q^4-1)}{(q^2-1)(q-1)}, \\ \mu = \frac{(q^6-1)(q^4-1)}{(q^3-1)(q-1)}. \end{cases}$$

The Hoffman bound for cocliques, that is, sets of pairwise non-adjacent vertices, is $q^8 + q^4 + 1$ (see [3]). A coclique of that maximal size will be called a *Cooperstein ovoid* of $\mathsf{E}_{6,1}(q)$. No Cooperstein ovoid is known to exist, and the goal of this paper is to establish some further restrictions on its existence.

We note an immediate consequence of the fact that the size of a Cooperstein ovoid reaches the Hoffman bound (see for instance [2], Proposition 1.1.7(i)).

Lemma 2. *Each point of $\mathsf{E}_{6,1}(q)$ outside a given Cooperstein ovoid O is collinear to exactly $q^3 + 1$ points of O.*

A similar result holds for ovoids of $\mathsf{D}_{4,1}(q)$, which are cocliques of the point graph reaching the Hoffman bound $q^4 + 1$. Such ovoids have a geometric definition as sets of points intersecting each maximal singular subspace in exactly one point (a definition in the same spirit does not exist for Cooperstein ovoids, but see Remark 3). The number $q^3 + 1$ below also follows from the fact that the points of the ovoid collinear to a given point outside the ovoid form an ovoid in the point residual.

Lemma 3. *Each point of $\mathsf{D}_{4,1}(q)$ outside a given ovoid O is collinear to exactly $q^3 + 1$ points of O.*

3. Proofs

3.1. Main Result 1

We begin with Main Result 1. Suppose O is a set of points of $\mathsf{E}_{6,1}(q)$ no two of which are collinear. We use the notation of above. The Hoffman bound implies:

$$|O| \leq q^8 + q^4 + 1. \tag{1}$$

Suppose each symplecton intersects O in at least one point. For each symplecton ξ, set $N_\xi := |O \cap \xi|$. Then, $1 \leq N_\xi \leq q^4 + 1$; the latter following from the Hoffman bound in ξ (also known as the *ovoid number in* ξ). Note that, by the principle of duality, the number of symplecta containing a given point is equal to the number of points in a given symplection, which is $\frac{(q^8-1)(q^5-1)}{(q^4-1)(q-1)}$. Counting the pairs $(x, \xi) \in O \times \Xi$, with $x \in \xi$, in two ways, we see that:

$$\sum_{\xi \in \Xi} N_\xi = |O| \frac{(q^8 - 1)(q^5 - 1)}{(q^4 - 1)(q - 1)}. \tag{2}$$

Now, we count the triples $(x, y, \xi) \in O \times O \times \Xi$ with $x, y \in \xi$ in two ways. We obtain:

$$\sum_{\xi \in \Xi} N_\xi(N_\xi - 1) = |O|(|O| - 1). \tag{3}$$

Multiplying Equation (2) by $q^4 + 1$, subtracting this from Equation (3), and adding $q^4 + 1$ times the total number of symplecta, we obtain, after some elementary calculation:

$$\sum_{\xi \in \Xi} (N_\xi - (q^4 + 1))(N_\xi - 1) = \\ |O|^2 - (1 + (q^4 + 1)^2(q^4 + q^3 + q^2 + q + 1))|O| + (q^4 + 1)(q^8 + q^4 + 1)(q^8 + q^7 + \cdots + q + 1), \tag{4}$$

which we can write as:

$$\sum_{\xi\in\Xi}(N_\xi-(q^4+1))(N_\xi-1)= \\ \big(|O|-(q^8+q^4+1)\big)\cdot\big(|O|-(q^4+1)(q^8+q^7+\cdots+q+1)\big). \quad (5)$$

Now, since $1\le N_\xi\le q^4+1$, the left-hand side of Equation (5) is non-positive. Hence, the right-hand side of Equation (5) is also non-positive, implying that:

$$q^8+q^4+1\le|O|\le(q^4+1)(q^8+q^7+\cdots+1),$$

which leads to $|O|=q^8+q^4+1$ (and O is a Cooperstein ovoid). However, then the right-hand side of Equation (5) is 0, which means that every term of the left-hand side is 0 (since each such term is non-positive). Consequently, every symplecton contains either exactly one point of O, or exactly q^4+1. Let $\mathcal{B}$ be the set of intersections of O with symplecta containing exactly q^4+1 points of O. Clearly, by the foregoing, the point-line geometry $\Gamma=(O,\mathcal{B})$ is a linear space. Let $B_1,B_2\in\mathcal{B}$ be two arbitrary lines of Γ. Let ξ_1,ξ_2 be the respective corresponding symplecta. By Lemma 1(ix), there exists $x\in\xi_1\cap\xi_2$. If $x\notin O$, then, by Lemma 3, x is collinear to q^3+1 points of B_1 and q^3+1 points of B_2. Since $\xi_1\cap\xi_2$ is at most a 4-space, x is collinear to at least $2q^3+1>q^3+1$ points of O, contradicting Lemma 2. Hence, Γ is a projective plane.

Now, for $x\in O$ and $B\in\mathcal{B}$, with $x\notin B$, the point x is opposite B for if not, then, by Lemma 1(vii), the point x is collinear to a $4'$-space of the corresponding symplecton ξ, which contains a member of O as B is an ovoid of ξ, contradicting the fact that points of O are non-collinear. Hence, the map that assigns to $b\in B$ the unique member of $\mathcal{B}$ containing x and b is the restriction of the map that assigns to a point y of ξ the symplecton through x and y. It follows that the projectivity group of a line B of Γ extends to a collineation group of the ambient symplecton ξ. Since the projectivity group of a line of a projective plane is 3-transitive, the ovoid $B\subseteq\xi$ is 2-transitive. If we allow ourselves to use the classification of finite simple groups, then ([10], Main Result) implies the non-existence of B as an ovoid of ξ. We can also bypass the classification of finite simple groups as follows. First, we note that Kleidman only uses it to have the list of 2-transitive groups. So, if we could prove that the simple socle of the projectivity group of B is some specific group, in particular $\mathsf{PSL}_2(q^4)$, then ([10], Section 6) proves non-existence without relying on the classification of finite simple groups. First, a lemma.

Lemma 4. *A Cooperstein ovoid O in $\mathsf{E}_{6,1}(q)$ which intersects each symplecton of $\mathsf{E}_{6,1}(q)$ in either one or exactly q^4+1 points, is determined by its points in a single symplecton ξ with $|O\cap\xi|=q^4+1$, and two additional points x,y.*

Proof. We first note that we showed in the paragraph preceding this lemma that, if $a\in O$ and $B\in\mathcal{B}$ with $a\notin B$, then a is opposite the symplecton ζ containing the points of B.

Next, we claim that, if ξ_1 and ξ_2 are two distinct symplecta both containing q^4+1 points of O, then $\xi_1\cap\xi_2$ is a unique point (which belongs to O by a previous observation). Indeed, by Lemma 1(ix), we may suppose for a contradiction that ξ_1 and ξ_2 intersect in a 4-space U. Select $x\in O\cap(\xi_1\setminus\xi_2)$. Then, x is collinear with the points of a 3-space of U and so x neighbors ξ_2, contradicting the first paragraph.

Now, let O' be another Cooperstein ovoid in $\mathsf{E}_{6,1}(q)$, which intersects each symplecton of $\mathsf{E}_{6,1}(q)$ in either one or exactly q^4+1 points, and suppose that $O'\cap\xi=O\cap\xi$ and $x,y\in O'$. Let $a\in O$ be arbitrary. First, assume $a\notin\xi(x,y)$. Then, a is the intersection of the symplecta $\xi(a,x)$ and $\xi(a,y)$. Both symplecta intersect ξ in points of $O\cap O'$ and, as such, both symplecta contain q^4+1 points of both O and O'. Therefore, $a\in O\cap O'$. If $a\in\xi(x,y)$, then we replace b with another point of $O'\setminus(\xi(x,y)\cup\xi)$ and apply the same argument. $\square$

The following lemma can be deduced from [11] for fields of characteristic different from 2 and 3. We are not aware of an explicit proof in the literature for general fields, and hence, we provide one. It uses an explicit, but well-known representation of $\mathsf{E}_{6,1}(\mathbb{K})$ in the projective space $\mathsf{PG}(26,\mathbb{K})$.

Lemma 5. *Let $\mathbb{K}$ be a field. For a given symplecton ξ of $\mathsf{E}_{6,1}(\mathbb{K})$, and two given non-collinear points a,b opposite ξ, there exists a collineation θ of $\mathsf{E}_{6,1}(\mathbb{K})$ pointwise fixing ξ, stabilizing every line, and hence, each symplecton through the intersection point $\xi \cap \xi(a,b)$, and mapping a to b.*

Proof. We use the following representation of $\mathsf{E}_{6,1}(\mathbb{K})$ in $\mathsf{PG}(26,\mathbb{K})$ described in [12]. We describe the points of $\mathsf{PG}(26,\mathbb{K})$ by 6-tuples $(x,y,z;X,Y,Z) \in \mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{O}'\times\mathbb{O}'\times\mathbb{O}'$, where $\mathbb{O}'$ is the (8-dimensional) split Cayley algebra over $\mathbb{K}$ with standard involution $\mathbb{O}' \to \mathbb{O}' : X \mapsto \overline{X}$. The points of $\mathsf{E}_{6,1}(\mathbb{K})$ are those of $\mathsf{PG}(26,\mathbb{K})$ satisfying:

$$\begin{cases} xy = Z\overline{Z}, \\ yz = X\overline{X}, \\ zx = Y\overline{Y}, \end{cases} \qquad \begin{cases} XY = z\overline{Z}, \\ YZ = x\overline{X}, \\ ZX = y\overline{Y}. \end{cases}$$

We denote this set of points by $\mathcal{E}_{6,1}(\mathbb{K})$. Each collineation of $\mathsf{PG}(26,\mathbb{K})$ preserving $\mathcal{E}_{6,1}(\mathbb{K})$ induces a collineation of $\mathsf{E}_{6,1}(\mathbb{K})$.

Now, one checks that $(0,*,*;*,0,0)$ defines a subspace spanned by a symplecton ξ with equations $yz = X\overline{X}$, $x = Y = Z = 0$. We may take $a = (1,0,0;0,0,0)$ and $b = (1,Y_0\overline{Y}_0,0;0,0,Y_0)$. Then, the following mapping:

$$(x,y,z,X,Y,Z) \mapsto (x, y + Y_0\overline{Y}_0x + ZY_0 + \overline{Y}_0\overline{Z}, z, X + Y_0\overline{Y}, Y, Z + \overline{Y}_0x)$$

preserves $\mathcal{E}_{6,1}(\mathbb{K})$. This follows from an easy calculation, keeping in mind that $\mathbb{O}'$ is an alternative algebra, that $X\overline{X} = \overline{X}X$ belongs to the center of $\mathbb{O}'$ and that the following identities hold in $\mathbb{O}'$:

$$X\overline{Y} + Y\overline{X} = \overline{X}Y + \overline{Y}X, X(YZ) + \overline{Y}(\overline{X}Z) = (XY)Z + (\overline{Y}\,\overline{X})Z.$$

Now, it is also readily checked that a point of $\mathcal{E}_{6,1}(\mathbb{K})$ is collinear to $(0,1,0;0,0,0)$ if, and only if, it has coordinates $(0,y,0;X,0,Z)$, with $Z\overline{Z} = X\overline{X} = ZX = 0$ (just express that all points of the line joining a generic point with $(0,1,0;0,0,0)$ belong to $\mathcal{E}_{6,1}(\mathbb{K})$). Such a point is mapped onto $(0, y + ZY_0 + \overline{Y}_0\overline{Z}, 0; X, 0, Z)$, which belongs to the line spanned by that point and $(0,1,0;0,0,0)$. Hence, all lines through $(0,1,0;0,0,0) = \xi \cap \xi(a,b)$ are stabilized (and consequently, all symplecta through that point are stabilized), all points in ξ are fixed, and a is mapped to b. The lemma is proved. □

Now we continue the proof of Main Result 1 without using the classification of finite simple groups. Suppose ξ is a symplecton that intersects O in q^4+1 points and let $x,y \in O$ be arbitrary and distinct points not in ξ. Set $b := \xi(x,y) \cap \xi$. Select $a \in \xi \cap O \setminus \{b\}$. Let ζ be an arbitrary symplecton through b distinct from both ξ and $\xi(x,y)$ and intersecting O in q^4+1 points. Then, $x' = \zeta \cap \xi(a,x)$ and $y' = \zeta \cap \xi(a,y)$ belong to O and $x'^{\theta} = y'$. Hence, O^{θ} contains y, y' and $\xi \cap O$. Hence, $O^{\theta} = O$ by Lemma 4. By the arbitrariness of ζ, x and y, we have shown that Γ is a Moufang plane. Since Γ is finite, it is isomorphic to $\mathsf{PG}(2,q^4)$. It is well known that the projectivity group of a line in that plane is isomorphic to $\mathsf{PGL}_2(q^4)$. Hence, the 2-transitive action on the set B (see above) is $\mathsf{PGL}_2(q^4)$, and hence, it has simple socle $\mathsf{PSL}_2(q^4)$.

This concludes the proof of Main Result 1.

Remark 2. *Let $\mathbb{O}_q$ be the split Cayley algebra over the finite field $\mathbb{F}_q$. Another way of concluding the proof of Main Result 1 without using the classification of finite simple groups, and even without using Kleidman's result on 2-transitive ovoids, would be to use the Moufang property of Γ and the* *coordinating structures $\mathbb{O}_q$ and $\mathbb{F}_{q^4}$ of $\mathsf{E}_{6,1}(\mathbb{K})$ and Γ, respectively, to show that $\mathbb{F}_{q^4}$ is a subalgebra of $\mathbb{O}_q$. However, $\mathbb{O}_q$ is quadratic over $\mathbb{F}_q$, whereas $\mathbb{F}_{q^4}$ is not, leading to a contradiction. This would perhaps conceptually be the most preferable way, but it requires rather many technicalities.*

3.2. Main Result 2

Let, with the above notation, θ be a collineation of $\mathsf{E}_{6,1}(q)$, whose fixed points form a Cooperstein ovoid O. Recall that we do not know anything about the sizes of the intersections of O with the symplecta. In fact, our goal is to prove that each symplecton has at least one point in common with O.

Claim 1. *We claim that for every triple of points $x, y, z \in O$, the symplecta $\xi(x,y)$ and $\xi(x,z)$ are never adjacent.* Indeed, suppose for a contradiction that $\xi(x,y) \cap \xi(x,z) = U$ is a 4-space U. Then, we infer from Lemma 1$(viii)$ that y together with the points of U collinear to y generate a $4'$-space U_y, which, due to Lemma 1(vi), is contained in a unique 5-space W_y. The latter is, by uniqueness, stabilized by θ. If some point $u \in O$ were far from W_y, then the unique point $u' \in W_y$ collinear to u would be fixed by θ, implying $u, u' \in O$, contradicting the collinearity of u and u'. Hence, all points of $O \setminus \{y\}$ are close to W_y. Likewise, each point of $O \setminus \{z\}$ is close to the analogously defined 5-space W_z. Since y and z are not collinear, and since W_y and W_z share the subspace $y^\perp \cap U \cap z^\perp$ of dimension at least 2, Lemma 1(x) implies that the 5-spaces W_y and W_z intersect precisely in a plane π, which is fixed by θ.

If for some point $u \in O$, the 3-space $u^\perp \cap W$ intersects π in a unique point, then this point is fixed by θ, and hence, it belongs to O, a contradiction again. Consequently, for each point $u \in O$, the 3-space $u^\perp \cap W$ either contains π, or intersects π in a line. In the latter case, only one line L can occur for all points of O as otherwise the intersection point of two of those lines is fixed, and would belong to O. Hence, $L \subseteq u^\perp$, for all $u \in O$. Also, notice that $u^\perp \cap W \neq v^\perp \cap W$, for distinct points $u, v \in O$, as otherwise u and v would be collinear. All this implies that O contains at most as many points as there are 3-spaces in W through L. The latter number is $(q^2+1)(q^2+q+1)$, which is far too small. This contradiction proves Claim 1.

Claim 2. *Each symplecton containing at least two points of O contains exactly q^4+1 points.* Indeed, fix an arbitrary point $x \in O$. Let $\Xi_x \subseteq \Xi$ be the set of symplecta through x containing at least one further point of O. By Claim 1, the members of Ξ_x form a coclique of points in the symplecton of $\mathsf{E}_{6,6}(q)$ corresponding to x. The Hoffman bound implies that Ξ_x contains at most q^4+1 members. By the same token, each of these symplecta contains, besides x, at most q^4 further points of O. Hence, we obtain at most $q^4(q^4+1)$ points of $O \setminus \{x\}$. Since O contains exactly $q^4(q^4+1)+1$ points, all abovementioned inequalities become equalities, and thus, there are exactly q^4+1 symplecta through x containing exactly q^4+1 points of O and every other symplecton through x intersects O in $\{x\}$. This proves Claim 2.

Claim 3. *Each symplecton of $\mathsf{E}_{6,1}(q)$ contains at least one point of O.* Indeed, let ξ be a symplecton and pick $x \in \xi$ arbitrary. Since x is collinear to exactly q^3+1 points of O, we can select two of them, and consider the symplecton ζ through them. Clearly, ζ contains x, and by Claim 2, ζ contains exactly q^4+1 points of O, and hence, an ovoid of ζ, which implies that x is collinear to q^3+1 points of $O \cap \zeta$. Hence, no other symplecton through x contains at least 2 points of O (as such a symplecton would generate another set of q^3+1 points of O collinear to x, a contradiction). Every symplecton ζ' through x adjacent to ζ already contains a point of O since $O \cap \zeta$ is an ovoid, and thus, there is a unique point of O in the 4-space $\zeta \cap \zeta'$. Hence, we may assume that ξ intersects ζ in x. There are exactly q^8 such symplecta, and there are also exactly q^8 points of O not in ζ. Also, no such symplecton contains at least two points. It follows that each such symplecton contains a unique point of O, and hence, so does ξ.

Now, Main Result 2 follows from Main Result 1.

3.3. Main Result 3

Let θ be a polarity (that is, a duality of order 2) of $\mathsf{E}_{6,1}(q)$ having a Cooperstein ovoid O as set of absolute points. Recall that an absolute points is a point p contained in its image p^θ (and the latter is a symplecton).

The fact that no pair of absolute points of θ is collinear, ([13], Corollary 2.4) implies that there is a point p mapped onto a neighboring symplecton $p^\theta = \xi$. Let W be the unique 5-space containing p and intersecting ξ in a $4'$-space U (cf. Lemma 1(vii)). The principle of duality and the uniqueness of W given $(p^\theta)^\theta$ and p^θ implies that $W^\theta = W$. Hence, the mapping from the point set of W to the set of hyperplanes of W given by $W \ni x \mapsto W \cap x^\theta$ defines a polarity of W. Since W is isomorphic to $\mathsf{PG}(5,q)$, there are at least two absolute points of that polarity, and they are obviously also absolute points of θ. This contradicts the fact that a Cooperstein ovoid by definition does not contain collinear points.

Remark 3. *Nobody knows a sensible definition of a Cooperstein ovoid in the infinite case. With "sensible" we mean a global definition that, specialized to the finite case, exactly yields Cooperstein ovoids. This is in contrast with ovoids of polar spaces. The reason is that, for polar spaces, there are natural subsets of points satisfying the Hoffman bound for maximal cliques, namely, the maximal singular subspaces. It then follows from ([2], Proposition 1.1.7(iii)) that the maximal cocliques attaining the Hoffman bound are precisely those cocliques that intersect every maximal singular subspace in exactly one point. Hence, the latter can be taken as general definition and it does not refer to any counting or bound. However, in $\mathsf{E}_{6,1}(q)$, there are no maximal singular subspaces whose size attains the Hoffman bound (which, by the way, equals $\frac{q^9-1}{q-1}$). Maybe, since the latter bound approximates the number of points of a symplecton (which is $q^4 + \frac{q^9-1}{q-1}$), requiring that a coclique intersects each symplecton non-trivially leads to a (special kind of) Cooperstein ovoid (which does not exist, however, as we proved; note that the notions are not equivalent). In the infinite case, the natural analogues of a Cooperstein ovoid in $\mathsf{E}_{6,1}(\mathbb{K})$ (containing in a certain sense* exactly *$|\mathbb{K}|^8 + |\mathbb{K}|^4 + 1$ points) do intersect each symplecton non-trivially. Furthermore, just like Γ above in the proof of Main Result 1, they carry the structure of a projective plane (over a quaternion and octonion division algebra, respectively).*

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