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Line Spreads That Produce Projective Planes

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Abstract: We explicitly classify those line spreads of projective 5-space over a field that have the property that the given spread induces a spread in the 3-space generated by any pair of spread lines. We determine their fix groups and conclude that there exist such spreads with a trivial fix group. Also, we characterise regular line spreads among all line spreads of projective 3-space by their projectivity group and also by a weakening of the regularity condition.

Keywords: line spread; projectivity group; regular spread

MSC: 51A40; 51E23

1. Introduction

Let $V/\mathbb{L}$ be a three-dimensional right vector space over the skew field $\mathbb{L}$. The 1- and 2-spaces of V form the points and lines of a Desarguesian projective plane $\mathsf{PG}(2,\mathbb{L})$. Suppose that $\mathbb{L}$ has a subfield $\mathbb{F}$ over which $\mathbb{L}$ is a natural vector space of dimension 2 (with "natural", we mean using the scalar multiplication given by the multiplication of $\mathbb{L}$). Then, we may regard V as a six-dimensional vector space $V/\mathbb{F}$ over $\mathbb{F}$, defining a five-dimensional projective space $\mathsf{PG}(5,\mathbb{F})$. The 1-spaces of $V/\mathbb{L}$ correspond to a selection $\mathfrak{L}$ of 2-spaces of $V/\mathbb{F}$ with the following properties:

(i) Every 1-space of $V/\mathbb{F}$ is contained in a unique member of $\mathfrak{L}$;

(ii) Two distinct members of $\mathfrak{L}$ generate a 4-space U of $V/\mathbb{F}$ with the property that every member of $\mathfrak{L}$ sharing at least a 1-space of $V/\mathbb{F}$ with U is entirely contained in U.

In $\mathsf{PG}(5,\mathbb{F})$, the set $\mathfrak{L}$ corresponds to a *line spread* (i.e., a set of lines, also denoted by $\mathfrak{L}$, partitioning the point set), which induces a line spread in every subspace spanned by two distinct but arbitrary members of $\mathfrak{L}$. We call such a line spread a *composition line spread*. The members of $\mathfrak{L}$ and all subspaces spanned by two of its members form the point set and line set, respectively, of the projective plane $\mathsf{PG}(2,\mathbb{L})$. We say that $\mathfrak{L}$ *arises from the extension* $\mathbb{L}/\mathbb{F}$.

We can now reverse the procedure. We start with the projective space $\mathsf{PG}(5,\mathbb{F})$ over the field $\mathbb{F}$ and try to find a composition line spread. One way of achieving this is to find a fixed point free collineation θ of $\mathsf{PG}(5,\mathbb{F})$ with the property that, for each point p, the line spanned by p and p^θ is stabilised. Note that every fixed point free involution has that property. Then, automatically, the fixed lines form a composition spread. In the present paper, we determine all composition line spreads of $\mathsf{PG}(5,\mathbb{F})$, with $\mathbb{F}$ a field, and determine their fix group. It is revealed that there exist such spreads whose fix group is trivial, that is, which can not be constructed as fix (line) structure of a fixed point free collineation of $\mathsf{PG}(5,\mathbb{F})$. More precisely, we show the following:

Theorem 1. *Let $\mathfrak{L}$ be a composition line spread of $\mathsf{PG}(5,\mathbb{F})$. Then, there exists a skew field $\mathbb{L}$ containing $\mathbb{F}$ such that $\mathfrak{L}$ arises from the extension $\mathbb{L}/\mathbb{F}$. Moreover, we have exactly one of the*

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following situations, where we denote by T the fix group of $\mathfrak{L}$, that is, the group of all collineations of $\mathsf{PG}(5,\mathbb{F})$ stabilising each member of $\mathfrak{L}$.

(i) *$\mathbb{L}$ is a (separable or inseparable) quadratic extension field of $\mathbb{F}$, and T is a group abstractly isomorphic to $\mathbb{L}^\times/\mathbb{F}^\times$, and as a permutation group acts sharply transitively on the set of points of each line of $\mathfrak{L}$;*

(ii) *$\mathbb{L}$ is a quaternion algebra over a subfield $\mathbb{F}'$ of $\mathbb{F}$. The latter is quadratic over $\mathbb{F}'$. If $\mathbb{F}/\mathbb{F}'$ is separable, then T has order 2, and its nontrivial member is a semi-linear involution corresponding to Galois descent. If $\mathbb{F}/\mathbb{F}'$ is inseparable, then T is trivial.*

In case (i) above, the line spread $\mathcal{S}$ induced in a subspace of dimension 3 is *regular*; that is, for each triple of lines L_1, L_2, L_3 of $\mathcal{S}$, every line intersecting each transversal of L_1, L_2, L_3 belongs to $\mathcal{S}$ (a *transversal* of a set of lines is a line intersecting each line of the set in a point). If this condition is only satisfied for given lines L_1, L_2 and L_3 of $\mathcal{S}$, then we say that the triple $\{L_1, L_2, L_3\}$ is *regular*. If the triple $\{L_1, L_2, L_3\} \subseteq \mathcal{S}$ is regular for given $L_1, L_2 \in \mathcal{S}$ and all $L_3 \in \mathcal{S} \setminus \{L_1, L_2\}$, then we say that the pair $\{L_1, L_2\}$ is *regular*. We will show the following:

Theorem 2. *A line spread $\mathcal{S}$ of $\mathsf{PG}(3,\mathbb{F})$ is regular if, and only if, there exists a regular pair $\{L_1, L_2\} \subseteq \mathcal{S}$ and a regular triple $\{L_1, L_3, L_4\}$ such that no point of L_2 is on any transversal of $\{L_1, L_3, L_4\}$.*

This is a substantial weakening of the condition in the definition of regular spread. It is, for instance, satisfied as soon as there exist two different regular pairs.

Let L_1, L_2, L_3 be three members of a line spread $\mathcal{S}$ of $\mathsf{PG}(3,\mathbb{F})$. We define the *perspectivity of L_1 to L_2 from L_3* as the map from the point set of L_1 to the point set of L_2 assigning to $x_1 \in L_1$ the unique point $x_2 \in L_2$ contained in the plane generated by x_1 and L_3 (or, in other words, such that the line x_1x_2 intersects L_3 in a point). The composition of a finite number of perspectivities is called a *projectivity*, and if a projectivity has domain L_1 and target L_1, then we call it a *self-projectivity of L_1*. The set of all self-projectivities of L_1 forms a (permutation) group, called the *projectivity group of L_1*, denoted $\Pi_{\mathcal{S}}(L_1)$. The projectivity groups of all members of $\mathcal{S}$ are isomorphic, and so we can speak about the projectivity group $\Pi(\mathcal{S})$ of $\mathcal{S}$. We will show the following:

Theorem 3. *A line spread $\mathcal{S}$ of $\mathsf{PG}(3,\mathbb{F})$ is regular if, and only if, $\Pi_{\mathcal{S}}(L)$ acts freely on L, for at least one and hence each $L \in \mathcal{S}$ if, and only if, $\Pi_{\mathcal{S}}(L)$ acts sharply transitively on L for at least one and hence each $L \in \mathcal{S}$ if, and only if, the restriction of the fix group T of $\mathcal{S}$ to the line L coincides with $\Pi(L)$, for at least one and hence each line $L \in \mathcal{S}$.*

Motivation—Firstly, we outline the motivation for the study of (regular) line spreads of projective 3-space, without going into details of the definitions of the various notions. In general, a line spread (say, $\mathcal{S}$) of a three-dimensional projective space $\mathsf{PG}(3,\mathbb{F})$ gives rise to a *translation plane* (which we can denote as $\mathsf{PG}(\mathcal{S})$) via the *André–Bose–Bruck* construction; see [1,2]. Then, one wants to know which properties of the spread induce higher transitivity properties of the translation plane, in particular, which properties of $\mathcal{S}$ are needed to put $\mathsf{PG}(\mathcal{S})$ in a certain class of the *Lenz–Barlotti* classification of projective planes. The highest such classes are the classes of *Moufang projective planes* and *Desarguesian projective planes*. The former are translation planes with respect to each line; the latter are Moufang planes that admit transitive homology groups. If these groups are abelian, then one refers to the plane as a *Pappian projective plane*. It is well known (see [3] (Satz 3)) that $\mathcal{S}$ is regular if, and only if, $\mathsf{PG}(\mathcal{S})$ is Pappian. So, it is natural to try to find alternative characterisations of regular line spreads in $\mathsf{PG}(3,\mathbb{F})$. Theorems 2 and 3 contribute to that (see also Section 5 for more explanation how these characterisations can help future research).

Secondly, we detail the motivation for the study of composition line spreads and explain where, more specifically, the interest in Theorem 1 derives from, again without going into details of the various notions (referring to [4]). Recently, the author, together with Yannick Neyt and James Parkinson, classified all automorphisms of spherical Tits buildings with the property that the Weyl distance between each chamber and its image lies in a given unique (possibly twisted) conjugacy class of the Weyl group (such automorphisms are called *uniclass*). For projective spaces, the uniclass collineations are exactly the members of the fix groups of line spreads, hence the interest in determining these explicitly. Also, it is interesting that there exist composition line spreads with a trivial fix group. That means that the geometric notion of composition line spread is not entirely equivalent with the notion of nontrivial uniclass collineation, in contrast to some other types of buildings. Note that our results carry over to projective spaces of arbitrary dimension (at least 5) in an obvious way (every composition line spread restricts to a composition line spread in each subspace generated by three of its members not contained in the same 3-space and hence generating a five-dimensional subspace).

2. Preliminaries

In the present paper, our main objects are the *Pappian projective spaces* $\mathsf{PG}(n,\mathbb{F})$, that is, projective spaces originating from vector spaces V_{n+1} of dimension $n+1$ defined over fields $\mathbb{F}$. Recall that the points of $\mathsf{PG}(n,\mathbb{F})$ are the 1-spaces of V_{n+1}. The set of 1-spaces in a given subspace of V_{n+1} is also called a *subspace* of $\mathsf{PG}(n,\mathbb{F})$. The (projective) dimension of a subspace is one less than its corresponding vector space dimension. The one-dimensional subspaces of $\mathsf{PG}(n,\mathbb{F})$ are also called *lines*, the two-dimensional ones *planes* and the three-dimensional ones *solids*. The one-dimensional subspaces are the *hyperplanes* and correspond to the points of the projective space defined by the dual vector space. If P is a set of points of $\mathsf{PG}(n,\mathbb{F})$, then the intersection of all subspaces containing P is called the *span* of P, denoted by $\langle P\rangle$, and we also say that P *generates* $\langle P\rangle$. If P has exactly two elements, then $\langle P\rangle$ is a line.

A *coordinatisation* of the projective space $\mathsf{PG}(n,\mathbb{F})$ consists of choosing a basis of V_{n+1} and attaching coordinates to each 1-space, determined up to a nonzero scalar multiple. Such a coordinatisation is equivalent to choosing $n+1$ points of $\mathsf{PG}(n,\mathbb{F})$ corresponding to $n+1$ distinct 1-spaces of V_{n+1} generated by a basis $(e_1,\ldots,e_{n+1})$, and a *unit* point, that is, a 1-space of V_{n+1} generated by a vector e that is linearly independent of every set of n basis vectors. Requiring that e has coordinates $(1,1,\ldots,1)$ determines the e_i up to a common scalar multiplicative constant. We say that $(e_1,\ldots,e_{n+1};e)$ is a *basic skeleton*.

A *(projective) line spread* $\mathfrak{L}$ of $\mathsf{PG}(n,\mathbb{F})$ is a partition of the point set into lines. The seminal paper by Bruck and Bose [2] contains many fundamental results and conjectures, some of which have been proved or refuted since. However, over the past decades, spreads have mainly been investigated over either the finite fields or the connected compact fields. Our results hold over arbitrary fields.

A *composition line spread* is a line spread with the property that the members of the spread contained in the subspace generated by any given pair of lines of the spread again form a line spread. Composition line spreads are sometimes also called *geometric* line spreads (but this would interfere with our notion of *geometric descent*; see Remark 1). For instance, composition line spreads in finite projective spaces of dimension at least 5 are classified; see [5]. Theorem 1 recovers this classification. Also, as shown in [2], the geometry with point set the lines of a composition line spread $\mathfrak{L}$ of $\mathsf{PG}(5,\mathbb{F})$ and line set the solids in which $\mathfrak{L}$ induces a line spread is a projective plane which we denote by $\mathsf{PG}(\mathfrak{L})$.

Let S_1, S_2 be two subspaces of $\mathsf{PG}(n, \mathbb{F})$ of the same dimension, and let T be a subspace complementary to both S_1, S_2; that is, T and S_i generate the whole space, but are disjointed, $i = 1, 2$. Then, we denote the map

$$S_1 \to S_2 : p \mapsto \langle p, T \rangle \cap S_2$$

by $S_1 \overline{\wedge}_T S_2$ and call it the *perspectivity of* S_1 *to* S_2 *from* T. A (finite) product of perspectivities $S_1 \overline{\wedge}_{T_1} S_2 \overline{\wedge}_{T_2} S_3 \overline{\wedge}_{T_3} \cdots$ is called a *projectivity*, and if the last subspace of that sequence is S_1 again, then we have a *self-projectivity*. The set of self-projectivities of S_1 is a group denoted by $\Pi(S_1)$ and called the *projectivity group of* S_1. If we restrict the subspaces $S_1, \ldots$ and $T_1, \ldots$ to the members of a given line spread $\mathcal{S}$ of $\mathsf{PG}(3, \mathbb{F})$, then we obtain the projectivity group $\Pi_{\mathcal{S}}(S_1)$, which is clearly a subgroup of $\Pi(S_1)$.

3. Proofs

Introduction of coordinates—Let $\mathfrak{L}$ be a composition line spread of $\mathsf{PG}(5, \mathbb{F})$, with $\mathbb{F}$ a field. Select a 3-space S spanned by two spread lines L_1, L_2 and a line $L_3 \in \mathfrak{L}$ outside S and a third line L_{12} of $\mathfrak{L}$ in S (meaning $L_1 \neq L_{12} \neq L_2$). Choose two points e_1, e_2 on L_1, and let e_{i+2} be the unique point of L_2 with the property that $\langle e_1, e_{i+2} \rangle$ intersects L_{12} nontrivally, say in the point $e_{i,i+2}$, $i = 1, 2$. In the solid $\langle L_2, L_3 \rangle$, we select a third spread line L_{23}, and we consider the points e_5 and e_6 on L_3 such that the line $\langle e_i, e_{i+2} \rangle$ intersects the line L_{23} nontrivally, say in the point $e_{i,i+2}$, $i = 3, 4$. Then, the line $L_{123} := \langle L_3, L_{12} \rangle \cap \langle L_1, L_{23} \rangle$ belongs to $\mathfrak{L}$. We may choose the unit point e on L_{123}, and then, taking $(e_1, e_2, \ldots, e_6; e)$ as a basic skeleton, we have (with self-explaining shorthand notation and with $L_{13} := \langle L_1, L_3 \rangle \cap \langle L_2, L_{123} \rangle$),

$$L_1 = \langle 100000, 010000 \rangle, L_2 = \langle 001000, 000100 \rangle, L_3 = \langle 000010, 000001 \rangle,$$

$$L_{23} = \langle 001010, 000101 \rangle, L_{13} = \langle 100010, 010001 \rangle, L_{12} = \langle 101000, 010100 \rangle.$$

Note that this coordinatisation depends on the choices for e_1 and e_2, and also on the choice of e. For instance, the following coordinate change preserves the above equalities:

$$(x_1', x_2', \ldots, x_6') = (x_1 + kx_2, x_2, x_3 + kx_4, x_4, x_5 + kx_6, x_6), k \in \mathbb{F}. \tag{1}$$

Now, let $\mathcal{S}$ be the line spread induced by $\mathfrak{L}$ in the 3-space $\langle L_1, L_2 \rangle$.

The spread $\mathcal{S}$ in coordinates—Let us represent the spread $\mathcal{S}$ in coordinates. For clarity, we leave out the last two coordinates. Every spread line distinct from L_1 intersects the plane $\langle L_1, e_3 \rangle$ in a unique point $(a, b, 1, 0)$, $a, b \in \mathbb{F}$, and every such point lies on a unique spread line $L(a, b)$. The line $L(a, b)$ intersects the plane $\langle L_1, e_4 \rangle$ in a unique point with coordinates $(f(a, b), g(a, b), 0, 1)$, where $f : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$ and $g : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ are two maps with $f(0, 0) = 0 = g(0, 0)$ and also $f(1, 0) = 0$ and $g(1, 0) = 1$.

Expressing that each point of $\langle L_1, L_2 \rangle \setminus L_1$ lies on a unique line $L(a, b)$, we obtain the following sufficient and necessary condition for a set of lines of the form $L(a, b)$, together with L_1, to be a spread of $\langle L_1, L_2 \rangle$: the system of equations

$$\begin{cases} ax + bf(x, y) = c \\ ay + bg(x, y) = d \end{cases} \tag{2}$$

has a unique solution for each $a, b, c, d \in \mathbb{F}$, $(a, b) \neq (0, 0)$.

Regularity— Suppose that each line of $\mathcal{S}$ that intersects $\langle (1, 0, 0, 0), (0, 0, 1, 0) \rangle$ also intersects $\langle (0, 1, 0, 0), (0, 0, 0, 1) \rangle$. Then, clearly, $f(a, 0) = 0$. If, moreover, each such line intersects each transversal of L_1, L_2 and L_{12}, then one calculates that $g(a, 0) = a$.

Set $L := L(0,1)$, and set $\alpha := f(0,1)$ and $\beta = g(0,1)$. We express that $\{L_1, L\}$ is a regular pair. An arbitrary line M_c in $\langle L_1, e_3\rangle$ through $(0,1,1,0)$ has a single-parameter description $(cr, 1-r, 1, 0)$ for some $c \in \mathbb{F}$, and we assume that $c \notin \{0,1\}$, and r is the parameter, taking all values in $\mathbb{F}$. For $r = 0$, we get $(0,1,1,0)$. For $r = 1$, we get the point $(c,0,1,0)$, which lies on the line $L(c,0)$, which intersects $\langle L_1, e_4\rangle$ in the point $(0,c,0,1)$. The transversal M'_c of L_1, L and $L(c,0)$ through $(\alpha, \beta, 0, 1)$ then goes through $(0,c,0,1)$ and has a single-parameter description $(\alpha - \alpha r, \beta - r(\beta - c), 0, 1)$, where $r = 0$ corresponds to $(\alpha, \beta, 0, 1)$ and $r = 1$ corresponds to $(0,c,0,1)$. It can be seen that, due to regularity, common values of r in the descriptions of M_c and M'_c above provide points on the same member of $\mathcal{S}$. Hence, we conclude that

$$\begin{cases} f(cr, 1-r) = \alpha - \alpha r, \\ g(cr, 1-r) = \beta - r\beta + cr, \end{cases}$$

which is, after setting $r = 1 - b$ and $cr = a$ (for $r \neq 0$), equivalent to

$$\begin{cases} f(a,b) = \alpha b, \\ g(a,b) = \beta b + a. \end{cases}$$

This holds for all $a, b \in \mathbb{F}$, except for $b = 1$ and $a \neq 0$. But these values correspond to the points $(t,1,1,0)$ of $\langle L_1, e_3\rangle$, and one can see that there is a unique line through such a point not intersecting any spread line obtained thus far, and it is given by setting $b = 1$ in the above expressions.

Now, one checks that the system of Equations (2) has always a unique solution if, and only if, the quadratic polynomial $x^2 + \beta x - \alpha$ is never zero and hence is irreducible. We will see in the next few paragraphs that such a spread admits a 3-transitive group; hence, each triple of the lines of the spread is regular, which yields a regular spread. This shows Theorem 2.

We now return to the general situation.

An additive automorphism group of $\mathcal{S}$—For every $a, b \in \mathbb{F}$, the line $L'(a,b) =: \langle L_3, L(a,b)\rangle \cap \langle L_1, L_{23}\rangle$ belongs to $\mathfrak{L}$. An elementary calculation shows that

$$L'(a,b) = \langle (a,b,1,0,1,0), (f(a,b), g(a,b), 0,1,0,1)\rangle.$$

Likewise, the line $L''(a,b) =: \langle L_2, L'(a,b)\rangle \cap \langle L_1, L_3\rangle$ belongs to $\mathfrak{L}$. In coordinates, we have the following:

$$L''(a,b) = \langle (a,b,0,0,1,0), (f(a,b), g(a,b), 0,0,0,1)\rangle.$$

Now, we define the following projectivity $\rho(a,b)$ of S: We project S onto $\langle L_1, L_{23}\rangle$ from the line L_3 and then project $\langle L_1, L_{23}\rangle$ back onto S from the line $L''(a,b)$. In coordinates, we have the following (leaving out the last two coordinates again):

$$\rho(a,b) : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -a & -f(a,b) \\ 0 & 1 & -b & -g(a,b) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Now, denote the matrix $\begin{pmatrix} a & f(a,b) \\ b & g(a,b) \end{pmatrix}$ by $M(a,b)$, and let $\mathcal{M}$ be the set of all such matrices. Then, $M(0,0)$ is the 0-matrix, also denoted by $\mathsf{0}$, and $M(1,0)$ the identity matrix, also denoted by I. Furthermore, since $\rho(a,b)$ preserves $\mathcal{S}$, the set of (4×4)-matrices

$$\begin{pmatrix} \mathsf{I} & -M(a,b) \\ \mathsf{0} & \mathsf{I} \end{pmatrix}, a,b \in \mathbb{F},$$

forms a group A acting on the left sharply transitively on the set $\{L(a,b)|a,b\in\mathbb{F}\}$. Applying $\rho(a,b)$ to $L(0,0)$, we deduce that $-M(a,b) = M(-a,-b)$. Consequently, A consists of the linear collineations with matrix $\begin{pmatrix} \mathsf{I} & M(a,b) \\ \mathsf{0} & \mathsf{I} \end{pmatrix}$. Since $\rho(a,b)(L(x,y)) = L(a+x,b+y)$, we see that $\mathcal{M}$ is an additive group isomorphic to A.

Additivity of f and g—Let $a,b,c,d\in\mathbb{F}$ be arbitrary. Expressing that $M(a,b)+M(c,d)\in\mathcal{M}$, we deduce that $f(a+c,b+d) = f(a,b)+f(c,d)$ and likewise for g. In particular, $f(a,d) = f(a,0)+f(0,d)$ and likewise $g(a,d) = g(a,0)+g(0,d)$, for all $a,d\in\mathbb{F}$. We may set $f(a,0) := f_1(a)$ and $f(0,b) = f_2(b)$; likewise, we set $g(a,0) = g_1(a)$ and $g(0,b) = g_2(b)$. Then, $f(a,b) = f_1(a)+f_2(b)$ and $g(a,b) = g_1(a)+g_2(b)$. Note that $f_1(a+c) = f(a+c,0) = f(a,0)+f(c,0) = f_1(a)+f_1(c)$; hence, f_1 is additive. Similarly, f_2, g_1, g_2 are additive maps.

A multiplicative automorphism group of $\mathcal{S}$—For every $a,b\in\mathcal{L}$, the line $L'''(a,b) =: \langle L_{13}, L(-a,-b)\rangle \cap \langle L_2, L_3\rangle$ belongs to $\mathfrak{L}$. With coordinates,

$$L'''(a,b) = \langle (0,0,1,0,a,b), (0,0,0,1,f(a,b),g(a,b)\rangle.$$

Now, we define the following projectivity $\xi(a,b)$ of S: We project S onto $\langle L_2, L_3\rangle$ from the line L_{13} and then project $\langle L_2, L_3\rangle$ back onto S from the line $L''(a,b)$. In coordinates, we have the following (leaving out the last two coordinates again):

$$\xi(a,b) : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} a & f(a,b) & 0 & 0 \\ b & g(a,b) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Hence, we obtain a group of (linear) collineations with matrices $\begin{pmatrix} M(a,b) & \mathsf{0} \\ \mathsf{0} & \mathsf{I} \end{pmatrix}$. We deduce immediately that all nontrivial members of $\mathcal{M}$ are nonsingular and that $\mathcal{M}$ is closed, not only for addition but also multiplication. Hence, it defines a skew field. This also implies that the automorphism group of $\mathcal{S}$ is triply transitive, as mentioned earlier.

g_1 is a field endomorphism—The fact that for all $a,x\in\mathbb{F}$ the matrix $M(a,0)M(x,0)$ belongs to $\mathcal{M}$ is equivalent to the identities

$$f_1(ax) = af_1(x)+g_1(x)f_1(a), \quad (3)$$
$$g_1(ax) = g_1(a)g_1(x). \quad (4)$$

Hence, g_1 is a field endomorphism. Since every member of $\mathcal{M}$ is invertible, g_1 is injective. For clarity, we denote $g_1(x) =: \overline{x}$. The identity automorphism of $\mathbb{F}$ shall be denoted by id.

***f_2* is a multiple of *g_1*—**The fact that for all $a, x \in \mathbb{F}$ the matrix $M(0,a)M(x,0)$ belongs to $\mathcal{M}$ is equivalent to the identities

$$f_2(ax) = f_2(a)\overline{x}, \tag{5}$$
$$g_2(ax) = af_1(x) + \overline{x}g_2(a). \tag{6}$$

It immediately follows from Identity (5), setting $f_2(1) = F$, that $f_2(x) = F\overline{x}$. Comparing Identities (3) and (6), we obtain the following, taking into account $f(1) = 0$ and setting $g_2(1) := G$:

$$g_2(x) = f_1(x) + G\overline{x}. \tag{7}$$

More identities—The fact that for all $a, x \in \mathbb{F}$ the matrix $M(a,0)M(0,x)$ belongs to $\mathcal{M}$ is equivalent to the following identities (taking into account the above expressions for f_2 and g_2 in function of f_1 and g_1):

$$f_1(xf_1(a)) + F\overline{\overline{a}}\,\overline{x} = Fa\overline{x} + f_1(a)f_1(x) + G\overline{x}f_1(a), \tag{8}$$
$$\overline{x}\overline{f_1(a)} + f_1(\overline{a}x) + G\overline{\overline{a}}\,\overline{x} = \overline{a}f_1(x) + G\overline{a}\,\overline{x}. \tag{9}$$

Finally, the fact that for all $a, x \in \mathbb{F}$ the matrix $M(0,a)M(0,x)$ belongs to $\mathcal{M}$ is equivalent to the following identities (taking into account the above expressions for f_2 in function of g_1):

$$f_1(Fx\overline{a}) + F\overline{x}\overline{g_2(a)} = F\overline{a}g_2(x), \tag{10}$$
$$\overline{x}\overline{F\overline{a}} + g_2(xg_2(a)) = \overline{x}Fa + g_2(a)g_2(x). \tag{11}$$

The case of $g_1 \equiv \mathrm{id}$—Suppose that $x = \overline{x}$, for all $x \in \mathbb{F}$. Then, Identities (3) and (9) imply that $2xf_1(a) = 0$, for all $x, a \in \mathbb{F}$. Hence, if $\operatorname{char}\mathbb{F} \neq 2$, then $f_1 \equiv 0$ and $\mathcal{S}$ is a regular spread with $f(a,b) = Fb$ and $g(a,b) = a + Gb$ (corresponding to the irreducible quadratic polynomial $x^2 - Gx - F$; the projective plane $\mathsf{PG}(\mathfrak{L})$ is isomorphic to $\mathsf{PG}(2,\mathbb{L})$, where $\mathbb{L} = \mathbb{F}(\alpha)$, with α a root of the said polynomial). This also holds if $\operatorname{char}\mathbb{F} = 2$ and $f_1 \equiv 0$.

We now claim that, in the above case, the spread $\mathcal{S}$, and hence also $\mathfrak{L}$, arises from the field extension $\mathbb{L}/\mathbb{F}$. First note that

$$\mathcal{S} = \{\langle e_1, e_2\rangle\} \cup \{\langle (a,b,1,0), (Fb, a+Gb, 0, 1)\rangle \mid a, b \in \mathbb{F}\}.$$

Write a generic member of $\mathbb{L}$ as $a + b\alpha$, $a, b \in \mathbb{F}$, and consider the 1-space $(a + b\alpha, 1)\mathbb{L}$. We select the two particular vectors

$$(a + b\alpha, 1) \text{ and } (a\alpha + b\alpha^2, \alpha) = (Fb + (a + Gb)\alpha, \alpha)$$

and write these as vectors of $\mathbb{F}^4$ with respect to the basis $((1,0),(\alpha,0),(0,1),(0,\alpha))$. This yields the two vectors $(a,b,1,0)$ and $(Fb, a+Gb, 0, 1)$, and the claim follows.

Now, suppose that $\operatorname{char}\mathbb{F} = 2$ and $f_1 \not\equiv 0$. Assume first that $G \neq 0$. Let $\mathbb{F}'$ be the set of all elements $x \in \mathbb{F}$ such that $f_1(x) = 0$. If $x, y \in \mathbb{F}'$, then from Identity (3), we see that $xy \in \mathbb{F}'$ and by linearity also $x + y \in \mathbb{F}'$. Hence, $\mathbb{F}'$ is a subfield of $\mathbb{F}$. Since we assume that $f_1 \not\equiv 0$, there exists $t \in \mathbb{F}$ with $t = f_1(t') \neq 0$ for some $t' \in \mathbb{F}$, and we fix such t and t'. Also, $\mathbb{F}^2 \subseteq \mathbb{F}'$ as $f(x^2) = xf(x) + xf(x) = 0$, from Identity (3).

Identity (8) says that $f_1(f_1(a)) = Gf_1(a)$. Let, for all $a \in \mathbb{F}$, the map $f : \mathbb{F} \to \mathbb{F}$ be defined as $f(a) = f_1(a)G^{-1}$. Then, one easily checks that $f(f(a)) = f(a)$, for all $a \in \mathbb{F}$.

Now, we can write an arbitrary element $x \in \mathbb{F}$ as $x = f(x) + (x + f(x))$. The element $x + f(x)$ lies in $\mathbb{F}'$, as $f(x + f(x)) = f(x) + f(f(x)) = f(x) + f(x) = 0$. Moreover, the element $f(x)t^{-1}$ belongs to $\mathbb{F}'$; indeed,

$$f(f(x)t^{-1}) = f(f(x))t^{-1} + f(x)f(t^{-1}) = f(x)(t^{-1} + f(t^{-1})) = 0,$$

since $f(t^{-1}) = t^{-2}f(t) = t^{-2}f(f(t')) = t^{-2}f(t') = t^{-2}t = t^{-1}$. Hence, we can write every element $x \in \mathbb{F}$ as $x = x' + x''t$, with $x', x'' \in \mathbb{F}'$. This decomposition is unique since if x would also be written as $x'_0 + x''_0 t$, with $x'_0, x''_0 \in \mathbb{F}'$, then $(x'' + x''_0)t \in \mathbb{F}'$, which means, again using Identity (3) (translated to f, i.e., $f(ab) = af(b) + bf(a)$), that $(x'' + x''_0)f(t) = 0$, implying $0 = f(t) = t$, a contradiction, or $x'' = x''_0$, which we had to prove. Hence, $\mathbb{F}$ is a quadratic extension of $\mathbb{F}'$; more exactly, $\mathbb{F} = \mathbb{F}'(t)$.

Note that $f(x)t^{-1} \in \mathbb{F}'$, for all $x \in \mathbb{F}$, implies, in particular, that $G^{-1} = (tG^{-1})t^{-1} = (f_1(t')g^{-1})t^{-1} = f(t')t^{-1} \in \mathbb{F}'$. Hence, $G \in \mathbb{F}'$. Also, putting $a = x = 1$ in Identity (10), we deduce that $f_1(F) = 0$; hence, $F \in \mathbb{F}'$.

We conclude that if we write every element $a \in \mathbb{F}$ as $a_1 + a_2 t$, with $a_1, a_2 \in \mathbb{F}'$, then

$$M(a,b) = \begin{pmatrix} a_1 + a_2 t & Ga_2 t + F(b_1 + b_2 t) \\ b_1 + b_2 t & a_1 + a_2 t + Gb_1 \end{pmatrix}.$$

The determinant of $M(a,b)$ is $(a_1^2 + Ga_1 b_1 + Fb_1^2) + t^2(a_2^2 + Ga_2 b_2 + Fb_2^2)$. Since $F, G \in \mathbb{F}'$, this is the norm of a quaternion algebra $\mathbb{H}$ over $\mathbb{F}'$, with basis $\{1, \alpha, t, \alpha t\}$, with α a root of $x^2 + Gx + F = 0$, and $\alpha t = t(\alpha + G)$. Writing out the multiplication explicitly, one indeed sees that $\mathcal{M}$ is a quaternion algebra over $\mathbb{F}'$ with the above norm form and given multiplication rule for α times t.

To see that $\mathcal{S}$, and hence $\mathfrak{L}$, is obtained from the extension of $\mathbb{F}$ to $\mathbb{H}$, we write every element of $\mathbb{H}$ in the form $(a_1 + b_1\alpha) + (a_2 + b_2\alpha)t$ and associate it with the vector $(a_1 + a_2 t, b_1 + b_2 t) \in \mathbb{F} \times \mathbb{F}$. The rest is similar to the arguments above for the case $f_1 \equiv 0$, taking into account that we must now multiply with α from the right to obtain the second vector.

Now, suppose that $\operatorname{char}\mathbb{F} = 2$ and $G = 0$. Identity (8) says that $f_1(f_1(a)) = 0$ for all $a \in \mathbb{F}$ (keeping in mind that $f_1(1) = 0$). This time, one calculates using Equation (3) that for an arbitrary $t \in \mathbb{F} \setminus \mathbb{F}'$ (where $\mathbb{F}'$ is again the subfield consisting of those elements x of $\mathbb{F}$ for which $f_1(x) = 0$), one has $f_1(tf_1(t)^{-1}) = 1$. So, we set $u =: tf_1(t)^{-1}$. Then, we can write every element a of $\mathbb{F}$ uniquely as a sum $a_1 + a_2 u$, with $a_1, a_2 \in \mathbb{F}'$. Moreover, $a_2 = f_1(a)$, since $f_1(a + f_1(a)u) = f_1(a) + f_1(a)f_1(u) = 0$. Hence, we have

$$M(a,b) = \begin{pmatrix} a_1 + a_2 u & a_2 + F(b_1 + b_2 u) \\ b_1 + b_2 u & a_1 + a_2 u + b_2 \end{pmatrix}.$$

This again defines a quaternion algebra $\mathbb{H}$ with the norm form

$$\left(a_1^2 + a_1 b_2 + (Fu^2)b_2^2\right) + u^{-2}\left((a_2 u^2)^2 + (a_2 u^2)b_1 + (Fu^2)b_1^2\right).$$

Similarly as before one shows that $\mathcal{S}$ is obtained from the extension of $\mathbb{F}$ to $\mathbb{H}$.

This completes the analysis for the case $g_1 \equiv \mathrm{id}$. From now, we assume that g_1 is not the identity.

Reduction—We start by reducing the number of identities. From Identity (3), it follows that $(x - \overline{x})f_1(a) = (a - \overline{a})f_1(x)$, for all $a, x \in \mathbb{F}$. Hence, there is a constant C such that $f_1(x) = C(x - \overline{x})$, for all $x \in \mathbb{F}$ (note that possibly $C = 0$). This determines all the maps

f_1, f_2, g_1, g_2 in function of the constants C, F, G and the (nontrivial) field endomorphism $x \mapsto \overline{x}$. Indeed,

$$\begin{cases} f_1(x) = C(x - \overline{x}), \\ f_2(x) = F\overline{x}, \\ g_1(x) = \overline{x}, \\ g_2(x) = C(x - \overline{x}) + G\overline{x}. \end{cases}$$

This replaces Identities (3)–(6) above.

We can now rewrite Identity (9) as

$$(\overline{C} + C - G)(\overline{a} - \overline{\overline{a}}) \quad = \quad 0, \tag{12}$$

which readily implies that $G = \overline{C} + C$, and hence, from Identity (7), $g_2(x) = Cx + \overline{C}\overline{x}$. Now, Identity (8) can be rewritten as

$$(F + C\overline{C})(a - \overline{\overline{a}}) \quad = \quad 0. \tag{13}$$

Identity (10) reduces to

$$C\overline{F} \quad = \quad \overline{\overline{C}}F, \tag{14}$$

whereas Identity (11) reduces to, taking into account Identity (13),

$$F + C\overline{C} \quad = \quad \overline{F} + \overline{C}\,\overline{\overline{C}}. \tag{15}$$

The case where g_1 is an involution—Suppose that $a = \overline{\overline{a}}$, for all $a \in \mathbb{F}$. Then, g_1 is surjective. Identity (15) implies that $F = \overline{F}$, and we have

$$M(a,b) = \begin{pmatrix} a & C(a - \overline{a}) + F\overline{b} \\ b & \overline{a} + Cb + \overline{C}\overline{b} \end{pmatrix}.$$

We perform the coordinate change mentioned in Formula (1) with $k = C$. This transforms $M(a,b)$ into (and we use the same notation $M(a,b)$ and set $K = F + C\overline{C}$)

$$M(a,b) = \begin{pmatrix} a & K\overline{b} \\ b & \overline{a} \end{pmatrix}.$$

Let $\mathbb{F}'$ be the fix field of g_1. Then, $F = \overline{F}$ belongs to $\mathbb{F}'$, and hence so does K. The latter cannot be written as $z\overline{z}$ for any $z \in \mathbb{F}$, as otherwise $M(z,1)$ is singular, a contradiction. Hence, this defines a quaternion algebra $\mathbb{H}$ over $\mathbb{F}'$ with the norm form $a\overline{a} - Kb\overline{b}$, with both $a, b \in \mathbb{F}$ considered as pairs of $\mathbb{F}'$ in the natural way with respect to the field extension $\mathbb{F}/\mathbb{F}'$.

Similarly (but even simpler) to before, one shows that $\mathcal{S}$ is obtained from the extension of $\mathbb{F}$ to $\mathbb{H}$.

The case where g_1 has order of at least 3—Hence, from now on, we may assume that g_1 is not an involution. This implies, following Identity (13), that $F = -C\overline{C}$. Identities (14) and (15) become redundant. So, we have $M(a,b) = \begin{pmatrix} a & C(a - \overline{a}) - C\overline{C}\,\overline{b} \\ b & \overline{a} + Cb + \overline{C}\,\overline{b} \end{pmatrix}$. Setting $a = -Cb$, we obtain $M(Cb,b) = \begin{pmatrix} -Cb & -C^2b \\ b & Cb \end{pmatrix}$, which has determinant 0 and hence does not define any legal member of $\mathcal{M}$.

Fix groups—We now determine the fix groups of the spreads found in the previous paragraphs.

Let $\theta(M,\sigma)$ be a semi-linear transformation in the vector space underlying S, with matrix M and field automorphism σ. Suppose that $\theta(M,\sigma)$ stabilises each line of $\mathcal{S}$. Then, e_1 and e_2 are mapped to points of $L_1 = \langle e_1, e_2\rangle$, and e_3 and e_4 are mapped to points of $L_2 = \langle e_3, e_4\rangle$. So, M is as follows

$$M = \begin{pmatrix} x & y & 0 & 0 \\ z & u & 0 & 0 \\ 0 & 0 & x' & y' \\ 0 & 0 & z' & u' \end{pmatrix}.$$

Expressing that $\theta(M,\sigma)$ stabilises each member $\langle (a,b,1,0),(f(a,b),g(a,b),0,1)\rangle$ of $\mathcal{S}$ results, by linear algebra, in the equalities

$$\begin{cases} a^\sigma x + b^\sigma y = ax' + f(a,b)z', \\ a^\sigma z + b^\sigma u = bx' + g(a,b)z', \\ f(a,b)^\sigma x + g(a,b)^\sigma y = ay' + f(a,b)u', \\ f(a,b)^\sigma z + g(a,b)^\sigma u = by' + g(a,b)u', \end{cases}$$

which must hold for all $a,b \in \mathbb{F}$. Setting $a = 1$ and $b = 0$, taking into account $f(1,0) = 0$ and $g(1,0) = 1$, we deduce that $(x,y,z,u) = (x',y',z',u')$. This implies that

$$\begin{cases} (a^\sigma - a)x + b^\sigma y - f(a,b)z = 0, \\ bx + (g(a,b) - a^\sigma)z - b^\sigma u = 0, \\ f(a,b)^\sigma x + (g(a,b)^\sigma - a)y - f(a,b)u = 0, \\ by - f(a,b)^\sigma z + (g(a,b) - g(a,b)^\sigma)u = 0, \end{cases}$$

for all $a,b \in \mathbb{F}$.

Suppose now first that $\sigma = \mathrm{id}$. Then, the first (and also the last) equation implies that if $b \neq 0$, then $f(a,b)b^{-1}$ is independent of a,b. This is only the case if $f_1 \equiv 0$, which in our examples only holds in Case (i) of Theorem 1 (if in the case $g_1 \neq \mathrm{id}$, $C = 0$, with the above notation, then the inverse coordinate change as given above transforms the matrices to a case where $f_1 \not\equiv 0$). Hence, $f(a,b) = Fb$ and $g(a,b) = a + Gb$. If $z = 0$, then $y = 0$ by the first equation, and $x = u$ by the second; hence, we have the identity. So, we may assume that $z = 1$. Then, $y = F$ by the first equation, and $u = x - G$. Hence, we get a group consisting of the identity and linear maps with 4×4 block matrices having two identical 2×2 blocks $\begin{pmatrix} x & F \\ 1 & x - G \end{pmatrix}$ on the diagonal, and 0 elsewhere. This group clearly acts sharply transitively on L_1 (and hence on every line of $\mathcal{S}$).

Now, suppose $\sigma \neq \mathrm{id}$. The second equality implies, setting $b = 0$, that either $z = 0$ or $\overline{a} = a^\sigma$ for all $a \in \mathbb{F}$. If $z = 0$, then the first equation implies first (setting $a = 0$) that $y = 0$ and then (for general a) $a = a^\sigma$ (as $x \neq 0$) for all $a \in \mathbb{F}$, a contradiction. Hence, $\overline{a} = a^\sigma$, for all $a \in \mathbb{F}$, and we are in the Galois case. Then, we may assume that $f(a,b) = K\overline{b}$ and $g(a,b) = \overline{a}$. With this it is now easy to calculate $x = u = 0$ and $y = Kz$. This yields a unique involution (the Galois involution).

This completes the proof of Theorem 1.

Remark 1. *(We again refer to [4] for undefined notions in the theory of buildings.) Theorem 1 illustrates three phenomena that can occur in order to construct subcomplexes of spherical buildings* *that are also buildings. The first phenomenon is* Galois descent, *where one considers the fixed complex of a Galois group (here, this group is the one generated by g_1). This phenomenon is completely understood; a classification can be found in [6]. The second is an analogue of this, but then using a linear group, one considers the fixed complex of a linear automorphism group. Usually, this group is larger than its Galois analogue (and, remarkably, the subcomplex is also—dimensionwise in the sense of algebraic groups—usually larger). Also, in the situation of the present paper, we can observe that in the linear case, the group acts transitively on each spread line. One could call this* linear descent. *This phenomenon is less well understood, and there is no classification but only partial results available. We refer to [7] for a substantial background and a systematic treatment of these two phenomena. The third does not use a group but is simply a subgeometry constructed in an algebraic (here using a subfield of a quaternion algebra) or geometric way; its fix group is trivial. We could call this* geometric descent. *As geometric descent seems to be a rare phenomenon, it would be interesting to determine other examples of the third phenomenon and perhaps classify under mild conditions. At present, and also inspired by the results of the present paper, the author is tempted to think that geometric descent is a characteristic* 2 *or* 3 *phenomenon. Is this really true?*

4. Groups of Projectivities

With the notation of Section 3, we have seen that an arbitrary regular line spread $\mathcal{S}$ of $\mathsf{PG}(3,\mathbb{F})$ can be represented as a set of lines

$$\{\langle(1,0,0,0),(0,1,0,0)\rangle\} \cup \{\langle(a,b,1,0),(\alpha b, a+\beta b,0,1)\rangle \mid a,b \in \mathbb{F}\},$$

for some constants $\alpha,\beta \in \mathbb{F}^\times$ such that the polynomial $x^2-\beta x-\alpha$ is irreducible over $\mathbb{F}$. We now determine the projectivity group $\Pi_{\mathcal{S}}(L_1)$, with $\langle 1,0,0,0),(0,1,0,0)\rangle = L_1 \in \mathcal{S}$ as before.

We first consider a special case. Let $L \in \mathcal{S}\setminus\{L_1,L_2\}$ be arbitrary. Then, we calculate $L_1 \overline{\wedge}_{L_{12}} L_2 \overline{\wedge}_L L_1$. The first perspectivity maps $(x,y,0,0)$ to $(0,0,x,y)$, and the second maps $(0,0,x,y)$ back to the point $(ax+\alpha by, bx+(a+\beta b)y,0,0)$. In binary coordinates, this yields the map

$$\begin{pmatrix} x\\ y\end{pmatrix} \mapsto \begin{pmatrix} a & \alpha b\\ b & a+\beta b\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix} =: M(a,b)\begin{pmatrix} x\\ y\end{pmatrix},$$

using similar notation as in Section 3. Note that this already defines a sharply transitive group G acting on the points of L_1.

Since the matrices $M(a,b)$ and $(a,b) \neq (0,0)$ form a multiplicative group, each projectivity $L_1 \overline{\wedge}_K L_2 \overline{\wedge}_L L_1$ and $K,L \in \mathcal{S}\setminus\{L_1,L_2\}$ has this form, and this can be written as $L_1 \overline{\wedge}_K L_2 \overline{\wedge}_{L_{12}} L_1 \overline{\wedge}_{L_{12}} L_2 \overline{\wedge}_L L_1$.

Moreover, since the matrices $M(a,b)$ form an additive group acting sharply transitively on $\mathcal{S}\setminus\{L_1\}$ (as deduced in Section 3), the same remains true if we substitute L_2 with an arbitrary member of $\mathcal{L}\setminus\{L_1\}$. Now, we can break up any sequence of projectivities

$$L_1 \overline{\wedge}_{K_1} M_1 \overline{\wedge}_{K_2} M_2 \overline{\wedge}_{K_3} M_3 \overline{\wedge}_{K_4} M_4 \overline{\wedge}_{K_5} \cdots$$

into subsequences of self-projectivities as follows

$$(L_1 \overline{\wedge}_{K_1} M_1 \overline{\wedge}_{K_2} L_1)\cdot(L_1 \overline{\wedge}_{K_2} M_2 \overline{\wedge}_{K_3} L_1)\cdot(L_1 \overline{\wedge}_{K_3} K_4 \overline{\wedge}_{K_5} L_1)\cdot\cdots,$$

which shows that the full group of projectivities $\Pi_{\mathcal{S}}(L_1)$ of L_1 is exactly G. This shows that if a line spread of $\mathsf{PG}(3,\mathbb{F})$ is regular, then all other conclusions in Theorem 3 hold.

Now, assume a spread $\mathcal{S}$ of $\mathsf{PG}(3,\mathbb{F})$ is not regular. Then, there exist four lines $K,L,M,N \in \mathcal{S}$ admitting a common transversal X such that the lines K,L,M admit a transversal Y not intersecting N. Clearly, the self-projectivity $K\overline{\wedge}_L M\overline{\wedge}_N K$ fixes $K \cap X$ but moves $K \cap Y$. Hence, $\Pi(K)$ does not act freely on K and hence also not sharply transitively. Moreover, if the fix group T of $\mathcal{S}$ fixed $x := K \cap X$, then we claim it is the identity. Indeed, T then fixes the plane $\langle L, x\rangle$ and hence fixes it pointwise as every point of the plane off L is fixed (because each such point is the intersection of $\langle L, x\rangle$ with a spread line). This holds for arbitrary $L \in \mathcal{S}$, and the claim follows.

This completes the proof of Theorem 3.

5. Concluding Remarks

Concerning Theorem 1—It was previously known that, for a composition line spread $\mathfrak{L}$ of $\mathsf{PG}(5, \mathbb{F})$, the projective plane $\mathsf{PG}(\mathfrak{L})$ satisfies the Moufang condition; that is, it is a translation plane with respect to every line (see [2]) or, equivalently, Desargues's little axiom holds. In algebraic terms, the plane is coordinatised by an alternative division algebra. The results of the present paper imply that $\mathsf{PG}(\mathfrak{L})$ is in fact always a Desarguesian projective plane; that is, Desargues's general axiom holds. In algebraic terms, the plane is coordinatised by an associative division ring, despite the fact that some alternative division rings $\mathbb{A}$ contain subfields $\mathbb{F}$ with $\dim_{\mathbb{F}} \mathbb{A} = 2$.

Concerning Theorems 2 and 3—The two characterisations of regular line spreads in $\mathsf{PG}(3, \mathbb{F})$ are meant to be applied in opposite circumstances. Indeed, Theorem 2 is designed to make it easier to prove that a certain line spread is regular, since the theorem weakens the regularity condition. Theorem 3, on the other hand, is designed to prove that certain line spreads are *not* regular. Indeed, as soon as some self-projectivity of the spread lines can be found that has some fixed point, the theorem implies that the line spread cannot be regular. This observation makes the results of the present paper particularly interesting in future research where line spreads will be used.

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