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Involutions of Exceptional Geometries of Type E_7

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Abstract

An involution of an exceptional geometry of type E_7 is called regular if its fix structure, viewed as simplicial complex, is a building. Involutions which do not act trivially on the underlying field correspond to Galois descent and were treated a long time ago by Jacques Tits. In the present paper, we classify the regular involutions of geometries of type E_7 that act trivially on the underlying (arbitrary) field, which we assume not to have characteristic 2. As a result, we discover new subgeometries of the exceptional geometry of type E_7.

Keywords: spherical buildings; involution; exceptional type E_7; generalised quadrangles; metasymplectic space; Moufang set

MSC: 51E24

1. Introduction

Involutions play a prominent and important role in both algebra and geometry. For instance, centralisers of involutions in known groups gave rise to new simple groups, cf. the simple Suzuki and Ree groups. Also, involutions in geometries usually give rise to interesting and large subgeometries, for instance Baer subspaces of projective spaces. These two examples connect in the theory of Tits buildings with each other by considering an involution in a spherical building: its centraliser may well be a simple algebraic group acting on the (large) fix structure, which is a spherical building itself.

Let Δ be a geometry of type E_7, defined over an arbitrary field $\mathbb{K}$ of characteristic not 2. To each involution of Δ is associated its fix geometry, that is, the geometry of the fixed vertices. The *fix diagram* encodes these types. The principal goal of the paper is to classify all linear involutions of Δ according to their fix diagrams, when the latter is not full, that is, not all types admit fixed vertices. In particular, we prove that each fix geometry is the geometry associated to a spherical building itself (possibly of rank 1, or even 0) and we provide an explicit list of all possibilities through their description/definition via Galois descent and the corresponding Tits index. If the geometry is empty, then we prove that the involution is anisotropic, that is, maps each simplex to an opposite simplex, and we also determine the *imaginary* fix building, yielding the (type of the) centraliser of the given involution.

Our method is geometric: We determine all fix structures of these linear involutions of spherical buildings of exceptional type E_7 that do not pointwise fix an apartment (or, equivalently, a chamber; see below for the motivation). As a corollary, we find a lot of simple algebraic groups inside the one of type E_7 which were not realised before. Since the root system of type A_7 is a subsystem of the root system of type E_7, it does not come

Academic Editor: Emil Saucan

Received: 10 August 2026
Revised: 6 September 2026
Accepted: 14 September 2026
Published: 17 September 2026

as a big surprise that forms of type A_7 turn up and play a central role. Recall that a linear involution is one that acts trivially on the underlying field. The non-linear involutions give rise to Galois descent, which is described in detail in [1]. The motivation of our restriction to involutions not fixing a chamber is mainly the analogy with Galois descent: each descent group corresponds to a Tits index (which can be viewed as a fix diagram) which determines the type and a thick spherical subbuilding of that type. If one fixes a chamber, then the fix structure is always a building of the same type, whereas otherwise the type changes, which is more interesting. However, in the Galois descent case, the fix diagram determines the type of the fixed building, whereas in the linear case, this does not seem to be always the case. The types of the fixed buildings of linear involutions fixing a chamber seem to be dictated by the Borel–de Siebenthal theory (see [2]; see also [3,4]) since in this case (and still in characteristic different from 2) one also fixes an apartment pointwise (see Proposition 1 below). When a linear involution does not fix a chamber, the type of the fixed building is not easily predictable, and cases occur where the same fix diagram gives rise to different types of fixed buildings (see Types I and II, and types III and IV in Theorem 1). Also, the study of involutions fixing a chamber requires different methods, and this shall be done elsewhere.

Let us also mention that the case of type E_7 is particularly interesting since it allows the largest number of Tits indices and fix diagrams for involutions among all exceptional types. This implies for instance that, in contrast to all other exceptional types, there are buildings of rank 3 which are the fix structure of an involution in a building of type E_7. See Figure 1 for the enumeration of all possible fix diagrams for involutions of a building of type E_7.

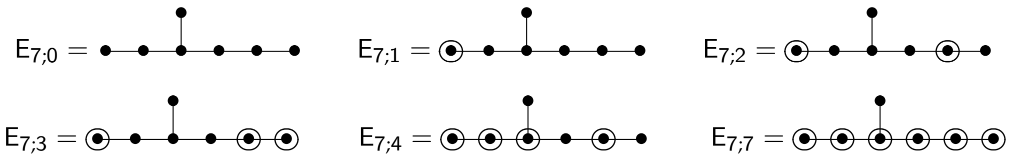


Figure 1. The possible fix diagrams for involutions in $E_7(\mathbb{K})$.

1.1. Motivation

In general, involutions play an important role in many theories. Perhaps the best example is in group theory itself, where the theory of centralisers of involutions is heavily used in the classification of finite simple groups; see [5,6]. Concrete examples are the old discoveries of new (classes of) finite simple groups as centrailisers of involutions, such as the Suzuki groups [7], the Ree groups [8,9] and the Steinberg groups [10]. Also, many sporadic groups were discovered via involutions, or at least described as centralisers of involutions in the Monster groups; see [11] (Chapter 1) and many references therein. Another example are the involutions in algebras; we refer to [12] for a book dedicated to the study of algebras with involutions in connection with (simple linear) algebraic groups. Many examples of Galois descent in algebraic groups come with involutions; see [1]. The linear counterpart for buildings of exceptional type E_6 has been studied recently in [13]; see also [14]. Involutions also play a crucial role in decomposition results for groups with a BN-pair, possibly related to twin buildings; see [15].

Hence, a study of linear involutions in buildings of type E_7 seems well motivated. The absolute geometry of a linear involution in a spherical building is in general larger than the one of a non-linear involution. So, our classification in particular exhibits relatively large subgeometries of geometries of type E_7. This contributes to our knowledge of the structure of such geometries, just like the subgroup structure of a group provides very useful information. Moreover, the centralisers of the involutions are large subgroups of the groups of type E_7 and our results contribute to the subgroup structure of those groups.

Let us finally mention that the assumption of characteristic not equal to 2 is necessary, as in characteristic 2, linear involutions behave completely different. In particular, there are linear involutions not fixing any chamber but yet their fix structure is not a subbuilding. This requires other, additional methods and shall be done elsewhere.

1.2. Structure of the Paper

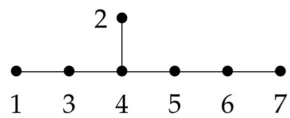
In Section 2, we introduce the necessary notions in order to be able to state our main results more precisely. Then, in Section 3, we recall some known results that we will use in our proofs. Most of them will concern the standard exceptional geometry of type E_7 over an arbitrary field $\mathbb{K}$, which fully describes the corresponding building of type E_7. Our method requires some knowledge about involutions in two other types of geometries: projective spaces and hyperbolic quadrics in projective spaces of dimension at most 11. We study the latter in Section 4. Then, in Section 5, we prove our main results. The last section is devoted to examples, which should also settle existence of the different classes of involutions turning up in our main result.

Sections 2 and 3 contain some overlap with [13], as the goals and background of the latter are very similar to those of the present paper. Instead of referring, we rather repeat for the convenience of the reader.

This paper grew out of the master thesis of the first author, written under the supervision of the second author at Ghent University, academic year 2025–2026.

2. Main Result

We assume the reader is familiar with the basics of (spherical) building theory; see [16,17], in particular with opposition, Coxeter and Dynkin diagrams and the Moufang condition. We will view a spherical building as a simplicial complex, numbering the types of the vertices with Bourbaki labelling [18]. In the present paper, we are concerned with buildings of type E_7. Their Coxeter or Dynkin diagram, including the Bourbaki labelling, looks like



It is proved in [17] that for each field $\mathbb{K}$, there exists a unique building of type E_7, which we denote by $\mathsf{E}_7(\mathbb{K})$, and call the *building of type* E_7 *over* $\mathbb{K}$. The latter is sometimes also referred to as the *ground field*. It means that each projective space that turns up as a residue is defined using a vector space over $\mathbb{K}$.

In order to state our main results, we need terminology and notation to describe the fix geometry of an involution within $\mathsf{E}_7(\mathbb{K})$, and its abstract isomorphism type within the world of spherical buildings. We will do so with the help of fix diagrams and Tits indices. The former, being more intuitive than the latter, helps to better understand the latter.

2.1. Fix Diagrams

Let Δ be an arbitrary irreducible spherical building, and let G be an automorphism group of Δ. Then, the *fix diagram* for G is the Coxeter/Dynkin diagram of Δ furnished with encircled orbits of nodes under the action of G on the diagram indicating the types of minimal simplices that are fixed by G. Note that, as soon as G preserves types and fixes a chamber, which is somehow the generic case, the fix diagram is trivial—all nodes are separately encircled; but in this case, the fix diagram obviously does not provide useful information. However, if the fix structure is a building again, then such diagrams are very useful, and we will only use fix diagrams in that case. Some examples related to E_7 are given in Figure 1, where we also introduce the names of the various fix diagrams, for later reference.

Fix diagrams in which each node is separately encircled are called *full*, e.g., $\mathsf{E}_{7;7}$.

2.2. Tits Indices

Tits indices are introduced in [1], and therein called simply *indices*, as a generalisation of the classical Witt index. They can be considered as a special case of a fix diagram in the following way. Recall that each simple algebraic group G gives rise to a unique irreducible spherical building $\Delta(G)$; see chapter 5 of [17]. Now, let G be defined over an algebraically closed field $\mathbb{F}$. Let Ψ be an automorphism group of $\mathbb{F}$. Denote the fixed field by $\mathbb{K}$. Note that Ψ is a Galois group for the extension $\mathbb{F}/\mathbb{K}$. Then, one can let Ψ act on G as an outer automorphism group. The fixed point group—also called the group of rational points in the literature—is a simple algebraic group H over $\mathbb{K}$. The spherical building $\Delta(H)$ arises from $\Delta(G)$ as the fix structure of the group Ψ. The corresponding Tits index is the fix diagram furnished with some other data regarding the algebraic groups, such as dimensions of anisotropic kernels and the like; see [1]. Since for our purposes, it is enough to understand the connection with fix diagrams, we refer to [1] for more precise information. We note that, if a spherical building arises from a Tits index, then the latter is unique for the building. A building is called *split* if it arises from a Tits index which is full as a fix diagram. Written in the symbols of [1], the Tits index is then of the form ${}^1\mathsf{X}_{n,n}^{(1)}$, if $\mathsf{X} \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$, and ${}^1\mathsf{X}_{n,n}^{0}$, if $\mathsf{X} \in \{\mathsf{E}, \mathsf{F}\}$. For instance, every building of type E_7 is split, as follows from the tables in [1]. The type of the building $\Delta(G)$ is called the *absolute type* of $\Delta(H)$, whereas the actual type of $\Delta(H)$ is sometimes referred to as the *relative type*. We say that $\Delta(H)$ arises from $\Delta(G)$ by *Galois descent*.

2.3. Moufang Buildings

Although we will not need a formal definition, it is instructive to mention the notion of a Moufang (spherical) building. This is a spherical building whose automorphism group satisfies certain transitivity properties; see [17] (Addendum). All spherical buildings of rank at least 3 are Moufang and they arise either from Galois descent, or from a classical group, or from a group of mixed type F_4. This is, for instance, well explained in [19]. However, in rank 2, there are exotic examples of buildings. Rank 2 simplicial complexes are just graphs and irreducible thick spherical buildings of rank 2 are graphs with diameter n and girth $2n$, $n \geq 3$, without vertices of valency 2. These are the thick generalised n-gons. The *Moufang condition* [20] distinguishes the exotic ones from the ones that behave just like the higher rank irreducible spherical buildings. For instance, the Moufang rank 2 buildings also arise either from Galois descent, or from a classical group, or from groups of mixed type $\mathsf{B}_2, \mathsf{F}_4$ or G_2 (this is the main result of [20]). We note that some Moufang buildings can arise both from Galois descent and from a classical group. A group of mixed type is a phenomenon in low characteristic using purely inseparable field extensions. It only applies to buildings of type B_2 and F_4 in characteristic 2 and type G_2 in characteristic 3. Examples of exotic rank 2 spherical buildings are finite non-Desarguesian projective planes (see [21]), and some finite generalised quadrangles, the first known ones contained in [22] (attributed to Jacques Tits there), later ones surveyed in [23].

The importance of the Moufang condition for the present paper is that, if the fix structure of an involution is a building, then it is a Moufang building; see [24] (Theorem 24.31). Hence, the fixed building either arises from Galois descent, or from a classical group, of from a group of mixed type. In our cases, it will always arise from Galois descent.

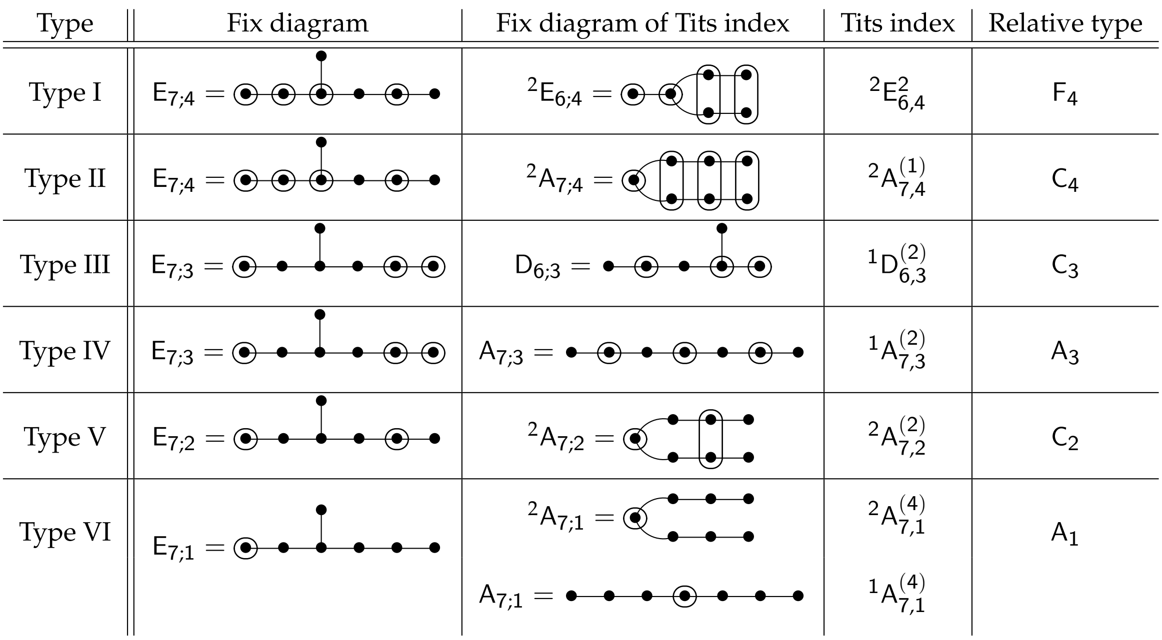
2.4. Preview of the Main Results

Finally, before stating our main result, we discuss the way we present it. Our theorem contains information about different types of linear involutions of buildings of type E_7. This information is given in tabular form. For a given involution, we provide two data. The first one is the fix diagram, as discussed in Section 2.1. This explains how the fix structure sits into the ambient building of type E_7 by revealing which types of simplices are fixed. The second one is the Tits index and corresponding relative type. This provides the abstract isomorphism type of the fixed building as explained above in Section 2.2. In one case, the fixed building has rank 1, which we did not yet discuss. In that case, ref. [24] (Theorem 24.31) asserts that the fixed building is a Moufang set, as defined and discussed by Tits in [25]. The ones that we will encounter will all arise from Galois descent, and hence, again, they correspond to Tits indices.

2.5. Main Result

We can now state our main result. We assign each class of involutions a type for further reference.

Theorem 1. *Let Δ be a building of type E_7 over a field $\mathbb{K}$ with $\mathrm{char}\,\mathbb{K} \neq 2$. Let ρ be a linear involution of Δ. Suppose ρ is not anisotropic, that is, it maps at least one object not to an opposite object. Then, its fix structure is a building (possibly of rank 1) and, if ρ does not pointwise fix an apartment, then the associated fix diagram and corresponding Tits index are given as in Figure 2. Along with the diagram, we mention the name of the corresponding fix diagram. We add a column with the name and symbol of the Tits index as introduced by Tits in [1], and a column with the relative type of the fixed building.*



Type	Fix diagram	Fix diagram of Tits index	Tits index	Relative type
Type I	$\mathsf{E}_{7;4} =$	${}^2\mathsf{E}_{6;4} =$	${}^2\mathsf{E}^2_{6,4}$	F_4
Type II	$\mathsf{E}_{7;4} =$	${}^2\mathsf{A}_{7;4} =$	${}^2\mathsf{A}^{(1)}_{7,4}$	C_4
Type III	$\mathsf{E}_{7;3} =$	$\mathsf{D}_{6;3} =$	${}^1\mathsf{D}^{(2)}_{6,3}$	C_3
Type IV	$\mathsf{E}_{7;3} =$	$\mathsf{A}_{7;3} =$	${}^1\mathsf{A}^{(2)}_{7,3}$	A_3
Type V	$\mathsf{E}_{7;2} =$	${}^2\mathsf{A}_{7;2} =$	${}^2\mathsf{A}^{(2)}_{7,2}$	C_2
Type VI	$\mathsf{E}_{7;1} =$	${}^2\mathsf{A}_{7;1} =$	${}^2\mathsf{A}^{(4)}_{7,1}$	A_1
		$\mathsf{A}_{7;1} =$	${}^1\mathsf{A}^{(4)}_{7,1}$	

Figure 2. Regular involutions of $\mathsf{E}_7(\mathbb{K})$ with $\mathrm{char}\,\mathbb{K} \neq 2$.

One case we did not yet discuss, but only briefly alluded to above, is the case in which the linear involution does not fix any vertex of the building. In this case, we will show that the involution is anisotropic (as mentioned above, this means that every simplex is mapped onto an opposite simplex). One can in that case not talk about the Tits index of the fixed building, but one can still determine the type of the so-called *imaginary* fixed building, that is, the building fixed over the algebraic closer of the base field. We shall prove the following result.

Theorem 2. *Let* Δ *be a building of type* E_7 *over a field* $\mathbb{K}$ *with* $\mathsf{char}\,\mathbb{K} \neq 2$*. Let* ρ *be a linear involution of* Δ *and suppose that* ρ *is anisotropic. Then, the type of the imaginary fixed building is* A_7*.*

We will refer to a linear anisotropic involution as one of Type VII.

2.6. Some More Discussion

It is interesting to note that almost all absolute types of the fix structures are A_7, and almost all possible Tits indices of the latter occur. As mentioned in the introduction, this has to do with the fact that the root system of type A_7 is a subsystem of the root system of type E_7 (and no other subsystem of roots is irreducible and has rank 7).

Let us mention once again that Tits [1] studied the case of non-linear involutions. The related Tits indices correspond to the fix diagrams given in Figure 1 and are $\mathsf{E}^{133}_{7,0}$, $\mathsf{E}^{66}_{7,1}$, $\mathsf{E}^{31}_{7,2}$, $\mathsf{E}^{28}_{7,3}$, $\mathsf{E}^{9}_{7,4}$ and $\mathsf{E}^{0}_{7,7}$. So, our paper can be considered as a complement to these results.

As already mentioned, we refer to Section 6 for (the construction of) examples. There, it will become clear which properties a field needs to have in order to admit a regular involution of a given type. Here, we mention that over a finite field, only involutions of types I and II exist, and over the real numbers, only Types I up to IV, and VII occur. Over the complex numbers, as well as over any algebraically closed field, every linear involution fixes a chamber as follows from the Density Theorem (see [26] (Section 22.2)).

3. Preliminaries

3.1. Buildings and Point-Line Geometries

As already mentioned, we assume familiarness with the basic notions in the theory of spherical buildings. We also use the standard notation such as $\mathsf{Res}_\Delta(x)$ for the residue in the building Δ of the element x (sometimes omitting Δ from this notation if no ambiguity can arise). Elements of Δ which belong to $\mathsf{Res}_\Delta(x)$ and which are opposite in $\mathsf{Res}_\Delta(x)$ are sometimes called *locally opposite at* x. We say that an automorphism is *anisotropic* if it maps every simplex to an opposite simplex. We have the following characterisation of anisotropic automorphisms. It basically says that it suffices to look at the images of vertices of any given type in order to check whether an automorphism is anisotropic or not.

Lemma 1 (Theorem 3.1 of [27])**.** *An automorphism of a spherical building maps every element of a given type to an opposite element if, and only if, it is anisotropic.*

The following proves a claim made in the introduction. In the proof, we use a lot of notions of the theory of buildings; since we do not need these notions in the sequel, we refer to [16] and especially the first three chapters of [17].

Proposition 1. *If a linear involution* ρ *of a building of type* E_7 *over a field* $\mathbb{K}$ *of characteristic different from* 2 *fixes a chamber, then it pointwise fixes an apartment.*

Proof. Let C be a fixed chamber. We prove by induction on the length of the shortest gallery stretched between C and a chamber D, that there exists a fixed chamber at arbitrary (gallery) distance from C, as long as this distance does not exceed the diameter of the chamber graph. Indeed, this is trivially true for distance 0. Suppose now that there exists a fixed chamber D at distance $d \geq 0$ from C. If d is the maximal distance, then there is nothing to prove. If not, then we can find a chamber E adjacent to D at distance $d+1$ from C. Then, every chamber through the panel $D \cap E$, except for D, has distance $d+1$ from C. Hence, it suffices to show that ρ fixes an additional chamber through $D \cap E$. The involution ρ acts on the set of chambers through $D \cap E$ as a linear automorphism of the projective line $\mathsf{PG}(1, \mathbb{K})$. Hence, we can represent the action of ρ on the set of chambers through $D \cap E$ as a matrix $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, with $a^2 - bc \neq 0$, and the chambers through $D \cap E$ can be represented by homogeneous coordinates $(x, y) \in \mathbb{K} \times \mathbb{K}$, determined up to a scalar multiple. We may assume action on the left. We may identify the chamber D with the coordinates $(1, 0)$. Then, we see that $c = 0$. Hence, we have the additional fixed coordinates $(b, -2a)$ and our claim that ρ fixes a chamber at essentially arbitrary distance from C is proved. Now, we consider a fixed chamber C' opposite C. Their convex closure is an apartment and the proposition is proved. □

The way we are going to approach buildings in the sequel of this paper is via their most standard geometry. More precisely, we turn them into point-line geometries using a well known recipe. Before quickly describing that recipe, we define point-line geometries.

Definition 1. *A point-line geometry Γ is a bipartite graph where the vertices of a given class X are called points and the vertices of the other class Y are called lines. Points x, y adjacent to a common line are called collinear and we denote $x \perp y$. The set of points collinear to a given point x is denoted as $x^\perp$. A repeated line is a set of points that is the neighbourhood of two distinct lines. A subgeometry is an induced subgraph, and it is called full if it contains the full neighbourhood in Γ of any of its lines. A full subgeometry is also called a subspace A subgeometry is called convex if it contains all vertices of each shortest path between every pair of its vertices. A subgeometry is called a geometric hyperplane if for each line L, the neighbour set $\Gamma(L)$ either is contained in it, or intersects the subgeometry in a unique point. It is called proper if it does not coincide with the whole geometry.*

For a given point-line geometry Γ with classes X and Y, we define $\mathcal{L} = \{\Gamma(y) \mid y \in Y\}$. Then, the bipartite graph Γ' with classes X and $\mathcal{L}$, where a point $x \in X$ is adjacent to any member L of $\mathcal{L}$ containing x, is a point-line geometry without repeated lines. We call Γ' the reduction *of Γ. The graph Γ' is completely determined by the sets X and $\mathcal{L}$.*

Definition 2. *Now, let Δ be an irreducible spherical building. Recall that, for the type set, we always use the Bourbaki labelling [18] of vertices of the corresponding diagram. Let X be the set of vertices of some given type, say i. Define Y to be the set of panels of cotype i. A vertex v of type i is adjacent to a panel P of cotype i if $P \cup \{v\}$ is a chamber. This defines a point-line geometry Γ the reduction of which is called the i-Grassmannian of Δ. If the diagram of Δ is simply laced, and Δ has rank $n \geq 3$, then Δ is determined by its diagram, say X_n, and a given skew field $\mathbb{K}$. In that case, we denote Δ by $\mathsf{X}_n(\mathbb{K})$ and its i-Grassmannian Γ by $\mathsf{X}_{n,i}(\mathbb{K})$. In general, we say that Γ is a Lie incidence geometry of type $\mathsf{X}_{n,i}$.*

3.2. Projective Spaces

With the conventions of Definition 2, the projective space of dimension n over a skew field $\mathbb{K}$, which is usually denoted $\mathsf{PG}(n, \mathbb{K})$, is also denoted by $\mathsf{A}_{n,1}(\mathbb{K})$. If $\mathbb{K}$ is a quaternion division algebra, then we call the projective space *quaternion*. Projective planes are Lie incidence geometries of type $\mathsf{A}_{2,1}$. A *collineation* of $\mathsf{PG}(n, \mathbb{K})$ is a permutation of the point set preserving the line set. An *involution* is a collineation of order 2. A *duality* is an automorphism of the underlying building $\mathsf{A}_n(\mathbb{K})$ that, using the usual Bourbaki labelling [18], interchanges the types i and $n-i+1$. A *polarity* is a duality of order 2. The projective dimension $\dim U$ of a subspace of $\mathsf{PG}(d, \mathbb{K})$ is its dimension as a projective subspace. Since $\mathsf{PG}(d,\mathbb{K})$ arises from a vector space V of dimension $d+1$ by taking for points the 1-dimensional subspaces of V, and for lines the 2-dimensional subspaces of V, the subspace U of $\mathsf{PG}(d,\mathbb{K})$ corresponds to a subspace of V of dimension $1+\dim U$. With this notation, a point of $\mathsf{PG}(d,\mathbb{K})$ has dimension 0 and the empty set has dimension -1. In the present paper, we will always use these projective dimensions unless explicitly mentioned otherwise. We will occasionally abbreviate a "subspace of (projective) dimension d" with a "d-space".

We will need the following two lemmas for projective spaces.

Lemma 2. *An involution of a projective space of even dimension fixes at least one point. An involution of a projective plane fixes at least three points and at least three lines.*

Proof. The first assertion follows from Proposition 3.3 of [28]. The second assertion follows from Theorem 4.4 of [21]. □

The next lemma immediately follows from [28] (Proposition 3.3).

Lemma 3. *Let σ be an involution of $\mathsf{PG}(d,\mathbb{K})$, $d\geq 2$, $\mathrm{char}\,\mathbb{K}\neq 2$. Suppose σ is induced by a linear transformation of the underlying vector space.*

(i) Suppose that either d is even, or σ admits at least one fixed point. Then, the set of fixed points is the union of two disjoint subspaces U and U', with $\dim U+\dim U'=d-1$.

(ii) Suppose that d is odd and σ does not fix any point. Then, σ fixes a spread of lines elementwise, that is, it globally stabilises a line through each point.

We also introduce the following notation: for a set S of points of a projective space, we denote by $\langle S\rangle$ the projective subspace generated by the elements of S. Note also Grassmann's identity, which states that, if U and W are two subspaces of $\mathsf{PG}(d,\mathbb{K})$, then $\dim U+\dim W=d+\dim(U\cap W)$.

3.3. Polar Spaces

A *polar space* is a point-line geometry for which $x^\perp$ is a geometric hyperplane. If this is always proper, then the polar space is called *non-degenerate*, otherwise *degenerate*. With the standard definition of residue at a point, we note that a polar space is degenerate as soon as the residue at some point is degenerate.

3.3.1. Hyperbolic Quadrics

In the present paper, we will approach buildings of type E_7 mainly via their Lie incidence geometries of type $\mathsf{E}_{7,7}$. An important ingredient to define such geometries and list their properties is the notion of a hyperbolic quadric and the related building of type D_n, for some natural $n\geq 3$.

Definition 3. *A hyperbolic quadric Q (over the field $\mathbb{K}$) is the set of points of a projective space $\mathsf{PG}(2n-1,\mathbb{K})$, $n\geq 2$, whose coordinates with respect to a suitable basis, satisfy the quadratic equation*

$$X_{-n}X_n+X_{-n+1}X_{n-1}+\cdots+X_{-2}X_2+X_{-1}X_1=0,$$

where we work with generic coordinates $(x_{-n},\ldots,x_{-1},x_1,\ldots,x_n)$. If $\mathrm{char}\,\mathbb{K}\neq 2$, then such a quadric also arises from a polarity η of $\mathsf{PG}(2n-1,\mathbb{K})$ by collecting the absolute elements, that is, the subspaces U of $\mathsf{PG}(2n-1,|K)$ contained in their own image under η. The maximum (projective) dimension of subspaces entirely contained in Q is $n-1$, and such subspaces are called generators. They fall into two classes in such a way that subspaces belonging to distinct classes intersect each other in a projective subspace of dimension $n-i$, with i even. We call these classes *the oriflamme classes. Subspaces contained in Q are usually called singular subspaces. The vertices of the corresponding building $\mathsf{D}_n(\mathbb{K})$ of type D_n are the singular subspaces of projective dimension $0, 1, \ldots, n-3, n-1$, and the maximal singular subspaces comprise two types of vertices according to the oriflamme class they are contained in. Two vertices corresponding to singular subspace of dimension $n-1$ form a simplex if they intersect in a subspace of dimension $n-2$ (see [17]). A duality of $\mathsf{D}_n(\mathbb{K})$ is a collineation of $\mathsf{PG}(2n-1,\mathbb{K})$ preserving Q and interchanging the two oriflamme classes. As usual, a polarity is a duality of order* 2.

Note that the points and lines of Q define the geometry $\mathsf{D}_{n,1}(\mathbb{K})$.

In general, quadratic equations define quadrics, and the Witt index of a quadric is one more than the maximum projective dimension of a singular subspace. A quadric is *non-degenerate* if there exist disjoint maximal singular subspaces. A non-degenerate quadric of Witt index n

(1) In $\mathsf{PG}(2n,\mathbb{K})$ is called a *parabolic quadric*;
(2) In $\mathsf{PG}(2n+1,\mathbb{K})$ is called an *elliptic quadric*;
(3) In $\mathsf{PG}(2n+d,\mathbb{K})$, $d \geq 2$, is called a *hyperelliptic quadric*.

We have the following property of involutions and polarities in $\mathsf{D}_{n,1}(\mathbb{K})$, $n \geq 2$.

Lemma 4. *Every polarity ρ of a hyperbolic quadric $\mathsf{D}_{2n,1}(\mathbb{K})$, $n \geq 1$, fixes at least one point. Every type-preserving involution ρ of a hyperbolic quadric $\mathsf{D}_{2n+1,1}(\mathbb{K})$, $n \geq 1$, fixes at least one point.*

Proof. Let U be an arbitrary maximal singular subspace. Then, either $U \cap U^\rho$ is a point (which is fixed under ρ), or $U \cap U^\rho$ is a projective space Π of even dimension. In the latter case, ρ induces an involution in Π. Then, the result follows from Lemma 2. $\square$

3.3.2. Small Hermitian and Quaternion Polar Spaces

Let $\mathbb{K}$ be a field, not of characteristic 2. Let σ be an involution of $\mathbb{K}$. Then, the Hermitian variety in $\mathsf{PG}(2n-1,\mathbb{K})$ defined, with respect to a suitable coordinate system, by the equation

$$X^\sigma_{-n}X_n + X^\sigma_{-n+1}X_{n-1} + \cdots + X^\sigma_{-2}X_2 + X^\sigma_{-1}X_1 = X^\sigma_n X_{-n} + X^\sigma_{n-1}X_{-n+1} + \cdots + X^\sigma_2 X_{-2} + X^\sigma_1 X_{-1},$$

is called a *small Hermitian polar space (of rank n)*. The same equation over a quaternion division algebra $\mathbb{H}$ of $\mathbb{K}$, where σ is the standard involution in $\mathbb{H}$, defines a polar space which we call a *quaternion polar space (of rank n)*.

3.4. Geometries of Type $\mathsf{E}_{7,7}$

Now, let Ω be a building of type E_7 over the field $\mathbb{K}$. We are going to work with the Lie incidence geometry $\mathsf{E}_{7,7}(\mathbb{K})$, the 7-Grassmannian of Ω, denoted by $\Delta = (X,\mathcal{L})$, and with $\mathsf{E}_{7,1}(\mathbb{K})$, the 1-Grassmannian of Ω, denoted by $\Delta^* = (X^*,\mathcal{L}^*)$. The respective sets of symps are denoted as Ξ and Ξ^*. A good reference is [29]; see also the later chapters of [30]. Additional properties of $\mathsf{E}_{7,7}(\mathbb{K})$ will be derived using an apartment of the corresponding building (since every pair of chambers are contained in an apartment, this method will be especially efficient when a property only involves two simplices). We gather them in this preliminary section as they are independent of an involution. This way, they do not interrupt the flow of the proof of the main results in Sections 4 and 5.

We begin with describing the elements of $\mathsf{E}_{7,7}(\mathbb{K})$ and linking them to the Coxeter diagram. The points of $\mathsf{E}_{7,7}(\mathbb{K})$ are the vertices of type 7 of the building $\mathsf{E}_7(\mathbb{K})$, by the very definition. The lines correspond to the vertices of type 6. Vertices of type 5 are planes of $\mathsf{E}_{7,7}(\mathbb{K})$, whereas vertices of type 4, 3 and 2 are projective subspaces of dimension 3, 5 and 6, respectively. We refer to them as 3-spaces, 5-spaces and 6-spaces, respectively. Finally, vertices of type 1 correspond to convex subspaces of $\mathsf{E}_{7,7}(\mathbb{K})$ isomorphic to $\mathsf{D}_{6,1}(\mathbb{K})$ and will be called *symplecta*, or briefly *symps*, as in the theory of parapolar spaces. However, we avoid introducing the latter theory here, and refer instead to the later chapters of [30].

We display the basic properties of $\mathsf{E}_{7,7}(\mathbb{K})$ (which can be found in [29]).

Proposition 2. *Let x, y be two points of $\mathsf{E}_{7,7}(\mathbb{K})$ and let ξ, ζ be two symps. Then, the following properties hold.*

(i) *Either $x = y$, or there is a unique line containing both x and y, or x and y are not collinear and there is a unique symp containing both x and y, or no symp contains both x and y;*

(ii) *Either $x \in \xi$, or x is contained in unique 6-space which intersects ξ in a maximal singular subspace distinct from a 5-space, or x is collinear to a unique point of ξ (in the latter case, we refer to this situation as x being far from ξ);*

(iii) *Either $\xi = \zeta$, or $\xi \cap \zeta$ is a 5-space (we then call them adjacent), or $\xi \cap \zeta$ is a line, or $\xi \cap \zeta = \varnothing$ and there is a unique symp adjacent to both ξ and ζ, or $\xi \cap \zeta = \varnothing$ and every point of ξ is far from ζ and vice versa.*

The intersection of the unique 6-space in (iii) with the symp ξ will be referred to as a $5'$-space. It follows from considering the diagram of type E_7 that 5-spaces and $5'$-spaces of a symp belong to different oriflamme classes. The unique symp through x and y in (i) is sometimes denoted as $\xi(x, y)$. Pairs of points x, y contained in a unique symp are referred to as *symplectic pairs* (we also say that x is symplectic to y and denote this as $x \perp\!\!\!\perp y$).

Definition 4. *As already mentioned in the propositions, two symps of $\mathsf{E}_{7,7}(\mathbb{K})$ are called adjacent if they intersect in a 5-space. A point x and a symp ξ are called far if exactly one point of ξ is collinear to x; they are called close if x is collinear to a $5'$-space of ξ.*

In fact, the properties in Proposition 2 can easily be checked inside an apartment, keeping in mind that every pair of simplices is contained in a common apartment. We now present an explicit construction of such an apartment Σ, based on [3] (§6), and we call it the *standard apartment*. Set $[m, n] = \{m, m+1, \ldots, n\}$, $m, n \in \mathbb{N}$. The point set of Σ is

$$\{S \subseteq [1,8] \mid |S| \in \{2,6\}\}.$$

Let S_1 be the set of members of S or size 2 and S_2 those of size 6. Lines, or vertices of type 6, in Σ are the pairs

$$\begin{cases} \{s_1, s_1'\} \subseteq S_1, & |s_1 \cap s_1'| = 1, \\ \{s_2, s_2'\} \subseteq S_2, & |s_2 \cup s_2'| = 7, \\ \{s_1, s_2\}, s_1 \in S_1, s_2 \in S_2, & s_1 \subseteq s_2. \end{cases}$$

This defines a graph Γ. The vertices of types 5, 4 and 2 correspond to the cliques of size 3, 4 and 7, respectively, those of type 3 to the maximal cliques of size 6, and those of type 1 to the so-called *hexacrosses*, which are induced subgraphs isomorphic to a clique of size 12 minus a perfect matching. The latter is an apartment of a building of type D_6. There are two kinds of hexacrosses in this description: those that consist of the elements of S_1 contained in a given 4-set T of $\{1, 2, \ldots, 8\}$ and the elements of S_2 containing T; and those that, for a given ordered pair (i, j), $i, j \in [1, 8]$, consist of the members of $S_1 \cup S_2$ containing i but not j.

Let us for instance check Proposition 2(ii). As symp ξ we can take the set of vertices, with obvious shorthand notation,

$$\xi = \{12, 13, 14, 23, 24, 34, 123456, 123457, 123458, 123467, 123468, 123478\}.$$

Now, the vertex $\{i, j\} \subseteq [5, 8]$ is adjacent to only $1234ij$; the vertex $\{i, j\}$ with $i \in [1, 4]$ and $j \in [5, 8]$ is adjacent to exactly the vertices $\{i, k\}$, $k \in [1, 4] \setminus \{i\}$, and the vertices $\{1, 2, 3, 4, j, \ell\}$, $\ell \in [5, 8] \setminus \{j\}$, which form a maximal 6-clique. "Complementarily", with the same notation as above, the vertex $[1, 8] \setminus \{i, j\}$, $\{i, j\} \subseteq [1, 4]$, is adjacent to only the vertex $[1, 4] \setminus \{i, j\} \in S_1$, and the vertex $[1, 8] \setminus \{i, j\}$ with $j \in [14]$ and $i \in [5, 8]$ is adjacent to exactly the vertices $[1, 8] \setminus \{i, k\}$, $k \in [5, 8] \setminus \{i\}$, and the vertices $[1, 4] \setminus \{j, \ell\}$, $\ell \in [1, 4] \setminus \{j\}$, which form a maximal 6-clique.

From now on, we leave such straightforward checks to the reader.

Similarly, one proves the following basic properties.

Lemma 5. *Let $x, x' \in X$, $\xi \in \Xi$ and U, U' two $5'$-spaces contained in ξ. If $x \perp U \subseteq \xi \supseteq U' \ni x'$, with $x, x' \notin \xi$ and $U \cap U' = \varnothing$, then $x \equiv x'$.*

Lemma 6. *For each symp ξ and each 6-space W, there exists at least one point $x \in W$ close to ξ. That point is unique if some symp zeta intersecting W in a 5-space is opposite ξ.*

3.5. Geometries of Type $\mathsf{E}_{7,1}$

The geometry $\mathsf{E}_{7,1}(\mathbb{K}) =: \Delta^* = (X^*, \mathcal{L}^*)$, by definition derived from Ω, can also be defined directly from $\Delta = \mathsf{E}_{7,7}(\mathbb{K})$ by declaring $X^* = \Xi$ and $\mathcal{L}^*$ to be the set of sets of symps containing a given 5-space. It is an example of what is called in the literature often a *long root subgroup geometry* [31], or also a *hexagonic geometry* [30]. Such geometries also have been characterised as *root filtration* or *root shadow spaces* [32]. We will not need these notions, but refer to these references for the properties that we will collect below, which can also be derived from either the diagram or a suitable representation of an apartment Σ^* with type 1 elements as points. The latter goes as follows.

Consider again the set $[1, 8]$ of integers from 1 to 8. The points of Σ^* are the ordered pairs of $[1, 8]$ and the subsets of size 4 of $[1, 8]$. The lines are the pairs $\{(a, b), (a, c)\}$ and $\{(a, c), (b, c)\}$, $a, b, c \in [1, 8]$, $|\{a, b, c\}| = 3$, the pairs $\{\{i, j, k, \ell\}, (i, m)\}$, $i, j, k, \ell, m \in [1, 8]$, $|\{i, j, k, \ell, m\}| = 5$, and the pairs $\{\{i, j, k, \ell\}, \{j, k, \ell, m\}\}$, $i, j, k, \ell, m \in [1, 8]$, $|\{i, j, k, \ell, m\}| = 5$. Again k-cliques of the corresponding graph represent singular subspaces of dimension $k - 1$, and the symps are the 756 induced subgraphs isomorphic to pentacrosses. They come in two flavours: for four distinct members $i, j, k, \ell \in [1, 8][$, the points $(i, k), (i, \ell), (j, k), (j, \ell)$ and all 4-subsets of $[1, 8]$ containing i, j and not containing k, ℓ form a pentacross (420 such), and, for each member $i \in [1, 8]$ and a given subset $\{i, j, k\} \subseteq [1, 8]$ of size 3, the point set consisting of the 4-subsets of $[1, 8]$ disjoint from $\{i, j, k\}$ together with the ordered pairs (ℓ, i), with $\ell \in [1, 8] \setminus \{i, j, k\}$ (168 such) defines a pentacross, and likewise the point set consisting of all 4-subsets of $[1, 8]$ containing $\{i, j, k\}$ together with the ordered pairs (i, ℓ), $\ell \in [1, 8] \setminus \{i, j, k\}$ (168 such) defines a pentacross.

We have the following properties, easy to check in an apartment. For brevity's sake, we introduce some notation in the properties.

Proposition 3. *Let $p, q \in \mathcal{P}^*$ be points of Δ^*. Then, exactly one of the following occurs.*

(i) $p = q$;
(ii) $p \perp q$ *and there is a unique line $L \in \mathcal{L}^*$ containing both p and q;*
(iii) *There is a unique symp $\xi \in \Xi^*$ containing both p and q (we denote $p \perp\!\!\!\perp q$ and say that p and q are synplectic);*
(iv) *There is a unique point $[p, q] \in \mathcal{P}^*$ collinear to both p and q (and we denote this as $p \bowtie q$);*
(v) *p is opposite q in Ω (denoted $p \equiv q$) and for every sequence $p \perp x \perp y \perp q$ holds $p \bowtie y$ and $x \bowtie q$.*

Conversely, regarding (v)*, if for some sequence* $p \perp x \perp y \perp q$ *holds* $p \bowtie y$ *and* $x \bowtie q$*, then* $p \equiv q$*.*

Proposition 4. *Let* $p \in \mathcal{P}^*$ *and* $\xi \in \Xi^*$ *be a point and a symp, respectively, of* Δ^**. Then, exactly one of the following occurs.*

(i) $p \in \xi$*;*
(ii) $p \perp U \subseteq \xi$ *with* U *a* $4'$*-space (every point of* $\xi \setminus U$ *is symplectic to* p*);*
(iii) $p \perp L \subseteq \xi$ *with* $L \in \mathcal{L}^*$*, every point* $x \in \xi \setminus L$ *collinear to* L *is symplectic* p *and every point of* $\xi \setminus L^{\perp}$ *is special to* p*;*
(iv) p *is symplectic to all points of* ξ *and is contained with* ξ *in a unique common para, which is a convex subspace isomorphic to* $\mathsf{E}_{6,1}(\mathbb{K})$*;*
(v) p *is symplectic to all points of a unique maximal singular subspace of* ξ*, which is a* 4*-space, and special to all other points of* ξ*;*
(vi) p *is symplectic to a unique point* x *of* ξ*, special to all points of* $(\xi \cap x^{\perp}) \setminus \{x\}$ *and opposite each point of* $\xi \setminus x^{\perp}$*.*

In Case (ii), we say that p is *max-collinear to* ξ, in Case (iii), we call *p line-collinear to* ξ, in Case (iv), we call *p para-collinear* to ξ, in Case (v) *max-symplectic*, and in Case (vi), finally, we call *p far* from ξ.

4. Linear Involutions of Hyperbolic Polar Spaces

This section is devoted to prove auxiliary results about linear involutions mainly in the polar spaces $\mathsf{D}_{5,1}(\mathbb{K})$ and $\mathsf{D}_{6,1}(\mathbb{K})$. We will phrase some results in full generality as their proofs are not more complicated in general than restricted to these two cases.

Each collineation φ of $\mathsf{D}_{n,1}(\mathbb{K})$, $n \geq 3$, is induced by a unique collineation of the ambient projective space $\mathsf{PG}(2n-1,\mathbb{K})$. We call φ *linear* if its extension to $\mathsf{PG}(2n-1,\mathbb{K})$ does not involve a field automorphism, that is, is induced by a linear map of the underlying vector space. Recall that a type-preserving involution of a hyperbolic quadric is an involutive collineation that preserves the oriflamme classes of maximal singular subspaces.

Lemma 7. *Let* ρ *be an involution of* $\mathsf{D}_{n,1}(\mathbb{K})$*,* $n \geq 3$*,* $\mathsf{char}\,\mathbb{K} \neq 2$*. Suppose that each singular subspace stabilised by* ρ *is a line and that* ρ *is not anisotropic. For each line* L*, fixed by* ρ*, there exists a natural bijection between the set of lines fixed by* ρ *and distinct from* L *and the set of subspaces of any given maximal singular subspace* U *through* L *complementary to* L*. In particular, each such singular subspace is collinear to a unique fixed line.*

Proof. First, we claim that each point $x \in U \setminus L$ is mapped onto an opposite point. Indeed, otherwise, the singular subspace $\langle x, x^{\rho}, L\rangle$, which is at least 2-dimensional, is stabilised, contradicting our assumptions. The claim follows.

This now implies that every subspace of U disjoint from L is mapped onto an opposite. Let S be any such a subspace having projective dimension $n-4$ (hence, S is the empty subspace if $n=3$). Then, in order to prove the lemma, it suffices to prove it in the rank 3 polar space $S^{\perp} \cap (S^{\rho})^{\perp}$. Hence, we are reduced to $n=3$.

Now, using the Klein correspondence, the assertion is equivalent to the assertion that in $\mathsf{PG}(3,\mathbb{K})$, each line containing at least one absolute point of a given elliptic polarity is either a tangent or has exactly two absolute points. However, since $\mathsf{char}\,\mathbb{K} \neq 2$, the set of absolute points is an elliptic quadric and the stated assertion is a well-known property of elliptic quadrics. □

Since the set of subspaces of dimension $n-3$ of a projective space of dimension $n-1$ has algebraic dimension $2n-4$ (for instance as a variety), we also say that the involution ρ

of the previous lemma *fixes a parameter variety of dimension* $2n-4$. (It is also the dimension of the root groups of the corresponding Moufang set, but we will not use that.)

In preparation to the determination of the fixed point sets of involutions in $\mathsf{D}_{5,1}(\mathbb{K})$ and $\mathsf{D}_{6,1}(\mathbb{K})$, we prove the following lemma, which we think belongs to folklore, but we sketch a proof. In that proof, a skeleton of a quadric is the set of points of an apartment of that quadric when viewed as a building.

Lemma 8. *Let* Σ *be a subspace of* $\mathsf{PG}(2n-1,\mathbb{K})$, $\mathsf{char}\,\mathbb{K}\neq 2$, *and let* Q *be a non-degenerate hyperbolic quadric in* $\mathsf{PG}(2n-1,\mathbb{K})$ *with associated polarity* η. *Suppose* Σ *has dimension* d *and* $P:=\Sigma\cap Q$ *is a (possibly empty) non-degenerate quadric with Witt index* w. *Then,* $P'=\Sigma^\eta\cap Q$ *is also a (possibly empty) non-degenerate quadric with Witt index* $n+w-d-1$. *If* P *is nonempty, then* $P'=P^\perp$.

Proof. The fact that P is non-degenerate implies immediately that Σ and Σ' are complementary subspaces, that is, $\Sigma\cap\Sigma'=\varnothing$ and $\dim\Sigma'=2n-d-2$. This implies on its turn that P' is non-degenerate. Set $\dim\Sigma'=d'$ and let w' be the Witt index of P'. We have to show $w'-w=n-d-1$. If both P and P' are empty, then $w=w'=0$ and $d=d'=n-1$ (indeed, every subspace of dimension at least n intersects each generator of Q nontrivially and hence has nonempty intersection with Q). Hence, $0=w-w'=n-d-1=0$ and the assertion follows in this case.

We now proceed by induction on n, the statement for $n=1$ being obvious (either $d=d'=0=w=w'$, or, with loss of generality, $d=1$, $d'=-1$, $w=1$ and $w'=0$).

Now, let $n\geq 2$. We may assume, by the first paragraph, that at least one of P,P' is nonempty. By the symmetry of the formula $w'-w=n-d-1$ (which is equivalent to $w-w'=n-d'-1$ as $d+d'=2n-2$), we may suppose $P\neq\varnothing$. We can then consider two non-collinear points $p,q\in P$ and consider $Q^*:=p^\perp\cap q^\perp$ in $W:=\langle p^\perp\cap q^\perp\rangle$. Since $\dim W=2n-3$, $\dim(P\cap W=d-2$, $P'\subseteq W$ and the Witt index of $P\cap W$ is $w-1$, the induction hypothesis yields $w'=(n-1)+(w-1)-(d-2)-1=n+w-d-1$.

This completes the proof of the lemma. $\square$

Proposition 5. *Let* ρ *be a linear type-preserving involution of* $\mathsf{D}_{5,1}(\mathbb{K})$, $\mathsf{char}\,\mathbb{K}\neq 2$. *Then, the fix structure of* ρ *is either*

(i) *An elliptic quadric in some* 7*-dimensional subspace;*
(ii) *The union of two non-collinear points and their common perp isomorphic to* $\mathsf{D}_{4,1}(\mathbb{K})$*;*
(iii) *A subquadric of Witt index* 1 *which is the intersection of* $\mathsf{D}_{5,1}(\mathbb{K})$ *with a* 5*-dimensional subspace of* $\mathsf{PG}(9,\mathbb{K})$*;*
(iv) *The union of a subquadric isomorphic to* $\mathsf{D}_{3,1}(\mathbb{K})$ *and its perp isomorphic to* $\mathsf{D}_{2,1}(\mathbb{K})$*;*
(v) *The union of an elliptic subquadric in a* 3*-dimensional subspace and its perp, an elliptic quadric in a* 5*-dimensional subspace;*
(vi) *The union of two disjoint maximal singular subspaces.*

Proof. We denote the involution of the ambient projective space $\mathsf{PG}(9,\mathbb{K})$ inducing ρ in $\mathsf{D}_{5,1}(\mathbb{K})$ also by ρ. Since $\mathsf{char}\,\mathbb{K}\neq 2$, there is a polarity η of $\mathsf{PG}(9,\mathbb{K})$ whose set of absolute points is precisely the hyperbolic quadric Q defined by $\mathsf{D}_{5,1}(\mathbb{K})$. Again, since $\mathsf{char}\,\mathbb{K}\neq 2$, Lemma 3 implies that the set of fixed points of ρ is the union of two disjoint subspaces U and U' with $\dim U+\dim U'=8$. We may assume $0\leq\dim U\leq\dim U'$. Note that, since ρ preserves Q, we always have $U\subseteq U^\eta$ or $U'=U^\eta$. We claim that this implies that U either is a singular subspace, or intersects Q in a non-degenerate quadric. Indeed, suppose $U\cap Q$ is degenerate but U is not contained in Q. Let R be the radical of $U\cap Q$. Then, $U\subseteq R^\eta$; so, $R\subseteq U^\eta$, which implies $U\subseteq U^\eta$. Then, U is a singular subspace after all.

Now, suppose that U is a singular subspace. We claim that U is a maximal singular subspace. Indeed, first suppose U is a point. Then, every line through U which is not a tangent line, is stabilised. Since that line intersects Q in a unique second point, that point is fixed. It follows that each point non-collinear to U is fixed, and it follows that ρ is the identity on Q, a contradiction. Now, let U be arbitrary, but not a maximal singular subspace. Take a hyperplane H of U. In the residue of H, we can repeat the exact same argument as for U a point since the image of U in the residue is a centre of the induced involution. We conclude that every singular subspace U' containing H is stabilised. Let M be a maximal singular subspace containing U. Then, it follows that ρ pointwise fixes $M \cap U'$, which is complementary to U in M, but each point of $M \setminus (U \cup U')$ is moved by ρ. Let x be such a point. Then, $\langle x, x^\rho \rangle \cap U$ is a point u, and choosing H above such that it does not contain u, we obtain the contradiction that ρ stabilises $\langle H, x \rangle$, which implies $u \in H$. The claim is proved.

We now review all possibilities for $\dim U$.

(0) $\dim U = 0$. By the claim, $U' = U^\eta$ and U does not belong to Q. By Lemma 8, $U' \cap Q$ is a parabolic quadric with Witt index 4. Since ρ is type-preserving, it stabilises the two maximal singular subspaces of Q through a given maximal singular subspace of $Q \cap U'$. Since $\mathrm{char}\,\mathbb{K} \neq 2$, Lemma 3 leads to the contradiction that each of them contains an additional fixed point. Alternatively, a reflection (in the sense of Dieudonné [33]) in a non-singular point and its polar hyperplane has the wrong determinant -1 and hence interchanges the two oriflamme classes of generators.

(1) $\dim U = 1$. By the previous claims, we have that $|U \cap Q| \in \{0, 2\}$. The case $U \cap Q = \varnothing$ leads, in view of Lemma 8, to (i), whereas $|Q \cap U| = 2$ leads to (ii).

(2) $\dim U = 2$. The previous claims imply that $U \cap Q$ is a non-degenerate conic. If that conic is non-empty, then Lemma 8 implies that $U' \cap Q$ is a parabolic quadric with Witt index 3. Consider a plane in $U' \cap Q$ and a point of $U \cap Q$; they generate a singular 3-space S. Since ρ is type-preserving, the two maximal singular subspaces of Q through S are stabilised and hence, by Lemma 3, contain further fixed points of ρ which clearly do not belong to $U \cup U'$, a contradiction. If $U \cap Q$ is empty, then by Lemma 8, $U' \cap Q$ is an elliptic quadric, with Witt index 2. Let M be a maximal singular subspace of Q containing some line L of $Q' \cap Q$. Then, $L \subseteq M^\rho$, and so either M is stabilised, or $M \cap N^\rho$ is a plane, which is stabilised. In either case, we find additional fixed points in Q outside U, a contradiction. We conclude that this case cannot occur.

(3) $\dim U = 3$. Here, the claims above imply that there are three cases: the case $U \cap Q = \varnothing$ leads to (iii) (use Lemma 8), the case $U \cap Q$ is a hyperbolic quadric and leads to (iv), and the case $U \cap Q$ is an elliptic quadric and leads to (v).

(4) $\dim U = 4$. The claim above implies that $U \cap Q$ and $U' \cap Q$ are either empty, hyperelliptic quadrics with Witt index 1, or parabolic quadrics with Witt index 2. The first case cannot occur since the intersection of each maximal singular subspace and its image is a subspace of even dimension stabilised by ρ, and so Lemma 3 yields at least one fixed point on Q. For the other case, we consider the singular subspace generated by respective maximal singular subspaces of $U \cap Q$ and $U' \cap Q$. This yields a subspace of odd dimension stabilised under ρ. The intersection of any maximal singular subspace containing such odd-dimensional subspace stabilised by ρ and its image is an even-dimensional singular subspace stabilised by ρ and hence, by Lemma 3, containing further fixed points not contained in $U \cup U'$, a contradiction. $\square$

Proposition 6. *Let ρ be a linear type-preserving involution of $\mathsf{D}_{6,1}(\mathbb{K})$, $\mathrm{char}\,\mathbb{K} \neq 2$. Then, either there are no fixed points, or the set of fixed points is one of the following.*

(i) *An elliptic quadric in some 9-dimensional subspace;*

(ii) *The union of two non-collinear points and their common perp isomorphic to* $\mathsf{D}_{5,1}(\mathbb{K})$*;*
(iii) *A hyperelliptic subquadric of Witt index* 2 *which is the intersection of* $\mathsf{D}_{6,1}(\mathbb{K})$ *with a* 7*-dimensional subspace of the ambient projective space* $\mathsf{PG}(11,\mathbb{K})$*;*
(iv) *The union of an elliptic subquadric in some* 3*-dimensional subspace of* $\mathsf{PG}(11,\mathbb{K})$ *and its perp, which is an elliptic quadric in some* 7*-dimensional subspace;*
(v) *The union of a subquadric isomorphic to* $\mathsf{D}_{2,1}(\mathbb{K})$ *and its perp isomorphic to* $\mathsf{D}_{4,1}(\mathbb{K})$*;*
(vi) *The union of two mutually orthogonal hyperbolic subquadrics in disjoint* 5*-dimensional subspaces of* $\mathsf{PG}(11,\mathbb{K})$*;*
(vii) *The union of two mutually orthogonal elliptic subquadrics in disjoint* 5*-dimensional subspaces of* $\mathsf{PG}(11,\mathbb{K})$*;*
(viii) *The union of two mutually orthogonal hyperelliptic subquadrics of Witt index* 1 *in disjoint* 5*-dimensional subspaces of* $\mathsf{PG}(11,\mathbb{K})$*;*
(ix) *The union of two disjoint maximal singular subspaces.*

Proof. We denote with η the polarity of $\mathsf{PG}(11,\mathbb{K})$ having absolute geometry a non-degenerate hyperbolic quadric Q. We may assume that ρ acts on Q and has fixed points in Q. Since ρ admits fixed points, the fix structure of the extension of ρ to $\mathsf{PG}(11,\mathbb{K})$ consists of two complementary subspaces U and U'. The first two paragraphs of the proof of Proposition 5 can be copied word-for-word (they hold for every hyperbolic quadric) and hence, we have that either both U and U' are (disjoint) generators (this is case (ix)) or $U' = U^\eta$ and both intersect Q in either the empty space or a non-degenerate subquadric (but by assumption not both at the same time in the empty space). Without loss of generality, we may assume $d := \mathsf{dim}\, U \leq 5 \leq \mathsf{dim}\, U'$.

Now, let w be the Witt index of the quadric $U \cap Q$ and let w' be the Witt index of $U' \cap Q$. Since w and w' are the vector dimensions of the maximal singular subspaces on these respective quadrics, and since these quadrics are contained in each other's perp, we see that ρ stabilises a singular subspace A of projective dimension $w + w' - 1$. If some singular subspace B of Q properly containing A were stabilised by ρ, then, by Lemma 3, the fixed points in B generate B, which contradicts the observation that the only fixed points of B lie in A, as A contains already the pointwise fixed singular subspaces of maximal dimension. Hence, since ρ is an involution, this implies that each generator M containing A satisfies $M \cap M^\rho = A$. Since ρ is type-preserving, this yields $n - 1 - \mathsf{dim}\, A = n - 1 - (w + w' - 1) = n - w - w'$ is even. By Lemma 8,

$$n - w - w' = n - w - (n + w - d - 1) = d + 1 - 2w.$$

We conclude that d must be odd.

We review all (odd) possibilities for $\mathsf{dim}\, U$.

(1) $\mathsf{dim}\, U = 1$. Here, either U is disjoint from Q, or intersects it in exactly two points (a quadric of Witt index 1 in a 1-dimensional subspace). These are Cases (i) and (ii), respectively.
(3) $\mathsf{dim}\, U = 3$. Here, there are three possibilities, according to whether U is disjoint from Q, intersects it in an elliptic quadric, or in a hyperbolic quadric. According to Lemma 8, U' then intersects Q in a quadric of Witt index 2, 3 and 4, respectively. These are the respective Cases (iii), (iv) and (v) of the proposition.
(5) $\mathsf{dim}\, U = 5$. Here are again three possibilities, since, if $U \cap Q = \varnothing$, then by Lemma 8, also $U' \cap Q = \varnothing$, contradicting the assumption that there really are fixed points. In fact, Lemma 8 implies that U and U' have the same intersection behaviour with respect to Q. Hence, either both $U \cap Q$ and $U' \cap Q$ have Witt index 1 (Case $(viii)$, or both have Witt index 2 (Case (vii)), or both have Witt index 3 (Case (vi)).

This completes the proof of the proposition. □

We have the following consequence.

Corollary 1. *Let ρ be a linear type-preserving involution of $\mathsf{D}_{6,1}(\mathbb{K})$, $\mathsf{char}\,\mathbb{K} \neq 2$. If ρ is not anisotropic, then it stabilises at least three lines.*

Proof. Since ρ is not anisotropic, Lemma 1 yields some point x mapped to a collinear point x^ρ. If $x = x^\rho$, then the involution ρ stabilises the residue at x, which is a hyperbolic quadric of Witt index 5. Proposition 5 implies that there are at least three stabilised lines through x. Now, assume $x \neq x^\rho$. Consider a singular 3-space U through $L := \langle x, x^\rho \rangle$. If $U^\rho = U$, then Lemma 3 proves the assertion. If $U \cap U^\rho$ is a plane, then Lemma 2 proves the assertion. If $U \cap U^\rho = L$, then any line $M \subseteq U \setminus L$ is mapped onto an opposite line M^ρ. Then, $Q_M := M^\perp \cap (M^\rho)^\perp$ is a stabilised subquadric of type $\mathsf{D}_{4,1}$ containing L. Re-running this argument with a singular 3-space $W \subseteq Q_M$ through L, and a line $K \subseteq W \setminus L$, we obtain a stabilised hyperbolic quadric of Witt index 2, hence a grid, containing L. Since the oriflamme class of lines containing L is a projective line over $\mathbb{K}$, there must be a second fixed line L_K. Repeating this argument with $K' \neq K$, we obtain another fixed line $L_{K'}$, which is distinct from L_K as otherwise L_K is collinear to at least a plane of W, and hence to at least one point of L, a contradiction. The assertion now follows. □

We can refine the previous proof slightly to analyse the case where a linear involution does not fix any point.

Lemma 9. *Let ρ be a linear involution of $\mathsf{D}_{6,1}(\mathbb{K})$, $\mathsf{char}\,\mathbb{K} \neq 2$, without fixed points. Then, exactly one of the following happens.*

(i′) *ρ is anisotropic;*

(ii′) *ρ does not stabilise any singular subspace of dimension distinct from 1 and fixes a parameter variety of dimension 8;*

(iii′) *ρ stabilises lines and 3-spaces, but no singular 5-spaces. The fix structure forms a generalised quadrangle isomorphic to an elliptic quadric over a quadratic extension of $\mathbb{K}$ and the corresponding field extension over $\mathbb{K}$ has degree 4 with Galois group elementary abelian of order 4 (Klein 4-group);*

(iv′) *ρ stabilises lines, 3-spaces and 5-spaces and the geometry induced has the structure of a top-thin polar space of rank 3, namely a hyperbolic quadric of Witt index 3 over a quadratic extension of $\mathbb{K}$;*

(v′) *ρ stabilises each line of a line spread, which forms a small Hermitian polar space of Witt index 3.*

Proof. First, note that ρ is type-preserving, as otherwise, a generator and its image intersect in a subspace of even dimension, yielding a fixed point by Lemma 3(i).

We may assume that ρ is not anisotropic and hence stabilises at least one line L. If ρ does not stabilise any subspace of dimension distinct from 1, then Lemma 7 yields (ii'). So, we may assume that ρ stabilises at least one 3-space Σ (as stabilising a plane or 4-space leads to a fixed point, and stabilising a 5-space leads to stabilising a 3-space). We may assume that $L \subseteq \Sigma$ (otherwise, we project L onto Σ and replace L by that projection). Then, by Lemma 3, ρ stabilises each member of a line spread of each stabilised 3-space. Considering the residue of a stabilised line, Lemma 7 yields at least three stabilised 3-spaces through each stabilised line. Clearly, the projection of a stabilised line onto a stabilised 3-space is a stabilised line. It follows that the fix structure is a generalised quadrangle. To see the isomorphism type of that quadrangle, one can introduce coordinates and find an explicit form. One calculates that, if $\mathsf{D}_{6,1}(\mathbb{K})$ is given by the equation

$$X_{-6}X_6 + X_{-5}X_5 + X_{-4}X_4 + X_{-3}X_3 + X_{-2}X_2 + X_{-1}X_1 = 0,$$

then without loss of generality, the involution ρ can be chosen as

$$(x_{-6}, \ldots, x_6) \mapsto (kx_{-5}, x_{-6}, kx_{-3}, x_{-4}, m_2x_2, m_1x_1, km_1^{-1}x_{-1}, km_2^{-1}x_{-2}, kx_4, x_3, kx_6, x_5),$$

where $k, m_1, m_2 \in \mathbb{K}$ such that k is a non-square in $\mathbb{K}$ and $-m_2m_1^{-1} \in \mathbb{K}$ is a non-square in $\mathbb{K}(\sqrt{k})$. One then also computes the fix structure on the quadric and obtains, with new coordinates in $\mathbb{K}(\sqrt{k})$ as follows,

$$\begin{cases} Y_1 = X_1 + m_1^{-1}X_{-1}\sqrt{k}, \\ Y_2 = X_2 + m_2^{-1}X_{-2}\sqrt{k}, \\ Y_3 = X_{-4} + X_{-3}\sqrt{k}, \\ Y_4 = X_3 + X_4\sqrt{k}, \\ Y_5 = X_{-6} + X_{-5}\sqrt{k}, \\ Y_6 = X_5 + X_6\sqrt{k}. \end{cases}$$

the equation $2Y_3Y_4 + 2Y_5Y_6 + m_1Y_1^2 + m_2Y_2^2 = 0$, which proves (iii'), since this equation splits over $\mathbb{K}(\sqrt{k})(\sqrt{-m_2m_1^{-1}}) = \mathbb{K}(\sqrt{k}, \sqrt{-m_2m_1^{-1}})$, which is a Galois extension of $\mathbb{K}$ of degree 4 with Galois group elementary abelian.

Suppose now that ρ stabilises lines, 3-spaces and 5-spaces, but not a spread of lines. It is easy (and similar to previous arguments) to deduce that the fix structure is a polar space of rank 3. To see that it is top-thin, we look at the residue of a fixed 3-space Σ and since there exist points mapped onto opposites, we may assume that such point is collinear to Σ. It then follows easily that ρ stabilises exactly two 5-spaces through Σ. Taking the quotient space with respect to the fixed spread, (iv') follows.

Finally, suppose that ρ stabilises all lines of a spread of lines. Then, ρ maps each point to a collinear point. In this case, ref. [28] (Theorem 3.22(ii)) shows that we are in case (v').

Since the fixed structures in the different cases of the lemma are not isomorphic to each other, the cases are mutually exclusive. □

We end this section with a result in the exceptional geometry $\mathsf{E}_{6,1}(\mathbb{K})$. The proof uses the fix diagrams of Figure 3.

Figure 3. The fix diagrams $^2\mathsf{E}_{6;1}$ and $^2\mathsf{D}^2_{5;1}$.

Proposition 7. *Let ρ be a polarity of $\mathsf{E}_{6,1}(\mathbb{K})$ stabilising 5-spaces but without absolute points. Let W be a fixed 5-space and let $L \subseteq W$ be an arbitrary line. For each symp ξ through L locally opposite W at L, we choose an arbitrary 4-space U_ξ through L in ξ. Then, there exists a natural bijection between the set of fixed 5-spaces of ρ distinct from W and the set of planes in the subspaces U_ξ disjoint from L, for ξ ranging over all symps through L locally opposite W at L. More exactly, for each fixed 5-space $W' \neq W$, there exists a unique symp ξ as above and a unique plane π in U_ξ disjoint from L such that $W' \cap \xi$ is a line collinear to π, and vice versa (all such planes π do occur).*

Proof. Let W, L and ξ be as stated. Set $p := \xi^\rho$. Since $\xi \cap W = L$ and $W^\rho = W$, we find $\langle p, p^\perp \cap W \rangle =: \Sigma$ is a 4-space. It follows that p is opposite ξ. Now, by Theorem 3.28 and

Proposition 3.29 of [17], it follows that the map assigning to each line $K \subseteq \xi$ the unique 4-space Σ_K through p, which is the intersection of all symps containing p and a point of K, is the restriction of an isomorphism between ξ and $\mathsf{Res}(p)$; in particular, it is bijective. It follows that $W_K := \langle K, K^\perp \cap \Sigma_K \rangle$ is a 5-space. The set Y of all such 5-spaces is hence canonically bijective with the line Grassmannian of ξ. By construction, ρ preserves Y. We can define ρ_ξ as the map that assigns to any line K of ξ the line $W_K \cap \xi$. Then, ρ_ξ is the composition of ρ and the projection of $\mathsf{Res}(p)$ to ξ, and hence induces an automorphism of ξ. It is now easy to see that the fact that the fix diagram of ρ is ${}^2\mathsf{E}_{6;1}$ translates to the fact that ρ_ξ only fixes lines of ξ; hence, its fix diagram is ${}^2\mathsf{D}_{5;2}$. Lemma 7 yields a unique fixed line collinear to each plane of U_ξ disjoint from L.

It remains to show that each fixed 5-space W' uniquely arises like that. Since W' is opposite W, there is a unique line L' of W' contained in a symp $\xi' \supseteq L$. Moreover, ξ' is clearly locally opposite W at L. This completes the proof of the lemma. □

Corollary 2. *Let ρ be a polarity of $\mathsf{E}_{6,1}(\mathbb{K})$ stabilising 5-spaces but without absolute points. Then, ρ fixes a parameter variety of dimension 10.*

Proof. The residue of a line L in $\mathsf{E}_{6,1}(\mathbb{K})$ is the line Grassmannian of a projective space of dimension 4 in which the 5-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$ through L play the role of points. Hence, the set of symps through L locally opposite a given 5-space through L forms a parameter variety of dimension 4. Since, with the notation of Proposition 7, the planes in the 4-space U_ξ disjoint from L form a parameter variety of dimension 6, the assertion follows. □

5. Proofs of Theorems 1 and 2

Now that we gathered all necessary general properties of Δ in Section 3 and studied involutions in polar spaces of types $\mathsf{D}_{5,1}$ and $\mathsf{D}_{6,1}$ in Section 4, we can embark on the proof of Theorem 1, and prove Theorem 2 underway. For the rest of this section, $\Delta = (X, \mathcal{L})$ is a Lie incidence geometry of type $\mathsf{E}_{7,7}$ over the field $\mathbb{K}$ with characteristic different from 2, and ρ is a given linear involution of Δ. We refer to the building corresponding to Δ as Ω.

We first show that the fix diagram of an involution is one of $\mathsf{E}_{7;0}$, $\mathsf{E}_{7,1}$, $\mathsf{E}_{7;2}$, $\mathsf{E}_{7;3}$, $\mathsf{E}_{7;4}$ or $\mathsf{E}_{7;7}$.

Proposition 8. *If ρ does not fix any symp, then it is anisotropic.*

Proof. Suppose for a contradiction that ρ is not anisotropic and does not fix any symp. Then, by Lemma 1, there is a symp ξ such that ξ^ρ is not opposite ξ. According to Proposition 2(iii), there are three possibilities. Note that $\xi^\rho \neq \xi$ by assumption. We show that each possibility leads to a fixed symp, which proves the assertion.

(i) *ξ and ξ^ρ are adjacent.*
Then, $\xi \cap \xi^\rho$ is a 5-space W, which is stabilised by ρ. Lemma 3 yields a fixed line L in W. Then, $\mathsf{Res}_\Omega(L)$ is isomorphic to $\mathsf{D}_5(\mathbb{K})$ whose elements of type 1 correspond to symps of Δ. Proposition 5 now yields a fixed symp of Δ.

(ii) *$\xi \cap \xi^\rho$ is a line L.*
Then, the same argument as in the previous case with $\mathsf{Res}_\Omega(L)$ yields a fixed symp of Δ.

(iii) *There is a unique symp ζ adjacent to both ξ and ξ^ρ.*
Clearly, ρ fixes ζ. □

We now show Theorem 2.

Proposition 9. *The absolute type of the imaginary fixed building of an anisotropic linear involution ρ of a building of type E_7 over a field $\mathbb{K}$ of characteristic not equal to 2 is A_7.*

Proof. Let $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$ and consider ρ in $\mathsf{E}_7(\overline{\mathbb{K}})$. Proposition 1 yields an apartment pointwise fixed by ρ. By Borel–de Siebenthal [2], ρ pointwise fixes a closed maximal subbuilding of either types $\mathsf{A}_1 \times \mathsf{D}_6$, D_6, E_6 or A_7. However, in the former three cases, ρ does not map any chamber to an opposite, as shown in [34] (Theorems 2 and 3). This contradicts the fact that, by the same reference, ρ does map chambers to opposites in $\mathsf{E}_7(\mathbb{K})$. Hence, only type A_7 remains and the proposition is proved. □

From now on, we assume that ρ is not anisotropic. We introduce some notation.

Notation 1.

- *Let ξ be a symp, fixed by ρ. Then, ρ induces an involution in ξ, and we denote the restriction of ρ to ξ as ρ_ξ. Proposition 6 is applicable to ρ_ξ.*
- *Suppose ρ fixes some line $L \in \mathcal{L}$. Then, ρ induces an involution in the residue $\mathsf{Res}_\Omega(L)$, which we can view as a polar space of type $\mathsf{D}_{5,1}$ where the points correspond to the symps of Δ. We denote that involution as ρ_L. A fixed point for ρ_L corresponds to a stabilised symp for ρ through L.*

Since we assume that ρ is not anisotropic, Proposition 8 implies that ρ fixes at least one symp. We now prove that either ρ fixes a lot of symps, or ρ also fixes at least one line.

Lemma 10. *Let ρ be a linear involution of Δ that is not anisotropic. Then, either ρ fixes a parameter variety of dimension 16 (the elements of which are symps), or ρ stabilises some line.*

Proof. Suppose ρ does not stabilise any line. Then, by Corollary 1, it acts anisotropically on every stabilised symp. Let ξ be such a stabilised symp (it exists by Proposition 8). Let p be any point close to ξ and set $U := \xi \cap p^\perp$. Then, p^ρ is collinear to U^ρ, which is ξ-opposite U. Lemma 5 yields $p \equiv p^\rho$.

Now, consider the map ρ_p that assigns to a line $L \ni p$ the symp ξ_L through p defined by p and the unique point of L^ρ symplectic to p. Then, ρ_p is the composition of ρ (acting on the lines through p) and the projection map from p^ρ onto p. Hence, we may view ρ_p as an automorphism of $\mathsf{Res}_\Omega(p)$, switching the types. With the above notation, ξ_L^ρ is the projection of $(L^\rho)^\rho = L$ from p onto p^ρ; hence, L is the projection of ξ_L^ρ from p^ρ onto p and so $L = \xi_L^{\rho_p}$. It follows that ρ_p is a polarity. If $\in L \subset L^{\rho_p}$, then there is a unique line M intersecting both L and L^ρ. Clearly, $M^\rho = M$, a contradiction, and so ρ_p has no absolute points if $\mathsf{Res}_\Omega(p)$ is viewed as $\mathsf{E}_{6,1}(\mathbb{K})$ with point set the set of lines of Δ through p. However, the 6-space $\langle p, U\rangle$ is stabilised, and so ρ admits fixed 5-spaces. It follows from Corollary 2 that ρ_p fixes a parameter variety of dimension 10 (the elements of which are 5-spaces, each of which corresponds to a fixed symp for ρ).

Letting p vary over $\langle p, U\rangle \setminus U$, we obtain a parameter variety of dimension 16 (whose elements are the fixed symps). The fact that this includes *all* fixed symps follows directly from the first statement of Lemma 6, whereas the fact that we count every fixed symp *only once* follows from the second statement of Lemma 6. □

Proposition 10. *If a linear involution of Δ is not anisotropic and only fixes symps and no lines, then the Tits index of the fixed building is either ${}^2\mathsf{A}_{7,1}^{(4)}$ or ${}^1\mathsf{A}_{7,1}^{(4)}$.*

Proof. The same arguments as in the proof of Proposition 9 yield the absolute type A_7. In order to know the fix diagram, and hence the Tits index, we play the same game with the residue of a fixed symp, which is a building of type D_6. There, involutions in characteristic different from 2 have as fix structure buildings of types D_5, A_5, $\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{D}_4$ and $\mathsf{A}_3 \times \mathsf{A}_3$, according to [2]. Only A_5 and $\mathsf{A}_3 \times \mathsf{A}_3$ are compatible with fix diagrams of A_7 where one orbit is encircled, and these correspond to the Tits indices ${}^1\mathsf{A}_{7,1}^{(4)}$, ${}^2\mathsf{A}_{7,1}^{(4)}$ and ${}^2\mathsf{A}_{7,1}^{(1)}$. However, in the latter case, the dimension of the root groups is the dimension of a Hermitian curve in $\mathsf{PG}(7, \mathbb{K}')$, which is odd over the fixed field of the corresponding (Galois) involution, contradicting Lemma 10. □

We will provide examples of both cases in Section 6.6.

Proposition 11. *The fix diagram of an involution ρ of $\Omega = \mathsf{E}_7(\mathbb{K})$ that does not fix any chamber is one of $\mathsf{E}_{7;0}$, $\mathsf{E}_{7;1}$, $\mathsf{E}_{7;2}$, $\mathsf{E}_{7;3}$ or $\mathsf{E}_{7;4}$.*

Proof. Assume that the fix diagram of ρ is not among $\mathsf{E}_{7;0}$, $\mathsf{E}_{7;1}$ and $\mathsf{E}_{7;2}$. Then, Lemma 10 yields at least one fixed symp ξ, at least one fixed line L and at least one fixed element x of another type. Note that, by Lemma 3, no 6-space is stabilised as otherwise a chamber is fixed. Likewise, no plane of ξ is fixed, as the residue of a plane contains an irreducible factor "isomorphic" to $\mathsf{PG}(4, \mathbb{K})$ in which the points are synps of Δ and the hyperplanes are 6-spaces of Δ. We distinguish some cases. Suppose first that x is a point. Since ρ does not stabilise any 6-space, we infer that x is either in or far from ξ, and so there is a fixed point in ξ. Since ρ does not fix any plane of ξ, the only possibilities for ρ_ξ from Proposition 6 are (iii) and $(viii)$. Hence, we have shown in any stabilised symp, ρ fixes points and lines, but no other subspaces. Suppose ρ fixes a 3-space Σ. The symps and 5-spaces through Σ form a projective plane (as shown by the diagram), implying, by Lemma 2, that ρ stabilises a symp ζ through Σ, contradicting the previous assertion that ρ only fixes points and lines in ζ. At last, if ρ stabilised a 5-space, then Lemma 3 would yield a stabilised 3-space, a contradiction. Hence, if some point is fixed, then the fix diagram is either $\mathsf{E}_{7;3}$ or $\mathsf{E}_{7;7}$.

From now on, we may assume that no point is fixed. By Lemma 2, no plane is stabilised. If a 5-space is stabilised, then, by Lemma 3, some 3-space is stabilised. Conversely, if some 3-space is stabilised, then, by Lemma 2, there are symps and 5-space through it stabilised (as these form a projective plane). So, we have shown that, if the fix diagram is not full, and no point is stabilised, but still an element of type distinct from the type of a symp or a line is stabilised, then the fix diagram is $\mathsf{E}_{7;4}$. □

Proposition 12. *If the fix diagram of a linear involution ρ of $\mathsf{E}_7(\mathbb{K})$ is $\mathsf{E}_{7;2}$, then the fix structure is a building of type B_2. More precisely, the fixed points of $\mathsf{E}_{7,1}(\mathbb{K})$ and stabilised symps form the points and lines of a (thick) generalised quadrangle Q.*

Proof. Let the point-line geometry Q consist of the fixed points (as point set) and a line is the set of fixed points in a stabilised symp. Let x be a fixed point of $\mathsf{E}_{7,1}(\mathbb{K})$ and let ξ be a fixed symp, and suppose $x \notin \xi$. Due to Proposition 4, we have the following possibilities.

(i) x is collinear to a unique line L of ξ, then L is stabilised by ρ, a contradiction to our hypothesis.
(ii) x is collinear to a unique maximal singular subspace U of ξ. Then, U is stabilised by ρ, again a contradiction.
(iii) x is symplectic to all points of a unique maximal singular subspace U of ξ, again, a contradiction, as in the previous case.
(iv) x is symplectic to all points of ξ. Then, by Proposition 4(iv), x and ξ are contained in a unique para, which must then be stabilised by ρ, a contradiction.
(v) x is symplectic to a unique point $y \in \xi$; then both y and $\xi(x, y)$ are stabilised.

Since only the last possibility does not lead to a contradiction, we verified the main axiom of a generalised quadrangle. Note that this also shows that each stabilised symp contains at least one fixed point, and each fixed point is contained in at least one stabilised symp.

Now, we show that each point is on at least three lines and each line contains at least three points. Since the residue of a fixed point is a building of type D_6 in which the symps are lines of the corresponding polar space $\mathsf{D}_{6,2}(\mathbb{K})$, and a line consists of the set of fixed points in a stabilised symp of $\mathsf{E}_{7,1}(\mathbb{K})$, it suffices, in view of Proposition 5 and Corollary 1, to show that the involutions induced in stabilised symps and residues of fixed points are never anisotropic. However, this follows from the last sentence of the previous paragraph. □

In Proposition 15, we will determine the isomorphism class of the fixed quadrangle by means of its Tits index. In order to be able to do so, we first need to handle some higher rank cases.

Proposition 13. *If the fix diagram of a linear involution ρ of $\mathsf{E}_7(\mathbb{K})$ is $\mathsf{E}_{7;3}$, then the fix structure is a building of type B_3. More precisely, if it is thick, the fixed points, symps and paras of $\mathsf{E}_{7,1}(\mathbb{K})$ form the points, lines and planes of a quaternion polar space Γ. If it is non-thick, then the corresponding polar space is top-thin and arises from the line Grassmannian of a quaternion projective space of dimension* 3.

Proof. Let Γ be the geometry with as point set the set of fixed points of ρ, and a typical line is the set of fixed points in a stabilised symp. We first check the one-or-all axiom. Let p be a fixed point and ξ a stabilised symp. According to Proposition 4, there are six possible mutual positions for p and ξ. Using the numbering of Proposition 4, we see that the Cases (ii), (iii) and (v), p and ξ determine either a unique maximal singular subspace of ξ, or a unique line in ξ, which has consequently to be fixed by ρ, contradicting the fix diagram. Hence, only Cases (i), (iv) and (vi) can occur. In Case (i), $p \in \xi$. In Case (iv), p and ξ are contained in a unique common para Π. Such a para is isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$ and has hence diameter 2. Clearly, p and ξ are opposite in Π as otherwise p is max-collinear to ξ and we are in Case (ii). Hence, p and each fixed point in ξ are contained in a unique symp, which must then also be fixed by ρ. So, p is collinear in Γ to all points of ξ. The last possibility is that p is far from ξ (that is, Case (vi)). Then, there is a unique symp ζ containing p and intersecting ξ in a unique point q. Given these uniquenesses, the point q is a fixed points and the symp ζ is stabilised by ρ. This completes the check of the one-or-all axiom. Note, however, that we did not yet show that each line of Γ really contains points. This will be done in the next paragraph.

From the previous paragraph follows that a stabilised symp ξ either contains at least one fixed point, and hence, by Proposition 5, a whole subquadric of Witt index 1 as set of fixed points, or it is contained in a stabilised para Π. In Π, no point is mapped onto a collinear one since otherwise, the joining line is stabilised, contradicting the fix diagram. Noting that ρ has at least one fixed point—the intersection of a non-fixed symp and its image—Theorem 4.1 of [34] implies that the fixed point set in each stabilised symp is again a subquadric of Witt index 1. Hence, all lines of Γ have at least three points.

Since p was arbitrary in the first paragraph, we deduce that the collineation induced by ρ in the residue of any fixed point p is not anisotropic, since it fixes at least one symp through p, or one para through it. In the latter case, the second paragraph implies that ρ also fixes at least one symp through p. We now switch to $\mathsf{E}_{7,7}(\mathbb{K})$. Then, the involution induced by ρ in any stabilised symp ξ fixes at least one line. The fix diagram implies that there is at least one fixed point, somewhere. If that point x is not in ξ, then, by Proposition 2(ii), either x is collinear to a unique $5'$-space, which must be stabilised and contradicts the fix diagram, or x is collinear to a unique point of ξ, proving that ξ contains, besides at least one stabilised line, also at least one fixed point. Since, by the fix diagram, ρ can only fix points and lines, Proposition 6 implies that the fix structure is either

Case (1) a hyperelliptic quadric of Witt index 2 in a 7-dimensional subspace of the ambient 11-dimensional space of ξ;

Case (2) the union of two mutually orthogonal hyperelliptic quadrics of Witt index 1, each spanning a 5-dimensional subspace.

Since these two cases both show that the perp of a point in Γ is a non-degenerate (possibly top-thin) generalised quadrangle, we deduce (using the note in the beginning of Section 3.3) that Γ is non-degenerate. Hence, by [35] (Theorem 1.7.1), Γ is either thick and the fix structure in each symp is as in Case (1), or top-thin and the fix structure in each symp is as in case (2) above.

Note that in Case (1), this implies that every maximal singular subspace of ξ contains a line of fixed points, since, by Grassmann's identity, a generator intersects the pointwise fixed 7-space in at least (and hence, by the fix diagram, exactly) a line. We now treat both cases separately.

Case (1). Let p be any point of $\mathsf{E}_{7,7}(\mathbb{K})$. We claim that p^ρ is not opposite p. Consider a stabilised symp ξ as in the previous paragraph. If $p \in \xi$, then the claim is obvious since $p^\theta \in \xi$. If p is close to ξ, then by the note of the previous paragraph, p is collinear to a fixed point x and hence $p \perp x = x^\rho \perp p^\rho$, implying the claim. Hence, we may assume that p is far from ξ. Let y be the unique point of ξ collinear to p. By the one-or-all axiom in ξ, we find a fixed point x of ρ in ξ collinear to y, and hence symplectic to p. Then, by the fix diagram and Proposition 2(iii), the symps $\xi(p,x)$ and $\xi(p,x)^\rho$ intersect in a unique line L, which is stabilised by ρ. Let M be any fixed line in ξ through x. Note that $M \neq L$ since p is collinear to some point z of L (inside the polar space $\xi(p,x)$), but $p^\perp \cap \xi = \{y\}$ and $y \notin M$. By the fix diagram, L and M are not contained in a plane. Hence, using Proposition 2(i), we deduce that they are contained in a unique (stabilised) symp ζ. Applying Proposition 6 to ζ, we see that all points on L are fixed and so p is collinear to the fixed point z. The claim is proved.

Hence, ρ is point-domestic and the proposition follows from Theorem 1 of [34].

Case (2). Since Γ has rank 3 and is top-thin, it arises from the line Grassmannian of a 3-dimensional projective space $\mathsf{PG}(3,\mathbb{A})$, for some skew field $\mathbb{A}$. We now determine $\mathbb{A}$. Therefore, we consider again the situation in $\mathsf{E}_{7,1}(\mathbb{K})$ and we let Π be a stabilised para. The fix structure of ρ in Π is a plane of Γ, which is isomorphic to $\mathsf{PG}(2,\mathbb{A})$. Noting again that ρ does not map any point x of Π to a collinear point $x^\rho \neq x$, and that ρ has fixed points in Π, Theorem 4.1 of [34] implies that $\mathbb{A}$ is either a quaternion division algebra over $\mathbb{K}$, or an inseparable field extension in characteristic 2. Since we assume $\mathsf{char}\,\mathbb{K} \neq 2$, this completes the proof of the proposition. □

In the next proposition, a *quadratic metasymplectic space* is the metasymplectic space arising from the Tits index ${}^2\mathsf{E}^2_{6,4}$. The points are chosen in such a way that they conform to vertices of type 2 in the diagram of the absolute type.

Proposition 14. *If the fix diagram of a linear involution of* $\mathsf{E}_7(\mathbb{K})$ *is* $\mathsf{E}_{7;4}$*, then the fix structure is a (possibly non-thick) building of type* F_4*. More precisely, the fixed points, lines, planes and symps of* $\mathsf{E}_{7,1}(\mathbb{K})$ *either form the points, lines, planes and symps, respectively, of a quadratic metasymplectic space (and then the fixed building is a thick building of type* F_4*), or form the symps, planes, lines and points, respectively, of the line Grassmannian of a small Hermitian polar space of rank* 4 *(and this then conforms to a non-thick building of type* F_4*).*

Proof. We divide this long proof into some parts to keep the overview.

Part 1. *The fix structure in a stabilised symp of* $\mathsf{E}_{7,1}(\mathbb{K})$.

We first show that the fix structure in any stabilised symp ξ of $\Delta^* = \mathsf{E}_{7,1}(\mathbb{K})$ is either an elliptic quadric in some 7-dimensional subspace of the ambient projective 9-space $\mathsf{PG}(9,\mathbb{K})$ of ξ (Case (i) of Proposition 5), or the union of an elliptic subquadric in a 3-dimensional subspace of $\mathsf{PG}(9,\mathbb{K})$ and its perp, an elliptic quadric in a 5-dimensional subspace of $\mathsf{PG}(9,\mathbb{K})$ (Case (v) of Proposition 5).

Due to the fix diagram, we only have to rule out Case (iii) of Proposition 5, that is, we have to show that ρ stabilises at least one line of ξ. The fix diagram ensures that there exists at least one line L stabilised by ρ. The residue of L is (isomorphic to the line Grassmannian of) a projective 5-space $\mathsf{PG}(5,\mathbb{K})$. On the diagram, we can read that lines of $\mathsf{PG}(5,\mathbb{K})$ correspond to planes of Δ^* and 3-spaces of $\mathsf{PG}(5,\mathbb{K})$ correspond to symps of Δ^*. Now, Lemma 3 implies that at least one 3-space of $\mathsf{PG}(5,\mathbb{K})$ is stabilised by ρ. This yields a stabilised symp ζ containing L. However, then, Proposition 5 yields at least two fixed points x, y on L. At least one of them is not contained in ξ as otherwise $L \subseteq \xi$ and we reached our goal. We may assume $x \notin \xi$. Since ρ does not fix any para, x is not para-collinear to ξ. Since ρ does not fix any 4-space, x is not max-symplectic or max-collinear to ξ. Hence, x is either far from ξ, or line-collinear to ξ. In the latter case, ρ stabilises the line $x^\perp \cap \xi$ and we are done. In the former case, y is not contained in ξ and hence, we may assume that y is also far from ξ. Let $\{x'\} = x^{\perp\!\!\!\perp} \cap \xi$ and $\{y'\} = y^{\perp\!\!\!\perp} \cap \xi$. If $x' \perp\!\!\!\perp y'$, then by Proposition 4(vi), $x' \equiv y$, contradicting Proposition 3(v). If $x' \perp y'$, then, since x' and y' are both fixed points, the line $\langle x', y' \rangle \subseteq \xi$ is stabilised. Hence, we may assume that $x' = y'$. There are three possibilities:

(i) $\xi(x,x') = \xi(y,y')$.
In this case, the unique point of $\langle x, y \rangle$ collinear to $x' = y'$ is also fixed, and similarly as above, it is line-collinear to ξ leading to a fixed line in ξ.

(ii) $\xi(x,x') \neq \xi(y,y')$ *and* x *is line-collinear to* $\xi(y,y')$.
Since y is not collinear to x' and x' is symplectic to x, this contradicts Proposition 4(iii).

(iii) $\xi(x,x') \neq \xi(y,y')$ *and* x *is max-collinear to* $\xi(y,y')$.
Then, ρ stabilises the maximal singular subspace $x^\perp \cap \xi(y,y')$ of $\xi(y,y')$, contradicting the fix diagram.

Hence, we have shown that the fix structure in any stabilised symp is either Case (i) or Case (v) of Proposition 5.

Part 2. *The case that the fix structure is a thick metasymplectic space.*

Suppose now first that Case (v) never occurs. We claim that in each stabilised symp ξ of $\Delta = \mathsf{E}_{7,7}(\mathbb{K})$, each point is mapped onto a collinear one. Indeed, let x^* be the point of Δ^* corresponding to ξ. Considering a fixed symp ξ^* in Δ^*, we see that either x^* is far from ξ^*, and then the unique symp through x^* intersecting ξ^* is stabilised, or x^* is line-collinear to ξ^*, and then there exists a fixed point $y^* \in \xi^*$ collinear to each point of $(x^*)^\perp \cap \xi^*$ and so $\xi(x^*, y^*)$ is stabilised, or $x^* \in \xi^*$. In all cases, we found a stabilised symp ζ^* in Δ^* through x^*. Let Q^* be the residue of x^* in ζ^*. Then, Q^* is a quadric of type $\mathsf{D}_{4,1}$. Our assumption implies that the fix structure of ρ in Q^* is an elliptic subquadric of Witt index 2 spanning a subspace of dimension 5 in the ambient projective space of Q^*. By Grassmann's identity, every generator of Q^* intersects that subspace of dimension 5 nontrivially, and hence contains a fixed point. It follows that no singular 3-space of Q^* is mapped onto an opposite one. Now, x^* corresponds to ξ in Δ, and ζ^* corresponds to a line L of Δ. In view of the Coxeter diagram of type E_7, the quadric Q^* corresponds to a quadric Q, where the points of Q^* become maximal singular subspaces of a given oriflamme type, and the points of Q are the maximal singular subspaces of Q^* of a given oriflamme type. It follows that ρ does not map any point of Q to an opposite. This means that ρ maps each plane π of ξ through L to a plane π^ρ contained in a 3-space together with π. This way, we obtain a stabilised 3-space Σ through L in ξ. Lemma 3 yields a stabilised line in Σ through each point of Σ.

Now, let p be any point in ξ. We claim that $p^\perp \cap \Sigma$ contains a stabilised line. Indeed, if not, then $p^\perp \cap \Sigma$ is a plane α. Then, $\alpha \cap \alpha^\rho$ is a stabilised line and the claim is proved. Without loss of generality, we may assume that p is collinear to L. Then, $\langle p, L\rangle$ is a plane, and we proved in the previous paragraph that this plane is mapped onto a collinear plane. This means that p is mapped onto a collinear point. Our claim that each point of ξ is mapped onto a collinear one, is proved.

Consequently, no point of Δ is mapped onto a symplectic one. Indeed, if $p \perp\!\!\!\perp p^\rho$, then $\xi(p, p^\rho)$ is stabilised, and the argument above shows $p \perp p^\rho$.

Now noting that ρ does not map any point to itself, and that it maps at least one point to a collinear one, Theorem 7.1 of [34] implies that the fix structure of ρ in Δ^* is a quadratic metasymplectic space.

Part 3. *The consistency property.*

From now on, we may assume that Case (v) does occur and let ξ^* be a symp of $\mathsf{E}_{7,1}(\mathbb{K})$ the fix structure of which is exactly Case (v). So, the fixed point set in ξ^* can be written as $Q \cup Q^\perp$, where Q is an elliptic subquadric of Witt index 1 in a 3-dimensional subspace of the ambient projective space of ξ^*, and $Q^\perp$, its orthogonal complement, is an elliptic subquadric of Witt index 2 in a 5-dimensional subspace. We claim that, for a point $p \in Q \cup Q^\perp$, the fix structure in $\mathsf{Res}(p)$, the latter viewed as a quadric $\mathsf{D}_{6,1}(\mathbb{K})$, conforms either to Case (v') of Lemma 9, or to Case (iv'), depending on whether $p \in Q$ or $p \in Q^\perp$.

Indeed, in any case, ρ stabilises 3-spaces and generators through the line L of Δ that corresponds to the symp ξ^* of Δ^*. Consequently, only Cases (iv') and (v') can occur. If $p \in Q$, then all generators through a stabilised 3-space are stabilised (since all points on a line of $Q^\perp$ are fixed), and so Case (v') occurs. If $p \in Q^\perp$, then exactly two generators through a stabilised 3-space are stabilised (these correspond to a point of Q and a point of $Q^\perp$). This shows the claim.

Now, let $p \in Q$ be arbitrary. The stabilised symps through p form the point set of a non-degenerate thick polar space and consequently, each pair of stabilised symps is locally opposite a common stabilised symp. Let ζ_1^* and ζ_2^* be two stabilised symps, locally opposite at p and suppose one of them conforms to Case (v). Let L_1 be a stabilised line of ζ_1^* through p. There is a unique line $L_2 \subseteq \zeta_2^*$, $p \in L_2$, collinear to L_1, and hence L_2 is stabilised. In $\mathsf{Res}(p)$, the plane $\langle L_1, L_2\rangle$ is a line, and since p conforms to Case (v'), each line in $\langle L_1, L_2\rangle$ through p is stabilised (take into account that in $\mathsf{Res}(p)$, points and lines correspond to 5-spaces and 3-spaces, respectively, of the corresponding symp in $\mathsf{E}_{7,7}(\mathbb{K})$). Lemma 3 implies that either both L_1 and L_2 are pointwise fixed, or none of them is. Hence, both symps ζ_1^* and ζ_2^* conform to Case (v). We conclude that all stabilised symps through p conform to Case (v). Moreover, since the local fix structures in these symps are isomorphic, we see that in each of these symps, p belongs to a pointwise fixed subquadric of Witt index 1 whose orthogonal complement in the symp is also pointwise fixed. We refer below to this property as the *consistence property*.

Part 4. *Identifying a small Hermitian polar space of rank* 4.

Now, define the following point-line geometry $\Gamma_\rho = (X, \mathcal{L})$. The point set is the set of stabilised symps of $\mathsf{E}_{7,7}(\mathbb{K})$ in which the fix structure conforms to Case (v'); the lines are obtained from the stabilised symps of $\mathsf{E}_{7,1}(\mathbb{K})$ in which the fix structure conforms to Case (v) as the fixed quadrics therein of Witt index 1 (contained in 3-spaces). Just like conics in $\mathsf{PG}(2, \mathbb{K})$ are representations of projective lines $\mathsf{PG}(1, \mathbb{K})$, these quadrics are representations of the projective line $\mathsf{PG}(1, Ł)$, for the quadric extension of $\mathbb{K}$ corresponding to the irreducible polynomial of degree 2 in two variables appearing in the standard equation of the elliptic quadric. Hence, the lines of Γ have the structure of $\mathsf{PG}(1, Ł)$, for a quadratic extension Ł of $\mathbb{K}$. Also, by the previous paragraph and Lemma 9(v'), the residue at each point of Γ is a small Hermitian polar space of rank 3 over Ł. We now claim that Γ is a small Hermitian polar space of rank 4. Since the lines of Γ contain at least three points, and no point residual is degenerate, we only have to verify the one-or-all axiom.

We consider the situation in $\mathsf{E}_{7,1}(\mathbb{K})$, where we are given a fixed point p and a stabilised symp ζ^*. We assume that $p \in X$ and ζ^* corresponds to a member of $\mathcal{L}$. As such, the fix structure of ρ in ζ^* can be written as $Q \cup Q^\perp$, with Q as before. Also, we may assume $p \notin \zeta^*$. Since ρ does not stabilise 4-spaces or paras, we see that p is either collinear to a line of ζ^*, or far from ζ^*.

Suppose first that p is collinear to a line L of ζ^*. Then, L is stabilised. However, we know that p corresponds in Δ to a fixed symp of Case (v'); hence, since in such a symp, all 5-spaces through a stabilised 3-space are stabilised, L (which corresponds in Δ to such a 3-space) is fixed pointwise (since points correspond to 5-spaces in Δ). However, then, $L \subseteq Q^\perp$ and every point $x \in Q$ is collinear to L. So, for each $x \in Q$, the symp $\xi(p,x)$ is stabilised. Since each such symp conforms to Case (v), it follows that all points of ζ^* in X are Γ-collinear to p.

Now, suppose that p is far from ζ^*. Let x be the unique point of ζ^* symplectic to p. Then, $x \in Q \cup Q^\perp$. Suppose for a contradiction that $x \in Q^\perp$. The symp $\xi(p,x)$ is locally opposite ζ^*, and so the local fix structure in $\xi(p,x)$ at x is isomorphic to the one in ζ^*. Since in ζ^*, it is the union of two subquadrics of Witt index 1, the same is true in $\xi(p,x)$. However, then, in $\xi(p,x)$, the point x belongs to a pointwise fixed subquadric of Witt index 2. Since p is not collinear to x, the point p also belongs to the same subquadric, a contradiction to the consistency property derived above. Hence, $x \in Q$ and p is Γ-collinear to a unique point of Q.

The claim that Γ is a small Hermitian polar space of rank 4, is proved.

Part 5. *Identifying the fix structure as a non-thick metasymplectic space.*

By definition, the lines of Γ can be identified with the stabilised symps of Δ^* which conform to Case (v) of Proposition 5. We now claim that no stabilised symp of Δ^* conforms to Case (i) of Proposition 5. Indeed, assume, for a contradiction, that ζ^* is such a symp in Δ^*. Let ξ^* be a symp conforming to Type (v), let Q be the pointwise fixed subquadric of Witt index 1 in ξ^*, and let x be an arbitrary point in Q. The consistency property implies that $x \notin \zeta^*$. Since no 4- or 4′-spaces are stabilised by ρ, and no para is stabilised by ρ, Proposition 4 implies that either $x^\perp \cap \zeta^*$ is a line L, or x is far from ζ. In the former case, we pick a fixed point $y \in \zeta^*$ collinear to all points of L and obtain the contradiction against the consistence property that the symp $\xi(x,y)$ contains a pointwise fixed plane $\langle y, L\rangle$. Hence, x is far from ζ^*. Set $\{z\} = x^{\perp\!\!\!\perp} \cap \zeta^*$. Then, $\xi(x,z)$ if stabilised and conforms to Type (v) (by the consistency property). Since z is not collinear to x, the point z belongs to the pointwise fixed subquadric of Witt index 1 in $\xi(x,z)$. By the consistence property again, ζ^* conforms to Type (v), a contradiction. This yields the claim.

The previous claim implies directly that each stabilised line of Δ can be identified with a(ny) line of Γ, hence with a point of the line-Grassmannian of Γ.

Now, let π^* be a plane of Δ^* stabilised by ρ. Let π be the corresponding 3-dimensional singular subspace of Δ. Then, by Lemma 3, π contains at least one stabilised line L. Let ξ^* be the symp of Δ^* corresponding to L, and let $Q \cup Q^\perp$ be the fix structure of ρ in ξ^*, with, as above, Q a subquadric of Witt index 1. Since no planes are contained in $Q \cup Q^\perp$, π^* contains a point $x^* \in Q$. If ξ is the symp of Δ corresponding to x^*, then in ξ, the 3-space π is a line of the small Hermitian polar space pointwise fixed by ρ. Hence, the pair $\{\xi, \pi\}$ is a point-plane pair of Γ. This shows that we can identify each stabilised singular 3-space of Δ with a(ny) line of the line-Grassmannian of Γ.

Now, let L^* be a line of Δ^* stabilised by ρ. Let U be the corresponding 5-dimensional singular subspace of Δ. As in the previous paragrah, we find a stabilised line in U, and hence L^* is some stabilised line in a stabilised symp ξ^* of Δ^*. Let $Q \cup Q^\perp$ be as in the previous paragraph. Then, since Q does not contain lines, L^* contains at least one point y^* of $Q^\perp$. If ζ is the symp of Δ corresponding to y^*, then in ζ, the 5-space U is a maximal singular subspace of the top-thin polar space defined in Lemma 9(iv'), and the oriflamme type of it depends on L^* being contained in $Q^\perp$ or instead having a unique point in Q (and both do occur). This top-thin polar space is the line-Grassmannian of a projective space $\mathsf{PG}(3,Ł)$, and as such can be identified with a maximal singular subspace of Γ. This shows that we can identify U with a(ny) plane of the line-Grassmannian of Γ.

Finally, the fix structure of each stabilised symp ξ of Δ is either a small Hermitian polar space of rank 3 (and hence can be identified with a point of Γ, defining a symp of the line-Grassmannian of Γ), or a top-thin polar space of rank 3, corresponding to a maximal singular subspace of Γ (and hence can be identified with a symp of the line-Grassmannian of Γ defined by that maximal singular subspace).

This shows that the points, lines, planes and symps of $\mathsf{E}_{7,1}(\mathbb{K})$ form the symps, planes, lines and points, respectively, of the line Grassmannian of Γ, which is a small Hermitian polar space of rank 4, and with that a non-thick metasymplectic space arising from a non-thick building of type F_4.

This completes the proof of the proposition. $\square$

We can now return to the rank 2 case.

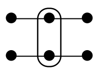
Proposition 15. *If the fix diagram of a linear involution ρ of $\mathsf{E}_7(\mathbb{K})$ is $\mathsf{E}_{7;2}$, then the fix structure is a building of type C_2, which is a form of absolute type A_7. More exactly, it is a Moufang quadrangle with Tits index ${}^2\mathsf{A}^{(2)}_{7,2}$. It also arises as the fixed structure of a polarity in a quaternion projective space of dimension 3.*

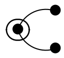
Proof. Similarly to the proof of Lemma 9, a fix point free involution of $Q := \mathsf{D}_{6,1}(\mathbb{K})$, with equation

$$X_{-6}X_6 + X_{-5}X_5 + X_{-4}X_4 + X_{-3}X_3 + X_{-2}X_2 + X_{-1}X_1 = 0,$$

stabilising singular lines but no singular 3-spaces can be represented as

$$\begin{cases} (x_{-6}, \ldots, x_6) \\ \mapsto (kx_{-5}, x_{-6}, m_4x_4, m_3x_3, m_2x_2, m_1x_1, km_1^{-1}x_{-1}, km_2^{-1}x_{-2}, km_3^{-1}x_{-3}, km_4^{-1}x_{-4}, kx_6, x_5), \end{cases}$$

where $k, m_1, m_2, m_3, m_4 \in \mathbb{K}$ such that k is a non-square in $\mathbb{K}$ and $N(x_1, x_2, x_3, x_4) := m_1x_1^2 + m_2x_2^2 + m_3x_3^2 + m_4x_4^2$ is anisotropic in $\mathbb{K}(\sqrt{k})$. If we extend the involution ρ (linearly) to $\mathsf{D}_{6,1}(\mathbb{K}(\sqrt{k}))$, then we see that the fixed point set consists of two hyperelliptic subquadrics of Witt index 1 in subspaces of dimension 5 (Case $(viii)$ of Proposition 6). Note that the Galois involution interchanges these quadrics. Since a single hyperelliptic quadric in projective 5-space splits as a polar space of type D_3—and hence has Tits index ${}^1\mathsf{A}^{(2)}_{3,1}$—the fix diagram of the Tits index of the pair of interchanged hyperelliptic quadrics (as a Moufang building of rank 1) is .

Now, we consider the residue of a fixed line, which is a polar space P isomorphic to $\mathsf{D}_{5,1}(\mathbb{K})$. The linear involution ρ fixes only points; hence, we are in case (iii) of Proposition 5, and the fixed point set is a hyperelliptic quadric R in projective 5-space. The action of ρ on the residue of a fixed point in P is described by the stabiliser of a fixed line in Q. Hence, extending P to $\mathsf{D}_{5,1}(\mathbb{K}(\sqrt{k}))$, the corresponding Galois involution interchanges types. Over a suitable quadric extension, R splits as a hyperbolic quadric, and so we see that the Galois group of the corresponding Tits index is a group of order 4 which does not act type-preserving. Hence, the fix diagram of the Tits index of R (viewed as a Moufang building of rank 1) is .

Now, there is only one Tits index whose residues conform to the two above fix diagrams, and that is ${}^2\mathsf{A}_{7,2}^{(2)}$.

For the last assertion, note that, in $\mathsf{E}_7(\mathbb{K}(\sqrt{k}))$, ρ extends to an involution of Type IV, fixing a 3-dimensional quaternion projective space $\mathsf{PG}(3,\mathbb{H})$, with $\mathbb{H}$ determined by the anisotropic norm form N given above. Since the Galois involution interchanges the hyperelliptic quadrics, it interchanges points and planes of $\mathsf{PG}(3,\mathbb{H})$. It follows that the fixed point set of ρ is given by the set of fixed lines and incident point-plane pairs of a polarity of a 3-dimensional quaternion projective space. $\square$

The generalised quadrangle of Proposition 15 is *not* a quaternion quadrangle, as the latter has Tits index $\mathsf{D}_{4,2}^{(1)}$.

Example 1. *We now provide an explicit example of such a generalised quadrangle with Tits index* ${}^2\mathsf{A}_{7,2}^{(2)}$*. We use the notation of [35] and refer to the latter for undefined notions. Let* $\mathbb{H}$ *be a quaternion division algebra over the field* $\mathbb{K}$*. Let* Ł *be a separable quadratic extension of* $\mathbb{K}$*, with* $L \subseteq \mathbb{H}$ *(it is the field* $\mathbb{K}(\sqrt{k})$ *in the previous proof). Denote the Galois involution of* Ł$/\mathbb{K}$ *by* Ł $\to$ Ł$: x \mapsto \overline{x}$*. Due to the Cayley–Dickson process, we can write the elements of* $\mathbb{H}$ *as pairs* $(a,b) \in$ Ł $\times$ Ł*. Let* $k \in \mathbb{K}$ *be such that it cannot be written as* $x\overline{x}$*, for* $x \in$ Ł*. The multiplication in* $\mathbb{H}$ *can then be given by* $(a,b)\cdot(c,d) = (ab + k\overline{b}d, \overline{a}d + bc)$*. Let* σ *be an involution of* Ł *that commutes with the Galois involution of* Ł$/\mathbb{K}$ *and such that the induced involution in* $\mathbb{K}$ *is nontrivial (that is,* σ *does not coincide with that Galois involution). Define the following map* $\theta : \mathbb{H} \to \mathbb{H} : (a,b) \mapsto (\overline{a}^{\sigma}, b^{\sigma})$*. Then, one checks that* θ *is an involution in* $\mathbb{H}$*.*

Let V *be a 4-dimensional right vector space over* $\mathbb{H}$*. Then, the isotropic points and lines of the* θ*-anti-Hermitian form*

$$F : V \times V \to \mathbb{H} : ((x_{-2},x_{-1},x_1,x_2),(y_{-2},y_{-1},y_1,y_2)) \mapsto x_{-2}^{\theta}y_2 + x_{-1}^{\theta}y_1 - x_1^{\theta}y_{-1} - x_2^{\theta}y_{-2}$$

define a generalised quadrangle with Tits index ${}^2\mathsf{A}_{7,2}^{(2)}$ *which also arises as fix structure of an involution in* $\mathsf{E}_7(\mathbb{K})$*.*

6. Concrete Constructions; Existence

6.1. A Construction of $\mathsf{E}_{7,7}(\mathbb{K})$ *in* $\mathsf{PG}(55,\mathbb{K})$

In order to construct examples of involutions of $\mathsf{E}_{7,7}(\mathbb{K})$, we view it as a full subgeometry of $\mathsf{PG}(55,\mathbb{K})$. This has been done before, using certain 4-forms, by Aschbacher [36] and Cooperstein [37] (see also [38]). Vavilov and Luzgarev [39] use intersections of quadrics. We here follow the approach of [40] (§10), which is slightly more explicit and combinatorial than [39].

We need an alternative description of the Gosset graph. Pick two symbols ∞ and ∞_*. Let $\widetilde{\Gamma}$ and $\widetilde{\Gamma}_*$ be two copies of the complement of the collinearity graph of the generalised quadrangle of order $(2,4)$. Let $\Gamma \to \Gamma_* : x \mapsto x_*$ be the corresponding identification. Then, we define the graph Γ as follows. The vertices are the symbols ∞ and ∞_*, together with the vertices of $\widetilde{\Gamma}$ and $\widetilde{\Gamma}_*$. The symbol ∞ is adjacent to all vertices of $\widetilde{\Gamma}$, whereas the symbol ∞_* is adjacent to all vertices of $\widetilde{\Gamma}_*$. Adjacency within $\widetilde{\Gamma}$ or $\widetilde{\Gamma}_*$ is natural. Finally, a vertex x of $\widetilde{\Gamma}$ is adjacent to all vertices y_* of $\widetilde{\Gamma}_*$ with the property that y_* is collinear to x_* in the underlying generalised quadrangle. Now, let $\mathcal{S}$ be a Hermitian spread of the quadrangle underlying $\widetilde{\Gamma}$ and $\mathcal{S}_*$ the corresponding partition of $\widetilde{\Gamma}_*$ in 3-cocliques.

Let V be a 56-dimensional vector space over $\mathbb{K}$ where the basis vectors—and hence also coordinates—are labeled using the vertices of Γ (vertex v corresponds to basis vector e_v and generic coordinate x_v). We define the following $126 + 63$ quadratic forms on V. (We denote the vertex set of Γ also with Γ.)

- Each hexacross H of Γ containing ∞ contains a unique vertex v_* of $\widetilde{\Gamma}_*$. Also, there is a unique pair $\{w, w'\}$ of vertices of $\widetilde{\Gamma}$ contained in a member of $\mathcal{S}$. Let P be the partition of the vertices of $H \setminus \{\infty, v_*, w, w'\}$ into non-adjacent pairs. Then, we define

$$Q_H : V \to \mathbb{K} : (X_u)_{u\in\Gamma} \mapsto X_\infty X_{v_*} + X_w X_{w'} - \sum_{\{a,b\}\in P} X_a X_b.$$

  A similar quadratic form is defined for each hexacross containing ∞_*.
- Each hexacross H of Γ not containing either of ∞ and ∞_* contains a 6-clique of $\widetilde{\Gamma}$ and a 6-clique of $\widetilde{\Gamma}_*$. Each such 6-clique has the property that it can be uniquely partitioned into two 3-sets $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ such that the members of $\mathcal{S}$ containing v_1, v_2, v_3 and v_4, v_5, v_6, respectively, form a regulus in the underlying generalised quadrangle (that is, the union of these lines is a quadrangle of order $(2, 1)$, which is a hyperbolic polar space of rank 2). Let w_i be the unique vertex of H not collinear to v_i, $1 \le i \le 6$ (then $w_i \in \widetilde{\Gamma}_*$). Then, we define

$$Q_H : V \to \mathbb{K} : (X_u)_{u\in\Gamma} \mapsto \sum_{i=1}^{3} X_{v_i} X_{w_i} - \sum_{i=4}^{6} X_{v_i} X_{w_i},$$

  where it does not matter which triplet of monomials gets the minus sign.

The quadratic forms defined up to now are referred to as the *short quadratic forms*. The null set of each of them defines a symp, which we shall denote by $\xi(Q_H)$, in the subspace spanned by the basis vectors corresponding to the coordinates that appear in the form. The next two classes of quadratic forms will be referred to as *long quadratic forms*.

- Now, let H and H' be two opposite hexacrosses (hence, each vertex of one hexacross is collinear to a unique vertex of the other and also opposite a unique vertex of the other). Suppose $\infty \in H$. Then, $\infty_* \in H'$. Let v be the unique vertex of H' in $\widetilde{\Gamma}$. Then, $v_* \in H$. We define

$$Q_{H,H'} : V \to \mathbb{K} : (X_u)_{u\in\Gamma} \mapsto X_\infty X_{\infty_*} + X_v X_{v_*} - \sum_{w\in H\setminus\{\infty, v_*\}} X_w X_{w_*}.$$

- Finally, let H and H' be two opposite hexacrosses not containing either ∞ or ∞_*. Let W be the set of vertices of H in $\widetilde{\Gamma}$ and let W' be the set of vertices of H' in $\widetilde{\Gamma}$. Then, we define

$$Q_{H,H'} : V \to \mathbb{K} : (X_u)_{u\in\Gamma} \mapsto \sum_{v\in W} X_v X_{v_*} - \sum_{v\in W'} X_v X_{v_*}.$$

It is shown in [40] (§10) that the common null set X of all the mentioned quadratic forms is the point set of a fully embedded $\mathsf{E}_{7,7}(\mathbb{K})$ in $\mathsf{PG}(V)$. The lines are just the lines of $\mathsf{PG}(V)$ contained in X.

Now, we provide an explicit construction of Γ. It suffices to give an explicit construction of $\widetilde{\Gamma}$ and $\mathcal{S}$. Set $T = \{1, 2, 3, 4, 5, 6\}$ and $T' = \{1', 2', 3', 4', 5'\}$. Let T'' be the set of pairs of elements of T. Then, the set of vertices of $\widetilde{\Gamma}$ is $T \cup T' \cup T''$. Adjacency is determined by declaring both T and T' to be cliques, and saying that $i \in T$ is adjacent to $i' \in T'$ and both are adjacent to all pairs $\{j, k\}$ with $i \notin \{j, k\}$. Furthermore, the pairs $\{i, j\}$ and $\{k, \ell\}$ are adjacent if $|\{i, j, k, \ell\}| = 3$.

The spread $\mathcal{S}$ consists of the following triples (with obvious shorthand notation):

$$\begin{array}{lll} 1\,12\,2' & 4'\,45\,5 & 14\,25\,36 \\ 2\,23\,3' & 5'\,56\,6 & 15\,26\,34 \\ 3\,13\,1' & 6'\,46\,4 & 16\,24\,35 \end{array}$$

This allows one to very explicitly write down the 126 short quadratic forms and 63 long quadratic forms (we will not do so, but the readers can easily do so by themselves).

6.2. Generalities About the Examples

We will now construct examples of involutions of all types. For Type I, we refer to [29], and Type III examples can be found in [34]. Now, let ρ be an involution of Type II, IV, V or VI. Then, ρ stabilises at least two opposite symps (by our analysis in the previous sections). We may assume that these symps are $\xi(Q_H)$ and $\xi(Q_{H^*})$, for two opposite hexacrosses H and H^* of Γ. We then provide explicit expressions for ρ restricted to these symps, making sure the fix structure in such symp is exactly the one dictated by the fix diagram and the type. Then, we extend this to all $\mathsf{PG}(55, \mathbb{K})$ by defining the involution on all basis vectors. We show that the obtained linear map ρ is an involution (this will always be obvious) and that it preserves $\mathsf{E}_{7,7}(\mathbb{K})$ (this will follow from the fact that it preserves the set of quadratic forms given above, up to a scalar).

We now claim that, once we have shown that ρ is an involution, it has the right intended type. Indeed, first assume ρ fixes some chamber. Then, the building-theoretic projection of that chamber onto $\xi(Q_H)$ is a chamber in $\xi(Q_H)$ that is fixed, a contradiction. So, the fix diagram of ρ is one of Type I up to VI. In case we start with Type II, no type other than I and II fixes elements of type 2. Also, every stabilised symp in Type I has a different fix structure from the one we start with to describe Type II. Similarly, if we start with a stabilised symp for Type IV, then it can not appear in Type III, and since no other type fixes vertices of type 7, the obtained involution has to have Type IV.

Now, we note that in all Types I up to IV, each stabilised symp fixes an element of type either 2 or 7, and no stabilised symp in Types V or VI has this property. Hence, if we have an involution which only fixes lines in a given stabilised symp, then it will automatically be of Type V. Finally, since for Type V, the fix structure defines a thick generalised quadrangle, each stabilised symp contains stabilised lines and so, every linear involution which stabilises a symp and induces an anisotropic involution therein, has Type VI.

Our claim is proved. We summarise what we showed so far (we add Type III in (b) below with similar proof as for the other cases; this does not work for Type I; hence, the latter is not included).

Proposition 16. *Let ρ be a linear involution of $\mathsf{E}_{7,7}(\mathbb{K})$ and suppose ρ stabilises at least one symp ξ. Let F be the fix structure of ρ in ξ.*

(a) If F conforms to Case (iv') of Lemma 9, then ρ has Type II.
(b) If F conforms to Case (iii) of Proposition 6, then ρ has Type III.
(c) If F conforms to Case $(viii)$ of Proposition 6, then ρ has Type IV.
(d) If F conforms to Case (ii') of Lemma 9, then ρ has Type V.
(e) If ρ is anisotropic over ξ (so, F is empty), then ρ has Type VI.

Concerning Type VII, we will have to use other methods to identify our examples as belonging to that type. There are two main steps for that.

Lemma 11. *Let p be a point of $\mathsf{E}_{7,7}(\mathbb{K})$ and let ρ be a linear involution. Then, ρ is anisotropic if, and only if, each point collinear or equal to p is mapped onto an opposite.*

Proof. If ρ is anisotropic, then every point of $\mathsf{E}_{7,7}(\mathbb{K})$ is mapped onto an opposite and so, in particular, this is true for every point equal or collinear to p. Now, suppose that each point collinear or equal to p is mapped onto an opposite and assume for a contradiction that ρ is not anisotropic. Then, by Proposition 8, ρ stabilises some symp ξ. Clearly, p is not contained in ξ. Proposition 2 yields at least one point $q \perp p$ contained in ξ. However, then, $q^\rho \in \xi$, contradicting $q \equiv q^\rho$. The lemma is proved. $\square$

In order to use Lemma 11 we will have to determine explicitly the coordinates of the points not opposite a given point collinear to a well-chosen point p. If we take for p the point corresponding to the vertex ∞ of the graph Γ, then the following lemma provides all information that we will need.

Lemma 12. *Let $\widetilde{W}$ be the subspace generated by all e_v, with $v \in \{\infty\} \cup V(\widetilde{\Gamma})$. Let*

$$p = x_\infty e_\infty + \sum_{v \in V(\widetilde{\Gamma})} x_v e_v$$

be a point belonging to $X \cap \widetilde{W}$. Then, a point $(y_v)_{v \in V(\Gamma)} \in X$ is opposite p if, and only if,

$$\sum_{v \in V(\widetilde{\Gamma})} x_v y_{v_*} - x_\infty y_{\infty_*} \neq 0.$$

Proof. Let $\widetilde{U}$ be the subspace of $\widetilde{W}$ generated by the e_v, with $v \in V(\widetilde{\Gamma})$, and let $\widetilde{U}_*$ be the subspace generated by the e_{v_*}, with $v \in V(\widetilde{\Gamma})$. By construction of X, the lemma is true for $p = e_v$, with $v \in V(\widetilde{\Gamma})$. Now, Definition 10.10 and Lemma 10.12 of [40] imply that, in view of Definition 7 of [13], each automorphism of X fixing e_∞ and e_{∞_*}, has the same action on the basis $(e_v)_{v \in V(\widetilde{\Gamma})}$ of $\widetilde{U}$ as it has on the basis dual to $(e_{v_*})_{v \in V(\widetilde{\Gamma})}$ of the subspace $\widetilde{U}_*$. Hence, the lemma follows for $p \in \widetilde{U}$.

Now, we claim that, for each vertex v of $\widetilde{\Gamma}$, the point $p = x_\infty e_\infty + x_v e_v$ is not opposite the point $p_* = x_v e_{\infty_*} + x_\infty e_{v_*}$. Indeed, this can be done by establishing a common collinear point, for each choice of v. Since all v play the same role, we content ourselves with taking $v = 46$. Let q be the point with all coordinates zero, except

$$\begin{cases} x_{35} = \lambda_{35} x_\infty, \quad x_{12} = \lambda_{12} x_\infty, \quad x_{25} = \lambda_{25} x\infty, \quad x_{23} = \lambda_{23} x_\infty, \quad x_{15} = \lambda_{15} x_\infty, \\ x_{13} = \lambda_{13} x_\infty, \quad x_6 = \lambda_6 x_\infty, \quad x_{4'} = \lambda_{4'} x_\infty, \quad x_4 = -\lambda_4 x_\infty, \quad x_{6'} = \lambda_{6'} x_\infty, \\ x_{12_*} = \lambda_{35} x_{46}, \quad x_{35_*} = \lambda x_{12}, \quad x_{13_*} = \lambda_{25} x_{46}, \quad x_{15_*} = \lambda_{23} x_{46}, \quad x_{23_*} = \lambda_{15} x_{46}, \\ x_{25_*} = \lambda_{13} x_{45}, \quad x_{4'_*} = \lambda_6 x_{46}, \quad x_{6_*} = \lambda_{4'} x_{46}, \quad x_{6'_*} = \lambda_{4'} x_{46}, \quad x_{4_*} = \lambda_{6'} x_{46}, \end{cases}$$

where the λ belong to $\mathbb{K}$ and satisfy

$$\lambda_4 \lambda_{6'} = \lambda_6 \lambda_{4'} + \lambda_{23} \lambda_{15} + \lambda_{13} \lambda_{25} + \lambda_{12} \lambda_{35}.$$

One now easily calculates that all of q, $p + q$ and $p_* + q$ belong to X. This shows the claim.

Now, it follows from the fact that a point $(x_v)_{v \in V(\Gamma)}$ of X satisfies $x_{w_*} = 0$ if, and only if, it is not opposite e_w, $w \in V(\Gamma)$, the symmetric statement interchanging w and w_*, and the previous claim that a point $(x_v)_{v\in V(\Gamma)}$ of X is not opposite the point $y_{w_*}e_{w_*} + y_{\infty_*}e_{\infty_*}$ if, and only if, $x_w y_{\infty_*} = y_{w_*}x_\infty$. It then follows that the point

$$p = x_\infty e_\infty + \sum_{v\in V(\widetilde{\Gamma})} x_v e_v$$

is not opposite the point $x_\infty e_{w_*} + x_w e_{\infty_*}$, for all $w_* \in V(\widetilde{\Gamma}_*)$. After an elementary calculation, this implies the lemma. □

We now explain how we present the examples, and why our presentation proves existence. We denote by H the hexacross in Γ defined by the vertices ∞ and 12_* (and remember that this is shorthand for $\{1,2\}_*$). Then, we see that H consists of the vertices $\infty, 12_*, 1, 1', 2, 2', 34, 35, 36, 45, 46, 56$. The corresponding symp ξ_H lives in the space $U = \langle e_\infty, e_{12_*}, e_1, e_2, e_{1'}, e_{2'}, e_{34}, e_{35}, e_{36}, e_{45}, e_{46}, e_{56}\rangle$ and has equation

$$-X_\infty X_{12_*} - X_1X_{2'} + X_{1'}X_2 + X_{34}X_{56} + X_{35}X_{46} + X_{36}X_{45} = 0.$$

We will also always assume that the opposite hexacross is stabilised, which we are allowed to, since we may assume that the apartment corresponding to Γ is stabilised. All involutions that we will present will hence induce a permutation of the basis vectors (or rather the corresponding projective points). By putting forward an involution in ξ_H, this permutation is already defined on H and H^*. Each other vertex of Γ is adjacent to a unique 6-clique of H and so the permutation of these 6-cliques determines the permutation of the projective points defined by all the remaining basis vectors. For a given collineation θ of $\mathsf{E}_{7,7}(\mathbb{K})$ stabilising the apartment corresponding to Γ, we denote this permutation with σ_θ (and we use the same notation for the corresponding automorphism of Γ).

First, suppose σ_θ is the identity. Then, θ belongs to the torus of the corresponding algebraic group (or Chevalley group).

Lemma 13. *Suppose, with above notation, that σ_θ is the identity. Let θ be given by $e_v \mapsto \lambda_v e_v$, for all vertices v of Γ. Then, θ is determined by the (free) choice the eight well-chosen parameters λ_v for $v = \infty, 1, 2, 3, 45, 46, 56, 34$.*

Proof. Indeed, let C be the chamber of the apartment Σ defined by Γ, expressed in elements of $\mathsf{E}_{7,7}(\mathbb{K})$ as follows: C contains the point $\langle e_\infty\rangle$, the line $\langle e_\infty, e_1\rangle$, the plane $\langle e_\infty, e_1, e_2\rangle$, the 3-space $\langle e_\infty, e_1, e_2, e_{45}\rangle$, the 5-space $\langle e_\infty, e_1, e_2, e_{45}, e_{46}, e_{34}\rangle$, the 6-space $\langle e_\infty, e_1, e_2, e_3, e_{45}, e_{46}, e_{5,6}\rangle$ and the symp $\xi(\langle e_\infty\rangle, \langle e_{12_*}\rangle)$.

The panel $P_7 := C \setminus \{\langle e_\infty\rangle\}$ is contained in the two chambers C and $P_7 \cup \{\langle e_1\rangle\}$ of Σ. All other chambers through P_7 can be written as $P_7 \cup \{\langle e_\infty + \lambda e_1\rangle\}$. Hence, the action of θ on the set of chambers through the panel P_7 is the same as its restriction to the projective line $\langle e_\infty, e_1\rangle$ and determined by the quotient λ_∞/λ_1. Similarly, the action of θ on the chambers through the panel $P_6 = C \setminus \{\{e_\infty, e_1\rangle\}$ is the same as its action on the projective line $\langle e_1, e_2\rangle$ and therefore determined by λ_1/λ_2. We can continue like this for all the panels P_i obtained from C by deleting the element of type i, and we obtain that the respective actions are determined by λ_2/λ_{45}, $\lambda_{45}/\lambda_{46}$, $\lambda_{46}/\lambda_{56}$, $\lambda_{46}/\lambda_{34}$, and λ_{56}/λ_3.. Let us do the least obvious example, namely, when $i = 1$ (then the element we delete is the symp $\xi(\langle e_\infty\rangle, \langle e_{12_*}\rangle)$). The other symp through P_1 contained in Σ is $\xi(\langle e_\infty\rangle, \langle e_{4'_*}\rangle)$ (that coincides with the unique other one in Σ containing the 5-space of C). The two symps of Σ through the panel P_1 are determined by the respective $5'$-spaces $\langle e_\infty, e_1, e_2, e_{45}, e_{46}, e_{56}\rangle$ and $\langle e_\infty, e_1, e_2, e_3, e_{45}, e_{46}\rangle$. Any other symp completing P_1 to a chamber contains a $5'$-space of the form $\langle e_\infty, e_1, e_2, e_{45}, e_{46}, e_{56} + \lambda e_3\rangle$. Hence, its image under θ is determined by λ_{56}/λ_3.

The fact that every choice of the λ_v, $v \in \{\infty, 1, 2, 3, 45, 46, 56, 34\}$ really leads to a collineation of $\mathsf{E}_{7,7}(\mathbb{K})$ can be derived from explicit calculations, but also follows from the observation that each choice defines a unique torus element in each rank 2 residue of the building $\mathsf{E}_7(\mathbb{K})$ containing C, and then use [17] (Proposition 4.16). $\square$

Now, let σ_θ be arbitrary (but of course such that it induces an automorphism of Γ). As noted above, we know that there exists at least one collineation θ' of $\mathsf{E}_{7,7}(\mathbb{K})$ inducing σ_θ on Γ. Then, $\theta\theta'^{-1}$ belongs to the torus, and hence, to obtain all possible θ, it suffices to look at all possible torus elements, and then multiply with θ'. By Lemma 13, this argument shows that θ is not only determined by the eight maps $e_v \mapsto \lambda_v e_{v'}$, with $v' = \sigma_\theta(v)$ and $v = \infty, 1, 2, 3, 45, 46, 56, 34$, but every choice of λ_v leads to a well-defined collineation of $\mathsf{E}_{7,7}(\mathbb{K})$.

However, we are not so much interested in general collineations, we want to find involutions. Suppose that we stabilise the symp ξ_H (which will always be the case, except for Type VII). If we define the scalars λ_v, for $v \in \{\infty, 1, 2, 45, 46, 56, 34\}$ in the right way, then θ will automatically be an involution on ξ_H. It then suffices to calculate λ_w, for $w = \sigma_\theta(3)$, and express that θ^2 is the identity on P_1. This is done by expressing that θ^2 is the identity on $\langle U, e_3 \rangle$. Then, Theorem 4.1.1 of [17] implies that θ is an involution, since θ^2 acts trivially on $E_1(C) \cap \Sigma$ (with the notation of [17])). This then (only) puts a condition on the already chosen λ_v. We will refer to this condition in general as Condition (Inv).

We are going to carry out the above procedure in detail for the most intricate case, namely for involutions of Type VI; see Section 6.6. For Type VII, we will argue in a different way; see Section 6.7

So, for each example ρ, we present the action on ξ_H (showing, using Proposition 16, it leads to the right fix diagram) and then note down the extension to the whole of Δ, given λ_3. We leave it to the reader to calculate λ_w, for $w = \sigma_\rho(3)$; we just note down the corresponding algebraic condition, except for Type VI, which we work out in detail, as an example. We will also provide the scalars λ_v for all vertices of Γ (as an extra service).

We note that the extension of ρ to Δ starting from its action on ξ_H is not unique. The algebraic condition we obtain is always quadratic in λ_3, leading to two (commuting) extensions. If we call them ρ and ρ', then $\rho \cdot \rho'$ is the unique involution fixing precisely every point in $\langle \xi_H, \xi_{H^*} \rangle$ and every point in $\langle e_v \mid v \in \Gamma \setminus (H \cup H^*) \rangle$.

6.3. Involutions of Type II

Let k be a non-square in the field $\mathbb{K}$. The involution ρ defined on the coordinates as

$$
\begin{aligned}
&(x_\infty, x_{12_*}, x_1, x_{2'}, x_{1'}, x_2, x_{34}, x_{56}, x_{35}, x_{46}, x_{36}, x_{45}) \mapsto \\
&\qquad (kx_{35}, -x_{46}, kx_2, -x_{1'}, -kx_{2'}, x_1, x_{36}, kx_{45}, x_\infty, -kx_{12_*}, kx_{34}, x_{56}).
\end{aligned}
$$

conforms to Case (iv') of Lemma 9. Here, $\langle e_3 \rangle$ is mapped onto $\langle e_{5'_*} \rangle$. Condition (Inv) is $\lambda_3^2 = k^2$. For $\lambda_3 = k$, we obtain the following complete description (to keep the oversight, we denote every coordinate separately, but since the images come in pairs, we only mention every pair once: if $x_v \mapsto k_v x_w$, then $x_w \mapsto k k_v^{-1} x_v$).

$$
\begin{array}{cccc}
x_\infty \mapsto kx_{35} & x_{\infty_*} \mapsto x_{35_*} & x_{14} \mapsto x_{14_*} & x_{15_*} \mapsto kx_{25_*} \\
x_1 \mapsto kx_2 & x_{1_*} \mapsto -x_{2_*} & x_{15} \mapsto -x_{25} & x_{34_*} \mapsto -kx_{36_*} \\
x_3 \mapsto kx_{5'_*} & x_{3_*} \mapsto x_{5'} & x_{16} \mapsto -kx_{16_*} & x_{45_*} \mapsto -kx_{56_*} \\
x_4 \mapsto kx_6 & x_{4_*} \mapsto -x_{6_*} & x_{24} \mapsto -kx_{24_*} & x_{1'} \mapsto -kx_{2'} \\
x_5 \mapsto -kx_{3'_*} & x_{5_*} \mapsto -x_{3'} & x_{26} \mapsto k^2 x_{26_*} & x_{4'} \mapsto -kx_{6'} \\
x_{12} \mapsto -kx_{46_*} & x_{12_*} \mapsto -x_{46} & x_{34} \mapsto x_{36} & x_{1'_*} \mapsto x_{2'_*} \\
x_{13} \mapsto -x_{23} & x_{13_*} \mapsto kx_{23_*} & x_{45} \mapsto x_{56} & x_{4'_*} \mapsto x_{6'_*}
\end{array}
$$

6.4. Involutions of Type IV

Let $m_1, m_2, m_3 \in \mathbb{K}$ be such that the quadratic form $x_0^2 + m_1x_1^2 + m_2x_2^2 + m_3x_3^2$ is anisotropic over $\mathbb{K}$ in x_0, x_1, x_2, x_3. The involution ρ defined on the coordinates as

$$\begin{aligned}&(x_\infty, x_{12_*}, x_1, x_{2'}, x_{1'}, x_2, x_{34}, x_{56}, x_{35}, x_{46}, x_{36}, x_{45}) \mapsto \\ &\qquad (-x_\infty, -x_{12_*}, -m_1x_{2'}, -m_1^{-1}x_1, x_2, x_{1'}, m_3x_{56}, m_3^{-1}x_{34}, x_{35}, x_{46}, m_2x_{45}, m_2^{-1}x_{36}).\end{aligned}$$

conforms to Case $(viii)$ of Proposition 6. Define $\lambda_3 = k \in \mathbb{K}^\times$. Since, by Proposition 13, we expect a quaternion field to arise somewhere, it is no surprise that Condition (Inv) states that $k^2m_1m_2m_3 = 1$. Note that there are indeed two choices for k, and both are valid. Then, ρ extends as follows.

$$\begin{array}{cccc} x_\infty \mapsto -x_\infty & x_{6'} \mapsto m_1m_3kx_4 & x_{13} \mapsto -kx_{25} & x_{34} \mapsto m_3x_{56} \\ x_1 \mapsto -m_1x_{2'} & x_{3'} \mapsto m_1kx_5 & x_{26} \mapsto m_2kx_{14} & x_{35} \mapsto x_{35} \\ x_2 \mapsto x_{1'} & x_6 \mapsto -m_3kx_{4'} & x_{23} \mapsto -m_2m_3kx_{15} & x_{36} \mapsto m_2x_{45} \\ x_3 \mapsto kx_{5'} & x_{12} \mapsto x_{12} & x_{24} \mapsto -m_3kx_{16} & x_{46} \mapsto x_{46} \end{array}$$

Here, the rule is that, if $x_v \mapsto k_vx_w$, then $x_w \mapsto k_v^{-1}x_v$. Also, if $x_v \mapsto k_vx_w$, then $x_{v_*} \mapsto -k_v^{-1}x_{w_*}$ and $x_{w_*} \mapsto -k_vx_{v_*}$. This determines the involution ρ completely.

6.5. Involutions of Type V

Let k be a non-square in $\mathbb{K}$ and set $\text{Ł} := \mathbb{K}(\sqrt{k})$. Let $m_1, m_2, m_3 \in \mathbb{K}$ be such that the quadratic form $x_0^2 + m_1x_1^2 + m_2x_2^2 + m_3x_3^2$ is anisotropic over Ł in x_0, x_1, x_2, x_3. The involution ρ defined on the coordinates as

$$\begin{aligned}&(x_\infty, x_{12_*}, x_1, x_{2'}, x_{1'}, x_2, x_{34}, x_{56}, x_{35}, x_{46}, x_{36}, x_{45}) \mapsto \\ &\quad (-kx_{35}, x_{46}, -m_1^{-1}kx_{2'}, -m_1x_1, x_2, kx_{1'}, m_3x_{56}, m_3^{-1}kx_{34}, -x_\infty, kx_{12_*}, m_2x_{45}, m_2^{-1}kx_{36}).\end{aligned}$$

conforms to Case (ii') of Lemma 9. For aesthetic reasons, we set $\lambda_3 = k\ell$, $\ell \in \mathbb{K}^\times$. Since, by Proposition 12, we again expect a quaternion field to arise somewhere, it is no surprise that Condition (Inv) becomes $\ell^2m_1m_2m_3 = k$. Note that there are two choices for ℓ, and both are valid. Then, ρ extends as follows.

$$\begin{array}{cccc} x_\infty \mapsto -kx_{35} & x_{12} \mapsto kx_{46_*} & x_{34} \mapsto m_3x_{56} & x_{56_*} \mapsto -m_3x_{34_*} \\ x_{2'} \mapsto -m_1x_1 & x_{25} \mapsto -k\ell x_{13} & x_{36} \mapsto m_2x_{45} & x_{35_*} \mapsto -x_{\infty_*} \\ x_{1'} \mapsto x_2 & x_{14_*} \mapsto m_2\ell x_{14} & x_{46} \mapsto kx_{12_*} & x_{45_*} \mapsto -m_2x_{36_*} \\ x_3 \mapsto k\ell x_{3_*} & x_{23} \mapsto -m_2m_3\ell x_{15} & x_{3'} \mapsto m_1k\ell x_{3'_*} & x_{1_*} \mapsto m_1x_{2'_*} \\ x_{6'} \mapsto m_1m_3\ell x_4 & x_{16_*} \mapsto m_3\ell x_{16} & x_{5'} \mapsto m_1m_2m_3\ell x_{5'_*} & x_{2_*} \mapsto -x_{1'_*} \\ x_5 \mapsto m_2m_3k\ell x_{5_*} & x_{24} \mapsto m_3k\ell x_{24_*} & x_{13_*} \mapsto k\ell x_{25_*} & x_{4_*} \mapsto -m_1m_2kx_{6'_*} \\ x_6 \mapsto -m_3k\ell x_{4'} & x_{26} \mapsto m_2k\ell x_{26_*} & x_{15_*} \mapsto m_2m_3\ell x_{23_*} & x_{6_*} \mapsto -m_1m_2\ell x_{4'_*} \end{array}$$

6.6. Involutions of Type VI

There are two types of such collineations. One type has fixed points in $\mathsf{PG}(55, \mathbb{K})$, and the other has no fixed points. We can take both cases together by introducing an extra parameter k. If $k = 1$ (or a square in $\mathbb{K}$), then we have fixed points, and if k is a non-square in $\mathbb{K}$, then we do not have fixed points. Furthermore, let $m_1, \ldots, m_5 \in \mathbb{K}$ be such that the form $x_0^2 + m_1x_1^2 + m_2x_2^2 + m_3x_3^2 + m_4x_4^2 + m_5x_5^2$ in $(x_0, x_1, \ldots, x_5)$ is anisotropic over $\mathbb{K}(\sqrt{k})$. Then, it is straightforward to check that the involution ρ on ξ_H given in coordinates as

$$\begin{aligned}(x_\infty, x_{12_*}, x_1, x_{2'}, x_{1'}, x_2, x_{34}, x_{56}, x_{35}, x_{46}, x_{36}, x_{45}) \mapsto \\ (kx_{12_*}, x_\infty, m_1^{-1}x_{2'}, m_1kx_1, -m_2^{-1}kx_2, -m_2x_{1'}, -m_3kx_{56}, -m_3^{-1}x_{34}, -m_5kx_{46}, \\ -m_5^{-1}x_{35}, -m_4kx_{45}, -m_4^{-1}x_{36}).\end{aligned}$$

is anisotropic over ξ_H. Hence, by Proposition 16, an involution on $\mathsf{E}_7(\mathbb{K})$ that extends this involution will be of Type VI. Now, we define $\ell \in \mathbb{K}^\times$ such that $x_3 \mapsto -\ell m_2 x_{3_*}$. Then, $x_{3_*} \mapsto -k\ell^{-1}m_2^{-1}$, so $\lambda_{3_*} = -k\ell^{-1}m_2^{-1}$. Now, the quadratic form

$$-X_2X_{3'} - X_\infty X_{23_*} + X_{2'}X_3 + X_{15}X_{46} + X_{14}X_{56} + X_{16}X_{45}$$

should be mapped onto a multiple of the quadratic form (respecting the order of the monomials in the image)

$$X_{1'}X_{3'_*} + X_{12_*}X_{23} + X_1X_{3_*} - X_{15_*}X_{35} - X_{14_*}X_{34} - X_{16_*}X_{36}.$$

This implies, in view of the third and fourth terms, that

$$\lambda_{2'}\lambda_3 = -\lambda_{15}\lambda_{46} \Rightarrow \lambda_{15} = -k\ell m_1m_2m_5.$$

The quadratic form

$$-X_{35}X_{16} - X_\infty X_{24_*} + X_{2'}X_4 + X_{15}X_{36} + X_{13}X_{56} + X_2X_{4'}$$

should be mapped onto a multiple of the quadratic form

$$X_{46}X_{16_*} + X_{12_*}X_{24} + X_1X_{4_*} - X_{15_*}X_{45} - X_{13_*}X_{34} - X_{1'}X_{4'_*},$$

which implies, in view of the fourth and fifth terms,

$$\lambda_{13}\lambda_{56} = \lambda_{15}\lambda_{36} \Rightarrow \lambda_{13} = \lambda_{15}(-km_4)(-m_3) = -k^2\ell m_1m_2m_3m_4m_5.$$

Finally, the quadratic form

$$X_{2'}X_{3'_*} + X_{12_*}X_{13} + X_2X_{3_*} - X_{25_*}X_{35} - X_{24_*}X_{34} - X_{26_*}X_{36}$$

should be mapped onto a multiple of the quadratic form

$$-X_3X_{1'} - X_\infty X_{13_*} + X_{3'}X_1 + X_{25}X_{46} + X_{24}X_{56} + X_{26}X_{45},$$

which leads, in view of the first and second terms, to the Condition (Inv):

$$\lambda_{3_*}\lambda_2 = \lambda_{13}\lambda_{12_*} \Rightarrow -k\ell^{-1}m_2^{-1}.(-m_2) = -k^2\ell m_1m_2m_3m_4m_5 \Rightarrow k\ell^2m_1m_2m_3m_4m_5 = -1.$$

Then, ρ extends as follows.

$$\begin{array}{cccc}
x_\infty \mapsto kx_{12_*} & x_{\infty_*} \mapsto -x_{12} & x_3 \mapsto -\ell m_2 x_{3_*} & x_{3'} \mapsto -\ell k m_1 x_{3'_*} \\
x_{35} \mapsto -km_5 x_{46} & x_{46_*} \mapsto km_5 x_{35_*} & x_4 \mapsto -\ell k m_2 m_4 m_5 x_{4_*} & x_{4'_*} \mapsto \ell m_2 m_3 x_{4'} \\
x_{36} \mapsto -km_4 x_{45} & x_{45_*} \mapsto km_4 x_{36_*} & x_5 \mapsto -\ell k m_2 m_3 m_4 x_{5_*} & x_{5'_*} \mapsto \ell m_2 m_5 x_{5'} \\
x_{34} \mapsto -km_3 x_{56} & x_{56_*} \mapsto km_3 x_{34_*} & x_6 \mapsto -\ell k m_2 m_3 m_5 x_{6_*} & x_{6'_*} \mapsto \ell m_2 m_4 x_{6'} \\
x_2 \mapsto -m_2 x_{1'} & x_{1'_*} \mapsto m_2 x_{2_*} & x_{23_*} \mapsto \ell m_1 m_2 x_{23} & x_{24} \mapsto -\ell k m_3 x_{24_*} \\
x_{2'} \mapsto km_1 x_1 & x_{1_*} \mapsto -km_1 x_{2'_*} & x_{25} \mapsto -\ell k m_5 x_{25_*} & x_{26} \mapsto -\ell k m_4 x_{26_*} \\
x_{13_*} \mapsto \ell x_{13} & x_{14_*} \mapsto \ell k m_4 m_5 x_{14} & x_{15_*} \mapsto \ell k m_3 m_4 x_{15} & x_{16_*} \mapsto \ell k m_3 m_5 x_{16}
\end{array}$$

We only explicitly wrote down half of the arrows, since, if $x_v \mapsto k_v x_w$, then $x_w \mapsto \frac{k}{k_v} x_v$.

Now, when we have fixed points, then the two subspaces consisting of fixed points are—naturally—fixed by ρ. However, these subspaces, of dimension 5, represent the types of the imaginary fixed subbuilding. Through Galois descent, the dimensions get halved, but the spaces are fixed. Hence, this gives rise to the Tits index ${}^1\mathsf{A}_{7,1}^{(4)}$. On the other hand, when k is a non-square, then over a quadratic extension we have similar fixed spaces, but this time, they are interchanged by the Galois involution of the quadratic extension. We see that we obtain the Tits index ${}^2\mathsf{A}_{7,1}^{(4)}$.

6.7. Involutions of Type VII

We first show that the following always describes an involution ρ of $\mathsf{E}_7(\mathbb{K})$. Let $k, \ell, \lambda_1, \ldots, \lambda_6 \in \mathbb{K}^\times$ be arbitrary. We map the coordinate x_v to $\lambda_v x_{v_*}$, $v \in V(\widetilde{\Gamma})$ and x_{v_*} to $\frac{k}{\lambda_v} x_v$. Setting $\mu := k_1 k_2 k_3 k_4 k_5 k_6$, we define the other λ_v as follows.

$$\begin{cases} \lambda_\infty = \frac{\ell^3}{k\mu}, \\ \lambda_{ij} = \frac{\ell}{\lambda_i \lambda_j}, & 1 \le i < j \le 6, \\ \lambda_{i'} = \frac{\ell^2 \lambda_i}{\mu}, & 1 \le i \le 6. \end{cases}$$

The "(v, v_*)-symmetry" in the long quadratic forms immediately implies that all these are mapped onto their k-fold. Hence, we only have to deal with half of the short quadratic forms (also because of the (v, v_*)-symmetry). Noting that the above formulae have a $\mathsf{Sym}(6)$-symmetry, that is, are invariant under an arbitrary permutation of the indices, we only have to check these forms up to a permutation of the indices. Consider the following short quadratic forms:

$$\begin{cases} -X_1 X_{2'} - X_\infty X_{12_*} + X_{1'} X_2 + X_{34} X_{56} + X_{35} X_{46} + X_{36} X_{45}, \\ -X_{12} X_1 - X_\infty X_{1'_*} + X_{13} X_3 + X_{14} X_4 + X_{15} X_5 + X_{16} X_6, \\ X_1 X_{2_*} + X_{1'} X_{2'_*} + X_{23} X_{13_*} - X_{24} X_{14_*} - X_{25} X_{15_*} - X_{26} X_{16_*}, \\ X_1 X_{1'_*} + X_2 X_{2'_*} + X_3 X_{3'_*} - X_4 X_{4'_*} - X_5 X_{5'_*} - X_6 X_{6'_*}, \\ X_1 X_{23_*} + X_2 X_{13_*} + X_3 X_{12_*} - X_{4'_*} X_{56} - X_{5'_*} X_{46} - X_{6'_*} X_{45}. \end{cases}$$

Applying the $\mathsf{Sym}(6)$-symmetry and (v, v_*)-symmetry, these short quadratic forms give rise to 30, 12, 30, 2 and 20, respectively, short quadratic forms (possibly the minus signs are in different places, but in view of the lack of minus signs in the formulae of the λ_v, we can ignore this). We obtain another $12 + 20$ forms by interchanging the subscripts i with i' in the second and last form. In total, this yields all 126 short quadratic forms. Hence, we only have to check that the above five forms are mapped onto multiples of their opposite ones (obtained by switching the stars in the indices). However, this is an elementary check using the equalities

$$\begin{cases} \lambda_{ij}\lambda_j = \lambda_{im}\lambda_m = \lambda_\infty\lambda_{i'_*}, \text{ for all } i \neq j \neq m \neq i; \\ \lambda_{ij}\lambda_{j'} = \lambda_{im}\lambda_{m'} = \lambda_\infty\lambda_{i_*}, \text{ for all } i \neq j \neq m \neq i; \\ \lambda_i\lambda_{j'} = \lambda_{i'}\lambda_j = \lambda_{mn}\lambda_{rs} = \lambda_\infty\lambda_{ij_*}, \{i,j,m,n,r,s\} = \{1,2,3,4,5,6\}. \end{cases}$$

Now, let, with the notation of Lemma 12,

$$p = x_\infty e_\infty + \sum_{v \in V(\widetilde{\Gamma})} x_v e_v$$

be a point belonging to $X \cap \widetilde{W}$. Then, its image is

$$p^\rho = \lambda_\infty x_\infty e_{\infty_*} + \sum_{v \in V(\widetilde{\Gamma})} \lambda_v x_v e_{v_*}$$

Lemma 12 asserts that p and p^ρ are opposite if, and only if,

$$\sum_{v \in V(\widetilde{\Gamma})} \lambda_v x_v^2 - \lambda_\infty x_\infty^2 \neq 0. \tag{1}$$

Hence, Lemma 11 implies that ρ is anisotropic if, and only if, Equation (1) holds for each point

$$p = x_\infty e_\infty + \sum_{v \in V(\widetilde{\Gamma})} x_v e_v$$

of $X \cap \widetilde{W}$. A sufficient condition for ρ to be anisotropic is clearly that Equation (1) holds for all points $p \in \widetilde{W}$. Explicit examples are given over $\mathbb{R}$ by taking all coefficients λ_v positive, except for $k < 0$ (this is achieved by choosing all of $\lambda_1, \lambda_2, \ldots, \lambda_6, \ell$ arbitrarily but positive, and k arbitrarily but negative). Note that the extension of ρ to $\mathsf{PG}(55, \mathbb{K})$ has fixed points in $\mathsf{PG}(55, \mathbb{K})$ if, and only if, k is a square in $\mathbb{K}$. However, for k positive there is no example over the real numbers, as one can check with the above relations.

7. Conclusions and Outlook

In this paper, we have classified the possible fix structures of linear involutions of spherical buildings of type E_7 in characteristic different from 2. This gave rise to full embedded geometries that are related to spherical buildings of other types, like $\mathsf{E}_6, \mathsf{D}_6, \mathsf{A}_7$. Since buildings of type E_7 are residues of buildings of type E_8, the current results will be very helpful to investigate and perhaps classify the (fix structures of) linear involutions in buildings of type E_8, which should conclude our knowledge about these involutions in exceptional spherical buildings of simply laced type. We expect that a linear involution in a building of type E_8 not in characteristic 2 will either be anisotropic, or fixes a vertex of type 8, in which case the involution acts on the residue of type E_7 of that vertex (and then we can exploit the results and methods of the present paper). Also, since the real numbers have characteristic 0, it would be worthwhile to see the bearing of our results on the different symmetric spaces of type E_7. Note that these symmetric spaces are defined themselves using an involution, but that involution fixes an apartment pointwise. Another problem arising from our results is to determine the full fix groups, meaning the full group of automorphisms of the building pointwise fixing the fixed subbuilding. This group acts as a sort of "linear Galois group". Aiming at a systematic understanding of the involutions and their fix structure, the eventual goal could be to derive diagrammatic rules for determining this group. Understanding the exceptional cases is crucial in this respect since they provide a variety of cases.

Author Contributions: Conceptualization, R.L. and H.V.M.; methodology, H.V.M.; software, R.L.; validation, R.L. and H.V.M.; formal analysis, R.L. and H.V.M.; investigation, R.L. and H.V.M.; resources, H.V.M.; writing—original draft preparation, R.L.; writing—review and editing, R.L. and H.V.M.; supervision, H.V.M.; project administration, H.V.M.; funding acquisition, H.V.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partly funded by the Fonds Wetenschappelijk Onderzoek (FWO - Vlaanderen) through Project Number G023121N.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors wish to thank the referees whose comments lead to a substantial improvement of the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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