Ovoids and windows in finite generalized hexagons

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Abstract

We characterize some finite Moufang hexagons as the only generalized hexagons containing "a lot of" thick ideal subhexagons or as the only hexagons containing ovoids all of whose points are regular.

1. Introduction

A generalized hexagon of order $(s,t), s,t \geq 1$ is a $1-(v,s+1,t+1)$ design $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ whose incidence graph has girth 12 and diameter 6 , also denoted by $\mathcal{S}(s,t)$. If $s = t$, $\mathcal{S}$ is said to have order s. The only known finite generalized hexagons with $s,t > 1$ arise from the Chevalley groups $G_2(q)$ and ${}^3D_4(q)$ and have respective order (q,q) and (q,q^3), q power of a prime. We denote the ${}^3D_4(q)$-hexagon by $H(q,q^3)$, its dual by $H(q^3,q)$ and we denote the $G_2(q)$-hexagon by $H(q)$, its dual by $H^*(q)$. An explicit description of these is given in Kantor [2].

Note that $H^*(q)$ is always a subhexagon of $H(q,q^3)$; dually $H(q)$ is a subhexagon of $H(q^3,q)$. A subhexagon $\mathcal{S}'$ of order (s',t') is called *ideal* if $t = t'$ (see Ronan [4]). Furthermore, $\mathcal{S}$ is called *thick* if $s,t > 1$. Note that $s = 1$ or $t = 1$ corresponds to the incidence graph of a projective plane. With these definitions, $H(q)$ is a thick ideal subhexagon of $H(q^3,q)$. Now consider the following configuration in a generalized hexagon $\mathcal{S}$. Let L_1 and L_2 be two lines at distance 6 (in the incidence graph) from each other and let p_1,p_2,p_3 be three distinct points on L_1. There are points p'_1,p'_2,p'_3 on L_2 at distance 4 from resp. p_1,p_2,p_3 and there are unique chains $p_i I M_i I p''_i I M'_i I p'_i$, $i = 1,2,3$. The configuration consisting of the lines L_1, L_2, M_i, M'_i, $i = 1,2,3$ and the points p_i, p'_i, p''_i, $i = 1,2,3$, is called a *window* of $\mathcal{S}$. By the transitivity of the collineation group of $H(q^3,q)$ there is a subhexagon isomorphic to $H(q)$ containing any given window. It is our aim to show the contrary, namely that if every window of a thick generalized hexagon $\mathcal{S}$ is contained in an ideal subhexagon, then $\mathcal{S}$ is isomorphic to $H(q^3,q)$.

Let $d(x,y)$ denote the distance between x and y in the incidence graph

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of a generalized hexagon $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$, $x, y \in \mathcal{P} \cup \mathcal{B}$. Denote by $\Gamma_i(x)$, with $x \in \mathcal{P} \cup \mathcal{B}$, the set of all elements of $\mathcal{P} \cup \mathcal{B}$ at distance i from x. If $d(x, y) = 4$ for $x, y \in \mathcal{P}$, then there is a unique point z collinear to both and we denote $z = x * y$. Define $\mathcal{W}(x, y) = \Gamma_6(z) \cap \Gamma_4(x) \cap \Gamma_4(y)$. If $u \in \mathcal{W}(x, y)$ then we denote by z^u the set of points collinear with z and at distance 4 from u. If this set is independent from the choice of u, then z^u is called *an ideal line* (see Ronan [4]) and is denoted by $\langle x, y \rangle$. Now fix a point $p \in \mathcal{P}$ and suppose that $\langle x, y \rangle$ is an ideal line for every pair $(x, y) \in \mathcal{P}^2$ such that $d(x, p) = d(y, p) = 2$ and $d(x, y) = 4$, then we call p *half-regular*. If moreover $\langle p, z \rangle$ is an ideal line for every point z at distance 4 from p, then we call p *regular*. This is motivated by the facts that (1) if all points of $\mathcal{S}$ are regular and $\mathcal{S}$ has order s, then $\mathcal{S} \cong H(q)$, see Ronan [4], (2) a derivation can be defined in a regular point of $\mathcal{S}$ and if $s = t$, then this is a generalized quadrangle (see Van Maldeghem - Bloemen [7]). These properties are very similar to properties of generalized quadrangles with regular points (see Payne - Thas [3]). In fact, for generalized quadrangles of order s, one can show that, if every point of an ovoid is regular, then the generalized quadrangle is classical and arises from a Chevalley group $S_4(2^e)$. In this paper, we extend this property to generalized hexagons, an ovoid of a generalized hexagon of order s being a set of $s^3 + 1$ points at distance 6 from each other. There is one difference though: the existence of ovoids in $H(q)$ is only proved for $q = 3^e$. For q even, there are no ovoids (see e.g. Thas [6]) and for other values of q, the question remains open.

Also our characterization of $H(q^3, q)$ has an analogue for generalized quadrangles, (see Payne - Thas [3], 5.3.5. ii, dual), a window in a generalized quadrangle being a quadrilateral with one more "transversal".

2. Proof of the results

2.1. Characterization by windows

Lemma 2.1 *Let $\mathcal{S}(s, t)$ be a finite generalized hexagon which contains a proper subhexagon $\mathcal{S}'(s', t)$, which in turn contains a proper subhexagon $\mathcal{S}''(s'', t)$. Then $s = t^3$, $s' = t$ and $s'' = 1$.*

Proof. From Haemers and Roos [1] it follows that $s \leq t^3$ (1).
From Thas [5] we have $s \geq s'^2 t$ (2) and $s' \geq s''^2 t$ (3).
So (1) and (2) gives $s'^2 \leq t^2$ or $s' \leq t$ (4).
Now (3) and (4) gives $s'' = 1$, and so $s' = t$.
From (1) and (2) it then follows that $s = t^3$. □

Theorem 2.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a finite generalized hexagon of order (s,t) with $s \geq 3$.*
There exists a proper ideal subhexagon through every window of $\mathcal{S}$ iff $\mathcal{S}$ is isomorphic to $H(q^3, q)$, $s = q^3$ and $t = q$.

Proof.

$\Leftarrow$ See introduction.

$\Rightarrow$ In order to proof that $\mathcal{S}$ is Moufang we have to proof that $\mathcal{S}$ has ideal lines. [4] So we must proof that for all $a, b \in \mathcal{P}$ with $d(a,b) = 4$ and $a * b = c$ we have $\langle a, b \rangle = c^z$ for all $z \in \mathcal{W}(a,b)$.

Step 1: We show that $c^z = c^{z'}$ for all $z, z' \in \mathcal{W}(a,b)$ such that c, z and z' form a window with the same two lines , say L and M. Suppose z_1 and z_2 are such elements of $\mathcal{W}(a,b)$.

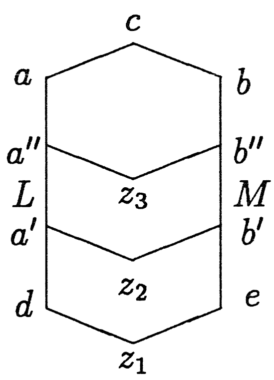

Figure 1.

Let $S_{12}(c, z_1, z_2)$ be the proper ideal subhexagon through the window c, z_1, z_2, L and M. Since $s \geq 3$ and S_{12} is proper, there exists another point b'' on M, with $b'' \notin S_{12}$. The shortest path between b'' and L gives rise to the point $z_3 \in \mathcal{W}(a,b)$ (see figure 1). Let $S_{13}(c, z_1, z_3)$ be the proper ideal subhexagon through the window c, z_1, z_3, L and M. Remark that $z_2 \notin S_{13}$. Finally, let $S_{23}(c, z_2, z_3)$ be the proper ideal subhexagon through the window c, z_2, z_3, L and M. Note that $z_1 \notin S_{23}$. We will now look at some intersections of those subhexagons. Let $\mathcal{D}_2 = S_{12}(c, z_1, z_2) \cap S_{23}(c, z_2, z_3)$, then $\mathcal{D}_2$ is a proper ($z_1 \notin S_{23}$ and $z_3 \notin S_{12}$) ideal subhexagon of S_{12} and S_{23}. Let $\mathcal{D}_3 = S_{13}(c, z_1, z_3) \cap S_{23}(c, z_2, z_3)$, then $\mathcal{D}_3$ is a proper ($z_1 \notin S_{23}$ and $z_2 \notin S_{13}$) ideal subhexagon of S_{13} and S_{23}. If we apply the lemma to

$$\begin{aligned} \mathcal{S}(s,t) &\supset S_{12} \supset \mathcal{D}_2, \\ \mathcal{S}(s,t) &\supset S_{13} \supset \mathcal{D}_3 \\ \text{and } \mathcal{S}(s,t) &\supset S_{23} \supset \mathcal{D}_3 \end{aligned}$$

we have that $\mathcal{S}$ has order (t^3, t), S_{12}, S_{13} and S_{23} have order (t,t) and $\mathcal{D}_2$ and $\mathcal{D}_3$ are thin ideal subhexagons. Now we can apply a corollary of the theorem of Thas [5] to the following pairs of generalized hexagons:

(1) $S_{12} \supset \mathcal{D}_2$.
$z_1 \in S_{12} \backslash \mathcal{D}_2$ and not collinear with a point of $\mathcal{D}_2$,
so z_1 is at distance 3 from $1+t$ lines of $\mathcal{D}_2 \subset S_{23}$.

(2) $S_{13} \supset \mathcal{D}_3$.
$z_1 \in S_{13} \backslash \mathcal{D}_3$ and not collinear with a point of $\mathcal{D}_3$,
so z_1 is at distance 3 from $1+t$ lines of $\mathcal{D}_3 \subset S_{23}$.

(3) $\mathcal{S} \supset S_{23}$.
$z_1 \in \mathcal{S} \backslash S_{23}$ and not collinear with a point of S_{23},
so z_1 is at distance 3 from $1+t$ lines of S_{23}.

From (1), (2) and (3) it follows that $\mathcal{D}_2$ and $\mathcal{D}_3$ have $1+t$ lines in common which are at distance 3 from z_1.

Case 1: Suppose that all those $1+t$ lines are at distance 3 from c. From the thinness of $\mathcal{D}_2$ and $\mathcal{D}_3$ it follows that $c^{z_1} = c^{z_2} = c^{z_3}$.

Case 2: Suppose at least one of those $1+t$ lines is at distance 5 from c, say L_2. Let $(c, L_0, l_0, L_1, l_1, L_2)$ denote the shortest path between the line L_2 and c. Since c and $L_2 \in \mathcal{D}_2$ $(\mathcal{D}_3)$ it follows that $L_0, l_0, L_1, l_1 \in \mathcal{D}_2$ $(\mathcal{D}_3)$. From $\mathcal{D}_2$ it then follows that $d(z_2, l_0) = 4$ and $d(z_2, l_1) = 6$. So there is a second point on L_2 at distance 4 from z_2. We deduce $d(a', l_1) = 4$. But l_1 and L lie in $\mathcal{D}_3$ so the shortest path between them is also in $\mathcal{D}_3$, a contradiction. So case 2 cannot occur and case 1 proves step 1.

Step 2:

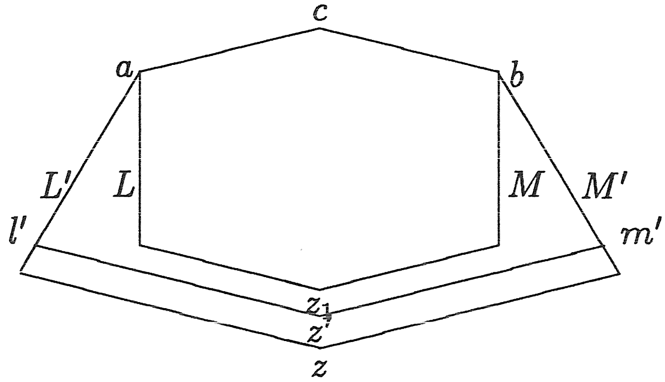

Figure 2.

Suppose $z \in \mathcal{W}(a,b)$ so that c, z_1, z, L and M do not form a window. There exists a thin ideal subhexagon $\mathcal{D}$ through c, z_1, L and M (see step 1). So L' and $M' \in \mathcal{D}$. Let l' and m' be the respective second points on L' and M' in $\mathcal{D}$ and denote $l' * m' = z'$. Then from the thinness of $\mathcal{D}$ we have $c^{z'} = c_1^z$. But applying step 1 we obtain $c^{z'} = c^z$. □

2.2. Characterization by subhexagons

Lemma 2.3 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a finite generalized hexagon of order (s,t). Through every 2 opposite, half-regular points there exists exactly one thin ideal subhexagon.*

Proof. Let p_1 and p_2 be two opposite, half-regular points of S. Since they are opposite, we can take all lines through p_1 and consider the unique $t+1$ shortest paths of length 4 between p_2 and those $t+1$ lines. So we get in an unique way, $t+1$ points collinear with p_1 and $t+1$ points collinear with p_2. Call them respectively $x_0, \ldots, x_t$ and $y_0, \ldots, y_t$ with $x_i \sim y_i$, $i = 0, \ldots, t$.

Because $d(x_i, y_{i+1}) = 6$, $i = 0, \ldots, t \pmod{t+1}$, we can do the same construction with each of these $t+1$ couples of opposite points to get for each x_i, in a unique way $t-1$ points collinear with x_i. We call them x_k^i, $k = 1, \ldots, t-1$. Similarly we obtain for each y_{i+1}, $t-1$ points collinear with y_{i+1} and call them y_k^{i+1}, $k = 1, \ldots, t-1$ and we can do this in such a way that $x_k^i \sim y_k^{i+1}$.

Now we have all the $2(t^2+t+1)$ points we need for a thin ideal subhexagon. We still have to consider the lines through x_k^i and through y_k^i, $i = 0, \ldots, t$ and $k = 1, \ldots, t-1$. Since p_1 and p_2 are half-regular, the hyperbolic lines $\langle x_i, x_j \rangle = \{x_0, \ldots, x_t\}$ and $\langle y_i, y_j \rangle = \{y_0, \ldots, y_t\}$ are ideal. Now each of the y_k^i belongs to $\mathcal{W}(x_i, x_{i-1})$, so $d(y_k^i, x_j) = 4$ for every $j \in \{0, \ldots, t\}$. So for each j, there is a line (x_j, x_l^j) containing a point collinear with y_k^i. But the point x_l^j on that line belongs to $\mathcal{W}(y_j, y_{j+1})$, so $d(x_l^j, y_i) = 4$. Since $y_i \sim y_k^i$ it must be that $x_l^j \sim y_k^i$. In this way we obtain all other lines of the thin ideal subhexagon.

Remark that for a fixed y_i, all y_k^i are collinear with some point x_l^j, $\forall j \in \{0, \ldots, t\} \backslash \{k\}$ and that no two of the y_k^i's can be collinear with the same x_l^j. Moreover, for two points y_k^i and y_l^j with $i \neq j$, there is exactly one x_n^m collinear with both.

Indeed, there cannot be more than one, otherwise we would have a quadrangle in $\mathcal{S}$. So if we look at all $t-1$ points $x_{n_m}^m \sim y_k^i$, $m \in \{0, \ldots, t\} \backslash \{i, j\}$, there is always one and only one y_s^j, $s = 1, \ldots, t-1$, collinear with one of those $x_{n_m}^m$'s. Since we have $t-1$ such y_s^j's and $t-1$ such $x_{n_m}^m$'s, there is exactly one of the $x_{n_m}^m$ collinear with y_l^j. The same arguments hold for the x_k^i.

It is now straightforward to check that there is always a path of length ≤ 6 between two of the constructed elements. So we indeed have a thin ideal subhexagon. □

Lemma 2.4 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a finite generalized hexagon of order s which contains an ovoid $\mathcal{O}$ for which all points are half-regular. Then every thin ideal subhexagon of $\mathcal{S}$ contains exactly 2 points of $\mathcal{O}$.*

Proof.

(1) From lemma 2.3 it follows that through every 2 points of $\mathcal{O}$ there is exactly one thin ideal subhexagon $\mathcal{D}$. Moreover $\mathcal{D}$ cannot contain more than 2 points of $\mathcal{O}$ since every other point of $\mathcal{D}$ is at distance ≤ 4 from one of those 2 points of $\mathcal{O}$. So in total there are $\frac{(s^3+1)s^3}{2}$ thin ideal subhexagons which contain two points of $\mathcal{O}$.

(2) Suppose there are α thin ideal subhexagons in $\mathcal{S}$. We count in two different ways the number of pairs $(x, \mathcal{D})$ with $x \in \mathcal{P}$, $\mathcal{D}$ a thin ideal subhexagon of $\mathcal{S}$ and $x \in \mathcal{D}$. It then follows that

$$\alpha \leq \frac{(1+s).(1+s^2+s^4).s^3}{2.(1+s+s^2)} = \frac{(s^3+1).s^3}{2}.$$

The lemma follows from (1) and (2). □

Corollary 2.5 *From the equality in the proof of lemma 2.4 it follows that through every point $x \in \mathcal{P}$ there are s^3 thin ideal subhexagons. This means that through every 2 points of $\mathcal{S}$ there exists a thin ideal subhexagon.*

Theorem 2.6 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a finite generalized hexagon of order s containing an ovoid. Every point of an ovoid $\mathcal{O}$ is regular iff $\mathcal{S}$ is isomorphic to $H(q)$, $q = s$.*

Proof.

$\Leftarrow$ This follows from Ronan [4].

$\Rightarrow$ Due to Ronan [4] we have to prove that $\mathcal{S}$ has ideal lines. So, for two points $x, y \in \mathcal{P}$ with $d(x,y) = 4$, $z = x * y$ we must prove that $\langle x, y \rangle = z^w$, $\forall w \in \mathcal{W}(a,b)$.

From lemma 2.4 it follows that there are s thin ideal subhexagons $\mathcal{D}_i$, $i = 1, \ldots, s$ containing x and y. They can be obtained by choosing a point y_i on a line through y at distance 5 from x and they all contain 2 points of $\mathcal{O}$. Since $z^w = z^{w'}$ $\forall w, w' \in \mathcal{W}(a,b) \cap \mathcal{D}_i$, we have to prove that $z^{w_1} = z^{w_2} = \ldots = z^{w_s}$ with $w_i \in \mathcal{D}_i$, $i = 1, \ldots, s$.

Case 1: $x \in \mathcal{O}$ or $y \in \mathcal{O}$ then it is immediate that $\langle x, y \rangle$ is ideal.

Case 2: x and z are collinear with the same unique point p_x of $\mathcal{O}$. Let p_y be the unique point of $\mathcal{O}$ collinear with y and denote the line through y and p_y by L. With every point p on $L \backslash \{y\}$ there corresponds a thin ideal subhexagon $\mathcal{D}_p$ through x, y and p.

First we look at $\mathcal{D}_{p_y}$ and the hyperbolic line $\langle x, y \rangle_{p_y}$ in $\mathcal{D}_{p_y}$. We will show that the hyperbolic lines $\langle x, y \rangle_p$ in the other $s-1$ $\mathcal{D}_p$'s are the same. Let k be a point of $L \backslash \{y, p_y\}$ and let $\mathcal{D}_k$ be the thin ideal subhexagon through x, y and k. From lemma 2.4 we know that $\mathcal{D}_k$ contains two points of $\mathcal{O}$. Since every point of $\mathcal{D}_k$ is at distance ≤ 4 from at least one of those two points of $\mathcal{O}$, the point z is at distance 4 from one of them, say p. Since p and L are in $\mathcal{D}_k$, also the shortest path between them lies in $\mathcal{D}_k$. Denote $p * k$ by a. Also the shortest path between a and the line through x and z lies in $\mathcal{D}_k$. Denote $a * x$ by b (see figure 3).

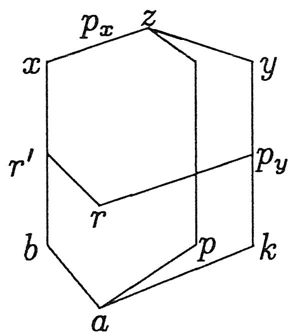

Figure 3.

Suppose that $\langle x, y \rangle_k = z^a$ is different from $\langle x, y \rangle_{p_y}$. So there is a line M through z on which the point c at distance 4 from a is different from the point d at distance 4 from r. Denote $c * a$ by e and $d * r$ by f. Since p_y and p are regular, we have ideal lines $\langle r', p_y \rangle$ and $\langle b, r \rangle = \langle b, k \rangle$ (see figure 3) . From $z \in \mathcal{W}(b, k)$ it follows that $e \in \langle b, k \rangle$, so $d(r, e)$ must be 4. Denote $r * e$ by g. From $a \in \mathcal{W}(r', p_y)$ it follows that $g \in \langle r', p_y \rangle$ and so $d(z, g)$ must be 4 which is a contradiction.

Case 3: y and z are collinear with the same unique point p_y of $\mathcal{O}$. This is similar to case 2.

Case 4: x, y and z are collinear to different points of $\mathcal{O}$, say respectively p_x, p_y and p_z.

(i) If $p_z \in z^w$ for some $w \in \mathcal{W}(x, y)$ then $\langle x, y \rangle$ is ideal since p_z is regular.

(ii) So suppose there is a point t on the line through z and p_z at distance 4 from a point $w \in \mathcal{W}(x, y)$. By case 2 we have that $\langle t, y \rangle$ is ideal , so $\langle x, y \rangle$ is ideal. □

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