CHAPTER 10

Some Classes of Rank 2 Geometries

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Introduction

In the preceding chapters a lot of rank 2 geometries, such as projective and affine planes, designs, linear spaces, generalized polygons, ... are studied in detail. In this chapter we will discuss some more rank 2 geometries that are merely generalizations of these geometries. A treatment of all the known rank 2 geometries is impossible. We have therefore made a choice which is rather restrictive and subjective. Although a lot of the basic definitions are given in other chapters (especially in Chapter 3) we will, for the sake of completeness, recall the most important ones. This will also give us the opportunity to fix the notations used in this chapter.

1. *Generalities on geometries*

Referring to definitions given in Chapter 3 we can say that a *rank* 2 *geometry* $\boldsymbol{S}$ is a $\{0,1\}$-geometry. The elements of type 0 will be called the *points* while the elements of type 1 will be called *lines*. In some other chapters a rank 2 geometry is called an *incidence structure* $\boldsymbol{S} = (P, B, \mathrm{I})$ with P $(\neq \varnothing)$ the set of points, B $(\neq \varnothing)$ the set of lines and a symmetric incidence relation $\mathrm{I} \subseteq (P \times B) \cup (B \times P)$. In this chapter both the sets P and B will be finite and the geometry will be connected. In a lot of cases, lines will be subsets of the point set P and the incidence I will be the natural incidence $(\in)$.

The *dual* of a rank 2 geometry $\boldsymbol{S} = (P, B, \mathrm{I})$ is the geometry $\boldsymbol{S}^D = (P^D, B^D, \mathrm{I}^D)$ with $P^D = B$, $B^D = P$, and $\mathrm{I}^D = \mathrm{I}$.

A rank 2 geometry $\boldsymbol{S}$ is called a *partial linear space*, if each point is on at least 2 lines, if all lines have at least two points and if any two distinct points in P are incident with at most one line, or equivalently, if any two distinct lines are incident with at most one point. Some authors call this a *semilinear space*. Lines incident with only 2 points, are called *thin lines*. If all lines are thin lines, then $\boldsymbol{S}$ is called a *thin partial linear space*. If all lines are incident with at least 3 points and if every point is incident with at least 3 lines, the partial linear space is called *thick*. Two points are said to be *collinear* if they are incident with a common line. Note that a point is collinear with itself. Dually, two lines are said to be *concurrent* if they are incident with a common point. We will denote collinear points x and y (resp., concurrent lines L and M) by $x \sim y$ (resp., $L \sim M$). On the other hand, we will sometimes use the standard notation for the set of points collinear to a point x: $x^\perp = \{y \in P\colon\ y \sim x\}$.

If any two different points are collinear, then $\boldsymbol{S}$ is called a *linear space*. For more details on linear spaces we refer to Chapter 6.

In this chapter we will mainly deal with quite special partial linear spaces. They will have the next two properties:

(S_1): Each point is incident with $t+1$ $(t \geqslant 1)$ lines.
(S_2): Each line is incident with $s+1$ $(s \geqslant 1)$ points.

A partial linear space $\boldsymbol{S}$ satisfying these two properties will be called a partial linear space of *order* (s,t).

Let $\boldsymbol{S}$ be a connected partial linear space. Let (x, L) be an antiflag of $\boldsymbol{S}$, i.e. x is a point and L is a line of $\boldsymbol{S}$, such that x is not incident with L. We denote by $\alpha(x, L)$ the number of points on L collinear with x, or equivalently the number of lines through x

concurrent with L. We will sometimes call $\alpha(x, L)$ the *incidence number* of the antiflag (x, L).

In this chapter we will mainly deal with connected partial linear spaces in which $\alpha(x, L)$ can take only a few values. For instance, if $\alpha(x, L)$ can only have the values 0 and $\alpha \neq 0$, then the connected partial linear space is called a $(0, \alpha)$-geometry in De Clerck and Thas [1983] and Thas, Debroey and De Clerck [1984]. One can easily check that if $\boldsymbol{S}$ is a $(0, \alpha)$ geometry with $\alpha > 1$ then there exist two integers s ($\geqslant 1$) and t ($\geqslant 1$) such that $\boldsymbol{S}$ is of order (s, t). The dual of a $(0, \alpha)$-geometry is of course again a $(0, \alpha)$-geometry. There are a lot of examples of $(0, \alpha)$-geometries. We will restrict ourselves to some special classes with extra regularity conditions.

A lot of the examples we will encounter in this chapter have points and lines in a projective or affine space. To be more precise, a geometry $\boldsymbol{S} = (P, B, \mathrm{I})$ is said to be *embedded* in a projective or an affine space if B is a subset of the set of lines of the space and if P is the set of all points of the space on these lines. We will always assume in what follows that the dimension of the space is the smallest possible dimension for an embedding. Some authors call this a *full embedding* or a *flat embedding*.

A special type of affine embedding is the so-called *linear representation* of a geometry of order (s, t) in $\mathrm{AG}(n+1, s+1)$. It is an embedding of $\boldsymbol{S} = (P, B, \mathrm{I})$ in $\mathrm{AG}(n+1, s+1)$ such that the line set B of $\boldsymbol{S}$ is a union of parallel classes of lines of $\mathrm{AG}(n+1, s+1)$ hence the point set P of $\boldsymbol{S}$ is the point set of $\mathrm{AG}(n+1, s+1)$. These lines of $\boldsymbol{S}$ define in the hyperplane at infinity Π_∞ a set of points $\mathcal{K}$ of size $t+1$. If $\boldsymbol{S}$ is a $(0, \alpha)$-geometry, then every line of Π_∞ intersects $\mathcal{K}$ in either $0, 1$ or $\alpha + 1$ points. A line intersecting $\mathcal{K}$ in m points will be called an *m-secant*. A 1-secant will also be called a *tangent line*, while a line not intersecting $\mathcal{K}$ will be called a *passant*.

Using standard notations, the linear representation of a geometry $\boldsymbol{S}$ in $\mathrm{AG}(n+1, s+1)$ will be denoted by $T_n^*(\mathcal{K})$. We shall give several examples in the next sections.

However we first need some graph theoretical definitions.

2. *Graphs and rank 2 geometries*

A finite *graph* $\Gamma = (X, E)$ is a structure consisting of a set X ($\neq \varnothing$) with v elements and a set E of unordered pairs of X. The elements of X are called the *vertices* of the graph Γ, while the elements of E are called the *edges*. If x and y are two different vertices such that $\{x, y\} \in E$, then x and y are called *adjacent* and we write $x \sim y$; if $\{x, y\} \notin E$ then we denote this by $x \not\sim y$; remark that $x \not\sim x$. If E is the set of all unordered pairs of X then Γ is called the *complete graph* denoted by K_v. The *complement* Γ^C of a graph $\Gamma = (X, E)$ is the graph $\Gamma^C = (X^C, E^C)$ with $X^C = X$ and $E^C = X^{|2|} \setminus E$. The *line graph* $\mathcal{L}(\Gamma)$ of a graph Γ is the graph with vertices the edges of Γ, two edges being adjacent if and only if they have a common vertex.

A *path* of length m from x to y, is a set of vertices $x = x_0, x_1, x_2, \ldots, x_m = y$ such that $x_i \sim x_{i+1}$, $0 \leqslant i \leqslant m-1$. If $x = y$ then any such path with $x_i \neq x_{i+2}$ ($0 \leqslant i \leqslant m-2$) will be called a *circuit*. Two vertices x and y of a graph Γ are at *distance* $d(x, y)$, provided there exists a path of length $d(x, y)$ between these vertices and there exists no shorter one. A vertex has distance 0 from itself and distance 1 from all its adjacent vertices. We will denote by $\Gamma_i(x)$ the set of all vertices of Γ at distance i from x. For convenience we will use $\Gamma(x)$ for the set $\Gamma_1(x)$. A graph is *connected* if and only if for any two distinct vertices x and y, there is at least one path connecting these 2 vertices. The *diameter* of a graph Γ is the maximum value of the distance function $d(x, y)$. The *girth* of Γ is the length of its shortest circuit.

Given a partial linear space $\boldsymbol{S}$, one may define the *point graph* or *collinearity graph* $\Gamma(\boldsymbol{S})$, by taking as vertices the points of $\boldsymbol{S}$. Two different vertices are adjacent whenever they are collinear. Remark that we are using the same symbol ($\sim$) for the collinearity relation as for the adjacency relation, although a point x is collinear to itself but not adjacent to itself. A geometry is connected whenever its point graph is.

On the other hand, the *incidence graph* $\mathcal{I}(\boldsymbol{S})$ is the graph with vertices the elements of $P \cup B$, and 2 vertices are adjacent if and only if the corresponding elements are incident, hence edges of $\mathcal{I}(\boldsymbol{S})$ are the flags of $\boldsymbol{S}$. Unlike the case of the collinearity graph, the geometry is completely determined by its incidence graph. Obviously, two vertices of the same type (i.e. either points or lines) in the incidence graph are connected by paths of even length. In particular, a circuit in an incidence graph has even length and hence the girth is an even positive integer, say $2g$. By definition, g is called the *gonality* of $\boldsymbol{S}$, and $2g$ is called *the* (*geometric*) *girth* of $\boldsymbol{S}$. Let x be a point or a line. A *geodesic* (*based at* x) is a path γ in the incidence graph starting in x and such that the length of γ is equal to the distance $d(x, y)$, where y is the last element of γ. A *maximal geodesic* is a geodesic that is not properly contained in another one. The *local diameter* $d(x)$ is the length of the longest geodesic based at x, whether x be a point or a line. The *point-diameter* d_p (resp., *line-diameter* d_l) of $\boldsymbol{S}$ is the greatest value taken by $d(x)$ for x a point (resp., a line). In Chapter 3, d_p is denoted by d_0 and is called the 0-diameter while d_l is denoted by d_1, the 1-diameter. The *diameter* d of a geometry $\boldsymbol{S}$ is the diameter of the incidence graph $\mathcal{I}(\boldsymbol{S})$, hence it is the largest of the two numbers d_p, d_l.

Finally, the *flag graph* of a rank 2 geometry $\boldsymbol{S}$ is the graph with vertices the maximal flags of $\boldsymbol{S}$ and 2 flags are adjacent whenever they share exactly one element. The *flag-diameter* d^* of $\boldsymbol{S}$ is the diameter of the flag graph.

A successful attempt to unify the study of all rank 2 geometries of this chapter (and some others) was made by Buekenhout [1982] (see also Chapter 3). He studied those geometries by considering their gonality g and the diameters d_p, d_l and d^*. He proved that a connected geometry $\boldsymbol{S}$ such that every element of $\boldsymbol{S}$ is incident with at least two other elements, and such that all points (resp., lines) have the same local diameter d_p (resp., d_l) has the property that d^* is the smallest of the two numbers d_p, d_l. Hence $\{d, d^*\} = \{d_p, d_l\}$ and we always have $g \leqslant d^* \leqslant d$. Moreover, $d - d^* \leqslant 1$. For the sequel we may assume that $d_p \leqslant d_l$, which is no loss of generality since one can always consider the dual geometry. With this terminology and under these conditions, $\boldsymbol{S}$ is called by Buekenhout [1982] a (g, d_p, d_l)-gon provided $d_p \leqslant g + 2$. The last condition is purely subjective and comes from the observation that almost all the 'nice' rank 2 geometries arising from finite simple groups – and especially the sporadic ones – satisfy this condition. Exceptions arise mainly from truncations of higher dimensional geometries. For instance, Suz acts on a $(3, 8, 8)$-gon which is a truncation of an extended generalized quadrangle (see Buekenhout [1985]).

In the last section we will return to the theory of the (g, d_p, d_l)-gons in order to explain how all the rank 2 geometries introduced in the other sections fit into this more global point of view.

3. *Distance regular graphs*

A graph Γ is called *regular* provided every vertex of Γ is adjacent to a constant number k of vertices, and this number k is called the *valency* or the *degree* of the graph.

A *distance regular graph* Γ with diameter d, is a regular and connected graph of valency k with the following property. There are natural numbers:

$$b_0 = k,\ b_1, \dots, b_{d-1}; \qquad c_1 = 1,\ c_2, \dots, c_d,$$

such that for each pair (x, y) of vertices at distance j, we have:

(1) $|\Gamma_{j-1}(y) \cap \Gamma_1(x)| = c_j$ $(1 \leqslant j \leqslant d)$;

(2) $|\Gamma_{j+1}(y) \cap \Gamma_1(x)| = b_j$ $(0 \leqslant j \leqslant d-1)$.

The *intersection array* of Γ is defined by

$$i(\Gamma) = \{k, b_1, \dots, b_{d-1}; 1, c_2, \dots, c_d\}.$$

For any graph Γ of diameter d, and vertex set $\{x_1, \dots, x_v\}$, the distance matrices A_h, $h = 0, \dots, d$, are the $v \times v$ matrices defined as follows:

$$(A_h)_{ij} = \begin{cases} 1 & \text{if } d(x_i, x_j) = h, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 1 (Damerell [1973]). *Let Γ be a distance regular graph with intersection array*

$$i(\Gamma) = \{k, b_1, \dots, b_{d-1}; 1, c_2, \dots, c_d\}.$$

For $1 \leqslant i \leqslant d-1$, put $a_i = k - b_i - c_i$; then

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (1 \leqslant i \leqslant d-1).$$

Moreover, the distance matrices form an algebra of dimension $d+1$, and $\{A_0 = I, A_1, \dots, A_d\}$ is a basis for this algebra.

This implies that we can use a lot of techniques from linear algebra, such as eigenvalue techniques, to find for instance restrictions on the intersection array. For all information on distance regular graphs and an extensive bibliography, we refer to Brouwer, Cohen and Neumaier [1989].

A distance regular graph of diameter 2 is better known as a *strongly regular graph*. For reasons of convenience we will recall the definition in order to introduce the notations that are mainly used for these graphs.

A regular graph Γ is called a strongly regular graph (notation srg(v, k, λ, μ)) provided:

(1) any two vertices x and y, $x \sim y$, are both adjacent to a constant number λ of vertices (independent of the choice of the adjacent pair $\{x, y\}$);

(2) any two vertices x and y, $x \not\sim y$, are both adjacent to a constant number μ of vertices (independent of the choice of the nonadjacent pair $\{x, y\}$).

We exclude disconnected graphs and their complements, hence we assume $0 < \mu < k < v-1$. It is easy to check that the complement of a srg(v,k,λ,μ) is a srg$(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$ and that a srg(v,k,λ,μ) is indeed equivalent to a distance regular graph with intersection array $\{k, k-1-\lambda; 1, \mu\}$.

The distance matrix $A_1 = A$ (or the $(0,1)$ adjacency matrix) of a srg(v,k,λ,μ) satisfies

$$AJ = kJ, \qquad A^2 + (\mu-\lambda)A + (\mu-k)I = \mu J,$$

where J is the all-one matrix. Hence A has the valency k as an eigenvalue with multiplicity 1, and two other eigenvalues r and l ($r > 0$ and $l < 0$) with $r + l = \lambda - \mu$ and $rl = \mu - k$.

There are some known necessary conditions for the existence of a srg(v,k,λ,μ). We will summarize the most important ones in the next theorem. For the proofs and more information on strongly regular graphs, we refer to Bose [1963], Van Lint and Seidel [1969], Cameron [1978], Seidel [1979], Cameron and Van Lint [1980], and Brouwer and Van Lint [1984].

THEOREM 2. *If Γ is a* srg(v,k,λ,μ) *then the following is true.*

(1) $v - 2k + \mu - 2 \geqslant 0$.

(2) $k(k-\lambda-1) = \mu(v-k-1)$.

(3) *The multiplicities of the eigenvalues r and l of A are, respectively,*

$$f = \frac{-k(l+1)(k-l)}{(k+rl)(r-l)} \quad \textit{and} \quad g = \frac{k(r+1)(k-r)}{(k+rl)(r-l)}.$$

They clearly have to be integers.

(4) *The eigenvalues $r > 0$ and $l < 0$ are both integers, except for one family of graphs, the so-called conference graphs, which are* srg$(2k+1, k, k/2-1, k/2)$. *In this case the number of vertices can be written as a sum of two squares, and the eigenvalues are* $(1 \pm \sqrt{v})/2$.

(5) *The Krein conditions*:

$$(r+1)(k+r+2rl) \leqslant (k+r)(l+1)^2,$$

$$(l+1)(k+l+2rl) \leqslant (k+l)(r+1)^2.$$

(6) *The absolute bound*:

$$v \leqslant \tfrac{1}{2} f(f+3), \qquad v \leqslant \tfrac{1}{2} g(g+3).$$

(7) *The claw bound*: *if $\mu \neq l^2$, $\mu \neq l(l+1)$, then* $2(r+1) \leqslant l(l+1)(\mu+1)$.

There are a lot of examples of strongly regular graphs known, see, e.g., Hubaut [1975] and Brouwer and Van Lint [1984]. We shall give here a short description of some examples which are important for the rest of this chapter. For information on the automorphism group of these graphs, we refer to Section 6.3 (see Table 1).

1. The pentagon Pn(5) is the unique srg$(5, 2, 0, 1)$.

2. The line graph of the complete graph K_n is called the triangular graph and is denoted by $T(n)$. This graph is a srg$(\frac{1}{2}n(n-1), 2(n-2), n-2, 4)$. If $n \neq 8$ then every strongly regular graph Γ with these parameters is indeed a triangular graph. If $n = 8$ there are exactly three nonisomorphic graphs with the same parameters but not triangular, these graphs are known as the graphs of Chang [1959], see also Seidel [1967].

3. The strongly regular graph $T(5)^C$ is better known as the Petersen graph Pe(10); it is the unique srg$(10, 3, 0, 1)$. This graph can also be constructed by taking as vertices the 10 points of a Desargues configuration, two vertices being adjacent if they are not on a line of the Desargues configuration.

4. The Clebsch graph Cl(16) is a srg$(16, 5, 0, 2)$.

There is only one graph with these parameters (easy exercise). The graph can be constructed as follows. Take a set C of cardinality 5. The vertices of the graph are the set C and the subsets of cardinality 1 and 2. The vertex C is adjacent to the 5 singletons, a singleton is adjacent to C and to all the pairs containing it, a pair is adjacent to the 2 singletons it contains and to the 3 pairs of C that are disjoint from it. Other constructions of this graph are known. Another simple construction goes as follows. The vertices of the graph are the elements of GF(16), two vertices are adjacent whenever their difference is a 3rd power in GF(16). The name comes from the fact that this graph corresponds to the 16 lines on the Clebsch quartic surface (see Clebsch [1868] or Coxeter [1950]).

5. The graph HoS(50) (see Hoffman and Singleton [1960]).

A lot of constructions of this graph are known. This graph is a srg(50,7,0,1), and is uniquely defined by its parameters. We shall give only one construction which will be useful later on. It is commonly known that there is a bijection between the 35 unordered triples of a 7-set and the 35 lines of PG$(3, 2)$, such that lines intersect if and only if the corresponding triples have exactly one element in common. The graph HoS(50) is constructed as follows. The vertices are the 15 points together with the 35 lines of PG$(3, 2)$. Points are mutually nonadjacent. A point is adjacent to a line whenever the point lies on that line. Two lines are adjacent whenever the corresponding two triples are disjoint.

6. The Higman–Sims family (see, e.g., Hubaut [1975]).

These graphs are constructed using the Steiner system $S(3, 6, 22)$. This Steiner system is the uniquely defined extension of the projective plane PG$(2, 4)$. Hence this Steiner system has as other parameters $b = 77, r = 21, \lambda_2 = 5$ and two different blocks intersect in 0 or 2 points.

(a) The Higman–Sims graph HS(100).

Take as vertices of the graph a symbol ∞, together with the 22 points and the 77 blocks of $S(3, 6, 22)$. The symbol ∞ is adjacent to all the 22 points but to no block. Points are never adjacent and a point is adjacent to a block whenever it is contained in that block. Two blocks are adjacent whenever they are disjoint. This graph is a srg$(100, 22, 0, 6)$, and is uniquely defined by its parameters, see Gewirtz [1970].

(b) The Higman–Sims graph HS(77).

This graph is the subgraph defined on the set $\Gamma_2(\infty)$ of the vertices of HS(100) that are not adjacent to ∞. It is a srg$(77, 16, 0, 4)$, and is uniquely defined by its parameters, see Brouwer [1983].

(c) The graph Gew(56) of Gewirtz [1969].

Delete from $S(3, 6, 22)$ all the 21 blocks through a fixed point. Take as vertices the 56 other blocks which are adjacent whenever they are disjoint. This graph is a srg(56,10,0,2), and is also uniquely defined by its parameters.

1. Partial geometries

1.1. *Definitions*

A (finite) *partial geometry* $\boldsymbol{S} = (P, B, \mathrm{I})$ is a partial linear space of order (s, t) such that for all antiflags (x, L) the incidence number $\alpha(x, L)$ is a constant α $(\neq 0)$. The numbers s, t and α are called the *parameters* of $\boldsymbol{S}$. This incidence structure was introduced by Bose [1963].

1.2. *Remarks*

1. If $\boldsymbol{S} = (P, B, \mathrm{I})$ is a partial geometry with parameters s, t, α, then the dual structure $\boldsymbol{S}^D = (P^D, B^D, \mathrm{I}^D) = (B, P, \mathrm{I})$, is a partial geometry with parameters $s^D = t$, $t^D = s$ and $\alpha^D = \alpha$.
2. $|P| = v = (s+1)\dfrac{(st+\alpha)}{\alpha}$ and $|B| = b = (t+1)\dfrac{(st+\alpha)}{\alpha}$.
3. The partial geometries can be divided into four (nondisjoint) classes.
   (a) The partial geometries with $\alpha = 1$, the generalized quadrangles. See Chapter 9 and Payne and Thas [1984].
   (b) The partial geometries with $\alpha = s+1$ or dually $\alpha = t+1$, i.e. the 2-$(v, s+1, 1)$ designs and their duals. See Chapter 8.
   (c) The partial geometries with $\alpha = s$ or dually $\alpha = t$. The partial geometries with $\alpha = t$ are the *Bruck nets of order* $s+1$ *and degree* $t+1$; Bruck [1963].
   (d) Finally, the so-called *proper* partial geometries with $1 < \alpha < \min(s, t)$. We shall mainly deal with this class of partial geometries.

1.3. *The point graph of a partial geometry*

THEOREM 3 (Bose [1963]). *The point graph $\Gamma(\boldsymbol{S})$ of a partial geometry $\boldsymbol{S}$ is a*

$$\mathrm{srg}\left((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

(If $\alpha = s+1$, the graph is a complete graph.) Each strongly regular graph Γ having parameters in this form with $t \geqslant 1$, $s \geqslant 1$, $1 \leqslant \alpha \leqslant s+1$ and $1 \leqslant \alpha \leqslant t+1$ is called a *pseudo-geometric* (t, s, α)-graph. If the graph Γ indeed is the point graph of at least one partial geometry, then Γ is called *geometric*. Given a pseudo-geometric (t, s, α)-graph Γ, the problem is to find a subset B of cliques of $s+1$ vertices of Γ, such that any two adjacent vertices of Γ are in exactly one element of B. In Bose [1963] the following condition for a pseudo-geometric (t, s, α)-graph to be geometric is proved.

THEOREM 4 (Bose [1963]). *A pseudo-geometric* (t, s, α)*-graph* Γ *is geometric if*

$$2(s+1) > t(t+1) + \alpha(t+2)(t^2+1).$$

This condition however is in general too strong in order to construct partial geometries from the graph Γ. In Cameron, Goethals and Seidel [1978] it is proved (using the Krein condition on the point graph of the dual geometry) that for a pseudo-geometric (t, s, α)-graph Γ satisfying the Bose inequality in the above theorem, $t \leqslant 2\alpha - 1$ holds.

Attempts to construct a partial geometry from a pseudo-geometric (t, s, α)-graph Γ were in most cases unsuccessful; we refer, e.g., to Spence [1992], De Clerck and Tonchev [1992], and De Clerck, Gevaert and Thas [1988], an exception however is the sporadic partial geometry of Haemers [1981] (see 1.4.5).

If we translate the necessary conditions for strongly regular graphs in Theorem 2 in terms of the parameters of a pseudo-geometric (t, s, α)-graph, then this yields the following theorem.

THEOREM 5. *If* Γ *is a pseudo-geometric* (t, s, α)*-graph, then* $r = s - \alpha$, $l = -t - 1$,

$$f = \frac{st(s+1)(t+1)}{\alpha(s+t+1-\alpha)}.$$

Moreover:

(1) v *is an integer, hence* $\alpha \mid (s+1)st$.

(2) *The multiplicities of the eigenvalues of the adjacency matrix are integers, hence*

$$\alpha(s+t+1-\alpha) \mid st(s+1)(t+1).$$

(3) *The Krein inequalities for strongly regular graphs are satisfied, hence*

$$(s+1-2\alpha)t \leqslant (s-1)(s+1-\alpha)^2.$$

REMARK. If $\alpha = 1$ (and $s \neq 1$), then the Krein inequality is better known as the Higman inequality $t \leqslant s^2$, Higman [1971]. Moreover, Cameron et al. [1978] proved that any pseudo-geometric $(s^2, s, 1)$-graph is geometric (see also Haemers [1980]). It is not known whether this theorem also holds for pseudo-geometric (t, s, α)-graphs satisfying the Krein equality in the case $\alpha > 1$.

OPEN QUESTION. There exists a pseudo-geometric $(27, 4, 2)$-graph, the McLaughlin graph, see, e.g., Goethals and Seidel [1975]. This graph does satisfy the Krein equality; however although several attempts have been made, e.g., by Van Lint [1984], it is not known whether this graph is geometric or not. For the moment, there is no known partial geometry with these parameters.

1.4. *The known models of proper partial geometries*

1.4.1. *The partial geometry* $\boldsymbol{S}(\mathcal{K})$

This infinite family was constructed by Thas [1973, 1974] and independently by Wallis [1973]. Let $\mathcal{K}$ be a maximal arc of degree d in a projective plane π of order q, i.e. a $\{qd-q+d;d\}$-arc (see Chapter 7 for the definitions and examples). We define the incidence structure $\boldsymbol{S}(\mathcal{K}) = (P, B, \mathrm{I})$. The points of $\boldsymbol{S}(\mathcal{K})$ are the points of π that are not contained in $\mathcal{K}$. The lines of $\boldsymbol{S}(\mathcal{K})$ are the lines of π that are incident with d points of $\mathcal{K}$. The incidence is the one of π. Then $\boldsymbol{S}(\mathcal{K})$ is a partial geometry with parameters $t = q - q/d$, $s = q - d$, $\alpha = q - q/d - d + 1$.

REMARKS. As there exist $\{2^{h+m} - 2^h + 2^m; 2^m\}$-arcs, whenever $0 < m < h$, in PG$(2, 2^h)$, there exists a class of partial geometries $\boldsymbol{S}(\mathcal{K})$ with parameters

$$s = 2^h - 2^m,\ t = 2^h - 2^{h-m},\ \alpha = (2^m - 1)(2^{h-m} - 1).$$

This is a generalized quadrangle if and only if $h = 2$, and then it is the unique quadrangle of order 2.

Suppose $m = h - 1, h \geqslant 2$. Then the point graph of $\boldsymbol{S}(\mathcal{K})$ is $T(2^h+2)^C$, the complement of the triangular graph $T(2^h+2)$. Hence $T(2^h+2)^C$ is a geometric $(2^h - 2, 2^{h-1}, 2^{h-1} - 1)$-graph. Although these graphs are uniquely defined by their parameters, this does not imply that the geometry is unique. For instance, Mathon [1981] proved by computer that there exist exactly two partial geometries with parameters $t = 6, s = 4, \alpha = 3$ (and point graph $T(10)^C$). All the complements of the triangular graphs $T(2n)$ are pseudo-geometric $(2(n-2), n-1, n-2)$-graphs. However it is possible to prove that $T(8)^C$ and the complements of the Chang graphs (having the same parameters as $T(8)^C$) are not geometric (De Clerck [1979]). Moreover, Lam, Thiel, Swiercz, and McKay [1983] proved that $T(12)^C$ is not geometric.

1.4.2. *The partial geometry* $T_2^*(\mathcal{K})$

Let $\mathcal{K}$ be a maximal arc of degree d in the projective plane PG$(2, q)$ over GF(q) ($q = p^h$, p prime). As $\mathcal{K}$ has only passants and d-secants, it will yield a linear representation of a partial geometry in AG$(3, q)$. This partial geometry $T_2^*(\mathcal{K})$ has parameters $t = (q+1)(d-1), s = q - 1, \alpha = d - 1$. This infinite family was constructed for the first time by Thas [1973, 1974].

REMARK. The partial geometry $T_2^*(\mathcal{K})$ using a maximal arc of degree 2^m, $0 < m < h$, in PG$(2, 2^h)$ has parameters $s = 2^h - 1$, $t = (2^h + 1)(2^m - 1)$, $\alpha = 2^m - 1$. This is a generalized quadrangle if and only if $m = 1$, i.e. if and only if $\mathcal{K}$ is a hyperoval. (See Chapter 9).

1.4.3. *The partial geometries* PQ$^+(4n-1, q)$*,* $q = 2$ *or* $q = 3$

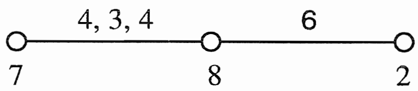
1. *Some properties of hyperbolic quadrics in* PG$(2m-1, q)$. Let $Q^+ = Q^+(2m-1, q)$, $m \geqslant 2$, be the hyperbolic quadric in PG$(2m-1, q)$ (the quadric with projective index

$m-1$). The set of maximal totally isotropic or singular subspaces on a hyperbolic quadric Q^+ is divided into two disjoint families $\mathcal{D}_1$ and $\mathcal{D}_2$. Two maximal totally isotropic or singular subspaces on the quadric are in the same family if and only if the codimension of their intersection has the parity of $m-1$ (see Chapter 2 for more details on quadrics).

Assume q is odd, let x and y be two points of $\mathrm{PG}(2m-1, q) \setminus Q^+$. Then x and y are called equivalent if and only if there exists a point $z \in \mathrm{PG}(2m-1, q) \setminus Q^+$ such that the lines xz and yz are tangent lines of Q^+. This relation can also be defined as follows. Embed Q^+ in the nonsingular hyperquadric Q of $\mathrm{PG}(2m, q)$. The pole of $\mathrm{PG}(2m-1, q)$ with respect to Q is denoted by p. Then x and y are equivalent if and only if the lines xp and yp are both secants or are both exterior lines of Q. The proof that this relation indeed is an equivalence relation was given by Thas [1981b]. There are two equivalence classes E_1 and E_2. For some i, $Q^+ \cup E_i$ is the projection of the nonsingular hyperquadric Q of $\mathrm{PG}(2m, q)$, from the point p onto $\mathrm{PG}(2m-1, q)$.

2. *The partial geometry* $\mathrm{PQ}^+(4n-1, 2)$. De Clerck, Dye and Thas [1980] constructed an infinite class of partial geometries as follows. Define a spread Σ of the nonsingular hyperbolic quadric $Q^+ = Q^+(4n-1, 2)$, $n \geqslant 2$, in $\mathrm{PG}(4n-1, 2)$ to be a (maximal) set of $2^{2n-1}+1$ disjoint $(2n-1)$-dimensional spaces on Q^+. Let Σ be a spread of $Q^+ = Q^+(4n-1, 2)$ and let Ω be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $\mathrm{PQ}^+(4n-1, 2) = (P, B, \mathrm{I})$ with P the set of points of $\mathrm{PG}(4n-1, 2)$ not on the quadric, $B = \Omega$ and $x\mathrm{I}L$, $x \in P$ and $L \in B$, if and only if x is contained in the polar space L^* of L with respect to Q^+. One can prove that $\mathrm{PQ}^+(4n-1, 2)$ is a partial geometry with parameters $s = 2^{2n-1}-1$, $t = 2^{2n-1}$, $\alpha = 2^{2n-2}$.

If $n = 2$, then the parameters of $\mathrm{PQ}^+(7, 2)$ are $s = 7$, $t = 8$, $\alpha = 4$. Cohen [1981b] was the first to construct a partial geometry with these parameters using the root system E_8. In Haemers and Van Lint [1982] a partial geometry with parameters $s = 8$, $t = 7$, $\alpha = 4$, was constructed using coding theory. Kantor [1982a] proved that $\mathrm{PQ}^+(7, 2)$ and the dual of the geometry of Haemers–Van Lint are isomorphic. Later on Tonchev [1984] showed with the help of a computer that the model of Cohen and the dual of the geometry of Haemers–Van Lint are isomorphic. In De Clerck et al. [1988] this isomorphism is proved without the use of a computer. Actually, L. Soicher (private communication) has checked by computer that $\mathrm{PQ}^+(7, 2)$ is uniquely determined by its point graph, as is its dual. Note that this partial geometry also appears as a residue of an element of type 2 in a rank 3 geometry for the Thompson group $F_3 = \mathsf{Th}$ (see Buekenhout [1985]) with diagram



(see also Chapter 22).

The residue of an element of type 0 in this rank 3 geometry is isomorphic to the (unique) generalized hexagon of order $(8, 2)$ (see Chapter 9).

Remark that nonisomorphic spreads of the quadric $\mathrm{PQ}^+(4n-1, 2)$ will produce nonisomorphic partial geometries. If $2n-1$ is composite then $\mathrm{PQ}^+(4n-1, 2)$ has nonisomorphic spreads, and probably this is true for all $n > 2$ (see Kantor [1982b]).

3. *The partial geometry* $\mathrm{PQ}^+(4n-1,3)$. For $q=3$ an analogous construction is given by Thas [1981b]. Again let Σ be a spread of $Q^+ = Q^+(4n-1,3)$ and let Ω be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $\mathrm{PQ}^+(4n-1,3) = (P,B,\mathrm{I})$ with P one of the sets E_i, $B=\Omega$ and with $x\mathrm{I}L$, $x\in P$ and $L\in B$, if and only if x is contained in the polar space $L^\star$ of L with respect to Q^+. One can prove that $\mathrm{PQ}^+(4n-1,3)$ is a partial geometry with parameters $s=3^{2n-1}-1$, $t=3^{2n-1}$, $\alpha = 2\cdot 3^{2n-2}$.

Up to now it is only known that $Q^+(7,3)$ has a spread. This yields a geometry $\mathrm{PQ}^+(7,3)$ with parameters $s=26$, $t=27$, $\alpha=18$ ($v=1080$, $b=1120$).

OPEN QUESTION. Do there exist partial geometries of type $\mathrm{PQ}^+(4n-1,3)$, $n>2$?

1.4.4. *The sporadic partial geometry of Van Lint–Schrijver*
Van Lint and Schrijver [1981] constructed the following sporadic proper partial geometry. We will sketch two constructions of this geometry. Let β be a primitive element of GF(3^4). Then $\gamma=\beta^{16}$ is a primitive 5-th root of the unity. Let $P=\mathrm{GF}(81)$, let B be the set

$$\{(b, 1+b, \gamma+b, \gamma^2+b, \gamma^3+b, \gamma^4+b)\colon\ b\in \mathrm{GF}(81)\},$$

I is the natural incidence, namely inclusion. Then $\boldsymbol{S}=(P,B,\mathrm{I})$ is a partial geometry with $s=t=5$ and $\alpha=2$. The point graph of this geometry has parameters $v=81$, $k=30$, $\lambda=9$, $\mu=12$, and is a graph which was not known before.

Another construction of this geometry is given by Cameron and Van Lint [1982]. Let C be the ternary repetition code of length 6, i.e.

$$C=\{(0,0,0,0,0,0),(1,1,1,1,1,1),(2,2,2,2,2,2)\}.$$

Any coset of C in GF(3)^6 has a well-defined type i in GF(3), i.e. the sum i of the coordinates of any vector in the coset. Let $\mathcal{A}_i$ be the set of cosets of type i. Define a tripartite graph Γ by joining the coset $C+v$ to the coset $C+v+w$ for each vector w of weight 1. Any element in $\mathcal{A}_i$ has 6 neighbours in $\mathcal{A}_{i+1}$ and 6 in $\mathcal{A}_{i+2}$ (indices taken mod 3).

Consider the incidence structure with point set $\mathcal{A}_i$, and line set $\mathcal{A}_{i+1}$, in which incidence is defined by adjacency in Γ. Then this incidence structure is the partial geometry of Van Lint–Schrijver.

For example, suppose $i=0$, and p is the point with coset representative $(0,0,0,0,0,0)$. The six lines incident with p have representatives of the form $(1,0,0,0,0,0)$, hence the 30 points ($\neq p$) collinear with p have representatives of the form $(1,2,0,0,0,0)$. It is immediately clear that two points are incident with at most one line. A line not incident with p has a representative of the form $(2,2,0,0,0,0)$ or $(2,1,1,0,0,0)$; in both cases, it is incident with two points collinear with p.

REMARK. Assume $\boldsymbol{S}$ is a proper symmetric partial geometry with $\alpha=2$, then the numerical conditions of Theorem 2 yield $s=5$. It is not known whether this geometry is unique. There is a great doubt that this geometry is a member of an infinite family. However the point graph is a member of an infinite family of so-called cyclotomic type (see Calderbank and Kantor [1986]).

1.4.5. *The sporadic partial geometry of W. Haemers*

Haemers [1981] constructed another sporadic proper partial geometry. It has parameters $s = 4$, $t = 17$, $\alpha = 2$. The point graph Γ however was known before (see, e.g., Hubaut [1975]). This graph Γ is constructed as follows. The vertices of Γ are the 175 edges of the Hoffman–Singleton graph HoS(50). Two vertices of Γ are adjacent whenever the corresponding edges of HoS(50) have distance two (i.e. the two edges are disjoint and there exists an edge connecting both). One can prove that this graph is a srg$(175, 72, 20, 36)$, moreover Γ is a pseudo-geometric $(17, 4, 2)$-graph. Haemers proved that Γ is indeed geometric. First of all we remark that a line of the partial geometry will be a set of 5 disjoint edges pairwise at distance two in the Hoffman–Singleton graph HoS(50). It is easy to see that in a Petersen graph there are 6 such sets. If we can find 105 Petersen graphs in the Hoffman–Singleton graph, then we have the right number of lines. However there are more than 105 Petersen graphs in HoS(50). W. Haemers was able to find a good subset of 105 special Petersen graphs in the Hoffman–Singleton graph, such that every pentagon of HoS(50) is contained in exactly one such special Petersen graph. Note that any two edges at distance two in HoS(50) are in a unique pentagon, so in a unique special Petersen graph, hence they define a unique set of 5 disjoint edges pairwise at distance two. In other words, the incidence structure of the 175 vertices of Γ and the 630 so-called 1-factors of the special Petersen graphs of HoS(50) has the property that any two adjacent vertices define a unique line. This is enough to conclude that the pseudo-geometric graph Γ indeed is geometric. The geometry is the unique one with this point graph (L. Soicher, private communication).

REMARKS.

1. The point graph of the dual of this geometry has parameters $v = 630$, $k = 85$, $\lambda = 20$, $\mu = 10$ and was not known before. This graph has exactly 175 cliques of size 18, and so this graph uniquely determines the dual of the Haemers geometry (L. Soicher, private communication).
2. Another construction of this geometry is related to the Steiner system $S(5, 8, 24)$ and was given by Calderbank and Wales [1984].
3. There are reasons enough to conjecture that this partial geometry is not a member of an infinite family.
4. Haemers [1991] proved that this partial geometry has a one point extension to a rank 3 geometry for the Mathieu group M_{22}.

1.5. *Some characterization theorems for partial geometries*

1.5.1. *The graphs with $\alpha = s - 1$*

If S is a partial geometry with $\alpha = s - 1$, then the complement of its point graph has a negative eigenvalue -2. All the strongly regular graphs with smallest eigenvalue -2 however are classified by Seidel [1968]. Using this theorem a classification of the pseudo-geometric $(t, s, s - 1)$-graphs can be given. Moreover a lot of information on geometric $(t, s, s - 1)$-graphs is known.

THEOREM 6 (De Clerck [1979]). *If Γ is a pseudo-geometric $(t, s, s-1)$-graph, then one of the following cases occurs.*

(1) $s = 2$ *and* $t = 1$, 2 *or* 4. *Then Γ is geometric and the corresponding generalized quadrangles are unique.*

(2) $s = 3$ *and* $t = 1, 2$ *or* 4.

(a) *If $t = 1$, then Γ is geometric and the corresponding partial geometry is the unique dual 2-design with these parameters.*

(b) *If $t = 2$, then Γ is geometric and there are exactly two possibilities, i.e. the corresponding nets are the two nets of order 4 and degree 3.*

(c) *If $t = 4$, then Γ never is geometric.*

(3) $s > 3$, $t = s - 1$ *and Γ is geometric if and only if there exists an affine plane of order $s + 1$. The corresponding net is obtained by deleting two parallel classes from the affine plane.*

(4) $s > 3$, $t = 2(s-1)$. *If there exists a hyperoval $\mathcal{O}$ in some projective plane of order $2s$ then Γ is geometric, the geometry being the dual of $\boldsymbol{S}(\mathcal{O})$.*

1.5.2. *Partial geometries and the axiom of Pasch*

Let us first introduce the *axiom of Pasch* ($\mathcal{P}$), also called the *axiom of Veblen and Young*.

If $L_1 \mathrm{I} x \mathrm{I} L_2$, $L_1 \neq L_2$, $M_1 \not\mathrm{I}\, x \not\mathrm{I}\, M_2$, $L_i \sim M_j$ for all $i, j \in \{1, 2\}$, then $M_1 \sim M_2$.

We remark that the dual axiom is called the diagonal axiom ($\mathcal{D}$). For a generalized quadrangle both ($\mathcal{P}$) and ($\mathcal{D}$) are satisfied in a trivial way. Evidently a 2-$(v, s+1, 1)$ design satisfies ($\mathcal{D}$). A 2-$(v, s+1, 1)$ design with $s > 1$ satisfying ($\mathcal{P}$), is an n-dimensional projective space ($n \geqslant 2$). The only known partial geometry with $\alpha \notin \{1, s+1, t+1\}$ and satisfying the axiom of Pasch is the dual net H_q^{n+1}. This dual net is constructed as follows. Let $H \cong \mathrm{PG}(n-1, q)$ be a subspace of a projective geometry $\Sigma \cong \mathrm{PG}(n+1, q)$. Then H_q^{n+1} is the incidence structure of points of $\Sigma \setminus H$ and lines of Σ skew to H, the incidence being the one of Σ. The parameters are $s = q$, $t = q^n - 1$, $\alpha = q$. In Thas and De Clerck [1977] it was proved that this dual net is the only one that satisfies the axiom of Pasch.

THEOREM 7 (Thas and De Clerck [1977]). *Let $\boldsymbol{S}$ be a dual net of order $s+1$ and degree $t+1$ ($t+1 > s$). If $\boldsymbol{S}$ satisfies ($\mathcal{P}$), then $\boldsymbol{S}$ is isomorphic to H_q^{n+1} (hence $q = s$, $t + 1 = q^n$).*

REMARKS.

1. Note that H_q^{n+1} may be seen as the complement of a singular symplectic geometry in $\Sigma \cong \mathrm{PG}(n+1, q)$ with radical H. Moreover the dual, $(H_q^{n+1})^D$, is also known as a *regulus net*, see De Clerck and Johnson [1992].

2. For a more general characterization theorem for partial geometries satisfying the axiom of Pasch, we refer to Thas and De Clerck [1977]. Moreover we remark that partial linear spaces satisfying both ($\mathcal{P}$) and ($\mathcal{D}$) are classified by Sprague [1981], see also Chapter 3.

1.5.3. *Partial geometries embedded in projective and affine spaces*
There exists a complete classification of partial geometries embedded in a projective space.

THEOREM 8 (De Clerck and Thas [1978]). *If* $\boldsymbol{S} = (P, B, \mathrm{I})$ *is a partial geometry with parameters* s, t, α, *which is embedded in a projective space* $\mathrm{PG}(n, s)$, *but not in a* $\mathrm{PG}(n', s)$, *with* $n' < n$, *then the following cases may occur.*

(1) $\alpha = s + 1$, *and* $\boldsymbol{S}$ *is the design of points and lines of* $\mathrm{PG}(n, s)$.

(2) $\alpha = 1$, *and* $\boldsymbol{S}$ *is a classical generalized quadrangle* (see Buekenhout and Lefèvre [1974]).

(3) $\alpha = t + 1, n = 2$ *and* $\boldsymbol{S}$ *is a dual design in* $\mathrm{PG}(2, s)$.

(4) $\alpha = s$ *and* $\boldsymbol{S} = H_s^n (n \geqslant 3)$.

There also exists a complete classification of partial geometries embedded in an affine space by Thas [1978]. For the case of a generalized quadrangle, we refer to Chapter 9, in this case some sporadic embeddings can occur. We will however restrict ourselves here to the case of proper partial geometries.

THEOREM 9 (Thas [1978]). *If* $\boldsymbol{S}$ *is a proper partial geometry embedded in an affine space* $\mathrm{AG}(n, s+1)$, *but not in an* $\mathrm{AG}(n', s+1)$ *with* $n' < n$, *then* $n = 3$ *and* $\boldsymbol{S} = T_2^*(\mathcal{K})$ *with* $\mathcal{K}$ *a maximal arc in the plane at infinity.*

COROLLARY. *If we combine the results on the affine embedding of generalized quadrangles in Chapter 9 and the above theorem we can conclude that if* $T_n^*(\mathcal{K})$ $(n > 1)$ *is a linear representation of a partial geometry of order* (s, t), *then either* $\mathcal{K}$ *is the complement of a hyperplane (hence* $\alpha = s$), *or* $n = 2$.

For more characterization theorems, especially regarding geometries of type $\boldsymbol{S}(\mathcal{K})$ and of type $T_2^*(\mathcal{K})$, we refer to De Clerck, De Soete and Gevaert [1987], Gevaert [1987] and De Clerck et al. [1988].

2. Semipartial geometries

2.1. *Definitions*

A *semipartial geometry* (Debroey and Thas [1978a]) with parameters s, t, α, μ is a partial linear space $\boldsymbol{S} = (P, B, \mathrm{I})$ of order (s, t), such that for each antiflag (x, L), the incidence number $\alpha(x, L)$ equals 0 or a constant $\alpha(> 0)$ and such that for any two points which are not collinear, there are μ $(\mu > 0)$ points collinear with both (μ-condition).

REMARKS.
1. A semipartial geometry is a $(0, \alpha)$-geometry such that, because of the μ-condition, the point graph is strongly regular. Besides the parameter μ, the other parameters of the

graph are

$$v = 1 + \frac{(t+1)s(\mu + t(s-\alpha+1))}{\mu}, \; k = (t+1)s, \; \lambda = s - 1 + t(\alpha - 1).$$

2. A semipartial geometry with $\alpha = 1$ is called a *partial quadrangle* and was introduced by Cameron [1974] as a generalization of the generalized quadrangles. Semipartial geometries generalize at the same time the partial quadrangles and the partial geometries. It is immediately clear that a semipartial geometry is a partial geometry if and only if $\mu = (t+1)\alpha$. If we want to exclude the partial geometries we will speak about *proper* semipartial geometries. In any case, for the rest of this section we will suppose, unless the contrary is stated, that $\boldsymbol{S}$ is not a 2-design, hence that $\alpha \leqslant \min(t+1, s)$.

3. The dual of a semipartial geometry again is a semipartial geometry if and only if either $s = t$ or $\boldsymbol{S}$ is a partial geometry (see Debroey and Thas [1978a]).

Using the fact that the point graph is strongly regular, and using other counting arguments, one can deduce a lot of conditions between the parameters of a semipartial geometry. We will give a summary of the most important ones in the next theorem.

THEOREM 10. *Let $\boldsymbol{S} = (P, B, \mathrm{I})$ be a proper semipartial geometry with parameters s, t, α, μ, then*

(1) $t \geqslant s$, *hence* $|B| = b = \dfrac{v(t+1)}{s+1} \geqslant v$;

(2) $D = (t(\alpha-1) + s - 1 - \mu)^2 + 4((t+1)s - \mu)$ *is either a square or equals* 5 (*then* $\boldsymbol{S}$ *is isomorphic to the pentagon*) *and*

$$\frac{2(t+1)s + (v-1)(t(\alpha-1) + s - 1 - \mu + \sqrt{D})}{2\sqrt{D}}$$

is an integer;

(3) $\alpha^2 \leqslant \mu \leqslant (t+1)\alpha$ *and* $\alpha \mid \mu$;

(4) $\mu \mid (t+1)st(s+1-\alpha)$;

(5) $\alpha \mid ts(t+1)$ *and* $\alpha \mid ts(s+1)$;

(6) $\alpha^2 \mid \mu st$;

(7) $\alpha^2 \mid t((t+1)\alpha - \mu)$;

(8) $2 \mid v(t+1)s$;

(9) $3 \mid v(t+1)s(s-1)$ *and* $3 \mid v(t+1)st(\alpha-1)$;

(10) $8 \mid v(t+1)s(s-1)(s-2)$;

(11) $8 \mid v(t+1)s(t(\alpha-1)((t-1)(\alpha-1) - (\alpha-2)) + t(s+1-\alpha)(\mu - 2\alpha + 1))$.

REMARK. The Krein inequalities for strongly regular graphs also yield some extra conditions, but these are rather complicated formulae.

2.2. *A first list of examples of proper semipartial geometries*

2.2.1. *The thin partial quadrangles*

Let Γ be a strongly regular graph with $\lambda = 0$. Then this graph is a partial quadrangle with $s = 1$ and $t = k-1$, hence a thin geometry. Up to now the only known examples of such graphs are the pentagon Pn(5), the Petersen graph Pe(10), the Clebsch graph Cl(16), the Hoffman–Singleton graph HoS(50), and the graphs from the Higman–Sims family (i.e. Gew(56), HS(77) and HS(100)). The parameter sets (v, k, μ) for these graphs are, resp., equal to $(5, 2, 1)$, $(10, 3, 1)$, $(16, 5, 2)$, $(50, 7, 1)$, $(56, 10, 2)$, $(77, 16, 4)$, $(100, 22, 6)$. All these graphs are uniquely defined by their parameters.

2.2.2. *The semipartial geometries $\overline{M(k)}$, $k \in \{2, 3, 7, 57\}$*

The three thin partial quadrangles with $\mu = 1$ are better known as *Moore graphs*. These graphs are the graphs with valency $k > 1$, girth 5 (i.e. they have no 3-cycles nor 4-cycles but they do have 5-cycles) and with the minimum number of vertices, which is $k^2 + 1$. It is known that necessarily $k \in \{2, 3, 7, 57\}$. However a Moore graph with $k = 57$ is not known to exist.

With each Moore graph Γ there is associated another semipartial geometry, which we will denote by $\overline{M(k)}$. The point set P is the set of vertices Γ, the line set B is the set $\{\Gamma(x)\colon\ x \in P\}$, with $\Gamma(x)$ the set of vertices adjacent to x, I is the natural incidence relation. Then $\overline{M(k)} = (P, B, \mathrm{I})$ is a semipartial geometry with parameters $s = t = \alpha = k - 1$, $\mu = (k - 1)^2$ (Debroey and Thas [1978a]).

2.2.3. *The semipartial geometries $U_{2,3}(n)$*

Let U be a set of cardinality n. Let P be the set of pairs, let B be the set of unordered triples of U, and let I be the inclusion relation. Then $U_{2,3}(n) = (P, B, \mathrm{I})$ is a semipartial geometry with parameters $s = \alpha = 2$, $t = n - 3$, $\mu = 4$ (Debroey and Thas [1978a]). The point graph of this geometry is the triangular graph $T(n)$.

2.2.4. *The semipartial geometries* LP(n, q)

Define P as the set of lines of PG(n, q) $(n \geqslant 4)$, B as the set of planes of PG(n, q), and I as the inclusion relation. Then (P, B, I) is a semipartial geometry with parameters

$$s = q(q + 1),\ t = \frac{q^{n-1} - 1}{q - 1} - 1,\ \alpha = q + 1,\ \mu = (q + 1)^2$$

(Debroey and Thas [1978a]). Remark that for $n = 3$ this construction yields the dual design of lines and planes of PG$(3, q)$.

2.2.5. *The semipartial geometries $\overline{W(2n + 1, q)}$*

Let σ be a symplectic polarity of PG$(2n + 1, q)$, $n \geqslant 1$. Let P be the point set of PG$(2n + 1, q)$, B the set of lines which are not totally isotropic (i.e. hyperbolic) with respect to σ, and I the incidence relation of PG$(2n+1, q)$. Then $\overline{W(2n + 1, q)} = (P, B, \mathrm{I})$ is a semipartial geometry with parameters

$$s = q,\ t = q^{2n} - 1,\ \alpha = q,\ \mu = q^{2n}(q - 1)$$

(Debroey and Thas [1978a]).

2.2.6. *The semipartial geometries* $\mathrm{NQ}^{\pm}(2n-1,2)$
Let Q be a (nonsingular) hyperquadric in $\mathrm{PG}(2n-1,2)$. Let P be the set of points off the quadric, let B be the set of nonintersecting lines of Q, and let I be the incidence of $\mathrm{PG}(2n-1,2)$. Then (P,B,I) is a semipartial geometry with parameters

$$s=\alpha=2,\ t=2^{2n-3}-\varepsilon 2^{n-2}-1,\ \mu=2^{2n-3}-\varepsilon 2^{n-1},$$

where $\varepsilon=+1$ for the hyperbolic quadric and $\varepsilon=-1$ for the elliptic quadric (we will denote these geometries by $\mathrm{NQ}^{+}(2n-1,2)$ and $\mathrm{NQ}^{-}(2n-1,2)$, respectively). This was first remarked by H. Wilbrink (private communication).

2.2.7. *The semipartial geometries* $H_q^{(n+1)*}$
This semipartial geometry is defined by taking as point set P the set of lines of a projective space $\Sigma\cong\mathrm{PG}(n+1,q)$ skew to a fixed projective space $H\cong\mathrm{PG}(n-1,q)$ and as line set B the set of the planes of Σ which intersect H in exactly one point. This semipartial geometry has parameters

$$s=q^2-1,\ t=\frac{q^n-1}{q-1}-1,\ \alpha=q,\ \mu=q(q+1).$$

REMARK. It is known that a (semi)partial geometry $\boldsymbol{S}$ satisfying the diagonal axiom, gives rise to a semipartial geometry $\overline{\boldsymbol{S}}$ satisfying the diagonal axiom (see De Clerck and Thas [1978] and Debroey [1979]). Indeed, let x and y be two collinear points. We denote by $D^1_{x,y}$ the set of points collinear with x and y but not on the line $L_{x,y}$ joining x and y and by $D^2_{x,y}$ the set of points of $L_{x,y}$ which are collinear with a point (hence with all points) of $D^1_{x,y}$. Then $D_{x,y}=D^1_{x,y}\cup D^2_{x,y}$ is a maximal set of pairwise collinear points, any such a set $D_{x,y}$ is called a *diagonal clique*. The incidence structure $\overline{\boldsymbol{S}}$ with the same point set as $\boldsymbol{S}$ and with line set, the set of diagonal cliques of $\boldsymbol{S}$ is a semipartial geometry with parameters

$$\bar{t}=s/(\alpha-1)-1,\ \bar{s}=(t+1)(\alpha-1),\ \overline{\alpha}=\alpha,\ \overline{\mu}=\mu$$

which satisfies the diagonal axiom. Note that $\overline{\boldsymbol{S}}$ has the same point graph as $\boldsymbol{S}$ and that $\overline{\overline{\boldsymbol{S}}}\cong\boldsymbol{S}$. In this way the dual of H_q^{n+1} is related to the semipartial geometry $H_q^{(n+1)*}$.

2.3. *The linear representations of semipartial geometries*

If $T_n^*(\mathcal{K})$ is a linear representation of a semipartial geometry, then one easily proves that $\mathcal{K}$ has to be a set of points in $\mathrm{PG}(n,q)$ such that each line of the projective space is either a passant, a tangent or an $(\alpha+1)$-secant and such that, because of the μ-condition, each point of $\mathrm{PG}(n,q)\setminus\mathcal{K}$ is on $\mu(\alpha(\alpha+1))^{-1}$ $(\alpha+1)$-secants. The following examples are known.

2.3.1. *Linear representations of proper partial quadrangles*
In this case the set $\mathcal{K}$ is a $(t+1)$-cap with the property that each point not in $\mathcal{K}$ is on $t+1-\mu$ tangents. Calderbank [1982] has given an almost complete classification of partial quadrangles with a linear representation. His proof is a number-theoretic proof. He lists the possible parameter values of the associated strongly regular graph.

The following cases occur.

1. $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid of the projective space $\mathrm{PG}(3,q)$. It is a partial quadrangle with parameters $s=q-1$, $t=q^2$, $\mu=q(q-1)$ and was first constructed by Cameron [1974].

2. Suppose $q=3$ and assume that $\mathcal{K}$ is not an ovoid. Then $\mathcal{K}$ is either an 11-cap in $\mathrm{PG}(4,3)$, see, e.g., Coxeter [1958] and Pellegrino [1974] for a description, the partial quadrangle $T_4^*(\mathcal{K})$ has parameters $s=2$, $t=10$, $\mu=2$, or $\mathcal{K}$ is the unique 56-cap in $\mathrm{PG}(5,3)$ in which case the partial quadrangle has parameters $s=2$, $t=55$, $\mu=20$. This 56-cap was first constructed by Segre [1965] but was also studied by several other authors, e.g., by Berlekamp, Van Lint and Seidel [1973], Bruen and Hirschfeld [1978], Hill [1973], McLaughlin [1969], and Thas [1981a].

3. Suppose $q=4$. Then either $\mathcal{K}$ is an ovoid in $\mathrm{PG}(3,4)$ or it is a 78-cap in $\mathrm{PG}(5,4)$ such that each external point is on 7 secants, or a 430-cap in $\mathrm{PG}(6,4)$ such that each external point is on 55 secants. If $\mathcal{K}$ is a 78-cap, the partial quadrangle $T_5^*(\mathcal{K})$ has parameters $s=3$, $t=77$, $\mu=14$. At least one example exists and was discovered by Hill [1976]. If $\mathcal{K}$ is a 430-cap then the partial quadrangle has parameters $s=3$, $t=429$, $\mu=110$. Up to now however, the existence of such a cap is not known.

4. Suppose $q\geqslant 5$. Then it was proved by Tzanakis and Wolfskill [1987] that the partial quadrangle has to be $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid.

REMARKS.

1. If $T_n^*(\mathcal{K})$ is a linear representation of a partial quadrangle with $q=2$, then the partial quadrangle coincides with its point graph, and Calderbank [1982] proved that there is only one solution, the strongly regular graph $\mathrm{srg}(v=16, k=5, \lambda=0, \mu=2)$ which is the Clebsch graph Cl(16) and is of type $T_3^*(\mathcal{O})$ with $\mathcal{O}$ the elliptic quadric in $\mathrm{PG}(3,2)$.

2. The existence of a cap $\mathcal{K}$ in $\mathrm{PG}(n,q)$, such that every exterior point is on a constant number of tangents, implies the existence of a uniformly packed $[|\mathcal{K}|, |\mathcal{K}|-n-1, 4]$ code C, which means that the dual code $C^\perp$ is a $[|\mathcal{K}|, n+1]$ code over $\mathrm{GF}(q)$ with exactly 2 weights (see Calderbank [1982] and Calderbank and Kantor [1986]).

2.3.2. *Linear representations of proper semipartial geometries with $\alpha>1$*
In this case the following models are known.

1. The set $\mathcal{K}$ is a unital $\mathcal{U}$ in the projective plane $\Pi_\infty=\mathrm{PG}(2,q^2)$ at infinity, and $T_2^*(\mathcal{U})$ has parameters $s=q^2-1$, $t=q^3$, $\alpha=q$, $\mu=q^2(q^2-1)$.

2. If $\mathcal{K}$ is a Baer subspace $\mathcal{B}$ of the projective space $\Pi_\infty=\mathrm{PG}(n,q^2)$ at infinity, then $T_n^*(\mathcal{B})$ has parameters

$$s=q^2-1,\ t=\frac{q^{n+1}-1}{q-1}-1,\ \alpha=q,\ \mu=q(q+1).$$

Note that this geometry is isomorphic to $H_q^{(n+2)*}$.

2.4. *Semipartial geometries and generalized quadrangles*

It is known that if $\boldsymbol{S} = (P, B, \mathrm{I})$ is a generalized quadrangle, then one can construct in the following way a $(0,1)$-geometry. Let p be any point of $\boldsymbol{S}$, let $p^\perp$ be the set of all points of $\boldsymbol{S}$ collinear with p (the trace of p) and let $B(p)$ be the set of lines of $\boldsymbol{S}$ through p, then the incidence structure $\boldsymbol{S}_p = (P_p, B_p, \mathrm{I}_p)$ with $P_p = P \setminus p^\perp$, $B_p = B \setminus B(p)$, and with $\mathrm{I}_p = \mathrm{I} \cap (P_p \times B_p)$ is clearly a $(0,1)$-geometry of order $(s-1, t)$. Moreover $\boldsymbol{S}_p$ satisfies the following property.

$(*)$ If L and M are two disjoint lines of $\boldsymbol{S}_p$ then there are either 0, $s-1$, or s lines of $\boldsymbol{S}_p$ concurrent to both L and M.

Note that this property of course is trivial in the case $s = 2$. The point graph $\Gamma(\boldsymbol{S}_p)$ of $\boldsymbol{S}_p$ will be a strongly regular graph with parameter μ if and only if for any 2 noncollinear points x and y in P_p, the set $\{p, x, y\}^\perp$ of points in $\boldsymbol{S}$ collinear with p, x and y has cardinality $t+1-\mu$. It is known (see Bose and Shrikhande [1972] and Cameron [1974]) that in a generalized quadrangle $\boldsymbol{S}$, $|\{x,y,z\}^\perp|$ is a constant for any triad $\{x,y,z\}$ of noncollinear points, if and only if $\boldsymbol{S}$ has order (s, s^2), moreover in this case $|\{x,y,z\}^\perp| = s+1$. Hence the only partial quadrangles of type $\boldsymbol{S}_p$ have parameters $(s-1, s^2, s(s-1))$.

There are a lot of generalized quadrangles of order (s, s^2) known. In all of them s is a prime power q and we will therefore in the sequel use q instead of s.

First of all there is the semiclassical example $T_3(\mathcal{O})$, constructed by Tits [1959], see Chapter 9. If p is the special point ∞ in $T_3(\mathcal{O})$ then the resulting partial quadrangle has a linear representation in $\mathrm{AG}(4,q)$; it is the partial quadrangle $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid in the hyperplane Π_∞. If p is any other point of $T_3(\mathcal{O})$ then the resulting partial quadrangle might be nonisomorphic to $T_3^*(\mathcal{O})$. On the other hand, any flock of a cone in $\mathrm{PG}(3,q)$ implies the existence of a generalized quadrangle of order (q, q^2) (see Chapter 9) and these generalized quadrangles give rise to a lot of nonisomorphic partial quadrangles with parameters $(q-1, q^2, q(q-1))$.

Ivanov and Shpectorov [1991] prove that every partial quadrangle with parameters $(q-1, q^2, q(q-1))$ is of type $\boldsymbol{S}_p$ and is uniquely extendible to a generalized quadrangle $\boldsymbol{S}$. For this they prove that such a partial quadrangle always satisfies property $(*)$. In fact they even prove a more general result: every strongly regular graph

$$\mathrm{srg}\left(q^4, (q^2+1)(q-1), q-2, q(q-1)\right),$$

such that every 2 adjacent vertices are contained in a clique of order q, is the point graph of a partial quadrangle of type $\boldsymbol{S}_p$, and this partial quadrangle is uniquely extendible to a generalized quadrangle of order (q, q^2). Remark that this implies that the $\mathrm{srg}(81, 20, 1, 6)$ is unique; see, e.g., Brouwer and Haemers [1992] where another proof of the result of Ivanov and Shpectorov is given.

Anyhow, it follows from the theorem of Calderbank that the linear representation of a partial quadrangle with parameters $(q-1, q^2, q(q-1))$ should be in $\mathrm{AG}(4,q)$, hence it should be $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an ovoid in Π_∞.

In De Clerck and Van Maldeghem [1994] the following theorem is proved.

THEOREM 11. *Let $T_n^*(\mathcal{K})$ $(n \geqslant 3)$ be a linear representation of a $(0,1)$-geometry, of order $(q-1,t)$, $q>2$, that satisfies $(*)$. If $\mathcal{K}$ spans the hyperplane Π_∞, then $T_n^*(\mathcal{K})$ is the partial quadrangle $T_3^*(\mathcal{O})$.*

REMARK. It is clear that the $(0,1)$-geometries of order $(q-1,t)$ of type $T_1^*(\mathcal{K})$ are the grids of order $q-1$, i.e. the generalized quadrangles of order $(q-1,1)$. If $\mathcal{S}$ is a $(0,1)$-geometry of type $T_2^*(\mathcal{K})$, then $\mathcal{K}$ is a set of points in the plane Π_∞ such that every line intersects in 0,1 or 2 points, i.e. $\mathcal{K}$ is an arc in Π_∞. $T_2^*(\mathcal{K})$ satisfies $(*)$ if and only if $|\mathcal{K}|$ is $q+1$ or $q+2$.

There is another way to construct semipartial geometries from generalized quadrangles. Suppose that $\overline{\mathcal{S}}$ is a generalized quadrangle embedded in a projective space $\mathrm{PG}(n,q)$, hence $\overline{\mathcal{S}}$ is classical. Suppose that p is a point of $\mathrm{PG}(n,q)$ and that Π is a hyperplane of $\mathrm{PG}(n,q)$ not containing p. Let $\overline{P}_1$ be the projection of the point set of $\overline{\mathcal{S}}$ from p on Π and let $\overline{P}_2$ be the set of points of Π on a tangent through p at $\overline{\mathcal{S}}$. Let $\mathcal{S}$ be the geometry with point set P the set $\overline{P}_1 \setminus \overline{P}_2$, whereas the line set B is the set of lines of Π which intersect $\overline{P}_1$ in at least two points. The incidence is the one of the projective space. It turns out that if $\overline{\mathcal{S}}$ is $Q^-(5,q)$ or $H(4,q^2)$ one gets semipartial geometries. Of course it depends on whether p is a point on $\overline{\mathcal{S}}$ or not.

If $\overline{\mathcal{S}} = Q^-(5,q)$ and p is a point on the quadric, the semipartial geometry $\mathcal{S}$ is $T_3^*(\mathcal{O})$, with $\mathcal{O}$ the elliptic quadric in $\mathrm{PG}(3,q)$. However if p is not on the quadric, it yields a semipartial geometry with parameters

$$s=q-1,\ t=q^2,\ \alpha=2,\ \mu=2q(q-1).$$

This construction is due to Hirschfeld and Thas [1980]. Another construction of this partial geometry was given by R. Metz (private communication). Let Q be a nonsingular hyperquadric of the projective space $\mathrm{PG}(4,q)$. If we define P as the set of 2-dimensional elliptic quadrics on Q, B as the set of bundles of such elliptic quadrics which are tangent to each other in a common point, and I as the natural incidence relation, then $\mathcal{S}=(P,B,\mathrm{I})$ is isomorphic to the one we just described.

If $\overline{\mathcal{S}} = H(4,q^2)$ and p is a point on $H(4,q^2)$, the semipartial geometry $\mathcal{S}$ is $T_2^*(\mathcal{U})$, with $\mathcal{U}$ the Hermitian unital in $\mathrm{PG}(2,q^2)$. However if p is not on $H(4,q^2)$, it yields a semipartial geometry with parameters

$$s=q^2-1,\ t=q^3,\ \alpha=q+1,\ \mu=q(q+1)(q^2-1).$$

This example is due to J.A. Thas (private communication).

2.5. *Semipartial geometries and SPG reguli*

In Thas [1983] a new construction method for semipartial geometries is introduced. We will give here a brief description of this construction but refer to Thas [1983] for the proofs and more details.

An SPG *regulus* is a set R of m-dimensional subspaces $\mathrm{PG}^{(1)}(m,q),\ldots,\mathrm{PG}^{(r)}(m,q)$ of $\mathrm{PG}(n,q)$, satisfying:

(1) $\mathrm{PG}^{(i)}(m,q) \cap \mathrm{PG}^{(j)}(m,q) = \varnothing$ for all $i \neq j$.

(2) If $\mathrm{PG}(m+1,q)$ contains $\mathrm{PG}^{(i)}(m,q)$, then it has a point in common with either 0 or α ($\alpha > 0$) spaces in $R \setminus \{\mathrm{PG}^{(i)}(m,q)\}$. If this $\mathrm{PG}(m+1,q)$ has no point in common with $\mathrm{PG}^{(j)}(m,q)$ for all $j \neq i$, then it is called a tangent $(m+1)$-space of R at $\mathrm{PG}^{(i)}(m,q)$.

(3) If the point x of $\mathrm{PG}(n,q)$ is not contained in an element of R, then it is contained in a constant number θ ($\theta \geqslant 0$) of tangent $(m+1)$-spaces of R.

By considering all the $(m+1)$-dimensional spaces through $\mathrm{PG}^{(i)}(m,q)$ we obtain that $\alpha(q-1)$ has to divide $(r-1)(q^{m+1}-1)$, and we see that the number of tangent $(m+1)$-spaces of R at $\mathrm{PG}^{(i)}(m,q)$ equals

$$\frac{q^{n-m}-1}{q-1} - \frac{r-1}{\alpha} \cdot \frac{q^{m+1}-1}{q-1}.$$

By counting the number of ordered pairs (M,x), with M a tangent $(m+1)$-space $\mathrm{PG}(m+1,q)$ of R, and x a point of M which is not in an element of R, we obtain:

$$\theta = \frac{\big(\alpha(q^{n-m}-1)-(r-1)(q^{m+1}-1)\big)rq^{m+1}}{\alpha\big((q^{n+1}-1)-r(q^{m+1}-1)\big)}.$$

Note that by $r > 1$ and by the first condition in the definition of R we have $n \geqslant 2m+1$. If $n = 2m+1$, then there are no tangent $(m+1)$-spaces, and $\alpha = r-1$. If $n = 2m+2$, then any two tangent $(m+1)$-spaces at distinct elements of R intersect.

Given an SPG regulus R, with $r > 1$, one can construct a semipartial geometry $\boldsymbol{S} = (P, B, \mathrm{I})$ as follows. Embed $\mathrm{PG}(n,q)$ as a hyperplane in $\mathrm{PG}(n+1,q)$. The points of $\boldsymbol{S}$ are the points in $\mathrm{PG}(n+1,q) \setminus \mathrm{PG}(n,q)$. The lines of $\boldsymbol{S}$ are the $(m+1)$-dimensional subspaces of $\mathrm{PG}(n+1,q)$ which contain an element of R but are not contained in $\mathrm{PG}(n,q)$. Incidence is that of $\mathrm{PG}(n+1,q)$. Then $\boldsymbol{S}$ is a semipartial geometry with parameters

$$s = q^{m+1}-1,\ t = r-1,\ \alpha = \alpha,\ \mu = (r-\theta)\alpha,$$

see Thas [1983].

REMARKS.

1. This geometry is a partial geometry if and only if $\theta = 0$, hence if $\theta \neq 0$, which implies that $r \geqslant q^{m+1}$, then $\boldsymbol{S}$ is a proper semipartial geometry.

2. If $n = 2m+1$, then $\boldsymbol{S}$ is a net of order $s+1 = q^{m+1}$ and degree $t+1 = r$.

SPG reguli and polar spaces

A spread R of the nonsingular elliptic quadric $Q^-(2m+3,q)$ ($m \geqslant 0$) contains $q^{m+2}+1$ elements (of dimension m) and is always an SPG regulus. The parameters of the corresponding semipartial geometry are

$$s = q^{m+1}-1,\ t = q^{m+2},\ \alpha = q^m,\ \mu = q^{m+1}(q^{m+1}-1).$$

For $m = 0$, this is the partial quadrangle $T_3^*(\mathcal{O})$. For $m = 1$, the semipartial geometry has parameters

$$s = q^2-1,\ t = q^3,\ \alpha = q,\ \mu = q^2(q^2-1)$$

which also are the parameters of the semipartial geometry $T_2^*(\mathcal{U})$. Indeed $T_2^*(\mathcal{U})$ is isomorphic to the semipartial geometry arising from a regular spread R (see Chapter 7) of $Q^-(5,q)$. However if the spread is nonregular, then the associated semipartial geometry is not isomorphic to $T_2^*(\mathcal{U})$. If $m > 1$, and q is even, then the quadric $Q^-(2m+3,q)$ has spreads, hence this yields new semipartial geometries. If q is odd, no spread of the quadric $Q^-(2m+3,q)$ $(m > 1)$ is known.

If the nonsingular quadric $Q(2m+2,q)$ (of $\mathrm{PG}(2m+2,q)$), $m \geqslant 0$, has a spread R, then it is not an SPG regulus.

If R is a spread of the quadric $Q^+(2m+1,q)$, $m \geqslant 1$, then necessarily m is odd, moreover this spread is an SPG regulus, but the associated semipartial geometry is a net.

Let $H(n,q^2)$ be a nonsingular Hermitian variety of $\mathrm{PG}(n,q^2)$, $n \geqslant 2$. If n is odd, the Hermitian variety has no spread (see Bruen and Thas [1976] for the case $n = 3$ and Thas [1989] for $n \geqslant 5$). Assume that n is even. Then R is always an SPG regulus with $m = (n/2) - 1$ and $|R| = q^{n+1} + 1$. Hence there corresponds a semipartial geometry $\boldsymbol{S}$ with parameters

$$s = q^n - 1,\ t = q^{n+1},\ \alpha = q^{n-1},\ \mu = q^n(q^n - 1).$$

However if $n = 2$ then this semipartial geometry is $T_2^*(\mathcal{U})$. Unfortunately for $n > 2$ no spread of $H(n,q^2)$, n even, is known. Brouwer (private communication, 1981) proved that $H(4,4)$ has no spread. For more details on spreads of polar spaces, we refer to Chapter 7.

2.6. *Some characterization theorems for semipartial geometries*

THEOREM 12 (Debroey and Thas [1978a]). *If $\boldsymbol{S}$ is a proper semipartial geometry with $\alpha = t$, then $\boldsymbol{S} \cong \overline{M(t+1)}$, hence $s = t$, $\mu = \alpha^2$ and $t \in \{1,2,6,56\}$.*

THEOREM 13 (Debroey [1979], Wilbrink and Brouwer [1984]). *Let $\boldsymbol{S}$ be a proper semipartial geometry with $\mu = \alpha^2$.*

(1) *If $\alpha = 2$, then $\boldsymbol{S} \cong U_{2,3}(n)$.*

(2) *If $2 < \alpha = s$ then $\alpha = t \in \{1,2,6,56\}$ and $\boldsymbol{S} \cong \overline{M(t+1)}$.*

(3) *If $2 < \alpha < s$ then $\boldsymbol{S} \cong \mathrm{LP}(n,q)$.*

THEOREM 14 (Debroey [1979]). *Let $\boldsymbol{S}$ be a proper semipartial geometry with parameters s, t, α (> 1) and $\mu = \alpha(\alpha+1)$. If $\boldsymbol{S}$ satisfies the diagonal axiom $(\mathcal{D})$, then $\boldsymbol{S}$ is isomorphic to a semipartial geometry of type $H_q^{(n+1)*}$.*

REMARK. In Wilbrink and Brouwer [1984] it is proved that all proper semipartial geometries with $\mu = \alpha^2$ and $2 \leqslant \alpha < s$ satisfy the diagonal axiom. Moreover, they proved that up to possibly a finite number of exceptions, all proper semipartial geometries with $\mu = \alpha(\alpha+1)$ satisfy the diagonal axiom. Cuypers [1992] observed that by adding some extra combinatorial conditions, their proofs can even be generalized to $(0,\alpha)$-geometries.

THEOREM 15 (Cuypers [1992]). *Let S be a finite $(0,\alpha)$-geometry with $\alpha \notin \{1,3\}$. Suppose that S satisfies the following conditions.*

(1) *If $\alpha \neq 2$ then $s > f(\alpha)$ where $f(4) = 12$, $f(5) = 6$, $f(6) = f(7) = 17$, $f(8) = 18$, $f(9) = 19$, $f(10) = 21$, $f(11) = 23$ and $f(\alpha) = 2\alpha$ for $\alpha \geqslant 12$;*

(2) $t \geqslant \max(s+1, \alpha(\alpha+1))$;

(3) *Two noncollinear points have either 0, α^2 or $\alpha(\alpha+1)$ common neighbours, and the last two cases both do occur;*

(4) *Let (x, L) be an antiflag, such that $\alpha(x, L) = \alpha$. Then for every two points y and z on $L \setminus x^\perp$, $|x^\perp \cap y^\perp| = |x^\perp \cap z^\perp|$.*

Then $\alpha = 2$ and S satisfies the diagonal axiom.

On the embedding of semipartial geometries in projective and affine spaces the following results are known.

THEOREM 16. *If S is a proper semipartial geometry with parameters s, t, α (> 1), μ, embedded in* $\mathrm{PG}(n, s)$, *$n \geqslant 3$ and $s > 2$, but not in* $\mathrm{PG}(n', s)$, *$n' < n$, then n is odd and S is the semipartial geometry $\overline{W(n, s)}$.*

REMARK. This theorem was proved by Debroey and Thas [1978b] for the case $n = 3$ and by Thas et al. [1984] for $n > 3$. If S is any semipartial geometry with $\alpha = s = 2$, then S is a cotriangle space and those are classified (see Theorem 19). A complete classification of the embedded cotriangle spaces exists for $n = 3$ (Debroey and Thas [1978b]) and for $n = 4$ (Thas et al. [1984]). In Lefèvre-Percsy [1983] an embedding of $U_{2,3}(n+2)$ in PG(n,2) is given. The lines of this geometry are hyperbolic lines, i.e. lines which are not totally isotropic, for some symplectic polarity. Also an embedding of $U_{2,3}(n+3)$ in PG(n,2) is described. The lines of this geometry are hyperbolic for some symplectic polarity if and only if n is odd. The problem of determining all embeddings of $U_{2,3}(n)$ in PG(d, 2) is equivalent to determining (up to equivalence) all binary codes of length n with all weights even and minimum weight greater than 4, see Hall [1983].

THEOREM 17 (De Clerck and Thas [1983]). *If S^D is the dual of a semipartial geometry S with $\alpha > 1$, and if S^D is embedded in a projective space* $\mathrm{PG}(n, s)$, *$n \geqslant 3$, but not in* $\mathrm{PG}(n', s)$, *$n' < n$, then $n = 3$ and S^D is the design of points and lines in* $\mathrm{PG}(3, q)$, *or $S^D = H_s^3$ or $S^D = \mathrm{NQ}^-(3, 2)$* (see 2.2.6).

OPEN QUESTION. Let $\mathcal{H}$ be a nonsingular Hermitian variety in PG($3, q^2$). The incidence structure $S = (P, B, \mathrm{I})$, defined by taking as point set P the point set of $\mathcal{H}$ and as line set B the set of lines of $\mathcal{H}$ minus all the lines concurrent with a given line L, is a dual partial quadrangle embedded in PG($3, q^2$). One can prove that the dual of this geometry is isomorphic to $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an elliptic quadric. It is not known whether this is the only proper dual partial quadrangle embedded in a projective space.

A complete classification of all proper semipartial geometries embedded in affine spaces is still open. However the problem is solved by Debroey and Thas [1977] for dimensions 2 and 3.

THEOREM 18. *A proper semipartial geometry $\mathcal{S}$ with parameters s, t, α, μ is not embeddable in an affine plane* AG$(2, s+1)$. *If $\mathcal{S}$ is embedded in* AG$(3, s+1)$, *then $\mathcal{S}$ is either the pentagon embedded in* AG$(3, 2)$ *(trivial case) or a linear representation and $\mathcal{S} = T_2^*(\mathcal{U})$ or $\mathcal{S} = T_2^*(\mathcal{B})$ (hence $s+1$ is a square).*

3. Copolar spaces

A *copolar space* (see Hall [1982]) is a partial linear space $\mathcal{S} = (P, B, \mathrm{I})$ such that for each antiflag (x, L), the incidence number $\alpha(x, L)$ equals 0 or $|L| - 1$.

A partial linear space with this property has been called a *proper Δ-space* by Higman [1979]. He observed that the above property is more or less the converse of the defining property of a polar space. This is the reason why J.I. Hall calls it a copolar space.

It is easily seen that the copolar spaces of order $(1,t)$ are precisely those graphs which contain no triangles. A copolar space of order $(2,t)$ is better known as a *cotriangle space*.

The copolar space $\mathcal{S}$ is called *indecomposable* if and only if $\mathcal{S}$ is not the union of two or more copolar spaces on disjoint point sets. A *reduced* copolar space is an indecomposable copolar space such that for all vertices x and y in the point graph $\Gamma(\mathcal{S})$, $\Gamma(x) = \Gamma(y)$ implies $x = y$.

Remark that a semipartial geometry with parameters s, t, $\alpha = s$ is indeed a copolar space of order (s, t). Of course the dual of a net is a copolar space, and since there is no hope to classify them, we assume from now on that there exists at least one antiflag (x, L) such that $\alpha(x, L) = 0$.

In Hall [1982] the finite reduced copolar spaces of order (s, t), $s \geqslant 2$, are classified up to isomorphism. It turns out that the reduced copolar space of order (s, t) is a (proper) semipartial geometry.

We summarize the results in the next theorem.

THEOREM 19 (Hall [1982]). *If $\mathcal{S} = (P, B, \mathrm{I})$, is a finite reduced copolar space of order (s, t) $s \geqslant 2$, then $\mathcal{S}$ is isomorphic to one of the following semipartial geometries*:

(1) $\overline{M(k)}$, $k \in \{2, 3, 7, 57\}$,

(2) $U_{2,3}(n)$,

(3) $\overline{W(2n+1, q)}$,

(4) $\mathrm{NQ}^{\pm}(2n-1, 2)$.

REMARK. The cotriangle spaces were in fact classified by Shult [1975], an earlier version of which was proved by Seidel [1973].

4. Near n-gons

4.1. *Definitions*

A partial linear space $\mathcal{S}$ is called a *near n-gon* if and only if the following axioms hold:

(1) n is an even integer and the diameter of the point graph is at most $n/2$,

(2) given any point p and any line L, L contains a unique point nearest to p.

This incidence structure was introduced by Shult and Yanushka [1980] because of their interest in line systems with few angles. See also Shad and Shult [1979].

Any generalized n-gon with n even can easily be seen to be a near n-gon. In the case $n = 4$, the converse is true: any near 4-gon is a generalized quadrangle. This is not true for $n > 4$.

A thin near n-gon (i.e. all lines have size 2) is just a connected bipartite graph. We will not discuss them here (see for instance Shad and Shult [1979] and Shad [1984]). We will assume from now on that the near n-gon is thick. However some of the next theorems still hold (possibly under some extra conditions) if one assumes that there are thin lines.

4.2. *Classical and sporadic near n-gons*

A subset Y of the point set P is called *geodetically closed* if for any two points $y_1, y_2 \in Y$ all the shortest paths between y_1 and y_2 are contained in Y. A *quad* is a geodetically closed subset of P of diameter 2 such that not all its points are adjacent to one fixed point. This quad is a near 4-gon, hence a generalized quadrangle. In Shult and Yanushka [1980] it is proved that for any thick near n-gon, any two points x and y at distance 2 with at least two common neighbours determine a unique quad $Q(x, y)$ containing them. Even more can be said.

THEOREM 20 (Brouwer and Wilbrink [1983a]). *If x and y are two points at distance i of a (thick) near n-gon, then they are contained in a unique geodetically closed sub-$2i$-gon.*

The existence of those geodetically closed sub-$2i$-gons has been very important for the characterization theorems, as we will illustrate by the next theorems.

THEOREM 21 (Shult and Yanushka [1980]). *Let Q be a quad in a thick near n-gon $\boldsymbol{S} = (P, B, \mathrm{I})$. Then for any point x not lying in Q, either*

(1) *x has distance d to exactly one point y in Q, distance $d+1$ to all points of Q collinear with y, and distance $d+2$ to all points of Q at distance 2 from y;*

(2) *x has distance d to all points of an ovoid in Q and distance $d+1$ to all remaining points.*

In the first case the pair (x, Q) is said to be of *classical* type, in the second case the pair (x, Q) is said to be of *ovoidal* type. A near n-gon is called *classical* if all its (nonincident) point-quad pairs are classical, otherwise it is called *sporadic*.

THEOREM 22 (Cameron [1982]). *A classical (thick) near n-gon with quads is a dual polar space (i.e. a partial linear space whose points and lines are respectively the maximal and second-maximal singular subspaces of a polar space of rank $n/2$, reverse containment signifying incidence).*

The embedding problem for near n-gons is settled by the next theorem.

THEOREM 23 (Cameron [1981]). *A (thick) near n-gon which is not a generalized n-gon, embedded in a projective space of order q, is classical and is of type $O_{n+1}(q)$ ($n > 4$) (i.e. the polar space is the quadric $Q(n, q)$).*

REMARK. For a detailed discussion on polar spaces, and dual polar spaces, we refer to Chapter 12.

4.3. *Regular near n-gons*

A near n-gon will be called *regular* with parameters $(s, t_2, t_3, \dots, t_n = t)$ if and only if (i) it is a thick near n-gon of order (s, t) and (ii) whenever two points x and y are at distance $d > 1$, exactly $t_d + 1$ lines through y contain points at distance $d - 1$ from x. The point graph $\Gamma(S)$ of a regular near n-gon is distance regular. Hence a lot of graph-theoretical results can be used in this case. For more details, we refer to Brouwer and Wilbrink [1983a] and Brouwer et al. [1989].

THEOREM 24 (Shult and Yanushka [1980]). *Let S be a (thick) regular near n-gon and let $\Gamma_d(x)$ be the set of points at distance d from x. Then*

$$|\Gamma_1(x)| = s(t+1),$$

$$|\Gamma_d(x)|\big(s(t - t_d)\big) = |\Gamma_{d+1}(x)|(t_{d+1} + 1).$$

Hence

$$|\Gamma_d(x)| = \frac{s^d(1+t)t(t-t_2)(t-t_3)\cdots(t-t_{d-1})}{(1+t_2)\cdots(1+t_d)}, \quad d \geqslant 2.$$

If $t_2 \neq 0$, then $t_d > t_{d-1}$, $d = 3, 4, \dots, n$, each point lies in $t(t+1)(t_2(t_2+1))^{-1}$ quads of order (s, t_2) and each line lies in t/t_2 quads (hence $t(t+1)(t_2(t_2+1))^{-1}$ and t/t_2 are integers). The set of all lines through a point p together with the set of all quads containing p form a 2-$(t + 1,\ t_2 + 1, 1)$-*design.*

The regular near hexagons (i.e. 6-gons) with $s = 2$, i.e. with parameters $(2, t_2, t)$ are completely classified. By the above theorem, if $t_2 \neq 0$, the near hexagons contain generalized quadrangles of order $(2, t_2)$, and hence (see Chapter 9), $t_2 = 1$, 2, or 4 and in each of the cases the generalized quadrangle is unique up to isomorphism.

We summarize the results in the next theorem.

THEOREM 25 (Shult and Yanushka [1980], Brouwer [1982a]). *If S is a regular near hexagon with $s = 2$ then the following cases occur.*

(1) *S is a generalized quadrangle, hence $t = t_2 = 1, 2$ or 4 (v is equal to $9, 15$ and 27, resp.).*

(2) *S is a generalized hexagon, $t_2 = 0$, and $t = 1, 2$ or 8 (v is equal to $21, 63$ and 819, resp.).*

(3) $\boldsymbol{S}$ *is a proper regular near hexagon and the following cases occur*:

(a) $(t, t_2) = (2, 1)$, $v = 27$.

(b) $(t, t_2) = (11, 1)$, $v = 729$.

(c) $(t, t_2) = (6, 2)$, $v = 135$.

(d) $(t, t_2) = (14, 2)$, $v = 759$.

(e) $(t, t_2) = (20, 4)$, $v = 891$.

REMARKS. These regular near hexagons with 3 points on a line are uniquely defined by their parameters. For the generalized quadrangles and hexagons with $s = 2$, we refer to Chapter 9. The proper regular near hexagons with $(t, t_2) = (6, 2)$, $v = 135$ and $(t, t_2) = (20, 4)$, $v = 891$ both are of classical type, they are the dual of the polar spaces Sp(6, 2) and U(6, 4). Each regular near hexagon with parameters $(s, t_2, t) = (s, 1, 2)$, for instance the one with $v = 27$, is of Hamming type (or a generalized cube): the point set is the set of all ordered triples from a set X, $|X| = s + 1$, the lines are the maximal cliques in the Hamming graph on X, i.e. two triples are collinear if and only if they differ in only one coordinate. More generally one can define in the same way near n-gons of Hamming type by taking the ordered n-tuples from a set X; they are uniquely defined by their parameters (see Shult and Yanushka [1980], Egawa [1981], Brouwer and Wilbrink [1983a]). The near hexagon with 729 points is derived from the extended ternary Golay code, its uniqueness is proved by Brouwer [1982a]. The uniqueness of the near hexagon on 759 points is also proved by Brouwer [1982b]. The points of this near hexagon are the 759 octads (i.e. the blocks of size 8) of the unique Steiner system $S(5, 8, 24)$. Two blocks are called collinear if they are disjoint. Since the complement of the union of two disjoint octads is an octad, a line of the near hexagon will be a set of 3 pairwise disjoint octads. This near hexagon has also a nice combinatorial characterization.

THEOREM 26 (Brouwer and Wilbrink [1983a]). *If a regular near hexagon satisfies* $s > 1$, $t_2 > 0$, *and* $1 + t = (1 + t_2)(1 + st_2)$ *then it is the unique regular near hexagon with* $s = t = 2$, $v = 759$.

This near hexagon cannot be a (geodetically closed) sub-hexagon of a regular near n-gon ($n \geqslant 8$), from which it follows that any sporadic regular near n-gon will have $1 + t_3 > (1 + t_2)(1 + st_2)$. For the regular near octagons one might even conjecture that they almost never exist (see Brouwer and Wilbrink [1983a] and Brouwer et al. [1989]). A sporadic regular near octagon with parameters $(s, t_2, t_3, t) = (2, 0, 3, 4)$ is constructed by Cohen [1981a]. The Hall–Janko group J_2 acts on this geometry, the point graph is a distance regular graph, which is uniquely defined by its parameters (see Cohen and Tits [1985]). This near octagon contains no quads but does contain generalized hexagons of type $\mathsf{G}_2(2)$. It has 315 points and 525 lines. It has generalized octagons of order (2,1) and generalized hexagons of order (2,1) and (2,2) as subgeometries. It is itself a subgeometry of the dual of the classical generalized hexagon of order (4,4) arising from $\mathsf{G}_2(4)$ (see Chapter 9).

REMARKS. Regular near hexagons with 4 or 5 points on each line and regular near octagons with 3 points on each line are discussed in Shad and Shult [1979]. For more constructions of regular near n-gons and their distance regular graphs, we refer to Brouwer et al. [1989].

We finally remark that nonregular near n-gons do exist. For instance, consider the graph $T(2n)^C$; the maximal cliques in this graph have n vertices. If one takes as points these maximal cliques and as lines the cliques with $n-2$ vertices, then the geometry with respect to the natural incidence is a near $2(n-1)$-gon of order $(2, n(n-1)/2-1)$ and is nonregular for $n \geqslant 4$ (see Brouwer and Wilbrink [1983b]).

In Theorem 25 the classification of the regular near-hexagons with line size 3 is given. However the nonregular hexagons with line size 3 having quads are also completely classified.

THEOREM 27 (Brouwer, Cohen and Wilbrink [1983], Brouwer [1985]). *Let S be a nonregular near hexagon with lines of size 3 having quads, then the following cases occur, and in each case the near hexagon is unique for given v, t, t_2.*

(1) $(v, t, t_2) = (45, 3, 1$ *or* $2)$,
(2) $(v, t, t_2) = (81, 5, 1$ *or* $4)$,
(3) $(v, t, t_2) = (105, 5, 1$ *or* $2)$,
(4) $(v, t, t_2) = (243, 8, 1$ *or* $4)$,
(5) $(v, t, t_2) = (405, 11, 1$ *or* 2 *or* $4)$,
(6) $(v, t, t_2) = (567, 14, 2$ *or* $4)$.

REMARK. The classification of the near hexagons with lines of size 3 has been republished by Brouwer, Cohen, Hall and Wilbrink [1994].

5. Moore geometries

5.1. *Moore graphs*

In Section 1 we have introduced the Moore graphs of diameter 2. In fact they can be defined for any diameter d. Indeed, for a regular graph of valency k and diameter d one has the inequality

$$v \leqslant 1 + k + k(k-1) + \cdots + k(k-1)^{d-1}$$

(proved by Moore, see Hoffman and Singleton [1960]), and graphs for which equality holds are called Moore graphs. The girth of a Moore graph is odd and satisfies $g = 2d+1$. Singleton [1968] proved that a connected graph with diameter d and girth $2d+1$ is necessarily regular and moreover is a Moore graph. As we have done in Section 1 for the special case of girth 5, one can also define Moore graphs with respect to a lower bound. Indeed, the number of vertices v of a regular graph of valency k and odd girth g satisfies

$$v \geqslant 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2},$$

and graphs for which the equality hold are the Moore graphs.

Remark that a $(2d+1)$-gon is a Moore graph of diameter d and valency 2. The fact that we restricted ourselves in Section 1 to the diameter 2 case, is not that restrictive, as we can notice in the next theorem.

THEOREM 28 (Damerell [1973]). *A Moore graph with valency $k = 2$ is a polygon. A Moore graph with valency $k \geqslant 3$ has diameter 2 and $k \in \{3, 7, 57\}$.*

As we already remarked in Section 2.2, no example with $k = 57$ is known, and if $k = 3$ the graph is the Petersen graph, whereas for $k = 7$, the graph is the Hoffman–Singleton graph.

5.2. (*Generalized*) *Moore geometries*

The concept of a Moore graph was generalized by Bose and Dowling [1971], they defined a Moore geometry of diameter d. This was even more generalized by Roos and Van Zanten [1982]: they introduced the concept of generalized Moore geometries.

A generalized Moore geometry of type $\mathrm{GM}_d(s, t, c)$ is a (finite) partial linear space of order (s, t), such that the point graph has diameter d, any two points at distance $i < d$ are joined by a unique shortest path, and any two points at distance d are joined by exactly c shortest paths. In order to exclude various trivial structures, $st > 1$ is assumed. These geometries include as special cases the Moore graphs ($s = c = 1$), the Moore geometries ($c = 1$), and the generalized $2d$-gons ($c = t + 1$). Another subfamily, namely, those with $c = s + 1$, is proved to exist only for small values of the diameter d, in a series of papers, the last of which by Damerell, Roos and Van Zanten [1989].

THEOREM 29. *A generalized Moore geometry of type* $\mathrm{GM}_d(s, t, s+1)$ *with* $st > 1$, *cannot exist for diameter* $d > 3$.

The proof is by using the fact, that if the geometry does exist, then the point graph is distance regular, and the eigenvalues of the adjacency algebra have to be rational.

Known examples of generalized Moore geometries of type $\mathrm{GM}_d(s, t, s+1)$ with $st > 1$ and $d \leqslant 3$ are the Clebsch graph ($d = 2$, $s = 1$, $t = 4$), the Gewirtz graph ($d = 2$, $s = 1, t = 9$), the odd graph O_4 ($d = 3$, $s = 1$, $t = 3$) (the vertices are the 35 unordered triples from a set X of cardinality 7, two triples being adjacent if and only if they are disjoint) and the generalized $2d$-gons, $d = 2$ or 3, with $s = t$.

Also, for other types of generalized Moore geometries it is proved that the diameter of the point graph (which is distance regular) should be small, but a discussion of these theorems would bring us too far. For more information and references we refer to Brouwer et al. [1989].

We only will state the theorem for the case $c = 1$, i.e. the Moore geometries as they were defined by Bose and Dowling [1971]. The proof again is a combination of several papers by Fuglister, Damerell and Georgiacodis, references of which can be found in Brouwer et al. [1989].

THEOREM 30. *A Moore geometry of diameter d is either a $(2d + 1)$-gon $(st = 1)$ or $d \leqslant 2$.*

Note that if $s = 1$ we have the Moore graphs. Moore geometries of diameter 1 are the Steiner systems $S(2, s + 1, v)$, there is no example of a nontrivial Moore geometry of diameter 2. We finally remark that also another generalization of the Moore graphs exists, see Kantor [1977].

6. (g, d_p, d_l)-gons

6.1. *Definitions*

Recall that one of the main motivations for studying geometries is provided by the fact that it gives ways to study groups by their flag-transitive action on geometries. This group action implies a certain regularity in the geometry S, such as the number of points on a line is constant, etc. If we have a group transitive on longer geodesics then we also have more regularity properties. Note that a flag is here considered as a geodesic of length 1.

From now on assume that S is a (g, d_p, d_l)-gon. Denote by Γ the incidence graph of S. Buekenhout and Van Maldeghem [1992] call S a *regular* (g, d_p, d_l)-gon if

$$\left|\Gamma_i(x) \cap \Gamma_j(y)\right| = \left|\Gamma_i(z) \cap \Gamma_j(u)\right|$$

for all positive integers i, j and all elements x, y, z, u whenever $d(x, y) = d(z, u)$ and x and z are either both points or both lines.

6.2. *Examples*

We first show how all this fits into the geometries of the preceding sections. Afterwards, we will give some other notable examples.

Some classical examples

Probably the most important class of (g, d_p, d_l)-gons is the class of *generalized polygons*, see Chapter 9. A generalized n-gon is an (n, n, n)-gon.

Another large class is the class of *linear spaces*; these are either projective planes or $(3, 3, 4)$-gons. Let us just mention a trivial example: every set P is the set of points of a linear space $L(P)$ by declaring all pairs of points to be the lines. In fact, this is a *circle* geometry in the sense of Buekenhout [1979].

A symmetric 2-(v, k, λ)-design with $1 < \lambda < k$ is a regular $(2, 3, 3)$-gon. If $\lambda = k$ it is a generalized digon. If $\lambda = 1$, then it is a projective plane, hence a regular $(3, 3, 3)$-gon.

In general, every design which is not a linear space, can be regarded as a $(2,3,d)$-gon with $d \in \{3,4\}$. The case $d=3$ corresponds exactly to the symmetric designs.

The diameter of a *partial geometry* $\boldsymbol{S}$ is at most 4 and we have the following possibilities.

(1) $\boldsymbol{S}$ is a regular $(3,3,3)$-gon, that is, a generalized triangle or a projective plane, hence $\boldsymbol{S}$ has parameters (s,s,s) for some positive integer s.

(2) $\boldsymbol{S}$ is a regular $(3,3,4)$-gon or its dual, i.e. $\boldsymbol{S}$ is a regular *proper* linear space or a regular proper dual linear space.

(3) $\boldsymbol{S}$ is a regular $(3,4,4)$-gon. Among these, we have the nets and the dual nets. The other members in this class are the proper partial geometries.

(4) $\boldsymbol{S}$ is a generalized quadrangle.

A partial quadrangle $\boldsymbol{S}$ with parameters s,t,μ is in general a $(4,5,6)$-gon, but if $\mu = t+1$, then we have a generalized quadrangle, hence a $(4,4,4)$-gon; if $\mu = 1$, then we have a $(5,5,6)$ or $(5,5,5)$-gon; if $\boldsymbol{S}$ is also a dual partial quadrangle, then it is a regular $(4,5,5)$-gon.

A *proper* Moore geometry is a $(g,g,g+1)$-gon for $g \geqslant 3$ and g odd. By Theorem 30, $g=3$ or 5. A generalized Moore geometry of type $\mathrm{GM}_d(s,t,c)$ which is not a Moore geometry or a generalized $2d$-gon can be a regular $(2d,2d+1,2d+1)$-gon (if $s=t$) or a regular $(2d,2d+1,2d+2)$-gon (in the other cases).

A *near n-gon* is in general a $(4,n,n)$-gon.

Some more examples

The near hexagon on 759 points (see Theorem 25 and its remarks) provides three examples of (g,d_p,d_l)-gons. The geometry itself is a $(4,6,6)$-gon (as mentioned above). If we take the quads as new lines and remove the old lines, then one obtains a $(3,4,4)$-gon of order $(14,34)$. We can also keep the old lines, remove the points and take as new points the quads. This constitutes a $(3,5,5)$-gon of order $(6,14)$.

The sporadic group J_1 of Janko acts on a regular graph of valency 11 with 266 vertices. Take as points the vertices of this graph. Define the lines to be the pairs of opposite vertices. Then we obtain a $(5,7,8)$-gon of order (1,11).

Consider the Steiner system $S(5,6,12)$. Take as points of a geometry the triads and as lines the *linked threes* (i.e. 4 triads every 2 of which form a hexad); incidence is the natural one. We obtain a (nonregular) $(5,6,6)$-gon of order (3,3) with the group M_{12} acting as an automorphism group. The geometry is self-dual (the outer automorphisms of M_{12} interchange points and lines).

The Hall–Janko group J_2 acts on the sporadic regular near octagon of Cohen–Tits which is a $(6,8,8)$-gon.

The group McL of McLaughlin acts on $\mathrm{U}_3(5)$ as a rank 5 group. The induced graph has triangles, but no 4-cliques with one edge removed. If we take as points the vertices of that graph and as lines the triangles, we obtain a (4,6,6)-gon of order (2,125) of 7128 points. A similar construction with Co_3 acting on HS yields a (4,6,6)-gon of order (2,175) consisting of 11178 points (due to Soicher, private communication).

6.3. *Characterizations by automorphisms*

Let G be a (type preserving) automorphism group of the (g, d_p, d_l)-gon S. We shall use the following terminology.

(1) Suppose G acts transitively on the set of pairs (x, y) of points at distance i from each other, for all even positive integers i. We call (S, G) a *point distance transitive* (g, d_p, d_l)-pair, dually a *line distance transitive* (g, d_p, d_l)-pair. If (S, G) is both point distance transitive and line distance transitive, then we call (S, G) a *weakly distance transitive* (g, d_p, d_l)-pair. If G acts transitively on each set of pairs of elements at distance j from each other and having fixed type, for all positive integers j, then (S, G) is called a *distance transitive* (g, d_p, d_l)-pair.

(2) Suppose G acts transitively on each set of geodesics based at some point x of S and ending in a point y at maximal distance, for all points $x \in P$, then we call (S, G) a *point geodesic transitive* (g, d_p, d_l)-pair. Similarly as above we can define *line geodesic transitive* (g, d_p, d_l)-pairs, respectively *weakly geodesic transitive* and *geodesic transitive*.

(3) If G acts transitively on each set of geodesics of length i based at some fixed variety x, for all varieties x, then (S, G) is called a *locally i-arc transitive* (g, d_p, d_l)-pair.

It is easy to see that, if S is a (g, d_p, d_l)-gon and if $2 \leqslant g \leqslant d_p \leqslant d_l \leqslant g + 1$, then each of the above assumptions on G implies that S is regular. Hence from now on we assume that all geometries are regular.

With this terminology, one can classify large classes of geometries with groups acting transitively on sets of relatively long geodesics. The following results are proved in Buekenhout and Van Maldeghem [1992, 1993], using the classification of finite single groups. The symbol q denotes a prime power and we follow the ATLAS (Conway, Curtis, Norton, Parker and Wilson [1985]) for the notation of the groups.

THEOREM 31. *Let (S, G) be a finite geodesic transitive (g, d_p, d_l)-pair, $2 \leqslant g \leqslant d_p \leqslant d_l \leqslant g + 1$; then one of the following holds.*

(1) *S is a generalized polygon related to an irreducible finite adjoint or twisted adjoint Chevalley group or Ree group of type ${}^2\mathsf{F}_4$ and $X_n(q) \leqslant G \leqslant \mathrm{Aut}(X_n(q))$, where $X_n(q)$ is the corresponding Chevalley group, or $G \cong A_6$ and S is the unique generalized quadrangle of order $(2, 2)$, or S is the flag complex of a self-dual classical generalized polygon and G is as above extended by a graph automorphism, or S is an ordinary polygon.*

(2) *S can be identified with the Petersen graph* Pe(10), *resp., the Hoffman–Singleton graph* HoS(50); *the lines are the edges of the graph and $G \cong S_5$, resp.,* $\mathrm{U}_3(5) \trianglelefteq G \leqslant \mathrm{U}_3(5) : 2$. *Here, S is considered as a Moore geometry, in particular a $(5, 5, 6)$-gon of order* $(1, 2)$, *resp.,* $(1, 6)$.

(3) *S is a $(3, 4, 4)$-gon. The following cases occur.*

(3.1) *S is a net of order q and degree q obtained from the Desarguesian projective plane* $\mathrm{PG}(2, q)$ *by deleting a flag (x, l) and all varieties incident with one of x, l, and G contains the stabilizer in* $\mathrm{PGL}_3(q)$ *of the flag (x, l) in* $\mathrm{PG}(2, q)$.

(3.2) $\boldsymbol{S}$ *is the net* $(H_q^{n+1})^D$ *of order* q^n *and degree* $q+1$ *and* G *contains a group isomorphic to the semidirect product of an elementary Abelian group* q^{2n} *with a group isomorphic to*

(a) $(\mathrm{SL}_2(q) \times \mathrm{SL}_n(q))/Z(\mathrm{SL}_2(q) \times \mathrm{SL}_n(q))$ *if* $n > 2$, *or*

(b) $(\mathrm{SL}_2(q) \times \mathrm{GL}_2(q))/Z(\mathrm{SL}_2(q) \times \mathrm{GL}_2(q))$ *if* $n = 2$, *or*

(c) $\mathrm{SL}_2(2) \times A_7$ *if* $(n, q) = (4, 2)$.

(3.3) $\boldsymbol{S}$ *is the dual of* 3.2.

(3.4) $\boldsymbol{S}$ *is a net of order* 16 *and degree* 9 *whose points can be identified with the points of an affine space* $\mathrm{AG}(8, 2)$ *and whose lines are the affine* 4-*subspaces whose* 3-*spaces at infinity constitute a* 2-*transitive spread of a hyperbolic quadric in* $\mathrm{PG}(7, 2)$; G *contains the full translation group of* $\mathrm{AG}(8, 2)$, *and its kernel 'at infinity' is* A_9.

(3.5) $\boldsymbol{S}$ *is the dual of* 3.4.

(4) $\boldsymbol{S}$ *is a* $(3, 3, 4)$-*gon. Three cases occur.*

(4.1) $\boldsymbol{S}$ *is the linear space consisting of the points and lines of* $\mathrm{PG}(d, q)$, $q \geqslant 3$, *and* $\mathrm{L}_{d+1}(q) \trianglelefteq G \leqslant \mathrm{P\Gamma L}_{d+1}(q)$.

(4.2) $\boldsymbol{S}$ *is the Desarguesian affine plane* $\mathrm{AG}(2, q)$; G *contains all translations and its kernel 'at infinity' contains* $\mathrm{L}_2(q)$.

(4.3) G *is a group acting* 4-*transitively on the set of points of* $\boldsymbol{S}$ *and the lines of* $\boldsymbol{S}$ *can be identified with the pairs of points.*

(5) $\boldsymbol{S}$ *is a* $(2, 3, 3)$-*gon. Here,* $\boldsymbol{S}$ *is a symmetric* 2-*design with* $\lambda > 1$ *and four cases occur* (see also Chapter 8).

(5.1) $\boldsymbol{S}$ *can be identified with* $\mathrm{PG}(d, q), d \geqslant 3$, *the blocks are either the hyperplanes or their complements and* $\mathrm{L}_{d+1}(q) \trianglelefteq G \leqslant \mathrm{P\Gamma L}_{d+1}(q) : 2$ *or* $G \cong A_7$ *or* S_7 (*if* $(d, q) = (3, 2)$ *and blocks are the hyperplanes*).

(5.2) $\boldsymbol{S}$ *is the* (*unique*) *Paley* (*or Hadamard*) *design on* 11 *points and* $\mathrm{L}_2(11) \trianglelefteq G \leqslant \mathrm{L}_2(11) : 2$.

(5.3) $\boldsymbol{S}$ *is isomorphic to one of Kantor's designs* $\mathcal{S}^{\pm}(n)$ *and* $G \geqslant 2^{2n} : \mathrm{Sp}_{2n}(2)$ (see Kantor [1975]).

(5.4) $\boldsymbol{S}$ *is the design with some point set* P *and with blocks the complements of the singletons* $\{p\}$ *in* P. *The group* G *acts* 3-*transitively on* P.

(6) $\boldsymbol{S}$ *is a generalized quadrangle of order* $(1, s)$ *or* $(s, 1)$ *and there is an almost simple* 2-*transitive group* T_O *of degree* $s+1$ *with socle* T *such that* $T \times T \trianglelefteq G \leqslant T_O \,\mathtt{wr}\, S_2$.

(7) $\boldsymbol{S}$ *is a generalized digon.*

For various subclasses of geometries, they obtain more general results by weakening the hypothesis on G. We start with the class of generalized polygons.

THEOREM 32. *Let* $\boldsymbol{S}$ *be a thick generalized* n-*gon,* $n \geqslant 3$, *and suppose* G *is a group of automorphisms acting point distance transitively on* $\boldsymbol{S}$; *then* $(\boldsymbol{S}, G)$ *is one of the thick examples in Theorem* 31(1) *above and* G *is the corresponding Chevalley group or its derived group, or* $\boldsymbol{S}$ *is the unique generalized quadrangle of order* $(3, 5)$ *and* G *contains a group isomorphic to* $2^6 : 3 : A_6$.

We say that the pair (S, G) has the *Tits property* if G acts transitively on the set of ordered circuits of minimal length (in which case the length is twice the girth). For a definition of the *Moufang property* and the *half Moufang property* for generalized polygons, we refer to Chapter 9. The next result is a corollary to Theorem 31.

THEOREM 33. *Let S be a finite thick generalized polygon and let $G \leqslant \mathrm{Aut}(S)$. Then the following conditions are equivalent.*

(1) *(S, G) has the Tits property.*

(2) *(S, G) has the Moufang property.*

For generalized hexagons and octagons every distance transitive group induces both the Tits and the Moufang property; for point distance transitive groups and for generalized quadrangles there are a few exceptions (namely, the smallest ones).

The following result is also proved by Buekenhout and Van Maldeghem [1993].

THEOREM 34. *Let S be a thick finite generalized hexagon or octagon and G an automorphism group of S. The pair (S, G) is half Moufang if and only if it is point distance transitive or line distance transitive, depending on the type of Moufang roots in S. In particular, half Moufang implies Moufang whenever $(s, t) \neq (2, 2)$ (for generalized hexagons) or $(s, t) \neq (2, 4), (4, 2)$ (for generalized octagons). Also, S is Moufang with respect to some automorphism group if and only if it is half Moufang with respect to some (possibly other) automorphism group. Finally, (S, G) is half Moufang if and only if G is flag-transitive on S.*

Note that for generalized quadrangles this result already was proved in Thas, Payne and Van Maldeghem [1991], without the classification of the finite simple groups, see also Chapter 9.

We now consider partial geometries and partial quadrangles. For proofs, see Buekenhout and Van Maldeghem [1992].

THEOREM 35. *Let S be a proper partial geometry and suppose that $G \leqslant \mathrm{Aut}(S)$ acts weakly distance transitively on S. Then the points of S can be identified with the points of an affine line $\mathrm{AG}(1, q)$ and $G \leqslant \mathrm{A\Gamma L}_1(q)$.*

No examples satisfying the hypothesis of the above theorem are known though.

THEOREM 36. *Let S be a partial quadrangle which is not a generalized quadrangle and let $G \leqslant \mathrm{Aut}(S)$ act point distance transitively on S. Then either the point set of S can be identified with the affine line $\mathrm{AG}(1, q)$ and $G \leqslant \mathrm{A\Gamma L}_1(q)$, or $\mathrm{Aut}(S)$ acts point geodesic transitively and one of the following possibilities occurs.*

(1) *$s = 1$ and S is a rank 3 strongly regular graph. The possibilities for (S, G) are listed in Table 1.*

(2) *S has a linear representation in the affine space $\mathrm{AG}(n, q)$, G acts point geodesic transitively, it contains the full translation group of $\mathrm{AG}(n, q)$ and the centre of*

the stabilizer of a point is an almost simple group M, (i.e. M contains a normal simple group and is included in its automorphism group) where the possibilities for $\boldsymbol{S}, n, q, M$ are given in Table 2.

Table 1
Point distance transitive partial quadrangles with $s = 1$

$\boldsymbol{S}$	G	(s, t, μ)	Remarks
Pn(5)	D_{10}	$(1, 1, 1)$	G is geodesic transitive;
Pe(10)	$A_5 \trianglelefteq G \leqslant S_5$	$(1, 2, 1)$	G is point geodesic transitive; S_5 is geodesic transitive
HoS(50)	$\mathrm{U}_3(5) \trianglelefteq G \leqslant \mathrm{U}_3(5) : 2$	$(1, 6, 1)$	G is point geodesic transitive and geodesic transitive
Gew(56)	$\mathrm{L}_3(4) \trianglelefteq G \leqslant \mathrm{P\Gamma L}_3(4)$	$(1, 9, 2)$	G is point geodesic transitive but *not* geodesic transitive
HS(77)	$\mathsf{M}_{22} \trianglelefteq G \leqslant \mathsf{M}_{22} : 2$	$(1, 15, 4)$	G is point geodesic transitive but *not* geodesic transitive
HS(100)	$\mathsf{HS} \trianglelefteq G \leqslant \mathsf{HS} : 2$	$(1, 21, 6)$	G is point geodesic transitive and geodesic transitive
Cl(16)	$2^4 : D_{10} \leqslant G \leqslant 2^4 : S_5$	$(1, 4, 2)$	$2^4 : (5 : 4)$ is point geodesic transitive; $2^4 : A_5$ is geodesic transitive

Table 2
Point geodesic transitive partial quadrangles with $s > 1$

$\boldsymbol{S}$	$\mathrm{AG}(n, q)$	M	Restrictions
$T_3^*(\mathcal{Q})$	$\mathrm{AG}(4, q)$	$\mathrm{L}_2(q^2)$	$\mathcal{Q}$ an elliptic quadric in $\mathrm{PG}(3, q)$
$T_3^*(\mathcal{O})$	$\mathrm{AG}(4, q)$	$\mathrm{Sz}(q)$	$\mathcal{O}$ the Suzuki–Tits ovoid in $\mathrm{PG}(3, q)$, $q = 2^{2e+1}$
$T_5^*(\mathcal{K})$	$\mathrm{AG}(5, 3)$	M_{11}	$\mathcal{K}$ the 11-cap in $\mathrm{PG}(5, 3)$ arising from M_{11}

This has the following immediate consequence:

THEOREM 37. *No locally* 4*-arc transitive* $(4, 5, 5)$*-pair exists.*

Note that from Theorem 31 it follows that there do not exist geodesic transitive $(g, g+1, g+1)$-pairs with $g \geqslant 4$. It is a conjecture that there exist no such geometries at all.

It is appropriate also to mention in this context a result which follows almost immediately from the classification of flag-transitive linear spaces by Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl [1990].

THEOREM 38. *Every linear space listed in Table* 3 *gives rise to a locally* 3*-arc transitive* $(3, 3, 4)$*-pair. Conversely, if* $(\boldsymbol{S}, G)$ *is a locally* 3*-arc transitive* $(3, 3, 4)$*-pair with* G *type* *preserving, then it is one of the examples of Table* 3. *If* (S, G) *is a geodesic transitive (or equivalently a weakly geodesic transitive)* $(3,3,4)$*-pair with* G *type preserving, then it is one of the first* 3 *examples in Table* 3.

Finally, we mention a characterization of a class of Moore geometries.

THEOREM 39. *If* (S, G) *is a point distance transitive* $(g, g, g+1)$*-pair,* $g \geqslant 5$*, with* G *type preserving, then it is one of the two examples of Table* 4. *Moreover, if* S *is* HoS(50) *then* G *acts geodesic transitively, if* S *is* Pe(10) *then* G *acts geodesic transitively if and only if* $G \cong S_5$.

Table 3
Geodesic transitive and locally 3-arc transitive linear spaces

S	G	Restrictions and remarks
$\mathrm{PG}(n,q)$	$\mathrm{L}_{n+1}(q) \trianglelefteq G \leqslant \mathrm{P\Gamma L}_{n+1}(q)$	$n \geqslant 3$
$\mathrm{AG}(2,q)$	$\mathrm{L}_2(q) \trianglelefteq G_0 \leqslant \Gamma\mathrm{L}_2(q)$	G contains all translations
$S(P)$	G	G is almost simple and acts 4-transitively on P
$\mathrm{PG}(3,q)$	A_7	G contains all translations
$\mathrm{U}_H(q)$	$\mathrm{PGU}_3(q) \trianglelefteq G \leqslant \mathrm{P\Gamma U}_3(q)$	Hermitian unital in $\mathrm{PG}(3,q^2)$
$\mathrm{AG}(n,q)$	$\mathrm{L}_n(q) \trianglelefteq G_0 \leqslant \Gamma\mathrm{L}_n(q)$	$n \geqslant 3$ and G contains all translations
$\mathrm{AG}(4,2)$	$G \cong 2^4 : A_7$	
$S(P)$	G	G is almost simple and acts 3-transitively on P

Table 4
Point distance transitive Moore geometries with $d \geqslant 2$

S	G	Remarks
Pe(10)	$A_5 \trianglelefteq G \leqslant S_5$	S_5 is geodesic transitive
HoS(50)	$\mathrm{U}_3(5) \trianglelefteq G \leqslant \mathrm{U}_3(5) : 2$	G is geodesic transitive

THEOREM 40. *If we allow diameter* 1 *for Moore geometries, then a point distance transitive* $(g, g, g+1)$*-pair is one of the examples in Tables* 3 *and* 4 *or it is one of the following linear spaces*:

- *the Hermitian unital with* $G \cong \mathrm{U}_3(q)$;
- *the Ree unital and* G *is an automorphism group of the corresponding Ree group*;
- *the affine space* $\mathrm{AG}(n,q)$ *with* $\mathrm{AL}_1(q^n) \leqslant G \leqslant \mathrm{A\Gamma L}_1(q^n)$.
- *the Hering plane of order* 27 *or the nearfield plane of order* 9 (see Chapter 5);
- *a Hering space* (see Chapter 22, 1.9.8);
- *a circle geometry with a* 2*-transitive almost simple group acting.*

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