The Classification of Finite Generalized Quadrangles Admitting a Group Acting Transitively on Ordered Pentagons

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Abstract

Let $S$ be a thick generalized quadrangle and let $G$ be a group of automorphisms of $S$. If $G$ acts transitively on the set of non-degenerate ordered pentagons, then $S$ is one of the classical generalized quadrangles $W(q)$, $Q(4,q)$, $Q(5,q)$ or $H(3,q^2)$. The possibilities for $G$ in each case are determined. We do not use the classification of the finite simple groups (from which this result also follows).

1 Introduction and Main Result

A finite generalized quadrangle (GQ) of order $(s, t)$, $s, t \in \mathbb{N} \setminus \{0\}$, is an incidence geometry $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint non-empty sets of objects called points and lines respectively, and for which $\mathcal{I}$ is a symmetric point-line incidence relation satisfying axioms (GQ1), (GQ2) and (GQ3).

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(GQ1) Each point is incident with \(1 + t\) lines and two distinct points are incident with at most one line.

(GQ2) Each line is incident with \(1 + s\) points and two distinct lines are incident with at most one point.

(GQ3) For every non-incident pair \((x, L) \in \mathcal{P} \times \mathcal{B}\), there exists a unique pair \((y, M) \in \mathcal{P} \times \mathcal{B}\) for which \(xIMIyIL\).

For terminology, notation, results, etc., concerning finite GQ, see the monograph [5], hereafter denoted by FGQ. We now introduce some further terminology. A finite GQ of order \((s, t)\) is thick if \(s, t \geq 2\) (the non-thick GQ are the grids (order \((s, 1)\)) and the dual grids (order \((1, t)\)). A pentagon in a GQ is a subconfiguration consisting of five distinct points and five distinct lines such that each line (respectively point) is incident with exactly two points (respectively lines). An ordered pentagon is a pentagon in which the elements are ordered in such a way that two consecutive elements are incident. A skeleton is a subconfiguration \(\Omega = (Q; L, p)\) where \(Q\) is a quadrilateral (i.e. a subquadrangle of order \((1, 1)\)) and \(L\) (respectively \(p\)) is a line (respectively point) not in \(Q\) but incident with a point \(p_1\) (respectively line \(L_1\)) of \(Q\), where \(p_1 I L_1\).

It follows easily from 9.8.3 of FGQ that the classical GQ \(W(q), Q(4, q), Q(5, q)\) and \(H(3, q^2)\) admit an automorphism group \(G\) acting transitively on the set of skeletons (the automorphism group of the GQ \(H(4, q^2)\) is not transitive on ordered triples of concurrent lines). We will see in 2.1.1 that this is equivalent with \(G\) acting transitively on the set of ordered pentagons. The converse is also true. Suppose the GQ \(S\) admits a group \(G\) acting transitively on the set of ordered pentagons. Then \(G\) is a group with a \((B, N)\)-pair of type \(B_2\) and using the classification of the finite simple groups one can show that \(S\) must be classical (for an explicit proof, see [1]), but in the present case different from \(H(4, q^2)\). The aim of this paper is to give a proof of this result without using the classification of the finite simple groups. We will also characterize the groups \(G\). Therefore, we need the following notation.

Let \(\text{PGO}_5(q)\) be the projective general orthogonal group in 4-dimensional projective space over the field \(GF(q)\) and let \(q\) be odd. This group has a unique normal subgroup of index 2, namely the simple group \(\text{PSO}_5(q)\), sometimes denoted by \(O_5(q)\). Now let \(\theta\) be a field automorphism of \(GF(q)\) and denote by \(g_\theta\) the element of \(\text{PGL}_5(q)\) corresponding to the semilinear
transformation with identity matrix and field automorphism \( \theta \). Let \( q \) be any element of \( PGO_5(q) \setminus PSO_5(q) \). Then we denote by \( PGO_5^{\theta}(q) \) the group generated by \( PSO_5(q) \) and \( g_{g_{gg}} \). Now consider the projective general unitary group \( PGU_4(q) \) (still assuming \( q \) odd). This group has a unique normal subgroup of index 2 (if \( q \equiv 1 \mod 4 \) or 4 (if \( q \equiv 3 \mod 4 \)), namely the simple group \( PSU_4(q) \), also denoted by \( U_4(q) \). The corresponding quotient group is cyclic and hence \( PGU_4(q) \) has a unique subgroup \( H \) of index 2 containing \( PSU_4(q) \). Let \( \theta \) be a field automorphism of \( GF(q^2) \) and let \( g_{g_{gg}} \in PGL_4(q^2) \) be defined similarly as above. Then we denote by \( PGU_4^{\theta}(q) \) the group generated by \( H \) and \( g_{g_{gg}} \), where \( g \in PGU_4(q) \setminus H \).

In this paper we will show:

**MAIN RESULT.** Let \( S \) be a finite thick generalized quadrangle and let \( G \) be a group of automorphisms of \( S \). Then \( G \) acts transitively on the set of ordered pentagons if and only if \( S \) is one of the classical generalized quadrangles \( W(q) \), \( Q(4,q) \), \( Q(5,q) \) or \( H(3,q^2) \), and \( G \) contains one of the following groups:

1. \( PGO_5(q) \), if \( S \cong W(\Pi) \) or \( S \cong Q(\triangle, \Pi) \),
2. \( PGO_5^{\theta}(q) \), if \( S \cong W(\Pi) \) or \( S \cong Q(\triangle, \Pi) \) with \( q \) an odd square and \( \theta \) a field automorphism of \( GF(q) \) of order a power of 2,
3. \( PGU_4(q) \), if \( S \cong Q(\wedge, \Pi) \) or \( S \cong H(\vee, \Pi^\vee) \),
4. \( PGU_4^{\theta}(q) \), if \( S \cong Q(\wedge, \Pi) \) or \( S \cong H(\vee, \Pi^\vee) \) with \( q \) odd and \( \theta \) a field automorphism of \( GF(q^2) \) of order a power of 2.

The groups will be further discussed in Section 2.4.

Part of the motivation of our proof is to stimulate people to aim for weaker hypotheses (such as the \((B,N)\)-pair hypothesis above), still giving a proof without relying on the classification of the finite simple groups.

## 2 Proof of the Main Result

In this section, we denote by \( S = (P, B, I) \) a finite thick generalized quadrangle of order \((s, t)\) and by \( G \) a group of automorphisms of \( S \) acting transitively on the set of ordered pentagons.
2.1 Some General Facts

2.1.1 Skeletons

Let $\Pi = (p_1, L_1, \ldots, p_5, L_5)$, with $p_1 I L_1 I p_2 I \ldots I p_5 I L_5 I p_1$, be an ordered pentagon in $\mathcal{S}$. Let $p'_i$ be the point of $L_{i+2}$ collinear with $p_i$ (where indices are taken modulo 5). Consider the skeleton $\mathcal{I} = (Q; L_0, p_0^5)$, where $Q$ is the quadrilateral $(p_1, L_1, p_2, L_2, p_3, L_3, p'_1, p_1 p'_1)$. Then $\Pi$ completely determines $\mathcal{I}$ and vice versa. Hence $G$ acts transitively on the set of skeletons.

2.1.2 Property (H)

Fix a point $u$ of $\mathcal{S}$ and let $(x, y, z)$ be a triad of points of $u^\perp$. Suppose $z \in \text{cl}(x, y)$. If $z \in \{x, y\}^{\perp\perp}$, then $y \in \{x, z\}^{\perp\perp}$. Now suppose $z \notin \{x, y\}^{\perp\perp}$ and let $v$ respectively $v'$ be a point in $\{x, y\}^{\perp} \setminus z^\perp$ respectively $\{x, z\}^{\perp} \setminus y^\perp$. This defines a unique skeleton $(Q; L_0, z')$ respectively $(Q', L'_0, y')$, where $uz \sim L_0 I v$ respectively $uy \sim L_0 I v'$, where $z \sim z' I vy$ respectively $y \sim y' I v'z$, and where $Q = (v, vy, y, yu, u, ux, x, xv)$ respectively $Q' = (v', v'y, z, zu, u, ux, x, xv')$. By the transitivity of $G$ on skeletons, we see that there is an automorphism mapping the ordered quadruple $(u, x, y, z)$ onto the ordered quadruple $(u, x, z, y)$. Indeed, $z$ is collinear with $z'$ and incident with the unique line $uy$ containing $u$ and concurrent with $L_0$; $z'$ is mapped onto $y'$, and the unique line containing $u$ and concurrent with $L'_0$ is the line $uy$. As $y' \sim y I uy$, the point $z$ is mapped onto $y$. Hence $y \in \text{cl}(x, z)$ and so $u$ has property (H). By transitivity, every point of $\mathcal{S}$ has property (H), and dually, every line has property (H). By 5.6.2 of FGQ $\mathcal{S}$ must satisfy, up to duality, one of the following conditions:

(i) $\mathcal{S}$ is isomorphic to $H(4, q^2)$, $q^2 = s$.

(ii) Every point and every line in $\mathcal{S}$ are regular.

(iii) Every hyperbolic line has exactly 2 points, and dually.

(iv) Every hyperbolic line in $\mathcal{S}$ has exactly 2 points and every line of $\mathcal{S}$ is regular.
It is readily seen that $H(4, q^2)$ does not admit a group of automorphisms acting transitively on the set of skeletons. Also, case $(ii)$ implies that $\mathcal{S}$ is isomorphic to $W(2^e)$ for some positive integer $e$ (see 5.2.1 and 3.2.1 of FGQ). So from now on, we may assume that every hyperbolic line has exactly 2 points. Using the same kind of argument to show property (H), one produces a collineation mapping $(u, x, y, z)$ onto $(u, x', y', z')$, where $(x, y, z)$ and $(x', y', z')$ are two arbitrary triads in $u^\perp$ (note that $z \notin \{x, y\}^\perp$ as required for the argument). So all triads in $u^\perp$ have a constant number of centers. By 1.7.1(i) of FGQ, this constant equals $1 + t/s$, so $s$ divides $t$.

Now we consider the cases $(iii)$ and $(iv)$ separately.

### 2.2 Case $(iii)$

In this case, also the dual of $\mathcal{S}$ has hyperbolic lines of length 2, hence also $1 + s/t$ is an integer. It follows that $s = t$. As each triad of points has either 0 or 2 centers, each point of $\mathcal{S}$ is antiregular (cf. 1.3.6(iii) of FGQ). Dually, each line of $\mathcal{S}$ is antiregular. Also $s = t$ is odd by 1.5.1(i) of FGQ.

Assume that $\mathcal{S}'$ is a thick subquadrangle of $\mathcal{S}$ of order $(s', t')$, and that $\mathcal{S}'$ does not admit a proper thick subquadrangle. Let $\Omega$ and $\Omega'$ be skeletons contained in $\mathcal{S}'$. Further, let $\theta$ be an automorphism of $\mathcal{S}$ mapping $\Omega$ to $\Omega'$. If $\mathcal{S}'^\theta \neq \mathcal{S}'$, then $\mathcal{S}' \cap \mathcal{S}'^\theta$ is a proper thick subquadrangle of $\mathcal{S}'$, a contradiction. Consequently $\mathcal{S}'^\theta = \mathcal{S}'$, and so the automorphism group $G'$ of $\mathcal{S}'$ acts transitively on the set of skeletons of $\mathcal{S}'$.

In $\mathcal{S}'$ any triad of points has at most two centers, that is, each point of $\mathcal{S}'$ is antiregular. Hence $s' \geq t'$ by 1.3.6(i) of FGQ. Similarly, $t' \geq s'$. Consequently, $s' = t'$ and then by 1.3.6(iii) of FGQ each triad of points respectively lines has 0 or 2 centers. Also, $s' = t'$ is odd. Let us now forget the GQ $\mathcal{S}$; so “$\perp$” means perpendicular in $\mathcal{S}'$, etc.

Since $\mathcal{S}'$ has no proper thick subquadrangles, the identity is the only automorphism of $\mathcal{S}'$ fixing a given skeleton of $\mathcal{S}'$. The group $H'$ of automorphisms of $\mathcal{S}'$ fixing an ordered quadrilateral $Q := (p_1, L_1, p_2, \ldots, L_4)$ and a line $L$ through $p_1$, $L_1 \neq L \neq L_1$, has even order $s' - 1$. Let $\sigma$ be an involution of the group $H'$.

By 1.3.2 of FGQ an affine plane $\pi(p_4, p_1)$ of order $s'$ may be constructed as follows. Points of $\pi(p_4, p_1)$ are the points of $p_1^\perp$ that are not on $L_4$. Lines
are the pointsets \( \{p_4, z\}^\perp \setminus \{p_4\} \), with \( p_4 \sim z \not\sim p_1 \), and \( \{p_4, u\}^\perp \setminus \{p_1\} \), with \( p_1 \sim u \not\sim p_4 \). Let us denote the pointset of a line \( M \) of \( S' \) by \( M^* \). The involution \( \sigma \) induces either the identity or an involution \( \sigma' \) in the plane \( \pi(p_4, p_1) \). In the latter case, \( \sigma' \) fixes the point \( p_3 \), the line \( L^*_3 \), the parallel class of lines defined by \( L^*_3 \), the line \( \{p_4, p_2\}^\perp \setminus \{p_1\} = U \) which contains \( p_3 \), the parallel class of lines defined by \( U \) (as \( L^*_4 = L_1 \)), and the parallel class containing the line \( \{p_4, v\}^\perp \setminus \{p_1\} = V \) with \( p_1 \neq v \not I \) \( L \) (as \( L^*_5 = L \)).

If \( \sigma' \) is the identity, then, by 2.4 of FGQ, \( \sigma \) is the identity, a contradiction; if \( \sigma' \) is a Baer involution, then, again by 2.4 of FGQ, the fixed elements of \( \sigma \) form a subquadrangle of order \( (\sqrt{\delta}', \sqrt{\delta}) \) of \( S' \), a contradiction. Hence \( \sigma' \) is a homothety of \( \pi(p_4, p_1) \) with center \( p_3 \). Consequently \( \sigma \) fixes every line through \( p_1 \) and every point of \( \{p_1, p_3\}^\perp \). As \( S' \) does not contain a proper thick subquadrangle, the fixed elements of \( \sigma \) are \( p_1, p_3 \), the points of \( \{p_1, p_3\}^\perp \), the lines through \( p_1 \), and the lines through \( p_3 \).

Now let \( x \) and \( y \) be distinct points of \( S' \) on the line \( N \) of \( S' \), with \( p_1 \) not incident with \( N \). By the transitivity properties of \( G' \) there is an involution \( \gamma \) of \( S' \) fixing \( p_1 \) linewise and fixing \( N \). Let \( r^\gamma \neq r \), with \( r \) a point of \( S' \) on \( N \). Further, let \( \delta \) be an element of \( G' \) fixing \( p_1, N \), and for which \( r^\delta = x \) and \( r^\gamma \delta = y \). Then \( \delta^{-1} \gamma \delta \) fixes \( p_1 \) linewise and maps \( x \) onto \( y \). Now it is clear that the subgroup of \( G' \) fixing \( p_1 \) linewise acts transitively on \( \mathcal{P}' \setminus \sqrt{\infty}^\perp \), with \( \mathcal{P}' \) the pointset of \( S' \). Then by 8.2.4 of FGQ the GQ \( S' \) is an EGQ (elation generalized quadrangle) with base point \( p_1 \) and elation group \( \overline{G} \).

Let \( u_1 \) and \( u_2 \) be distinct points of the plane \( \pi(p_4, p_1) \). If \( \zeta \) is the elation in \( \overline{G} \) mapping \( u_1 \) onto \( u_2 \), then \( \zeta \) induces a translation \( \zeta' \) in \( \pi(p_4, p_1) \) mapping \( u_1 \) onto \( u_2 \). Consequently \( \pi(p_4, p_1) \) is a translation plane. Interchanging the roles of \( p_1 \) and \( p_3 \), we see that \( \pi(p_1, p_4) \) is also a translation plane. It follows that, if \( \infty \) is the point at infinity of \( \pi(p_4, p_1) \) defined by the parallel class of lines containing \( L^*_3 \), then the projective completion \( \overline{\pi(p_4, p_1)} \) of \( \pi(p_4, p_1) \) is a dual translation plane with translation point \( \infty \).

Next, let \( x_1 \) and \( x_2 \) be points of \( \pi(p_4, p_1) \) not on \( L^*_3 \) and not on \( \{p_2, p_4\}^\perp \setminus \{p_1\} = U \). By the transitivity of \( G' \) on the skeletons of \( S' \), there is an automorphism \( \eta \) in \( G' \) fixing \( p_1, p_2, p_3, p_4 \) mapping \( x_1 p_4 \) onto \( x_2 p_4 \) and mapping \( x_1 \) onto \( x_2 \). Then \( \eta \) induces an automorphism \( \eta' \) of \( \pi(p_4, p_1) \) fixing \( L^*_3, U \) and mapping \( x_1 \) onto \( x_2 \). Hence \( \pi(p_4, p_1) \) is Desarguesian [4]. Then by 5.2.7 of FGQ, the GQ \( S' \) is isomorphic to \( Q(4, s') \). Consequently every line of \( S' \) is regular, a contradiction.
We conclude that case (iii) cannot occur.

2.3 Case (iv)

2.3.1 Some general observations

We first fix the notation.

Throughout, \( \Omega \) will denote the skeleton \(((p_1, L_1, p_2, L_2, p_3, L_3, p_4, L_4); L, p)\), where \( Q = (p_1, \ldots, L_4), L I p_1 \) and \( p I L_1 \). The group \( G_\Omega \) of automorphisms in \( G \) fixing \( \Omega \) has order \( k \in \mathbb{N} \setminus \{0\} \).

The line \( L_1 \) is regular in \( S \). For any two lines \( M \) and \( M' \) meeting \( L_1 \) in different points, we call the collection \( \{M, M'\} \) of \( s + 1 \) lines the regulus through \( M \) and \( M' \); it is denoted by \( MM' \). Every line of the regulus \( MM' \) meets \( L_1 \). The set of all lines through a point \( p \) will be denoted by \( p^* \).

Consider an involution \( \sigma \) that fixes \( p_1, L_1 \) and \( p_2 \). Suppose that \( \sigma \) does not fix all points on \( L_1 \). Let \( x \) be such a point on \( L_1 \) which is not fixed by \( \sigma \), and let \( M \) be a line through \( x \) different from \( L_1 \). The regulus \( MM' \) meets \( p_1^* \) respectively \( p_2^* \) in exactly one line \( R_1 \) respectively \( R_2 \). Clearly the regulus \( MM' \) is fixed by \( \sigma \), and since \( p_1 \) and \( p_2 \) are fixed by \( \sigma \), also \( R_1 \) and \( R_2 \) are fixed. Varying \( M \) through \( x \), one sees that the number of fixed reguli \( MM' \) is \( t \). But this number also equals \( t_1 t_2 \), where \( t_i \) is the number of lines through \( p_i \), different from \( L_1 \) and fixed by \( \sigma \), \( i = 1, 2 \). If moreover \( \sigma \) fixes a third point \( u \) on \( L_1 \), then the number \( t_u \) of fixed lines through \( u \) (different from \( L_1 \)) satisfies \( t_1 t_u = t_2 t_u = t_1 t_2 = t \) implying \( t_1 = t_2 = t_u = \sqrt{t} \).

2.3.2 A useful lemma

Let \( \theta \) be an automorphism in \( G \) fixing all points on \( L_1 \) and the lines \( L_2, L_4 \) and \( L \). Let \( r I L_1, p_2 \neq r \neq p_1, p_1 I M, M \neq L_1, r I N, N \neq L_1 \). Then the regulus \( MN \) is fixed by \( \theta \) if and only if \( M^\theta = M \) and \( N^\theta = N \). Let \( t_i + 1 \) be the number of fixed lines through \( p_i \), \( i = 1, 2 \), and let \( t' + 1 \) be the number of fixed lines through \( r \). If \( w \) is the number of fixed reguli consisting of \( s + 1 \) lines concurrent with \( L_1 \), then clearly \( w = t_1 t' \). Analogously, \( w = t_2 t' \) and \( w = t_1 t_2 \). Hence \( t_1 = t_2 = t' \). So \( \theta \) fixes a constant number \( t' + 1 \) of lines through every point of \( L_1 \). Let \( P' \) be the union of the points on all
these lines and let $B'$ be the set of lines meeting $P'$ in at least two points. Clearly, every element of $B'$ is incident with exactly $s + 1$ elements of $P'$ (by the regularity of $L_1$). By 2.3.1 of FGQ, $S' = (P', B', T')$, with $T'$ the restriction of $I$ to $P' \times B' \cup B' \times P'$, is a subquadrangle of order $(s, t')$. If $t' < t$, then by 5.3.5 of FGQ and the transitivity properties of $G$, the GQ $S$ is the classical GQ $Q(5, s)$. Now let $t = t'$. As $\theta \neq 1$, we have $p_3^2 \neq p_3$ by 2.4.1 of FGQ. For every point $x$ on $L_2$ with $p_2 \neq x \neq p_3$, the conjugate $\theta^\eta$ of $\theta$, where $\eta$ is a collineation fixing $L_1, L_2, L_4, p_3$ and mapping $p_3^k$ to $x$, fixes every line concurrent with $L_1$ and maps $p_3$ to $x$. Therefore $S$ is half Moufang and hence Moufang by [7]. Since all lines are regular, we have $S \cong Q(\Delta, f)$ or $S \cong Q(\triangledown, f)$ by 3.3.1 of FGQ.

Now we distinguish between several numerical cases.

### 2.3.3 $k$ and $s$ both odd

The subgroup $H$ of $G$ fixing $Q$ and $L$ has order $(s - 1)k$ and hence contains some involution $\sigma$. If $\sigma$ would fix a third point $u$ on $L_1$, then $\sigma$ would be inside $G_{\Pi}$ (taking without loss of generality $u = p$), contradicting the hypothesis $k$ odd. Hence $\sigma$ has no fixed points, different from $p_1$ and $p_2$, on $L_1$. From Section 2.3.1 follows immediately that $t_1t_2 = t$, where $t_i + 1$ is the number of fixed lines (for $\sigma$) through $p_i$, $i = 1, 2$.

Suppose $t_i > 1$ for $i = 1, 2$. Let $M_i \neq L_1$ be any line fixed by $\sigma$ and incident with $p_i$ and let $x_i$ be the point of $M_i$ collinear with $p_{i+2}$, $i = 1, 2$. Clearly $x_i$ is also fixed by $\sigma$ and considering the point $x'_2$ of $M_2$ collinear with $x_1$, one sees that $\sigma$ fixes a skeleton unless $x'_2 = x_2$. Since $k$ is odd, $\sigma$ cannot fix a skeleton and hence there arises a dual $(t_1 + 1) \times (t_2 + 1)$-grid having as points the points $p_1, p_2$, the $t_1$ points $x_1$ and the $t_2$ points $x_2$ (obtained by letting vary $M_1$ and $M_2$). So there are three points with at least $t_1 + 1$ centers and three points with at least $t_2 + 1$ centers. By the observation in 2.1.2, $t_i \leq t/s$ and hence $t = t_1t_2 \leq t^2/s^2$, implying $s^2 \leq t$, so $t = s^2$ (cf. 1.2.3 of FGQ) and $t_1 = t_2 = t/s = s$. So we have $|\{p_1, p_3, x_2\}| = 1 + s$, hence the left hand side is actually equal to $1 + s$. By the transitivity on triads in $x_1^+$, we have that $x_1$ is 3-regular. By transitivity every point of $S$ is 3-regular, and consequently, by 5.3.2(i) of FGQ, $S$ is the classical GQ $Q(5, s)$.

Next we consider the case $t_2 = 1$. Then of course $t_1 = t$ and so $\sigma$ fixes all lines through $p_1$. Consider two arbitrary points $y, z$ on $L_2$ with $z \neq p_2 \neq y$
and $y \neq p_3$. The subgroup $H^*$ of $G$ fixing $p_1, p_2, y, L_4$ and $L$ has also order $(s - 1)k$ and acts transitively on the points of $L_2$ different from $p_2$ and $y$. Let $\theta \in H^*$ map $y^\sigma$ (which is different from $y$) to $z$. Then the map $\theta^{-1}\sigma\theta$ maps $y$ to $z$ and fixes all lines through $p_1$. Varying the point $p_2$ on $L_1$ and varying afterwards the line $L_4$ through $p_1$, we see that there exists a group $K$ of automorphisms of $S$ fixing all lines through $p_1$ and acting transitively on $P \setminus \frac{1}{\infty}$. By 8.2.4(iv) of FGQ, there is a unique subgroup $E \trianglelefteq K$ of order $s^2t$ acting regularly on $P \setminus \frac{1}{\infty}$; this normal subgroup is the set of all automorphisms of $S$ fixing all lines through $p_1$ and having no fixed point in $P \setminus \frac{1}{\infty}$. Let $E_{L_2}$ be the subgroup of $E$, of order $s$, fixing $L_2$.

Now consider the subgroup $H_1$ of $G$ which fixes $L_4, p_1, L_1, p_2, L_2$ and $L$. This group acts transitively on the points of $L_1$ different from $p_1$ and $p_2$. Clearly $E_{L_2} \leq H_1$. If $\eta \in E_{L_2} \setminus \{1\}$ and $\delta \in H_1$, then $\delta^{-1}\eta\delta$ fixes $p_1$ linewise, fixes $L_2$, and has no fixed point collinear with $p_1$. Hence $\delta^{-1}\eta\delta \in E_{L_2}$, and so $E_{L_2} \trianglelefteq H_1$. Hence $E_{L_2}$ acts on the points of $L_1$ different from $p_1$ and $p_2$ in orbits of equal length $d$. So $d$ has to divide both $s - 1$ and $s$, hence $d = 1$. It follows that every element of $E_{L_2}$ fixes all points on $L_1$. By the regularity of $L_1$ it is immediately clear that every element of $E_{L_2}$ fixes each line of $\{L_4, L_2\}^1$. Now let $M$ be a line meeting $L_1$, with $p_1$ not incident with $M$, $p_2$ not incident with $M$, and $M \not\in \{L_4, L_2\}^{1,1}$. Let $N$ be the line through $p_1$ which belongs to $\{M, L_2\}^{1,1}$. Each element of $E_{L_2}$ fixes $\{N, L_2\}^{1,1}$ and the common point of $M$ and $L_1$. Hence each element of $E_{L_2}$ fixes $M$. It easily follows that each element of $E_{L_2}$ fixes each line concurrent with $L_1$. By transitivity of $G$ on the lines of $S$, $S$ is half Moufang and hence Moufang by [7]. So, by [3], $S$ is classical (cf. 9.1 of FGQ). Since all lines of $S$ are regular, we have $S \cong Q(\Delta, f)$ or $S \cong Q(\nabla, f)$.

This completes the case $k$ and $s$ both odd.

2.3.4 $k$ odd and $s$ even

We consider here the subgroup $H_1$ of $G$ fixing $L_4, p_1, L_1, p_2, L_2$ and $p$. This group has order $sk$ and hence it contains an involution $\sigma$. If $\sigma$ would fix a point not incident with $L_1$, then it would fix a skeleton and this contradicts the fact that $k$ is odd. So the only fixed points for $\sigma$ are on $L_1$. Let $x$ be any point on $L_1$ and let $y$ be any point collinear with $x$ but not on $L_1$. If
y^\sigma \sim y$, then clearly $x$ is fixed by $\sigma$. If $y \not\sim y^\sigma = y'$, then $\{y, y'\}^{++}$ contains a fixed point (indeed, it contains $t+1$ elements and since $s$ divides $t$, $t$ is even). Hence there is a point collinear with $y$ which is fixed by $\sigma$. This must clearly be $x$ and so we showed that $\sigma$ fixes every point on $L_1$. By Lemma 2.3.2, $\mathcal{S} \cong \mathcal{Q}(\triangle, f)$ or $\mathcal{S} \cong \mathcal{Q}(\nabla, f)$.

### 2.3.5 $k$ and $s$ both even

From these assumptions follows the existence of an involution $\sigma$ fixing the skeleton $\Omega$. This involution has as fixed elements the points and lines of a thick proper subquadrangle $\mathcal{S}'$ of order $(s', t')$. If $s = s'$, then by the transitivity properties of $G$, it follows from 5.3.5 of FGQ that $\mathcal{S}$ is the classical GQ $Q(5, s)$. Hence suppose $s \neq s'$. By Section 2.3.1, $t' = \sqrt{t}$.

Let $R$ be a line of $\mathcal{S}'$ and let $z$ be a point of $R$ not in $\mathcal{S}'$. Then any line through $z$, different from $R$, has no point in common with $\mathcal{S}'$; so there exists a line $M$ external to $\mathcal{S}'$. The line $M' = M^\sigma$ does not meet $M$ and the $s+1$ lines of $\{M, M'\}^{++}$ are permuted by $\sigma$. Since $s+1$ is odd, $\{M, M'\}^{++}$ contains at least one fixed line $N$, which contains in turn exactly $s'+1$ fixed points. A line containing any of these $s'+1$ points and concurrent with both $M$ and $M'$ is fixed by $\sigma$. Hence there are exactly $s'+1$ lines of $\mathcal{S}'$ concurrent with a given external line. By Proposition 2.2.1 of FGQ, we have $1 + s' = 1 + t'$. But again by 2.2.1 of FGQ, we also have $s \geq s't' = t$. Hence $s = t$ and $\mathcal{S}$ is the classical GQ $Q(4, q)$ (by the regularity of every line).

### 2.3.6 $k$ even and $s$ odd

We subdivide this case in two subcases: $t$ odd and $t$ even.

**Subcase $t$ even.** Here we consider the subgroup $H_2$ of $G$ fixing $L_4, p_1, L_1, p_2, L$ and $p$. This group has order $stk$ and hence contains a non-trivial Sylow 2-subgroup $S$. Suppose $2^n$ respectively $2^m$ divides $t$ respectively $k$ and $2^{n+1}$ respectively $2^{m+1}$ does not. Then $|S| = 2^{n+m}$. Let $\sigma$ be an involution in the center of $S$ and suppose that $\sigma$ fixes a second line through $p_2$, say $L_2$. Since $s$ is odd $\sigma$ fixes a second point on $L_4$ respectively $L_2$. If $\sigma$ would fix all points on $L_1$, then its fixed elements would be the points and lines of a thick proper subquadrangle of order $(s, t')$. Then, by the transitivity properties of $G$, it
again follows from 5.3.5 of FGQ, that $S \cong Q(\vartheta, f)$. Hence we assume now that $\sigma$ does not fix all points of $L_1$. By Section 2.3.1, $\sigma$ fixes $\sqrt{t}$ lines different from $L_1$ through $p_2$. The group $S$ acts on these lines and the stabilizer $S_M$ in $S$ of any of these lines $M$ has at most order $2^m$, since $S_M$ is a subgroup of a conjugate of the group $G_{1\ell}$ of order $k$. So the length of the orbit of $M$ under $S$ has at least order $|S|/2^m = 2^n$. Since $M$ was arbitrary, this means that $\sqrt{t}$ must be divisible by $2^n$, hence $2^{2n}$ divides $t$, a contradiction. Hence $\sigma$ does not fix any line through $p_2$ besides $L_1$. But then, by 2.3.1, $\sigma$ fixes every point on $L_1$. On $L_4$ there is a second fixed point $p_4$. If $\sigma$ does not fix all points of $L_4$, then, by 2.3.1, $p_4$ is on at least two fixed lines, a contradiction as every fixed line contains $p_1$. Hence $\sigma$ fixes every point of $L_4$, and similarly it fixes every point of $L$. Now consider the group $T$ generated by all conjugates of $\sigma$ by elements of the subgroup $H_3$ of $G$ fixing $p_1, L_1, p_2$. Every element of $T$ fixes each point of $L_1$. Let $M$ be a line different from $L_1$ containing $p_1$, and let $M', M''$ be distinct lines different from $L_1$ containing $p_2$. Then there is a $\delta \in H_3$ such that $L^\delta_1 = M$, $N^\delta = M''$, $N'^\circ \delta = M'$, where $N$ is any line through $p_2$ different from $L_1$. Then $\delta^{-1}\sigma\delta \in T$ fixes each point of $M$ and maps $M'$ onto $M''$. Now let $x$ be a point collinear with $p_2$ and not incident with $L_1$. Denote by $O_x$ the orbit of $x$ under $T$. Clearly, we have $|O_x| \geq t$. If $|O_x| = t$, then by the transitivity property just mentioned, all points of $O_x$ are collinear to one single point of $M$, $M \neq L_1$ and $M$ through $p_1$. But $M$ is arbitrary, so we obtain a dual $(t + 1) \times (t + 1)$-grid. Consequently we have a regular pair of points and by 1.3.6(i) of FGQ, $s \geq t$. But all lines are regular, so $t \geq s$ implying $s = t$, which means that $S$ is the classical GQ $Q(4, s)$. Hence we may assume $|O_x| > t$. This means that there exists a collineation $\gamma$ in $T$ preserving $L_2$ and not inducing the identity map on the set of points of $L_2$. By interchanging roles of $p_1$ and $p_2$, we see that there exists an element $\xi \in G$ fixing all points of $L_1$, all points of $L_2$ and mapping $L^\xi_1$ onto $L_4$. Then $\gamma \xi = \xi$ fixes $L_2$, $L_4$, all points of $L_1$, but not every point of $L_2$. Let $x^\xi \neq x$, $x L_2$. Choose on $L_2$ distinct points $y, z$, not on $L_1$. By the transitivity properties of $G$, there is an element $\delta \in H_3$ which fixes $L_4$, maps $x$ onto $y$, and $x^\delta$ onto $z$. Then $\delta^{-1} \gamma \xi \delta$ fixes all points of $L_1$, fixes $L_2$ and $L_2$, and maps $y$ onto $z$. Consequently the group $T_0$ generated by all conjugates of $\xi$ by elements of $H_3$ fixes all points of $L_1$, fixes $L_4$ and $L_2$, and acts transitively on the $s$ points of $L_2$ not on $L_1$. Hence $s$ divides $|T_0|$. Assume that no element of $T_0 \setminus \{1\}$ fixes a third line through $p_1$. Then $T_0$ acts faithfully and semi-regularly on the lines through $p_1$, different from $L_1$ and $L_4$. Hence $|T_0|$ divides $t - 1$, so $s$ divides $t - 1$. As $s$ divides $t$, we have a contradiction.
Hence there is an element $\theta \in T_0$ fixing all points of $L_1$, fixing $L_2$ and $L_4$, and fixing a third line, say $L$, containing $p_1$. By Lemma 2.3.2, $S \cong Q(\Delta, f)$ or $S \cong Q(\triangledown, f)$, so $s$ and $t$ have the same parity, a contradiction.

**Subcase** $t$ odd. Here we consider the subgroup $H_4$ of $G$ fixing $Q$ and $p$, which has order $(t - 1)k$. We consider a Sylow 2-subgroup $S$ in $H_4$. Suppose $2^n$ respectively $2^m$ divides $t - 1$ respectively $k$ and $2^{n+1}$ respectively $2^{m+1}$ does not. Then $|S| = 2^{n+m}$. Let $\sigma$ be an involution in the center of $S$. If $\sigma$ does not fix every point on $L_1$, then it fixes exactly $\sqrt{t} - 1$ lines different from $L_1$ and $L_4$ through $p_1$. Let $M$ be one of these lines. The group $S$ acts on these $\sqrt{t} - 1$ fixed lines and similarly as in the previous case, we obtain that the size of the orbit of $M$ is divisible by $2^n$. Hence $2^n$ divides $\sqrt{t} - 1$, so $2^{n+1}$ divides $(\sqrt{t} - 1)(\sqrt{t} + 1) = t - 1$, a contradiction. Hence $\sigma$ fixes all points on $L_1$. Assume there is no element of $G$ fixing all points of $L_1$, the lines $L_2$, $L_4$, and at least three lines containing $p_1$ or $p_2$. Let $T'$ be the group generated by the conjugates of $\sigma$ by the elements of the subgroup $H_5$ of $G$ fixing $L_4$, $p_1$, $L_1$, $p_2$ and $L_2$. Then the elements of $T'$ fix all points of $L_1$, fix $L_2$ and $L_4$. Suppose first that $\sigma$ does not fix all points of $L_2$. Then $T'$ acts transitively on the set $A$ of points of $L_2$ different from $p_2$. By an argument similar to the one in the case $t$ even, we obtain a collineation of $S$ fixing all points of $L_1$, the line $L_2$ and at least three lines containing $p_1$. Hence $\sigma$ fixes all points of $L_2$. Then the fixed elements of $\sigma$ are all elements of the $(s + 1) \times (s + 1)$-grid containing the lines $L_2$ and $L_4$. Let $N$ be a line through $p_2$ different from $L_1$ and $L_2$. Then $N^\sigma \neq N$. Let $H_5$ be the subgroup of $G$ fixing $p_1$, $L_1$, $p_2$, let $M$ be a line different from $L_1$ containing $p_1$, and let $M'$, $\overline{M}$ be distinct lines different from $L_1$ containing $p_2$. Then there is a $\delta \in H_3$ such that $L_4^\delta = M$, $N^\delta = \overline{M}$, $N^\sigma \delta = M'$. So $\delta^{-1} \sigma \delta$ fixes each point of $L_1$, fixes each point of $M$ and maps $M'$ onto $\overline{M}$. Let $T''$ be the subgroup of $G$ generated by the conjugates of $\sigma$ by the elements of the group $H_3$. Then the elements of $T''$ fix all points of $L_1$. Using an argument analogous to the one in the case $t$ even (replacing $T$ by $T''$), we find that necessarily $S \cong Q(\Delta, f)$. So we may assume that $S$ admits a collineation fixing all points of $L_1$, the lines $L_2$, $L_4$, and at least three lines containing $p_1$ or $p_2$. Then by Lemma 2.3.2, the GQ $S$ is isomorphic to either $Q(4, s)$ or $Q(5, s)$.
2.4 The possible groups

Let $\mathcal{S}$ be one of the generalized quadrangles $W(q), Q(4, q), Q(5, q)$ or $H(3, q^2)$ and let $G$ be a group of automorphisms acting transitively on the set of ordered pentagons. By duality, we may restrict ourselves to $Q(4, q)$ and $H(3, q^2)$. First let $\mathcal{S} \neq Q(\Delta, \varepsilon)$. By a theorem of Seitz [6], $G$ has to contain the simple group $O_5(q)$ ($q \neq 2$), respectively $U_3(q)$. If $q$ is even, then $O_5(q)$ respectively $U_3(q)$ coincides with $PGO_5(q)$ respectively $PGU(4, q)$ and each of these groups contains all generalized homologies of the corresponding GQ; hence these groups have the desired transitivity properties. If $\mathcal{S} = Q(\Delta, \varepsilon)$, then $O_5(2)$ is the full automorphism group of $\mathcal{S}$ and $|O_5(2)|$ is the number of ordered pentagons in $\mathcal{S}$. As $O_5(2)$ acts semiregularly on the set of ordered pentagons, and consequently also regularly, it follows that $G$ must coincide with $O_5(2)$ in this case. Suppose now that $q$ is odd.

First let $\mathcal{S} \cong Q(\Delta, \Pi)$. The full group of automorphisms of $Q(4, q)$ is $PGO_5(q).h = PTO_5(q)$, where $q = p^h$ with $p$ a prime number. If $h$ is odd, then the group $O_5(q).h$ does not act transitively on the ordered pentagons since its order is not divisible by the number of ordered pentagons. This number of ordered pentagons equals $2|O_5(q)|$, so $O_5(q)$ must have even index in $G$. Since the outer automorphism group of $O_5(q)$ has a unique involution (this is the diagonal automorphism, see [2]), every extension of $O_5(q)$ in $PTO_5(q)$ in which $O_5(q)$ has even index must contain $PGO_5(q)$. Suppose now that $h$ is even. Consider a skeleton $(Q; L, x)$ in $\mathcal{S}$. The stabilizer in $O_5(q)$ of $Q$ acts transitively on the points not in $Q$ and incident with the line of $Q$ containing $x$; but the stabilizer in $O_5(q)$ of $Q$ and $x$ acts on the set $V$ of lines not in $Q$ but incident with the point of $Q$ on $L$ as the permutation group on $GF(q) \setminus \{0\}$ consisting of the elements $\gamma_a : x \mapsto a^2x$, $a, x \in GF(q) \setminus \{0\}$. If $G$ does not contain $PGO_5(q)$, then it contains an element involving a field automorphism $\theta$, $\theta \neq 1$. So $G$ contains the group $PGO_5^\theta(q)$ generated by $O_5(q)$ and $g \theta g$, where $g \in PGO_5(q)$ and where $g \theta$ is the element of $PTL_5(q)$ corresponding to the semilinear transformation with identity matrix and field automorphism $\theta$. Clearly $PGO_5^\theta(q)$ contains an element that fixes $Q$ and $x$, and acts on $V$ as the permutation $\gamma : x \mapsto bx^\theta$, with $b$ a non-square in $GF(q)$, that is, we may assume that $g \in PGO_5(q) \setminus O_5(q)$. Also, $\theta$ has even order $n$ (otherwise $(g \theta g)^n \in PGO_5(q) \setminus O_5(q)$ and then $G$ contains $PGO_5(q)$). Conversely, if $g \theta$ is an element of $PGO_5(q)$ acts transitively on the set of skeletons of $\mathcal{S}$. Let $n = 1.2^l$ with $l$ odd and put $\theta' = \theta^l$. Then $\theta'$ has
order $2^e$, $PGO^e_5(q) \leq PGO^e_5(q)$, and $PGO^e_5(q)$ also acts transitively on the set of skeletons, and hence on the set of ordered pentagons. This shows (i) and (ii) of the main result.

Now suppose $S \cong H_3(\Pi^e)$, $q$ odd. First suppose that $q \equiv 3 \mod 4$. Then the Sylow 2-subgroup $P$ of the outer automorphism group of $U_4(q)$ is a semi-direct product of a cyclic group $C_4$ (of diagonal automorphisms) with a group of order 2 (the unique involutory field automorphism). Now the number of ordered pentagons equals $4|U_4(q)|$ and so $G$ must contain $U_4(q)$ as a subgroup of index divisible by 4. Hence $G/U_4(q)$ contains a group isomorphic to a subgroup of $P$ of order at least 4. But every such subgroup clearly contains the unique involutory diagonal automorphism (the unique involution of $C_4$). So $G$ contains the group $H$ (with the notation of the introduction). This is trivially true if $q \equiv 1 \mod 4$. But now, a similar reasoning as in the case $S \cong Q(\triangle, II)$, $q$ odd and $h$ even, of the preceding paragraph shows that only cases (iii) and (iv) of the main result are possible.

This completes the proof of our main result.

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References


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