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Arcs Fixed by a Large Cyclic Group (**).

Abstract. – *The classification of all complete $(k+1)$-arcs K in* PG$(2,q)$, *fixed by a cyclic projective group of order k stabilizing one point r of K and acting regularly on $K\backslash\{r\}$, is presented. It is shown that no other examples, than the known ones, exist.*

1. – Introduction [3].

A k-arc in PG$(2,q)$ is a set of k points, no 3 of which are collinear. A point p of PG$(2,q)$ extends a k-arc K to a $(k+1)$-arc if and only if $K \cup \{p\}$ is a $(k+1)$-arc. A k-arc K is complete if and only if it is not contained in a $(k+1)$-arc. Otherwise, K is called incomplete.

The point set of a conic C is a $(q+1)$-arc of PG$(2,q)$. If q is odd, then C is complete. When q is even, then C can be extended in a unique way to a $(q+2)$-arc by its nucleus.

In order to find new k-arcs, various methods have been used. Many geometers were led by the following idea by Lombardo-Radice [7] and Segre [8]:

«*Many points of the arc are chosen, with some exceptions, among the points of a conic or a cubic curve*».

This led to the discovery of various examples of arcs which have a

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lot of points in common with a conic or cubic curve. For instance, the complete $(1/2)(q+3)$-arc in PG(2, q), $q \equiv 1 \pmod 4$, $q \geqslant 9$, by Fisher, described in Section 2.3, is of this type. For a survey on the known k-arcs, we would like to refer to the bibliographies of [3-5] which contain a large number of papers in which arcs of PG(2, q) are constructed.

We used a different approach. We started with a group G and classified all complete k-arcs fixed by G. This approach was applied in [9] where all types of complete k-arcs of PG(2, q), fixed by a cyclic projective group of order k, were determined. This led to the discovery of k-arcs fixed by the semi-dihedral group of order $2k$ and containing $k/2$ points of 2 disjoint conics, which only have 2 conjugate points in PG(2, q^2)\PG(2, q) in common.

A variation to the approach, described in [9], is now used to classify all complete $(k+1)$-arcs K of PG(2, q) stabilized by a cyclic projective group of order k, fixing exactly one point of the arc and acting regularly on the remaining points. Three distinct infinite classes of such $(k+1)$-arcs are known (Section 2). It is proved that no other infinite classes exist.

2. – Known examples.

This paragraph lists the known complete $(k+1)$-arcs K of PG(2, q) stabilized by a cyclic projective group G of order k which fixes one point of K and which acts regularly on the remaining k points.

2.1. *A conic in* PG(2, q), *q an odd prime.*

The conic C: $X_0^2 = X_1X_2$ in PG(2, q), q odd, $q > 3$, is a complete $(q+$ $+1)$-arc, fixed by a sharply 3-transitive projective group, isomorphic to PGL(2, q) [4, p. 233].

The stabilizer group of a point of C is a group of order $q(q-1)$ and has an elementary abelian subgroup G acting regularly on the q remaining points of C. This group G is cyclic if and only if q is prime and so in this case, C is a complete $(q+1)$-arc fixed by the cyclic group G.

A conic in PG(2, 3) is a 4-arc. This arc is fixed by the symmetric group S_4.

2.2. *A regular hyperoval in* PG(2, q), *q even* [3].

A regular hyperoval K in PG(2, q), q even, is the union of a conic C and its nucleus.

Consider the subgroup G of PGL$(2, q)$, acting on C, which fixes two conjugate points r_1, r_2 of C in a quadratic extension of PG$(2, q)$. This group G, of order $q+1$, is cyclic and acts regularly on the points of C.

So, K is a complete $(q+2)$-arc fixed by the cyclic group G of order $q+1$.

2.3. *Arcs of size* $(q+3)/2$ [2].

Consider a conic C in PG$(2, q)$, q odd, and fix an internal point r of C. The subgroup H of PGL$(2, q)$, acting on C, which fixes C and r, is a dihedral group of order $2(q+1)$ and fixes also the polar line L of r with respect to C.

The cyclic subgroup G, of order $(q+1)/2$, of H, fixes the 2 points of $C \cap L$ in PG$(2, q^2)$ and partitions C into 2 orbits of size $(q+1)/2$. Let O be one of these orbits. Then, if $q \geqslant 9$ and $q \equiv 1 \pmod 4$, $O \cup \{r\}$ is a complete $(1/2)(q+3)$-arc fixed by the cyclic group G which acts regularly on O.

3. – Introductory result.

In the following sections, K always denotes a complete $(k+1)$-arc of PG$(2, q)$, fixed by the cyclic projective group $G = \langle \alpha \rangle$ of order k which fixes a unique point r of K and which acts regularly on $K \backslash \{r\}$.

PROPOSITION 1. *The group G fixes exactly one point and one line L of* PG$(2, q)$.

PROOF. The group G fixes r. Suppose it fixes a second point s, then s does not extend K, so s must belong to a bisecant of K. From the transitive action of G on $K \backslash \{r\}$, s belongs to $k/2$ bisecants of $K \backslash \{r\}$ and so k is even.

Suppose that G only fixes the 2 points r and s, then G fixes exactly 2 lines $L = rs$ and M [6, p. 256]. Moreover $r \in M$ or $s \in M$, otherwise G fixes a third point on L. If q is even, since G does not fix a second point on M, the action of α on M is a mapping of type $t \mapsto t + b$, $b \in \mathrm{GF}(q)$, so α^2 fixes M point by point and hence α^2 is a perspectivity. Since only an involutory perspectivity can fix a complete arc, $\alpha^4 = \mathbf{1}$. This implies that K is a complete 5-arc but this is false since no such arc exists [3, Chap. 14].

Assume now q odd. Since k is even and $|\mathrm{PG}(2, q) \backslash \{r, s\}|$ is odd, there is an orbit O with $1 < |O| < k$. So $\alpha^{|O|}$ fixes at least 4 points. But

$\alpha^{|O|}$ cannot fix a quadrangle, so $\alpha^{|O|}$ must be a (involutory) perspectivity. Hence $2|O| = k$, $|O| = k/2$ and O is contained in the axis of $\alpha^{k/2}$. Since G has a unique involutory perspectivity $\alpha^{k/2}$, G fixes the center and axis of $\alpha^{k/2}$. So this axis is L or M. If $O \subset L$, since q is odd, the center l of $\alpha^{k/2}$ does not belong to L. Then G fixes the point l not on L, which is false. Therefore $O \subset M$. Since we assume that G only stabilizes, on M, the intersection point with L, the action of α on M is of type $t \mapsto t + b$ which has order $k/2 = p$, $q = p^h$, p prime. So $k = 2p$ and $q \geqslant p^2$. Since G only fixes 2 points on L, $k|(q-1)$ since all orbits of G on $L\backslash\{r, s\}$ must have size k. So $2p|(q-1)$ which is false.

So G fixes a third point t of PG$(2, q)$ and as for s, also t belongs to $k/2$ bisecants of $K\backslash\{r\}$.

Consider the action of G on the lines through s. Let β be the unique involution of G, then β fixes all $k/2$ bisecants through s. If $k/2 > 2$, then this involution is a perspectivity with center s [1, p. 172]. But analogously, t must be the center of β. This is impossible. If $k/2 \in \{1, 2\}$, then K is a complete 3- or 5-arc, but no complete 3- or 5-arc exists [3, Chap. 14].

So G does not fix a second point of PG$(2, q)$. Then G stabilizes exactly one line L of PG$(2, q)$ [6, p. 256]. ∎

4. – q odd.

In this section, $q \geqslant 5$.

REMARK 1. It will first be assumed that $r \notin L$ where L is the unique line stabilized by G. Choose the reference system such that $r = (1, 0, 0)$ and L: $X_0 = 0$.

Since G does not fix a point of L, it fixes 2 conjugate points r_1, r_2 of L in PG$(2, q^2)\backslash$PG$(2, q)$. From the calculations in [9, Remark 4.2], it can be assumed that G is generated by

$$\alpha\colon \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & d_1 b & a \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

with d_1 non-square in GF(q). Then α fixes the 2 conjugate points $r_1 = (0, 1, i)$, $r_2 = (0, 1, -i)$, $i^2 = d_1$, of L in PG$(2, q^2)\backslash$PG$(2, q)$.

The conics C which contain r_1, r_2 and for which r and L are a polar point-line pair with respect to the polarity defining C, have equation C: $X_0^2 - cd_1X_1^2 + cX_2^2 = 0$, $c \neq 0$, and they are fixed by α if and only if $\det A = 1$ [9, Remark 4.2].

LEMMA 1. $k \mid (q+1)$ and $\det A = 1$.

PROOF. Since r extends $K \setminus \{r\}$ and since $r \notin L$, the cyclic group G has a faithful representation G_1 acting on the line L. This group G_1 is a subgroup of order k of the cyclic group of order $q+1$ of PGL$(2,q)$, acting on L, which fixes r_1 and r_2, and which acts regularly on the $q+1$ points of L. So $k \mid (q+1)$.

From $\alpha^k = \mathbf{1}$, $(\det A)^k = 1$, so $(\det A)^{q+1} = 1 = (\det A)^2$. Equivalently, $\det A = \pm 1$.

If $\det A = -1$, since $(\det A)^k = 1$, the order k is even. Let

$$\alpha^{k/2}: \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & e \\ 0 & d_1 e & f \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = A' \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

Then $A'^2 = I_3$, the identity matrix, implies

$$f^2 + e^2 d_1 = 1 \quad \text{and} \quad 2fe = 0,$$

so $e = 0$ and $f = -1$.

This would mean that $\alpha^{k/2}$ is the perspectivity with center r and axis L and would imply that the arc consists of r and k points on $k/2$ lines through r, which is false.

Therefore, $\det A = 1$. ■

PROPOSITION 2. *The set* $K \setminus \{r\}$ *consists of* $(q+1)/2$ *points of a conic* C. *The point* r *is internal to* C *and* r *and* L *are a polar point-line pair with respect to* C.

Furthermore, $q \equiv 1 \pmod 4$ *and* K *is complete if* $q \geqslant 9$.

PROOF. From Lemma 1, $\det A = 1$, so α fixes all conics C: $X_0^2 - cd_1X_1^2 + cX_2^2 = 0$, $c \neq 0$, and r and L are a polar point-line pair with respect to C (Remark 1).

The arc $K \setminus \{r\}$ therefore consists of k points of one of those conics and the group G is a subgroup of the cyclic group H of order $q+1$, fixing C and fixing the 2 conjugate points r_1, r_2 of C on the external line L (Remark 1).

If $|G| = k < (q+1)/2$, there will be a point on C, extending the arc K to a larger arc. Therefore $k = (q+1)/2$ or $k = q+1$. The last possibility cannot occur since r does not extend the conic C to a $(q+2)$-arc. This shows that $k=(q+1)/2$.

If $q \equiv -1 \pmod 4$, then k is even and so $\alpha^{k/2}$ exists. As in the proof of Lemma 1, $\alpha^{k/2}$ would be the involutory perspectivity with center r and axis L. This is false, hence $q \equiv 1 \pmod 4$. The arc K is of the type described in 2.3. This ends the proof. ∎

REMARK 2. The remaining part of this section treats the case $r \in L$. Let $r=(1,0,0)$, L: $X_2=0$ and let G be generated by

$$\alpha\colon \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & c \\ 0 & b & d \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

Since r is the only fixed point on L, $b=1$.

LEMMA 2. *The group G acts faithfully on $L \backslash \{r\}$, so $k \mid q$.*

PROOF. Suppose $L \backslash \{r\}$ has, under G, an orbit of size $t<k$. Then $t \geqslant 2$ since G only fixes r (Proposition 1), and so α^t fixes L point by point which means that α^t is a perspectivity with axis L. Only an involutory perspectivity can fix an arc, so $\alpha^{2t}=1$, $2t=k$ and k is even.

As in the proof of Proposition 1, G must fix the center and axis of the involutory perspectivity $\alpha^{k/2}$. Since G only fixes r and L, r is the center and L is the axis of $\alpha^{k/2}$ and since q is odd, $r \notin L$. This contradicts the assumption $r \in L$.

Therefore, G acts faithfully on $L \backslash \{r\}$ and $k \mid q$. ∎

LEMMA 3. *$k=p$ prime.*

PROOF. From $k \mid q$, if $q=p^h$, p prime, $p>2$, then $k=p^s$. So $\alpha^k=1$ with

$$A^k = \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & d' \\ 0 & 0 & e^k \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which gives $e=1$.

The action on L is defined by

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

and this is cyclic if and only if $k=p$. ∎

PROPOSITION 3. *The arc K is a conic in* $\mathrm{PG}(2,q)$, *q prime.*

PROOF. From Lemma 3, G is generated by

$$\alpha\colon \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

Using induction on n,

$$A^n = \begin{pmatrix} 1 & na & nc + n(n-1)\,ad/2 \\ 0 & 1 & nd \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose $(0,0,1) \in K$; the orbit of $(0,0,1)$ consists of the points $(nc + n(n-1)\,ad/2,\ nd,\ 1)$. If $d = 0$, then all images of $(0,0,1)$ belong to the line $X_1 = 0$. Hence K is not an arc, so $d \neq 0$.

A conic C through $(0,0,1)$ and tangent to L: $X_2 = 0$ at $(1,0,0)$ has equation C: $X_0X_2 + fX_1^2 + gX_1X_2 = 0$. This conic contains all the images of $(0,0,1)$ under G if and only if

$$c - \frac{cd}{2} + gd = 0 \quad \text{and} \quad \frac{ad}{2} + fd^2 = 0\,.$$

Since $d \neq 0$, f and g are uniquely defined. The orbit of $(0,0,1)$ is a subset of a conic C tangent to L in $(1,0,0)$. This is a complete arc if and only if $K = C$. So $|K| = p + 1 = |C| = q + 1$ which shows that q is prime. ∎

5. – q even.

Let $q \geqslant 4$.

LEMMA 4. *The point r fixed by G does not belong to the line L fixed by G.*

PROOF. Suppose $r \in L$. Let $r = (1,0,0)$, L: $X_2 = 0$. The group G must act faithfully on L. For, proceeding as in the proof of Lemma 2, if there is on L an orbit of size $t < k$, then $t = k/2$ and $\alpha^{k/2}$ is an involutory perspectivity with center r and axis L. This is impossible, otherwise r is incident with $k/2$ bisecants of $K \backslash \{r\}$.

Using the calculations of the proof of Lemma 3, it is proved that $k = p = 2$. So K is a 3-arc which is false. ∎

REMARK 3. As in Remark 1, let $r = (1, 0, 0)$, L: $X_0 = 0$. Since α does not fix a point of L in PG$(2, q)$, it fixes 2 conjugate points r_1, r_2 of L in PG$(2, q^2)\setminus$PG$(2, q)$. Assume that α fixes the points $r_1 = (0, 1, a)$, $r_2 = (0, 1, a+1)$ with $a^2 + a + d_1 = 0$ and $\mathrm{Tr}(d_1) = 1$.

Then

$$\alpha: \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & cd_1 & b+c \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

The conics with nucleus $(1, 0, 0)$, containing r_1 and r_2, are C: $X_0^2 + fd_1X_1^2 + fX_2^2 + X_1X_2 = 0$ with $f \neq 0$. They are fixed by α if and only if $\det A = 1$.

PROPOSITION 4. *The arc K is a regular hyperoval of* PG$(2, q)$. *The fixed point of K, under G, is the nucleus of the conic C contained in K.*

PROOF. Proceeding as in the proof of Lemma 1, $\det A = \pm 1$ and $k \mid (q+1)$. Since q is even, $\det A = 1$. It follows from Remark 3 that $K \setminus \{r\} \subseteq C$, where the conic C is described in Remark 3. Since K is complete, $K \setminus \{r\} = C$.

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