A NEW LOOK AT THE CLASSICAL GENERALIZED QUADRANGLES

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Both authors are supported by the National Fund of Scientific Research of Belgium (NFWO).

1. INTRODUCTION

In this paper we calculate explicitly the coordinatizing algebraic structures of the classical generalized quadrangles and give some applications. Although we restrict to finite structures, similar results for the infinite case can be derived.

A (finite) generalized quadrangle (GQ) is an incidence structure $S = (\mathcal{P},\mathcal{B},I)$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (nonempty) sets of objects called points and lines (respectively), and for which I is a symmetric point-line incidence relation satisfying the following axioms :

(i) each point is incident with 1+t lines ($t \geq 1$) and two distinct points are incident with at most one line ;

(ii) each line is incident with 1+s points ($s \geq 1$) and two distinct lines are incident with at most one point ;

(iii) if x is a point and L is a line not incident with x, then there is a unique pair $(y,M) \in \mathcal{P} \times \mathcal{B}$ for wich x I M I y I L.

Generalized quadrangles were introduced by J. Tits. [7]

The integers s and t are the parameters of the GQ and S is said to have order (s,t); if s = t, S is said to have order s.

Given two points p and q of S, we write $p \perp q$ and say that p and q are collinear provided that there is some L incident with both. If this is not the case, we write $p \not\perp q$.

For $p \in \mathcal{P}$, put $p^{\perp} = \{q \in \mathcal{P} \mid q \perp p\}$ and note that $p \in p^{\perp}$. Is $A \subset \mathcal{P}$, we write $A^{\perp} = \bigcap p^{\perp}, p \in A$. For distinct points p and q, $\{p,q\}^{\perp}$ is called the trace of p and q and $\{p,q\}^{\perp\perp}$ the span.

For details we refer to [5].

We recall briefly how coordinates are introduced in [2], for a finite GQ of order $(s,t), s,t > 1$.

We use a set R_1 (resp. R_2) of cardinality s (resp t) not containing the symbol ∞, but containing two distinguished elements 0 and 1.

We choose a point (∞) (resp line $[\infty]$) of S, $(\infty) I [\infty]$, and take $(a), a \in R_1$ (resp. $[k]$, $k \in R_2$) as remaining points on $[\infty]$ distinct from (∞) (resp. remaining lines on (∞) distinct from $[\infty]$).

Now we complete the elements $(\infty), [\infty], (0), [0]$ to a non-degenerate quadrangle $(\infty), [\infty], (0), (0)A, A, AB, B, [0]$. Like before we choose a bijection between R_1 and the points of the line (0)A with the only restriction that A corresponds to 0. The point of (0)A corresponding to $a' \in R_1$ will have coordinate $(0,0,a') \in R_1 \times R_2 \times R_1$ Dually, we give coordinates $[0,0,k'] \in R_2 \times R_1 \times R_2$ to the lines on B different from $[0]$, with the restriction that BA has coordinate $[0,0,0]$.

We define next the points with two coordinates : a point P collinear with (∞), but not lying on $[\infty]$ has coordinate $(k,a) \in R_2 \times R_1$ if and only if P lies on $[k]$ and is collinear with $(0,0,a)$. Dually lines meeting $[\infty]$ not passing through (∞) are given coordinates $[a,k] \in R_1 \times R_2$.

Finally, consider a point P not collinear with (∞). Because S is a GQ, there is exactly one line on P meeting $[\infty]$. This line must have two coordinates, say $[a,l]$. On the other hand, P is collinear with exactly one point $(0,a')$ on $[0]$. Now P is given the coordinate (a,l,a').

Conversely, let (a,l,a') be any element of $R_1 \times R_2 \times R_1$, then we construct a point P having this element as coordinate. Indeed, given the line $[a,l]$ and the point $(0,a')$ not incident with it, then there is exactly one point collinear with $(0,a')$ and lying on $[a,l]$ for S is a GQ. The coordinate of a line $[k,b,k']$ is defined dually.

We define two quaternary operations Q_1 and Q_2 as follows : if $a,a',b \in R_1$ and $k,k',l \in R_2$,

$$Q_1(k,a,l,a') = b$$

$$Q_2(a,k,b,k') = l$$

if and only if (a,l,a') is on $[k,b,k']$.

If the GQ S is coordinatized in this way we call (R_1,R_2,Q_1,Q_2) a quadratic quaternary ring.

We mention here that this coordinatization of the classical examples for s = t is equivalent to that given by S.E. Payne [4] . A detailed comparison will be worked out in a forthcoming paper.

2. THE SYMPLECTIC GENERALIZED QUADRANGLE W(q)

2.1. Definition

The points of PG(3,q), together with the totally isotropic lines with respect to a symplectic polarity, form the GQ W(q) with parameters s = t = q.

2.2. Coordinatization

We can choose our simplex in PG(3,q) such that the symplectic polarity has the following bilinear form :

$$x_0y_1-x_1y_0+x_2y_3-x_3y_2 = 0$$

Choose $(\infty) = (1,0,0,0)$,

$(0) = (0,0,1,0)$.

Let $R_1 = R_2 = GF(q)$ then we can take

$(a) = (a,0,1,0)$,

$[k]$ the line on $(1,0,0,0)$ and $(0,0,k,1)$.

Finally, we choose

$(0,0,a') = (0,1,a',0)$,

$[0,0,k']$ the line on $(0,0,0,1)$ and $(k',1,0,0)$.

We compute the points and lines with two coordinates :

$(k,a) = (a,0,k,1)$,

$[a,k]$ the line on $(a,0,1,0)$ and $(k,1,0,-a)$,

and next those with three coordinates :

$(a,l,a') = (l+aa',1,a',-a)$,

$[k,b,k']$ the line on $(b,0,k,1)$ and $(k',1,b,0)$.

2.3. Quadratic quaternary ring.

We will find Q_1 and Q_2 by expressing that for some $\lambda \in GF(q)$ there holds :

$$(l+aa',1,a',-a) = \lambda(b,0,k,1)+(k',1,b,0)$$

and solving for b and l. We obtain $\lambda = -a$ and

$$l+aa' = -ab+k'$$
$$a' = -ak+b,$$

hence $Q_1(k,a,l,a') = b = ka+a'$,

$Q_2(a,k,b,k') = l = a^2k+k'-2ab.$

3. THE QUADRICS Q(4,q) and Q(5,q).

3.1. Definition

Let Q(4,q) resp. Q(5,q) be a non singular quadric of projective index 1 of the projective space PG(4,q) resp. PG(5,q). Then the points of Q together with the lines on Q (subspaces of maximal dimension on Q) form a GQ with parameters (s,t) = (q,q) resp. (q,q^2).

3.2. Coordinatization of Q(5,q)

Recall that the quadric Q(5,q) has the following canonical equation :

$$-f(x_0,x_1)+x_2x_3+x_4x_5 = 0,$$

where f is an irreducible binary quadratic form, say $x_0^2+px_0x_1+rx_1^2$.

For $\bar{x} = (x_0,x_1) \in (GF(q))^2$, write $f(\bar{x})$ in place of $f(x_0,x_1)$, and $\bar{x}.\bar{y}$ for the bilinear form associated with the quadratic form f, i.e.

$$\bar{x}.\bar{y} = 2x_0y_0+p(x_0y_1+x_1y_0)+2rx_1y_1$$

for $\bar{x} = (x_0,x_1)$ and $\bar{y} = (y_0,y_1)$.

Choose $(\infty) = (0,0,1,0,0,0)$
$(0) = (0,0,0,0,1,0)$.

Let $R_1 = GF(q)$ and $R_2 = (GF(q))^2$, then we take
(a) = (0,0,-a,0,1,0),
$[\bar{k}]$ the line through (0,0,1,0,0,0) and $(\bar{k},0,0,f(\bar{k}),1)$.

Finally, we choose

(0,0,a') = (0,0,0,1,a',0),
$[0,0,\bar{k}']$ the line through (0,0,0,0,0,1) and $(\bar{k},f(\bar{k}),1,0,0)$.

From these follow the points with two coordinates ; namely $(\bar{k},b)$ is the point on the line described by

$$(\bar{k},\lambda,0,f(k),1),\ \lambda \in GF(q)$$

collinear with $(0,0,0,1,-b,0)$, i.e. lying in

$$x_2-bx_5 = 0.$$

We find :

$$(\bar{k},b) = (\bar{k},-b,0,f(\bar{k}),1).$$

The lines with two coordinates follow analogously :

$[a,\bar{l}]$ the line through $(0,0,-a,0,1,0)$ and $(\bar{l},f(\bar{l}),1,0,a)$.

Now we can look for elements with three coordinates :

$(a,\bar{l},a') = (\bar{l},f(\bar{l})-aa',1,a',a)$,

$[\bar{k},b,\bar{k}']$ the line through $(\bar{k},-b,0,f(\bar{k}),1)$ and $(\bar{k}',f(\bar{k}'),1,b+\bar{k}.\bar{k}',0)$.

3.3. Quadratic quaternary ring for Q(5,q)

We express now that $(a,1,a')$ and $[\bar{k},b,\bar{k}']$ are incident, i.e. for some $\mu \in GF(q)$ we have :

$$\bar{l} = \mu\bar{k}+\bar{k}',$$
$$f(\bar{l})-aa' = -\mu b+f(\bar{k}'),$$
$$a' = \mu f(\bar{k})+b+\bar{k}.\bar{k}',$$
$$a = \mu,$$

or equivalently

$$\bar{l} = a.\bar{k}+\bar{k}',$$
$$f(\bar{l})-aa' = -ab+f(\bar{k}'),$$
$$a' = af(\bar{k})+b+\bar{k}.\bar{k}'.$$

Solving for b and $\bar{l}$ we get :

$$b = af(\bar{k})+a'-\bar{k}.\bar{l},$$
$$\bar{l} = a.\bar{k}+\bar{k}'.$$

3.4. Coordinatization of Q(4,q)

If we intersect Q(5,q) with the hyperplane $x_1 = 0$, we get a quadric Q(4,q) with equation :

$$x_0^2-x_2x_3-x_4x_5 = 0.$$

We obtain a coordinatization of Q(4,q) by taking $R_1 = R_2 = GF(q)$ and projecting $\bar{k} = (k_0,k_1)$ onto its first coordinate k_0.

Writing k in place of k_0, and

$$f(\bar{k}) = f(k,0) = k^2,$$
$$\bar{k}.\bar{l} = 2kl,$$

it is clear that, with the natural adjustments, the QQR is given by

$$b = Q_1(k,a,l,a') = ak^2+a'-2kl,$$
$$l = Q_2(a,k,b,k') = ak+a'.$$

3.5. Corollary

It is clear from the corresponding QQR that Q(4,q) is isomorphic to the dual of W(q).

4. THE HERMITIAN VARIETIES $H(3,q^2)$ and $H(4,q^2)$

4.1. Definition

Let H be a nonsingular hermitian variety of the projective space $PG(3,q^2)$ resp. $PG(4,q^2)$. Then the points of H together with the lines on H form a GQ with parameters $(s,t) = (q^2,q)$ resp (q^2,q^3).

4.2. Coordinatization of $H(4,q^2)$

We may suppose that H has the following canonical equation :

$$x_0x_2^q+x_0^qx_2+x_1x_3^q+x_1^qx_3+x_4^{q+1} = 0.$$

We choose

$$(\infty) = (1,0,0,0,0)$$

$$(0) = (0,0,0,1,0).$$

Let $R_1 = GF(q^2)$, then we put

$$(a) = (a,0,0,1,0).$$

For R_2 we take $GF(q^2)\times K$ where K is the set of q solutions of the equation

$$t^q+t = 0.$$

Further, θ denotes a fixed non zero element of $GF(q^2)$ satisfying the equation $(1+\theta)^{q+1} = 1$.

We choose then :

$[k] = [(k_0,k_1)]$ the line on $(1,0,0,0,0)$ and $(0,1,0,\theta k_0^{q+1}+k_1,\theta k_0)$.

Next, take

$$(0,0,a') = (0,0,1,-(a'\theta)^q,0)$$

$[0,0,k'] = [0,0,(k_0',k_1')]$ the line on $(0,1,0,0,0)$ and
$(\theta k_0'^{q+1}+k_1',0,1,0,-\theta k_0')$.

We find now for the points and lines with two coordinates :

$(k,b) = (b\theta,1,0,\theta k_0^{q+1}+k_1,\theta k_0)$

$[a,\ell]$ the line on $(a,0,0,1,0)$ and $(\theta\ell_0^{q+1}+\ell_1,-a^q,1,0,-\theta\ell_0)$.

Finally we obtain those with three coordinates :

$(a,1,a') = (\theta 1_0^{q+1}+1_1-a(a'\theta)^q,-a^q,1,-(a'\theta)^q,-\theta 1_0)$

$[k,b,k'] = [(k_0,k_1),b,(k_0',k_1')]$ the line on

$(b\theta,1,0,\theta k_0^{q+1}+k_1,\theta k_0)$ and

$(\theta k_0'^{q+1}+k_1',0,1,-(b\theta)^q+\theta^{q+1}k_0^q k_0',-\theta k_0')$.

4.3. Quadratic quaternary ring for $H(4,q^2)$

As before, we obtain Q_1 and Q_2 by elminating $\lambda \in GF(q^2)$ in :

$$(\theta 1_0^{q+1}+1_1-a(a'\theta)^q,-a^q,1,-(a'\theta)^q,-\theta 1_0)$$

$$= \lambda(b\ \ ,1,0,\theta k_0^{q+1}+k_1,\theta k_0)$$

$$+ (\theta k_0'^{q+1}+k_1',0,1,-(b\theta)^q+\theta^{q+1}k_0^q k_0',-\theta k_0')$$

It is easily seen that

$$\lambda = -a^q,$$

hence

$$\theta 1_0^{q+1}+1_1-a(a'\theta)^q = -a^q b\theta+\theta k_0'^{q+1}+k_1',$$

$$-(a'\theta)^q = -a^q(\theta k_0^{q+1}+k_1)-(b\theta)^q+\theta^{q+1}k_0^q k_0',$$

$$-\theta 1_0 = -a^q\theta k_0-\theta k_0'.$$

We can write this as follows :

$$1_0 = a^q k_0 + k_0',$$

$$b = a' - \frac{1}{\theta} a(\theta^q k_0^{q+1} - k_1) + \theta^q k_0 (1_0^q - a k_0^q),$$

$$1_1 = a(a^q(\theta k_0^{q+1} + k_1) + (b\theta)^q - \theta^{q+1} k_0^q k_0') - \theta a^q b + \theta k_0'^{q+1} + k_1' - \theta(a^q k_0 + k_0')(a k_0^q + k_0'^q).$$

Remark that $\theta + \theta^{q+1} = -\theta^q$, then we obtain :

$$b = a(k_o^{q+1} + \frac{k_1}{\theta}) + a' + \theta^q k_0 1_0^q$$

$$1_0 = a^q k_0 + k_0'$$

$$1_1 = a^{q+1} k_1 + k_1' - \theta a^q (b + k_0 k_0'^q) + \theta^q a (b + k_0 k_0'^q)^q$$

4.4. Coordinatization of $H(3,q^2)$.

We obtain the non singular hermitian variety $H(3,q^2)$ in three dimensions if we intersect $H(4,q^2)$ having canonical equation as in 4.2 with $x_4 = 0$.

Let μ be a fixed element of $GF(q^2)$ satisfying $\mu \neq 0$ and

$$\mu^q + \mu = 0,$$

then we can identify the set K in 4.2 with GF(q) since $\mu k \in K$ for $k \in GF(q)$.

On the other hand, we can see $GF(q^2)$ as a quadratic extension of GF(q), i.e.

$$GF(q^2) = GF(q)(\lambda) \text{ with } \lambda^2 - p\lambda + r = 0.$$

It follows that both λ and λ^q are roots of the equation $x^2 - px + r = 0$, hence,

$$\lambda + \lambda^q = p$$
$$\lambda . \lambda^q = r.$$

An element a of $GF(q^2)$ can be written as

$$a = a_0 + a_1 \lambda,$$

This allows us to take $R_1 = GF(q)$ and $R_2 = GF(q)^2$. Following analogous computations as in the previous case, but for $k_0 = k'_0 = 0$, we obtain :

$$b = a^q \mu k + a',$$
$$\mu l = a^{q+1} \mu k + \mu k' + a^q b^q - ab.$$

Normalizing this QQR we get

$$b = ak + a'$$
$$1 = a^{q+1} k + k' - a^q b - ab^q.$$

4.5. Corollary

We can work out the above expressions in the following way :

$$a^{q+1} = (a_0+a_1\lambda)(a_0+a_1\lambda^q)$$

$$= a_0^2+pa_0a_1+qa_1^2$$

$$= f(a),$$

$$a^qb+ab^q = (a_0+a_1\lambda^q)(b_0+b_1\lambda)+(a_0+a_1\lambda)(b_0+b_1\lambda^q)$$

$$= 2a_0b_0+pa_0b_1+pa_1b_0+2a_1b_1$$

$$= \bar{a}.\bar{b}.$$

So, $\bar{b} = \bar{a}k+\bar{a}'$,

$1 = f(a)k+k'-\bar{a}.\bar{b}$

This shows that $H(3,q^2)$ is isomorphic to the dual of $Q(5,q)$.

5. SOME APPLICATIONS

5.1. Regularity

A pair of points (p,q) is called <u>regular</u> if $p \perp q$ and $p \neq q$, or if $p \not\perp q$ and for any pair of distinct points $a,b \in \{p,q\}^{\perp}$ we have that each point of $\{p,q\}^{\perp}$ is collinear with each point of $\{a,b\}^{\perp}$. If the GQ S is finite with parameters (s,t), then (p,q) is regular if and only if $|\{p,q\}^{\perp\perp}| = t+1$ provided $p \not\perp q$

The point p is <u>regular</u> if (p,q) is regular for all points $q \neq p$.

5.2. Proposition ([2],[4])

Let S be a GQ coordinatized by a QQR (R_1,R_2,Q_1,Q_2). Then the point (∞) is regular if and only if Q_1 is independent of the third argument, i.e.

$$Q_1(k,a,1,a') = Q_1(k,a,0,a')$$

for all $a,a' \in R_1$ and $k,1 \in R_2$. Dually the line $[\infty]$ is regular if and only if Q_2 is independent of the third argument.

Let S be a finite GQ of order s having (∞) as regular point. Then the incidence structure $\pi_{(\infty)}$ with pointset $(\infty)^{\perp}$, with lineset the set of spans $\{p,q\}^{\perp\perp}$, where $p,q \in (\infty)^{\perp}$, $p \neq q$, and with the natural incidence, is a projective plane of order s coordinatized by the PTR $T(k,a,a') = Q_1(k,a,0,a')$ (Coordinatization Method of Hall [1]), if R_1 and R_2 are identified by $k = Q_1(k,1,0,0)$.

5.3. Regularity in the classical GQ.

It is a well known fact that the group of automorphisms of a classical GQ acts transitively on the points and on the lines. Therefore every point resp. line is regular if and only if (∞) resp. $[\infty]$ is regular.

Using proposition 5.2 it follows immediately that W(q) has only regular points, and has regular lines if and only if the characteristic of the field is two. In particular W(q) is self-dual if and if the characteristic is two. Also, Q(5,q) has regular lines but no regular point, while $H(4,q^2)$ has no regular points nor lines. The results for Q(4,q) resp. $H(3,q^2)$ are the dual of W(q) resp. Q(5,q).

5.4. Ovoids, spreads and polarities.

An ovoid of the GQ S is a set **O** of points of S such that each line of S is incident with a unique point of **O**. A spread of S is a set R of lines of S such that each point of S is incident with a unique line.

It is not so hard to show that the set of all absolute points (resp. lines) of a polarity (i.e. an anti-automorphism of order two) of S is an ovoid (resp. a spread), (see also [4]).

5.5. Proposition

The QQR of W(q) as computed in section 2 is independent of the choice of the quadrangle (∞),(0),(0,0,0),(0,0) and the point (1) and line [1].

Proof.

It is always possible to choose in PG(3,q) the simplex as follows

$$(\infty) = (1,0,0,0)$$

$$(0) = (0,0,1,0)$$

$$(0,0,0) = (0,1,0,0)$$

$$(0,0) = (0,0,0,1).$$

Moreover the unit point can be chosen such that

$$(1) = (1,0,1,0)$$

[1] the line on (1,0,0,0) and (0,0,1,1).

However, the bilinear form defining the symplectic polarity remains the same. So the computations of section 2 also remain equal and hence the QQR.

5.6. Theorem ([6])

The GQ W(q) is self-polar if and only if $q = 2^{2h+1}$, $h > 0$.
The ovoid that arises from the polarity has the following explicit form in W(q), with $\sigma \in$ Aut GF(q) such that $x^\sigma = x^{2^{h+1}}$,

$$O = \{(a, a^{\sigma+2} + b^\sigma, b) \mid a, b \in GF(q)\} \cup \{(\infty)\}.$$

Proof

If W(q) is self-polar then in particular it is self-dual, hence the characteristic of the field is two. In view of the foregoing proposition, we can assume that the polarity θ switches (∞) and $[\infty]$, $(0,0,0)$ and $[0,0,0]$, (1) and $[1]$.

We define a permutation σ of GF(q)

$$(a)^\theta = [a^\sigma]$$

so $$[k]^\theta = (k^{\sigma^{-1}}).$$

Because $(1)^\theta = [1]$, $(1,0,0)^\theta = [1,0,0]$, and the elements $(a) \perp p \perp (0,a)$ with $p \in [1,0,0]$ are transformed into $[a^\sigma] \perp P \perp [0,a^\sigma]$ for $(1,0,0) \in P$. Hence,

$$(0,a)^\theta = [0,a^\sigma]$$

$$[0,k]^\theta = (0,k^{\sigma^{-1}}).$$

It follows that

$$(0,0,a)^\theta = [0,0,a^\sigma].$$

Now $[k]$ I $(k,a) \perp (0,0,a)$ is transformed into $(k^{\sigma^{-1}})$ I $[k^{\sigma^{-1}}, a^\sigma] \perp [0,0,a^\sigma]$, so

$$(k,a)^\theta = [k^{\sigma^{-1}}, a^\sigma]$$

$$[a,k]^\theta = (a^\sigma, k^{\sigma^{-1}}).$$

Finally, we get from $[a,1]$ I $(a,1,a') \perp (0,a')$ that $(a^\sigma, 1^{\sigma^{-1}})$ I $[a^\sigma, 1^{\sigma^{-1}}, a'^\sigma] \perp [0, a'^\sigma]$, and $(a,1,a')^\theta = [a^\sigma, 1^{\sigma^{-1}}, a'^\sigma]$

$$[k,b,k']^\theta = (k^{\sigma^{-1}}, b^\sigma, k'^{\sigma^{-1}}).$$

Expressing incidence of these elements in W(q), we obtain for all $a,b,a',k \in GF(q)$:

$$b^\sigma = a^\sigma(k^{\sigma-1})^2+a'^\sigma$$

with $b = ak+a'$.

Hence, $(ak+a')^\sigma = a\ (k^{\sigma-1})^2+a'^\sigma$.

Putting $a = 1$ and $a' = 0$, we see that

$$k^\sigma = (k^{\sigma-1})^2.$$

Substituting this expression, it follows from

$$(ak+a')^\sigma = a^\sigma k^\sigma+a'^\sigma$$

that σ is an automorphism of GF(q) satisfying

$$\sigma = 2\sigma^{-1},$$

or $\sigma^2 = 2$.

Therefore $q = 2^{2h+1}$,

and $x^\sigma = x^{2^{h+1}}$, for all $x \in GF(q)$

Apart from (∞), all absolute points of θ have three coordinates satisfying

$$(a,1,a')\ I\ [a^\sigma,1^{\sigma-1},a'^\sigma]$$

or $1^{\sigma-1} = a^\sigma a+a'$

$$1 = a^2 a^\sigma+a'^\sigma.$$

We conclude that the ovoid consists of the point (∞) together with the points $(a,a^{\sigma+2}+b^\sigma,b)$ for $a,b \in GF(q)$.

5.7 Equation of the ovoid in PG(3,q)

We obtain the equation in PG(3,q) using 3.2.

$$(a,a^{\sigma+2}+b^\sigma,b) = (a^{\sigma+2}+b^\sigma+ab,1,b,a)$$

It is an easy exercise to see that this cannot represent a quadric.

J. Tits also proves that the associated inversive plane admits the Suzuki group Sz(q) as automorphism group. Finally, the spread that arises by dualizing the ovoid is the Lüneburg-spread giving rise to the non-desarguesion Lüneburg-plane [3].

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