Elation generalized quadrangles of order \((p, t), p\) prime, are classical

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Abstract

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1. Introduction and definitions

1.1. Generalized quadrangles

A finite generalized quadrangle (GQ) of order \((s, t)\) is a finite incidence structure \(\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)\) in which \(\mathcal{P}\) and \(\mathcal{B}\) are disjoint (nonempty) sets of objects called points and lines respectively, and for which \(I\) is a symmetric point–line incidence relation satisfying the following properties:

GQ1. Each point is incident with \(t + 1\) lines \((t \geq 1)\) and two distinct points are incident with at most one line.

GQ2. Each line is incident with \(s + 1\) points \((s \geq 1)\) and two distinct lines are incident with at most one point.

GQ3. If \((x, L)\) is a nonincident point–line pair, then there is a unique point–line pair \((y, M)\) for which \(xI M I yL\).

If \(s = t\), then \(\mathcal{S}\) is said to have order \(s\). For any point \(x\) of \(\mathcal{S}\), the set of all points lying on the lines through \(x\) is denoted by \(x^\perp\). Generalized quadrangles were introduced by Tits (1959). For terminology, notation, results, etc., concerning finite GQ, see the monograph by Payne and Thas (1984).

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The classical GQ

1. Consider a nonsingular quadric $Q$ of projective index 1 of the projective space $PG(d, q)$, with $d = 3, 4$ or $5$. Then the points of $Q$ together with the lines of $Q$ (which are the subspaces of maximal dimension on $Q$) form a GQ $Q(d, q)$ of order $(q, 1)$ when $d = 3$, order $(q, q)$ when $d = 4$, and order $(q, q^2)$ when $d = 5$.

2. Let $H$ be a nonsingular hermitian variety of the projective space $PG(d, q^2)$, $d = 3$ or $4$. Then the points of $H$ together with the lines on $H$ form a GQ $H(d, q^2)$ of order $(q', q)$ when $d = 3$ and order $(q^2, q^3)$ when $d = 4$.

3. The points of $PG(3, q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a GQ $W(q)$ of order $q$.

Isomorphisms. The GQ, $Q(4, q)$ is isomorphic to the dual of $W(q)$, and $Q(4, q)$ (and hence $W(q)$) is self-dual if and only if $q$ is even. The GQ $H(3, q^2)$ is isomorphic to the dual of $Q(5, q)$.

1.2 Whorls, elations and translations

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order $(s, t)$, $s \neq 1$ and $t \neq 1$. A collineation $\theta$ of $\mathcal{S}$ is a whorl about the point $x$ if $\theta$ fixes each line incident with $x$. A whorl $\theta$ about $x$ is said to be an elation about $x$ if $\theta$ is the identity or if $\theta$ fixes no point of $\mathcal{P} - x$. The set of elations about a point does not necessarily form a group (see Payne, 1985). If there is a group $G$ of elations about $x$ acting regularly on $\mathcal{P} - x$, we say $\mathcal{S}$ is an elation generalized quadrangle (EGQ) with elation group $G$ and base point $x$. Briefly we say $(\mathcal{S}_x, G)$ is an EGQ. If the group $G$ is abelian, then $(\mathcal{S}_x, G)$ is a translation generalized quadrangle (TGQ) with translation group $G$ and base point $x$; in such a case $G$ is the set of all elations about $x$. (See Payne and Thas, 1984.)

Example. The classical GQ $Q(4, q)$, $W(q)$, $Q(5, q^2)$ are EGQ with base point $x$, for any point $x$.

It follows that any classical GQ of order $(p, t)$, $t \neq 1$ and $p$ prime, is an EGQ.

In this paper we will prove the converse.

Main result. Any elation generalized quadrangle of order $(p, t)$, $p$ prime and $t \neq 1$, is classical.

1.3 Elation generalized quadrangles as group coset geometries

Let $(\mathcal{S}_x, G)$ be an EGQ of order $(s, t)$, $s \neq 1 \neq t$, and let $y$ be a fixed point of $\mathcal{P} - x$. Let $L_0, L_1, \ldots, L_t$ be the lines incident with $x$. For each $L_i$, $i \in \{0, 1, \ldots, t\}$,
there is a unique point–line pair \((z_i, M_i)\) for which \(yIM_iinz_i\). Define \(S_i, S_i^*\) and \(J\) as follows: 
\[S_i = \{\theta \in G \parallel M_i = M_i\}, \quad S_i^* = \{\theta \in G \parallel z_i^0 = z_i\}\]
and \(J = \{S_i \parallel 0 \leq i \leq t\}\). The groups \(G, S_i, S_i^*\) have respective orders \(s^{2t}, s\) and \(st\). They also have the following two properties:

**K1:** \(S_i \cap S_j = \{1\}\) for distinct \(i, j, k\).

**K2:** \(S_i^* \cap S_j = \{1\}\) for distinct \(i, j\).

It was first shown by Kantor (1980) that the converse is also true, i.e., given a group \(G\) of order \(s^{2t}\) \((s > 1, t > 1)\) with \(1 + t\) subgroups \(S_i\) of order \(s\) and \(1 + t\) subgroups \(S_i^* \supseteq S_i\) of order \(st\) satisfying properties \(K1\) and \(K2\), then one constructs as follows an EGQ \(\mathcal{Q}(G, J)\), with \(J = \{S_i \parallel 0 \leq i \leq t\}\). There are three kinds of points:

(i) the elements of \(G\),
(ii) the right cosets \(S_i^* g, g \in G, i \in \{0, 1, \ldots, t\}\),
(iii) a symbol \((\infty)\).

There are two kinds of lines:

(a) the right cosets \(S_i^*g, g \in G, i \in \{0, 1, \ldots, t\}\),
(b) symbols \([S_i]\), \(i \in \{0, 1, \ldots, t\}\).

A point \(g\) of type (i) is incident with each line \(S_i^* g, 0 \leq i \leq t\). A point \(S_i^* g\) of type (ii) is incident with \([S_i]\) and with each line \(S_i^* h\) contained in \(S_i^* g\). The point \((\infty)\) is incident with each line \([S_i]\) of type (b). There are no further incidences. Then \(\mathcal{Q}(G, J)\) is an EGQ of order \((s, t)\) with base point \((\infty)\). Each EGQ \((S^{(a)}, G)\) with \(J\) defined as above is isomorphic to \(\mathcal{Q}(G, J)\).

2. Elation generalized quadrangles of order \((p, t)\), \(t \neq 1\) and \(p\) prime, are classical

Let \(\mathcal{Q} = (\mathcal{Q}^{(a)}, G) = (\mathcal{Q}, B, I)\) be an EGQ of order \((p, t)\), \(t \neq 1\) and \(p\) prime. We will use the notation of the previous sections.

**Proposition 1.** If \(t = p^2\), then \(\mathcal{Q} \cong Q(5, p)\).

**Proof.** Consider a triple \(\{x, x_1, x_2\}\) of pairwise noncollinear points. As \(t = p^2\) the set \(\{x, x_1, x_2\}^{1+} = \{x, x_1, \ldots, x_r\}\) contains at most \(p + 1\) points. Let \(\delta_i\) be the elation of \(G\) mapping \(x_i\) onto \(x_i, i = 1, 2, \ldots, r\). As \(\delta_i\) fixes \(\{x, x_1, x_2\}^{1+}\), it also fixes \(\{x, x_1, x_2\}^{1+}\). It is clear that \(G_1 = \{\delta_1, \delta_2, \ldots, \delta_r\}\) is the subgroup of \(G\) fixing \(\{x, x_1, x_2\}^{1+}\). As \(|G| = p^4\), either \(|G_1| = 1\) or \(|G_1| = p\). As \(p \geq r > 2\), we necessarily have \(r = p\). So the triple \(\{x, x_1, x_2\}\) is 3-regular, and consequently \(x\) is 3-regular (cf. 1.3 of Payne and Thas (1984)). If \(p = 2\), then \(\mathcal{Q}\) is the unique GQ \(Q(5, 2)\) of order \((2, 4)\); if \(p > 2\), then by 5.3.3(i) of Payne and Thas (1984), the GQ \(\mathcal{Q}\) is isomorphic to \(Q(5, p)\). □

**Proposition 2.** We have \(p \leq t\). For given lines \(L_i, L_j\) with \(i \neq j\), the pair \(\{L_j, M'\}\) is regular for every line \(M'\) with \(xIM'\) and \(L_i \sim M'\), if \(|\{L_j, M, N\}^{1+}| \geq 3\) for at least two
lines $M, N$ of $L^1_1$ with $xIM, xIN, M \not\parallel N$; if $\{L_j, M\}$ is not regular (that is, if $|\{L_j, M, N\}| \leq 2$ for every two lines $M, N$ of $L^1_1$, with $xIM, xIN, M \not\parallel N$) then $|\{L_j, M, N\}| = 2$ for every two lines $M, N$ of $L^1_1$, with $xIM, xIN, M \not\parallel N$, if and only if $p = t$.

Proof. Let $M, N$ be lines of $L^1_1$ with $xIM, xIN, M \not\parallel N$. Assume that $U \in \{L_j, N, M\}^1$, $U \neq L_i$. If $\{L_j, N, M\}^1$ contains a third line $U'$, then put $U'Uu'IM$ and $U'uIM$, and let $\delta$ be the element of $G$ mapping $u$ onto $u'$. The subgroup of $G$ fixing $M$ is the group $\langle \delta \rangle$ of order $p$. This group fixes the point $z = L_iM$ and acts on the set $V$ of points of $L_i$ different from $x$ and $z$. As $|V| = p - 1 < p = |\langle \delta \rangle|$, the group $\langle \delta \rangle$ fixes every point of $L_i$. It easily follows that $N^\delta = N$. Hence the elements of $\langle \delta \rangle$ fix $N$ and map $U$ onto all lines of $\{M, L_i\}^1 - \{L_i\}$. It follows that $N \in \{L_j, M\}^1$. Analogously every line of $\{L_j, M\}^1 - \{L_i\}$ belongs to $\{U, L_i\}^1$. Consequently $\{L_i, U\}$ and $\{L_j, M\}$ are regular. Let $M' \in L_i^1, xIM', M \neq M'$. As there is an element of $G$ mapping $M$ onto $M'$, also the pair $\{L_j, M'\}$ is regular.

If $\{L_j, M\}, M \in L_i^1, xIM$, is regular, then $p \leq t$ by 1.3.6 (i) of Payne and Thas (1984). Now suppose that $\{L_j, M\}$ is not regular, and count in two ways the ordered triples $\{U', N', M'\}$ with $zIL_iu, x \neq z \neq u \neq x, zIM', uIN', U' \in \{M', N', L_j\}^1 - \{L_i\}$. We obtain $pt \leq t^2$, i.e., $p \leq t$, with equality if and only if $|\{L_j, M', N'\}^1| = 2$ for every choice of $M'$ and $N'$. 

Proposition 3. If $t = p$, then $\mathcal{P} \cong Q(4, p)$ or $\mathcal{P} \cong W(p)$.

Proof. Let $p = t$. First assume that for no pair $(L_i, L_j), i \neq j$, we have $|\{L_j, M, N\}| \leq 2$ for each pair $(M, N)$ in Proposition 2. Then each line $L_i$ is regular, $i = 0, 1, \ldots, p$. By 8.3.3 of Payne and Thas (1984) $(\mathcal{P}(n), G)$ is a TGQ of order $p$. Then by 8.7.3 of Payne and Thas (1984) we have $\mathcal{P} \cong Q(4, p)$.

Next assume that for at least one pair $(L_i, L_j), i \neq j$, we have $|\{L_j, M, N\}| \leq 2$ for each pair $(M, N)$ in Proposition 2. By Proposition 2 $|\{L_j, M, N\}| = 2$ for every two such lines $M, N$. If $p = t = 2$, then $\mathcal{P} \cong Q(4, 2)$, so each line is regular, a contradiction. Hence $p$ is odd. Now we introduce the following incidence structure $\mathcal{A} = (\mathcal{P}', \mathcal{B}', I')$. The elements of $\mathcal{P}'$ are the lines of $L_i^1$ not incident with $x$. The elements of $\mathcal{B}'$ are

(i) the points of $L_i$ different from $x$,
(ii) the lines of $L_i^1$ not incident with $x$.

An element of $\mathcal{P}'$ is incident (I') with an element of type (i) of $\mathcal{B}'$ if it is incident with it in $\mathcal{P}$; an element of $\mathcal{P}'$ is incident (I') with an element of type (ii) of $\mathcal{B}'$ if it is concurrent with it in $\mathcal{P}$. Then $\mathcal{A}$ is a $2 - (p^2, p, 1)$ design, that is, an affine plane of order $p$.

Let $M$ and $N$ be as before, let $U \in \{L_j, M, N\}^1 - \{L_i\}$, and let $U1m1M$ and $U1n1N$. If $\delta$ is the element of $G$ mapping $m$ onto $n$, then $\delta$ fixes all points of $L_i$ and induces a translation in the affine plane $\mathcal{A}$. It follows easily that $\mathcal{A}$ is a translation plane. As $p$ is prime, the translation plane $\mathcal{A}$ is desarguesian.
Let C be a line not concurrent with $L_i$ or $L_j$. The elements of $G$ fixing C form a group $\langle \sigma \rangle$ of order $p$. In $\mathcal{A}$ the group $\langle \sigma \rangle$ induces an automorphism group $\Sigma$ of order $p$ fixing the parallel class consisting of lines of $A$ of type (i). If the matrix $A$ represents a generator of $\Sigma$, then we may take for $A$ the matrix

$$
\begin{bmatrix}
a & b & c \\
0 & d & e \\
0 & 0 & 1
\end{bmatrix}
$$

with $A^p = \ell \mathcal{A}$. It follows that $a = d = \ell = 1$, and so

$$
\begin{bmatrix}
1 & rb & rc + \frac{r(r - 1)}{2}be \\
0 & 1 & re \\
0 & 0 & 1
\end{bmatrix}, \quad r = 0, 1, \ldots, p - 1
$$

Since $x$ is the only point of $L_i$ or $L_j$ which is fixed by $\sigma$, the group $\Sigma$ fixes no line of $\mathcal{A}$ and fixes exactly one point at infinity of $\mathcal{A}$. Hence $e \neq 0 \neq b$. The orbit under $\Sigma$ of the point $(x_0, y_0, 1)$ of $\mathcal{A}$ is the parabola

$$\frac{1}{2}(Y - y_0)[(Y - y_0) - e]b + (Y - y_0)(c + by_0) - e(X - x_0) = 0.$$ 

Its point at infinity is exactly the point at infinity of the lines of $\mathcal{A}$ of type (i). The parabola having as elements the $p$ lines $M \in \{L_i, C\}^{\perp}$, with $xM$, will be denoted by $\tilde{C}$.

Now we consider distinct nonconcurrent lines $C$ and $C'$, which are not concurrent with $L_i$. First suppose that $C' \sim L_j \sim C$. As $\{L_i, C\}$ is not regular we necessarily have $|\{L_i, C, C'\}^{\perp}| \leq 2$. Next, let $C' \not\parallel L_j \sim C$. As the line $C$ of $\mathcal{A}$ contains at most two points of the parabola $\tilde{C}$, we have $|\{L_i, C, C'\}^{\perp}| \leq 2$. Finally, let $C' \not\parallel L_j \parallel C$. Then the parabolas $\tilde{C}$ and $\tilde{C}'$ either coincide or have at most two points in common. Hence, either for some $k$ we have $C \sim L_k \sim C'$ with $\{C, L_k\}$ regular and $C' \in \{C, L_i\}^{\perp}$ or $|\{L_i, C, C'\}^{\perp}| \leq 2$. Assume by way of contradiction that $\tilde{C} = \tilde{C}'$. If $w \not\parallel L_j$, $w \neq x$, then the line $W$ defined by $w \ell W \sim C$ is a tangent line of $\tilde{C}$. In this way we obtain the $p$ tangent lines of $\tilde{C}$. As $\tilde{C} = \tilde{C}'$, $W$ is also a tangent line of $\tilde{C}'$, hence $W \sim C'$. It follows that $L_j \in \{C, C'\}^{\perp}$. Hence $L_i = L_j$, a contradiction. So again we have $|\{L_i, C, C'\}^{\perp}| \leq 2$. We conclude that the line $L_i$ is antiregular.

The affine plane $\mathcal{A}$ is the affine plane $\pi(L_i, L_j)$ defined by the antiregular line $L_i$ and the line $L_j$; cf. 1.3.2 of Payne and Thas (1984). As $\pi(L_i, L_j) = \mathcal{A}$ is desarguesian, then by 5.2.7 of Payne and Thas (1984), the GQ $\mathcal{F}$ is isomorphic to $Q(4, p)$. \[ \square \]

**Proposition 4.** We have $t \in \{p, p^2\}$.

**Proof.** We already know that $p \leq t \leq p^2$.

Let us show that $p$ divides $t$. By way of contradiction assume that $p$ does not divide $t$. Consider the subgroup $S_0^*$ of $G$ which fixes the common point $z_0$ of $L_o$ and $M_o$. We
have \(|S_0^g| = pt\), and as \(p \nmid t\) the group \(S_0^g\) has \(1 + kp\) Sylow \(p\)-subgroups of order \(p\), with \(t = r(1 + kp)\). Let \(V\) be the set of all lines through \(z_0\), but different from \(L_0\).

If \(M, M'\) are lines of \(V\) and if \(S_M, S_{M'}\) are the subgroups of \(S_0^g\) fixing \(M, M'\) respectively, then \(S_M = oS_M o^{-1}\) with \(o\) any element of \(S_0^g\) which maps \(M\) onto \(M'\). Also, any Sylow \(p\)-subgroup of \(S_0^g\) is of the form \(S_M\) with \(M\) some line of \(V\). For given \(M \in V\) the number \(s\) of lines \(M' \in V\) for which \(S_M = S_{M'}\) divided by the order of \(S_M\). As \(|N(S_M)|(1 + kp) = |S_0^g| = pt\), we have \(|N(S_M)| = pr\), and as \(x \neq u \neq z_0\). The group \(S_0\) fixes \(u\), and as \(p \nmid t\) it also fixes at least one line \(U\) through \(u\), \(U \neq L_0\).

Let \(M_1, M_2, \ldots, M_r\) be the lines of \(V\) fixed by \(S_0\). Then \(\{U, M_j\}\) is fixed by \(S_0\), and all lines of \(\{U, M_j\}\) are concurrent with some line \(L_{ij}\), \(i \neq 0\). Now it follows from Proposition 2 that \(\{L_{ij}, M_j\}\) is regular, \(j = 1, 2, \ldots, r\). Notations are chosen in such a way that \(i_j - j, j = 1, 2, \ldots, r\). By Proposition 2 all pairs \(\{L_{ij}, M\}\) are regular, \(M\) any line with \(xM\) and \(L_0 \sim M\). Now it is clear that every line of \(\{L_{ij}, M_i\}\) is fixed by \(S_0\). The \(r^2\) sets \(\{L_{ij}, M_i\}^{1,1}, i, j = 1, 2, \ldots, r\), contain in total exactly \(r(1 + p)\) lines (through each point of \(L_0\) different from \(x\), there are \(r\) lines \((\neq L_0)\) fixed by \(S_0\)). It easily follows that each pair \(\{M', N'\}, M' \sim L_0 \sim N', xIM', xIN', M' \neq N', M' and N' fixed by \(S_0\), belongs to exactly one of the sets \(\{L_{ij}, M_i\}^{1,1}\); also, each pair \(\{L_{ij}, M_i\}, M' \sim L_0, L_j \neq M', j \in \{1, 2, \ldots, r\}\) and \(M'\) fixed by \(S_0\), belongs to exactly one of the sets \(\{L_{ij}, M_i\}^{1,1}\). Then by 2.3.1 of Payne and Thas (1984), the set \(\mathcal{P}\) consisting of the points on the lines of the sets \(\{L_{ij}, M_i\}^{1,1}\), together with the lines of \(\mathcal{P}\) containing at least two (and then exactly \(p + 1\)) points of \(\mathcal{P}\), form a subquadrangle \(\mathcal{P}'\) of order \((p, r)\) of \(\mathcal{P}\). As \(\mathcal{P}'\) also contains a subquadrangle of order \((p, 1)\), by 2.2.2 of Payne and Thas (1984), we have \(r = t = 1\) or \((r, t) = (p, p^2)\). But \(p \nmid t\), and so \(r \in \{1, t\}\). If \(r = 1\), then \(t = 1 + kp\). By 1.2.2 of Payne and Thas (1984), \((1 + k)p + 1\) divides \(p(1 + kp)(p + 1)(2 + kp)\), so divides \(p^2 - 1\), so divides \(kp(p - 1)\), so divides \(k(p - 1)\). Hence \(p(1 + k) + 1 \leq k(p - 1)\), a contradiction. Consequently, we may assume that \(S_0^g\) has exactly one Sylow \(p\)-subgroup, and so \(S_0\) fixes all lines of \(L_1\), \(i = 0, 1, \ldots, t\). It follows that \(\mathcal{P}\) is a translation GQ (cf. 8.2 of Payne and Thas, (1984)) and so \(p \mid t\) by 8.5.2 of Payne and Thas (1984). We conclude that \(t = np, 1 \leq n \leq p\).

Assume that \(n < p\). The elation group \(G\) of order \(p^3\) has \(1 + bp\) Sylow \(p\)-subgroups of order \(p^3\), where \(1 + bp\) divides \(n\). Hence \(b = 0\), and so \(G\) has a unique Sylow \(p\)-subgroup \(G'\) of order \(p^3\). If \(y_1 \sim y_2, y_1 \nmid x \nmid y_2\), then the elation \(\delta \in G\) mapping \(y_1\) onto \(y_2\) has order \(p\). Hence \(|\delta| = G'\). Let \(u_1\) and \(u_2\) be any two noncollinear points of \(G'\). If \(x \notin \{u_1, u_2\}\), then choose \(u_3\) with \(u_1 \sim u_3 \sim u_2\). If \(u_1\) \(u_2\) = \(u_3\), \(u_2^2 = u_2\), with \(\delta_1, \delta_2 \in G\), then \(\delta_1, \delta_2 \in G'\). Hence the elation of \(G\) mapping \(u_1\) onto \(u_2\) belongs to \(G'\). If \(x \notin \{u_1, u_2\}\), then, as \(x \neq 1\), we can choose points \(u_3, u_4\) in \(G'\) such that \(u_1 \sim u_3 \sim u_4 \sim u_2\). Again it is clear that the elation of \(G\) mapping \(u_1\) onto \(u_2\) belongs to \(G'\). Consequently \(G \leq G'\), that is, \(n = 1\). So \(t = p\).

We conclude that \(t \in \{p, p^2\}\).
Theorem 5. Elation generalized quadrangles of order \((p, t)\), \(t \neq 1\) and \(p\) prime, are classical.

Proof. This follows immediately from Propositions 1, 3 and 4. 

References


