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COORDINATIZATION OF GENERALIZED QUADRANGLES

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A coordinatization method for any thick generalized quadrangle is worked out, using a new algebraic structure, i.e. a quadratic quaternary ring.

1. INTRODUCTION

A *(thick) generalized quadrangle* (GQ) is an incidence structure $\mathbf{S} = (\mathbf{P}, \mathbf{L}, I)$ with point set $\mathbf{P}$ and line set $\mathbf{L}$, satisfying the following axioms :

(i) each point is incident with $1 + t$ lines ($t \geqslant 2$) and two distinct points are incident with at most one line ;

(ii) each line is incident with $1 + s$ points ($s \geqslant 2$) and two distinct lines are incident with at most one point ;

(iii) if P is a point and L is a line not incident with P, then there is a unique pair $(Q,M) \in \mathbf{P} \times \mathbf{L}$ for which P I M I Q I L.

We say that $\mathbf{S}$ has order (s,t), where $s,t \in \mathbb{N} \cup \{\infty\}$. In view of the point-line duality for GQ, we assume that the dual of a given definition or theorem has also been given implicitly. It is a nice exercise to show that axioms (i) and (ii) can be replaced by :

(i)' each point is incident with at least three lines ;

(ii)' each line is incident with at least three points ;

(iv)' there is a non-incident point-line pair.

Given two points P and Q of $\mathbf{S}$, we write $P \perp Q$ and say that P and Q are collinear provided that there is some line L incident with both. If this is not the case, we write $P \not\perp Q$.

For $P \in \mathbf{P}$, put $P^{\perp} = \{Q \in \mathbf{P} \mid Q \perp P\}$ and note that $P \in P^{\perp}$. If $A \subset \mathbf{P}$, we write $A^{\perp} = \cap\{P^{\perp} \mid P \in A\}$. For distinct points P and Q, $\{P,Q\}^{\perp}$ is called the trace of P and Q and $\{P,Q\}^{\perp\perp}$ the span.

Generalized quadrangles were introduced by J. Tits and appeared first in [1]. They arose as natural objects "succeeding" the projective planes, which could be viewed as generalized triangles.

One of the most powerful concepts in the modern theory of projective planes is that of coordinatization. This is certainly the case in constructing non-classical planes and to determine whether two given projective planes are isomorphic. For details we refer to [2] and [8].

It is surprising that an analogous general coordinatization theory for GQ is not yet available. The known non-classical GQ are constructed using geometrical methods or matrices. Especially in the last case, one gets the impression that there might be an underlying coordinatization without being explicit.

S. Payne [9] worked out a preliminary version of such a theory for a special class of GQ of order (s,s), $s > 1$, namely those having an axis of symmetry. He essentialy uses the coordinatization by a planar ternary ring of an underlying projective plane.

To be useful such a general coordinatization theory for GQ should satisfy some "beauty" conditions.

Firstly it must provide an easy, algebraically more concrete description of the existing GQ, and if possible, also of their automorphism groups [3].

Secondly, the fact of having certain automorphism groups for the GQ should be reflected by "nice" properties of the corresponding algebraic structure [4].

Finally, important geometrical conditions should have a simple algebraic reformulation (see § 3 and [5]).

We mention here also that the proposed method of coordinatization works in the case of generalized n-gons [6] as well. The theory of quadratic quaternary rings should also be the crucial tool in describing an affine building of type $\tilde{C}_2$ in terms of its (infinite) generalized quadrangle at infinity (see also [7] for further references and comments).

2. COORDINATIZATION BY QUADRATIC QUATERNARY RINGS

2.1. Introduction of coordinates

Let **S** be a GQ of order (s,t), $s,t > 1$. Choose an arbitrary point (∞) and an arbitrary line $[\infty]$ incident with it. Let R_1 be a set of cardinality s not containing the symbol ∞, and assign bijectively a coordinate (a), with $a \in R_1$, to every point on $[\infty]$ different from (∞).

Dually, let R_2 be a set of cardinality t not containing the symbol ∞, and give every line on (∞) different from $[\infty]$ a coordinate [k] with $k \in R_2$, such that there is a bijection between the lines on (∞) and R_2.

We pick out of R_1 resp. R_2 two distinguished elements denoted by 0 and 1.

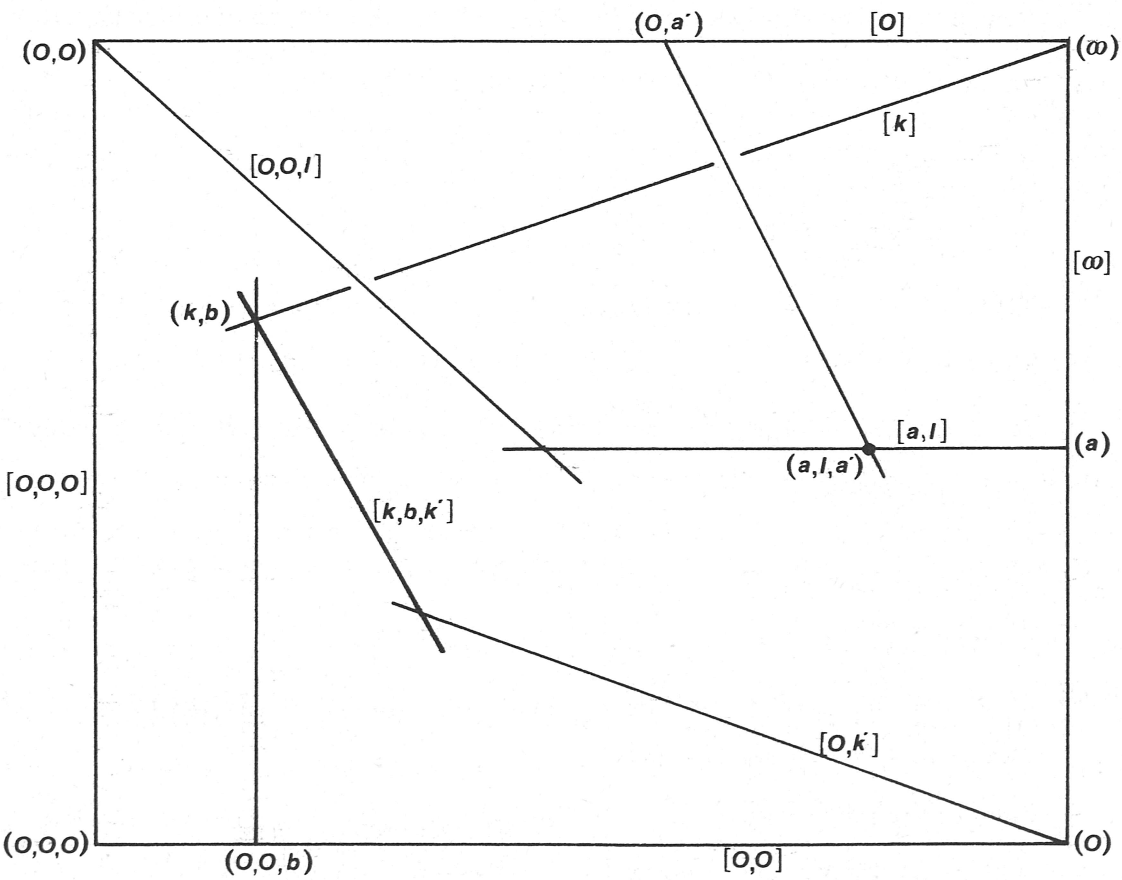
Now we choose a point A not on $[\infty]$ and collinear with (0), and call B the point of [0] collinear with A. Like before we choose a bijection between R_1 and the points of the line (0)A with the only restriction that A corresponds to 0. The point of (0)A corresponding to $a' \in R_1$ will have coordinate $(0,0,a') \in R_1 x R_2 x R_1$. Dually, we give coordinates $[0,0,k'] \in R_2 x R_1 x R_2$ to the lines on B different from [0], with the restriction that BA has coordinate [0,0,0].

We define next the points with two coordinates : a point P collinear with (∞), but not lying on $[\infty]$ has coordinate $(k,a) \in R_2 x R_1$ if and only if P lies on [k] and is collinear with (0,0,a). Dually, lines meeting $[\infty]$ not passing (∞) are given coordinates $[a,k] \in R_1 x R_2$.

Finally, consider a point P not collinear with (∞). By axiom (iii) there is exactly one line on P meeting $[\infty]$. This line must have two coordinates, say [a,l]. On the other hand, P is collinear with exactly one point (0,a') on [0]. Now P is given the coordinate (a,l,a'). Conversely, let (a,l,a') be any element of $R_1 x R_2 x R_1$, then we construct a point P having this element as coordinate. Indeed, given the line [a,l] and the point (0,a') not incident with it, then there is exactly one point collinear with (0,a') and lying on [a,l] by axiom (iii).

The coordinate of a line [k,b,k'] is defined dually. It is easy to check that there arises no ambiguity for the coordinates (0,0,a') and [0,0,k']. To avoid complications, we identify each element with its coordinate.

In this way we have coordinatized every point and line of **S** (see fig.1). Now we must build into our coordinatization system a criterium for determining whether two given elements are incident.



2.2. Quadratic quaternary rings

If a GQ **S** has been coordinatized by the elements of the sets R_1 and R_2 as described in section 2.1, then we use incidences of **S** to define two quaternary operations Q_1 and Q_2 as follows : if $a,a',b \in R_1$ and $k,k',l \in R_2$,

$$Q_1(k,a,l,a') = b \quad \text{if and only if } (k,b) \perp (a,l,a'), \qquad (1)$$

$$Q_2(a,k,b,k') = l \quad \text{if and only if } [a,l] \perp [k,b,k'] \qquad (2)$$

The operations are uniquely determined, for given the point (a,l,a') and the line [k], there is exactly one point (k,b) on [k] which is collinear with (a,l,a'). Dually, given the line [k,b,k'] and the point (a), there is exactly one line [a,l] on (a) meeting [k,b,k'].

It is clear that (a,l,a') I [k,b,k'] only if (1) and (2) are satisfied. Conversely, suppose that (1) and (2) hold, then (a,l,a') is collinear with $(k,b) \in [k,b,k']$ and [k,b,k'] meets [a,l] containing (a,l,a'). Because this meeting point of [a,l] and [k,b,k'] cannot be (k,b), it follows that (a,l,a') is incident with [k,b,k']. We will call the quadruple (R_1,R_2,Q_1,Q_2) a *coordinatization* of the GQ **S**.

We remark that Q_1 and Q_2 are dual operations.

2.2.1. Theorem

Let **S** be a GQ coordinatized by (R_1,R_2,Q_1,Q_2), then the following properties hold :

(O) $Q_1(k,0,0,a) = a$

$Q_1(0,a,k,a') = a'$.

($\overline{\text{O}}$) $Q_2(a,0,0,k) = k$

$Q_2(0,k,a,k') = k'$.

(A) If $a,b \in R_1$ and $k,l \in R_2$, then there is a unique $x \in R_1$ such that $Q_1(k,a,l,x) = b$.

($\overline{\text{A}}$) If $a,b \in R_1$ and $k,l \in R_2$, then there is a unique $p \in R_2$ such that $Q_2(a,k,b,p) = l$.

(B) If $a,b \in R_1$ and $k,l,k' \in R_2$ with $k \neq l$, then there is a unique pair $(x,y) \in R_1^2$ such that

$Q_1(k,x,Q_2(x,k,a,k'),y) = a$

$Q_1(l,x,Q_2(x,k,a,k'),y) = b$.

($\bar{B}$) If $a,b,a' \in R_1$ and $k,l \in R_2$ with $a \neq b$, then there is a unique pair $(p,q) \in R_2^2$ such that

$$Q_2(a,p,Q_1(p,a,k,a'),q) = k$$

$$Q_2(b,p,Q_1(p,a,k,a'),q) = l.$$

(C) If $a,a',b \in R_1$ and $k,k',l \in R_2$ then the system of equations in the unknowns x,p,x',p'

$$Q_1(k,x,Q_2(x,k,b,k'),x') = b$$

$$Q_1(p,x,Q_2(x,k,b,k'),x') = Q_1(p,a,l,a')$$

$$Q_2(a,p,Q_1(p,a,l,a'),p') = l$$

$$Q_2(x,p,Q_1(p,a,l,a'),p') = Q_2(x,k,b,k'),$$

has a unique solution $(x,p,x',p') \in R_1 x R_2 x R_1 x R_2$ if $Q_1(k,a,l,a') \neq b$ and $Q_2(a,k,b,k') \neq 1$ and none if one of both equalities holds.

Proof

We have only to show one of each pair of dual properties marked with a same letter. As to property (0), it follows from the coordinatization that $(0,0,a)$ is incident with $[k,a,0]$ and (a,k,a') with $[0,a,l]$ for some $l \in R_2$.

In order to derive the others, it suffices to express the axiom (iii) of the generalized quadrangle **S** with respect to the following point-line pairs :

(A) : (k,b) and $[a,l]$

(B) : (l,b) and $[k,a,k']$

(C) : (a,l,a') and $[k,b,k']$.

We remark that (C) is self-dual.

2.2.2. Definition

Let R_1, (resp.R_2) be a set containing distinguished elements 0 and 1 but not ∞ and Q_1(resp.Q_2) quaternary operations from $R_2 x R_1 x R_2 x R_1$ to R_1 (resp.$R_1 x R_2 x R_1 x R_2$ to R_2). We call the quadruple (R_1,R_2,Q_1,Q_2) satisfying the properties (0), ($\bar{0}$), (A), ($\bar{A}$), (B), ($\bar{B}$), and (C) of theorem 2.2.1 a *quadratic quaternary ring*, which we shall abbreviate to QQR.

We denote the unique solution of $Q_1(k,a,l,x) = b$ by $Q_1^*(a,k,b,l) = x$, and dually $p = Q_2^*(k,a,l,b)$ if p satisfies $Q_2(a,k,b,p) = 1$.

2.2.3. Theorem

If (R_1, R_2, Q_1, Q_2) is a QQR then the structure S defined as follows is a generalized quadrangle. The points of S are elements of $R_1 x R_2 x R_1$, $R_2 x R_1$ or R_1, denoted by parenthesis, together with (∞) where is a symbol not contained in R_1 or R_2. Lines are represented in square brackets by elements of $R_2 x R_1 x R_2$, $R_1 x R_2$ or R_2 together with $[\infty]$. Incidence is defined in the following manner :

$$(a,l,a') \text{ is on } [k,b,k'] \Leftrightarrow Q_1(k,a,l,a') = b$$

$$Q_2(a,k,b,k') = l,$$

(a,l,a') is on $[a,l]$,

(k,a) is on $[k]$ and $[k,a,k']$ for all $k' \in R_2$,

(a) is on $[\infty]$ and $[a,k]$ for all $k \in R_2$,

(∞) is on $[\infty]$ and $[k]$ for all $k \in R_2$,

and there are no further incidences.

Proof

It suffices to show that for any point P and any line L not incident with P, there is a unique line on P meeting L and a unique point on L collinear with P.

To begin with, suppose $P = (\infty)$, then L has to be $[k,b]$ or $[k,b,k']$. It is easy to check that for the first case only $(\infty)I[\infty]I(a)I[a,l]$ is possible whereas for the second $(\infty)I[k]I(k,b)I[k,b,k']$.

Next, take $P = (a)$, then the cases where L is $[k]$ or $[b,k]$ with $b \neq a$ are straightforward. So assume $L = [k,b,k']$. A line on (a) has to be of the form $[a,p]$ with $p \in R_2$, or $[\infty]$, ruled out by $[\infty] \not\perp L$, and a point on that line of the form (a,p,x). But if this point has to be incident with $[k,b,k']$ then $p = Q_2(a,k,b,k')$ is known and x is uniquely determined by the equation

$$Q_1(k,a,p,x) = b$$

in view of (A).

Now we suppose $P = (l,a)$. The cases where L is $[\infty]$ or $[k]$ with $k \neq l$ are already proved dually. Let L be $[b,k]$. Then any point on L has the form (b,k,x) or (b') and any line on p the form $[l,a,p]$ or $[l]$. These elements are incident iff $a = Q_1(l,b,k,x)$ and $k = Q_2(b,l,a,p)$, which uniquely determines x and p by (A) and $(\bar{A})$. Let L be of the form $[k,b,k']$ with $b \neq a$ or $k \neq l$.
Consider a chain of elements :

$$(l,a)I[l,a,q]I(x,p,y)I[k,b,k']$$

then we have

$$Q_1(l,x,p,y) = a \quad (1)$$

$$Q_2(x,l,a,q) = p \quad (2)$$

$$Q_1(k,x,p,y) = b \quad (3)$$

$$Q_2(x,k,b,k') = p \quad (4)$$

Suppose first that $k = l$, then $(k,a)I[k]I(k,b)I[k,b,k']$. On the other hand, (1) and (3) imply that $a = b$, contradicting our assumption. Hence, $[k]$ is the only line on P meeting L and (k,b) is the only point on L collinear with P. Suppose now $k \neq l$, then it follows from (B) that the equations

$$Q_1(l,x,Q_2(x,k,b,k'),y) = a$$

$$Q_1(k,x,Q_2(x,k,b,k'),y) = b$$

with $k \neq l$ uniquely determine x and y. From these p follows by (4) and q by (2) in view of $(\bar{A})$. Hence, in both cases the chain exists and is unique.

Finally, let $P = (a,l,a')$. Considering the dual, we can assume that $L = [k,b,k']$ with $Q_1(k,a,l,a') \neq b$ or $Q_2(a,k,b,k') \neq l$. If $Q_1(k,a,l,a') = b$ then

$$(a,l,a')I[k,b,p]I(k,b)I[k,b,k']$$

where p follows from $Q_2(a,k,b,p) = l$. Dually, if $Q_2(a,k,b,k') = l$, then

$$(a,l,a')I[a,l]I(a,l,x)I[k,b,k']$$

where x follows from $Q_1(k,a,l,x) = b$.

Consider now a chain

$$(a,l,a')I[p,y,p']I(x,q,x')I[k,b,k']$$

then this is equivalent to

$$Q_1(p,a,l,a') = y$$

$$Q_2(a,p,y,p') = l$$

$$Q_1(p,x,q,x') = y$$

$$Q_2(x,p,y,p') = q$$

$$Q_1(k,x,q,x') = b$$

$$Q_2(x,k,b,k') = q$$

and also to

$$Q_1(k,x,Q_2(x,k,b,k'),x') = b$$

$$Q_2(a,p,Q_1(p,a,l,a'),p') = l$$

$$Q_1(p,x,Q_2(a,k,b,k'),x') = Q\ (p,a,l,a')$$

$$Q_2(x,p,Q_1(p,a,l,a'),p') = Q_2(x,k,b,k').$$

If $Q_1(k,a,l,a') = b$ or $Q_2(a,k,b,k') = l$, then this set of equations has no solution, whereas in the opposite case it has exactly one by (C).
This proves the theorem.

2.2.4. Remark

Note that if we coordinatize **S** again in the obvious way, we get a QQR identical to the one we started with.

2.2.5. Definition

If (R_1,R_2,Q_1,Q_2) and (R_1',R_2',Q_1',Q_2') are two quadratic quaternary rings we say that they are isomorphic if there is a bijection α from R_1 onto R_1' and a bijection β from R_2 onto R_2' such that

$$(Q_1(a,k,b,l))^{\alpha} = Q_1'(a^{\alpha},k^{\beta},b^{\alpha},l^{\beta})$$

and

$$(Q_2(k,a,l,b))^{\beta} = Q_2'(k^{\beta},a^{\alpha},l^{\beta},b^{\alpha})$$

for all $a,b \in R_1$ and $k,l \in R_2$.

2.3. Normalization of a QQR

Let S be a GQ coordinatized by a QQR (R_1,R_2,Q_1,Q_2). As we have seen in 2.1 there does not have to be a connection between the bijections from R_1 onto $[\infty]-\{(\infty)\}$ and onto $(0)A-\{(0)\}$.

We now demand that the bijections are chosen such that $(a,0,0)$ is collinear with $(1,a)$.

For the QQR we get :

$$Q_1(1,a,0,0) = a \qquad \text{(N)}$$

We can do the same dually and obtain :

$$Q_2(1,k,0,0) = k \qquad (\overline{\text{N}})$$

If (R_1,R_2,Q_1,Q_2) is a QQR not satisfying (N) and $(\overline{\text{N}})$, then define permutations α of R_1 and β of R_2 by :

$$Q_1(1,a,0,0) = a^{\alpha}$$

$$Q_2(1,k,0,0) = k^{\beta}$$

Defining moreover

$$\tilde{Q}_1(k,a,l,a') = [Q_1(k,a,l^{\beta},a'^{\alpha})]^{\alpha^{-1}}$$

and

$$\widetilde{Q}_2(a,k,b,k') = [Q_2(a,k,b^{\alpha},k'^{\beta})]^{\beta^{-1}}$$

it is easy to check that $(R_1,R_2,\widetilde{Q}_1,\widetilde{Q}_2)$ is a QQR satisfying (N) and $(\overline{N})$. We call such a QQR *normalized*, and the procedure just described, *normalization*.

2.4. Algebraic properties of quadratic quaternary rings

2.4.1. Definition

Let (R_1,R_2,Q_1,Q_2) be a (normalized) QQR.
We introduce a binary operation of addition into R_1 and R_2, and a "twisted" multiplication as follows : for any $a,b \in R_1$ and $k,l \in R_2$, we define

$$a + b = Q_1(1,a,0,b) \in R_1$$

$$k + l = Q_2(1,k,0,l) \in R_2$$

$$k \,.\, a = Q_1(k,a,0,0) \in R_1$$

$$a \,.\, k = Q_2(a,k,0,0) \in R_2$$

We denote $R_i - \{0\}$ by R_i^*, $i = 1,2$.

2.4.2. Theorem

If (R_1,R_2,Q_1,Q_2) is a normalized QQR, then the following properties hold :

(i) $a + 0 = 0 + a = a$ for all $a \in R_1$

(ii) $k + 0 = 0 + k = k$ for all $k \in R_2$

(iii) $a + x = b$ has a unique solution for any $a,b \in R_1$

(iv) $k + p = l$ has a unique solution for any $k,l \in R_2$

(v) $1 \,.\, a = a$ for all $a \in R_1$

(vi) $1 \,.\, k = k$ for all $k \in R_2$

(vii) $k \,.\, x = a$ has a unique solution for any $a \in R_1$ and $k \in R_2^*$

(viii) $a \,.\, p = k$ has a unique solution for any $a \in R_1^*$ and $k \in R_2$

Proof

We have $a + 0 = Q_1(1,a,0,0) = a$ by (N), and $0 + a = Q_1(1,0,0,a) = a$ by (O). This proves (i), and dually (ii).

If we apply (A) for $k = 1$ and $l = 0$, we obtain (iii) and dually (iv). By (N) and $(\overline{N})$, we get (v) and (vi) respectively.

Now we apply (B) for $k = k' = 0$ and $a = 0$, and obtain a unique pair $(x,y) \in R_1^2$ such that

$$Q_1(0,x,Q_2(x,0,0,0),y) = 0$$

$$Q_1(1,x,Q_2(x,0,0,0),y) = b$$

But $Q_2(x,0,0,0) = 0$ by $(\bar{0})$ and $Q_1(0,x,0,y) = 0$ forces $y = 0$ by (0). Hence,

$$l \cdot x = b$$

has a unique solution $x \in R_1$ for $l \neq 0$. Of course, (viii) can be proved dually.

2.5. Case of a finite QQR

Theorem

Let (R_1,R_2,Q_1,Q_2) be a quadruple where R_1 and R_2 are distinguished finite sets containing the symbol 0, and Q_1 resp. Q_2 a quaternary operation from $R_1 x R_2 x R_1 x R_2$ resp. $R_2 x R_1 x R_2 x R_1$ to R_1 resp. R_2. Suppose that a weaker version of the properties (0)-(C) of 2.2.1, namely where "a unique" is replaced by "at most one", holds. Then (R_1,R_2,Q_1,Q_2) is QQR.

Proof

Put $|R_1| = s$ and $|R_2| = t$, and construct an incidence structure as was done in theorem 2.2.3. It is straightforward to check that the following hold :

(i) each point is incident with $1 + t$ lines and two distinct points are incident with at most one line ;

(ii) each line is incident with $1 + S$ points and two distinct lines are incident with at most one point ;

(iii) if P is a point and L is a line not incident with P, then there is at most one pair $(Q,M) \in \mathbf{P} \times \mathbf{L}$ for which P I M I Q I L.

(iv) $|\mathbf{P}| = v = (s+1)(st+1)$

(v) $|\mathbf{L}| = b = (t+1)(st+1)$

Now the number of points collinear with at least one point of a line L, but not lying on L, equals $(s+1)ts$. But this is exactly the number of points not lying on L, proving that in property (iii) we can replace "at most one" by "a unique". It follows that (R_1,R_2,Q_1,Q_2) is a QQR.

3. REGULARITY

3.1. Definition

A pair of points $(p,q), p \neq q$, is called *regular* if $p \perp q$, or if $p \not\perp q$ and for any pair of distinct points $a,b \in \{p,q\}^{\perp}$, we have that each point of $\{p,q\}^{\perp}$ is collinear with each point of $\{a,b\}^{\perp}$. If the GQ **S** is finite with parameters (s,t), then (p,q) is regular iff $|\{p,q\}^{\perp\perp}| = t + 1$ provided $p \not\perp q$. The point p is *regular* if (p,q) is regular for all points $q \neq p$.

3.2. Theorem

Let **S** be a GQ coordinatized by a QQR (R_1,R_2,Q_1,Q_2). Then the point (∞) is regular if and only if Q_1 is independent of the third argument, i.e.

$$Q_1(k,a,l,a') = Q_1(k,a,0,a')$$

for all $a,a' \in R_1$ and $k,l \in R_2$. Dually the line $[\infty]$ is regular if and only if Q_2 is independent of the third argument.

Proof

Suppose (∞) is a regular point. We have $(a), (0,a) \in \{(\infty),(a,m,a')\}^{\perp}$, and $\{(a),(0,a')\}^{\perp} = \{(a,l,a') \mid l \in R_2\} \cup \{(\infty)\}$. Now we express that $(k,b) \perp (a,l,a')$ for some $p \in R_2$, $[k,b,p]I(a,l,a')$, so

$$b = Q_1(k,a,l,a')$$

By the regularity of (∞) this must hold for all $l \in R_2$, so

$$Q_1(k,a,l,a') = b = Q_1(k,a,0,a')$$

for all $a,a' \in R_1$ and $k,l \in R_2$. Conversely, if $Q_1(k,a,0,a') = Q_1(k,a,l,a')$ then

$$(a,l,a') \perp (k,Q_1(k,a,0,a'))$$

hence, (∞) is regular if Q_1 does not depend on the third argument.

3.3. Theorem

Let S be a finite GQ of order s having (∞) as regular point. The incidence structure $\pi_{(\infty)}$ with pointset $(\infty)^{\perp}$, with lineset consisting of spans $\{p,q\}^{\perp\perp}$, where $p,q \in (\infty)^{\perp}$, $p \neq q$, and with the natural incidence, is a projective plane of order s coordinatized by the PTR $T(k,a,a') = Q_1(k,a,0,a')$ (Coordinatization Method of Hall) if R_1 and R_2 are identified by $k = Q_1(k,1,0,0)$ and the QQR is normalized.

Proof

If $s = t$ it is possible to identify R_1 and R_2. We can do that in the following way provided (∞) is regular : given $k \in R_2$, we denote the point on $[k]$, collinear with $(1,0,0)$ by (k,k^{σ}).

This means that $k^{\sigma} = Q_1(k,1,0,0)$, and in particular $0^{\sigma} = 0$ and $1^{\sigma} = 1$.
We show that the map σ from R_2 tot R_1 is injective. In order to derive a contradiction, let for $k \neq l$, (k,a) and (l,a) be collinear with $(1,0,0)$. Then (∞), $(1,0,0)$, $(0,0,a) \in \{(k,a),(l,a)\}^{\perp}$, and by regularity of (∞), $(0,0,a) \perp (1)$, clearly a contradiction.

The fact that the incidence structure $\pi_{(\infty)}$ is a projective plane is known [9].

The lineset is thus given by the set of spans $\{(a), (0,a')\}^{\perp\perp}$ together with the lines $[k]$, $k \in R_2$ and $[\infty]$. In view of the proof of theorem 3.2, we have :

$$\{(a),(0,a')\}^{\perp\perp} = \{(\infty),(a,0,a')\}^{\perp}$$

$$= \{(k,Q_1(k,a,0,a')) \mid k \in R_2\} \cup \{(a)\}$$

Define a ternary operation on R_1, $T(k^\sigma,a,a') = Q_1(k,a,0,a')$, and give the span $\{(a),(0,a')\}^{\perp\perp}$ the coordinate $[[a,a']]$, and the point (k,b) the coordinate $((k^\sigma,b))$. Clearly $((x,y))$ is on $[[m,k]]$ if and only if $(x^{\sigma^{-1}},y)$ is on the span $\{(m),(0,k)\}^{\perp\perp}$ and this holds if and only if $y = Q_1(x^{\sigma^{-1}},m,0,k)$, i.e. $y = T(x,m,k)$. Moreover there hold :

(A) $T(x,0,b) = Q_1(x^{\sigma^{-1}},0,0,b) = b$

$T(0,x,b) = Q_1(0,x,0,b) = b$

for all $x,b \in R_1$.

(B) $T(x,1,0) = Q_1(x^{\sigma^{-1}},1,0,0) = x$

$T(1,x,0) = Q_1(1,x,0,0) = x$

for all $x \in R_1$.

This proves the theorem.

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