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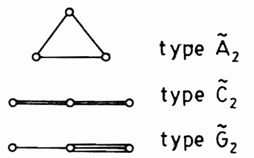
VALUATIONS ON PTR'S INDUCED BY TRIANGLE BUILDINGS

ABSTRACT. A valuation v (in the sense of [10]) is defined on all coordinatizing planar ternary rings (R, T) of any projective plane which can be thought of as the spherical building at infinity of a triangle building. It is shown that (R, T) is complete with respect to v.

INTRODUCTION

By a recent work of J. Tits [9], all affine buildings of rank $\geqslant 4$ are classified and are known to arise from algebraic groups over a local field. Affine buildings not arising in that way are called 'non-classical'. So the only candidates are the rank 3 affine buildings whose Buekenhout diagram [1] is one of



For each of these three types there are examples of non-classical buildings ([4], [6], [10]). They arise in different ways: as universal covering of certain (finite) geometries; as free constructions; or via pure construction. The main idea behind the present paper is the use of the spherical building at infinity of buildings of type $\tilde{A}_2$ (the so-called 'triangle buildings'). This notion of 'building at infinity of an affine building' was introduced by Bruhat and Tits in [1], and in [9] it plays a crucial role. The purpose of the present work is to characterize triangle buildings by means of their building at infinity which, in this case, is the building of a projective plane. More exactly, by [10] we know that a planar ternary ring (R, T) (briefly a PTR) which admits a valuation $v: R^2 \to Z \cup \{\infty\}$ gives rise to a triangle building. In this paper we show that there is a reverse procedure and that both operations are mutually inverse, granted that (R, T, v) is complete (for definition see Section 2.2). (R, T) is in fact a coordinatizing PTR of the projective plane corresponding to the building at infinity of the triangle building in question. This leads us to the main result of this paper:

MAIN THEOREM. *There is a bijective correspondance between the class of triangle buildings* (*with a complete set of apartments*) *and the class of projective planes coordinatized by a complete PTR with valuation.*

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The proof of this theorem will be completed in subsection 5.3.

The paper is organized as follows: In Section 1 we define triangle buildings and list some properties. We closely follow the terminology of Tits ([1], [9]). In Section 2, we briefly repeat the coordinatization of a projective plane according to Hughes and Piper [3]. We define the notion of a (complete) valuation v on a PTR (R, T). Section 3 is an abstract of [10] to which the present paper is a sequel. Section 4 is an important step in the proof of our main result. We show that any coordinatizing PTR of the 'projective plane at infinity' of a triangle building admits a valuation. We do not yet show completeness. In Section 5 we prove that the operations of Sections 3 and 4 are mutually inverse and we finish off the proof of the main theorem by showing:

THEOREM (I). *Any coordinatizing PTR of the projective plane at infinity of a triangle building (with a complete set of apartments) admits a complete valuation.*

Section 6 finally explains the connection with Tits' projective valuation.

1. Triangle buildings

1.1. *The Apartment*

Suppose $I = \{1,2,3\}$ and let $\mathscr{A}$ be the real affine plane provided with the usual Euclidean distance $d_{\mathscr{A}}$ and an origin O. Also let $R_2 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2, -\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_1 - \mathbf{e}_2\}$, with $|\mathbf{e}_1| = |\mathbf{e}_2| = 1$ and $\mathbf{e}_1 \cdot \mathbf{e}_2 = -\frac{1}{2}$, be the set of vectors of the root system A_2 (e.g. $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and W_0 the group generated by the reflexions about the vector lines perpendicular to the vectors of R_2 (note that $W_0 \cong S_3$). Consider the line L with equation $\mathscr{X} \cdot \mathbf{e}_1 = 1$. Then the group generated by W_0 and the reflexion about L is the affine Weyl group W; W can be written as the semi-direct product $W = W_0 \rtimes T$, where T is the group of translations generated by $2 \cdot R_2$ (see [1]). The reflexions of W are precisely those about the lines with equation $\mathscr{X} \cdot \mathbf{e} = n$, where $\mathbf{e} \in R_2$ and $n \in \mathbb{N}$. These lines are called *walls*. The topological closure of a connected component of the complement in $\mathscr{A}$ of all walls is called a *chamber*. If C is a chamber, then there are exactly three walls having non-empty intersection with C. Let L be one of them, then $C \cap L$ is called a *panel*. Any intersection point of two non-parallel walls is a *vertex*. Vertices and panels which are subsets of a chamber C are called *faces* of C. Note that, in contradistinction to [1], all our chambers and panels are closed in the topological sense: this has only a practical reason (it sometimes provides a shorter formulation). Let S_v be the set of walls through a vertex v ($|S_v| = 6$). Then the topological closure of a connected component of $A - \bigcup(S_v)$ is called a *quarter*. A quarter Q is bounded by two closed halflines, called the *panels of the quarter* Q, which we shall abbreviate to *pennels*, to make the difference with panels of chambers. A half plane bounded by a wall is a *half apartment*. If Q is a quarter and p_1, p_2 are the pennels bounding Q, then the common vertex v of p_1 and p_2 is called the *source* of Q, p_1, p_2. If H is a half apartment whose boundary L meets both p_1 and p_2, and the vertex v is not in $H - L$, then $Q \cap H$ is called a *truncated quarter*. If two quarters Q_1, Q_2 with common source meet in a pennel then $Q_1 \cup Q_2$ is called a *double quarter*. Suppose H_1 and H_2 are two half apartments with parallel boundaries the walls L_1, resp. L_2. Suppose also $L_1 \subset H_2$ and $L_2 \subset H_1$, then $H_1 \cap H_2$ is called a *strip*. If D is a double quarter bounded by the pennels p_1 and p_2, and S is a strip whose

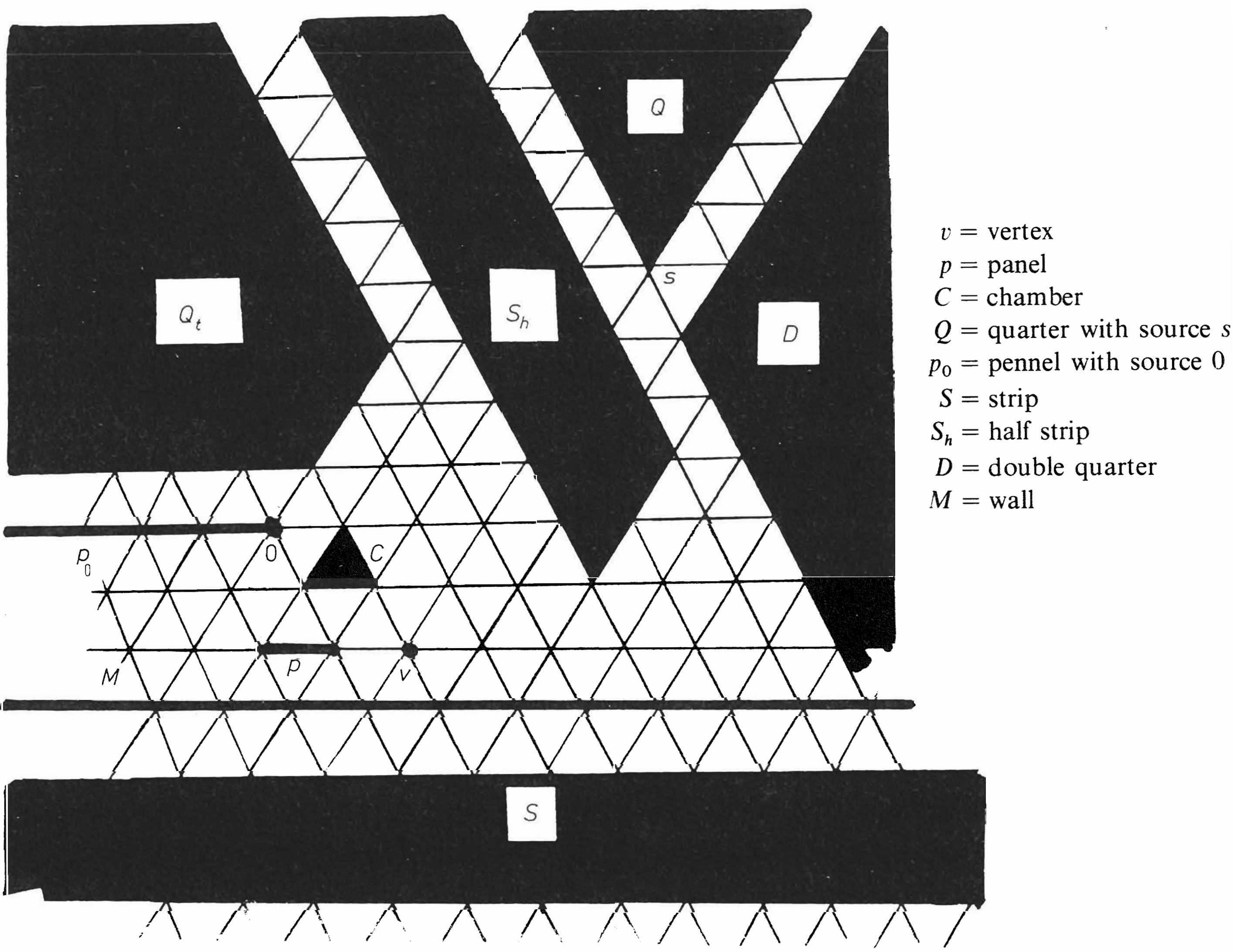

Fig. 1.

boundaries are not parallel to the wall containing p_1 or p_2, then $D \cap S$ is called a *half strip*. For an illustration of these definitions, see Figure 1. There is a type map 'typ' which assigns to any face of a chamber a subset of I such that $|\text{typ}(\text{vertex})| = 1$ and typ(panel) is the union of the types of the vertices on that panel. Also, any one-subset of I is type of a vertex of any chamber. Finally, W is type-preserving (see [1]).

1.2. *System of Apartments*

1.2.1. For the definition of a *discrete system of apartments* of type $\tilde{A}_2$ we refer to Tits [9]. Let Δ be such a discrete system of apartments, then Δ is the (non-disjoint) union of copies of the apartment $\mathscr{A}$ of (1.1). Recall that a vertex, panel, chamber, quarter, etc., in Δ is a subset of Δ which is a vertex, resp. a panel, chamber, quarter, etc., in some apartment of Δ. Denote the set of vertices of Δ by Ve(Δ). We now show how we can see Δ as an abstract building. We define the simplicial chamber complex $|\Delta|$ as follows: the set of vertices Ve($|\Delta|$) is the set Ve(Δ). The 1-dimensional complexes are the pairs of vertices of Δ belonging to the a common panel. The chambers of $|\Delta|$ are the triples of Ve(Δ) belonging to a common chamber of Δ. Similar definitions for apartments, quarters, pennels, etc., in $|\Delta|$. It follows now that $|\Delta|$ is an abstract building of type $\tilde{A}_2$. Viewed as a chamber system (see [5], [8]), the rank is 3, and so [5, Cor.(2.3)] implies:

PROPOSITION (1.2.1). *A 2-dimensional simplicial complex is a(n abstract) triangle building if it is simply connected and there is a suitable type map on the set of vertices such that any residue* (*in the usual sense of* [1]) *is a spherical building of type* $\tilde{A}_2$.

By Tits [9, Th. 1], $|\Delta|$ admits a complete (or a maximal) set of apartments. In this paper, we assume that any triangle building (and equivalently any corresponding discrete system of apartments Δ) is endowed with a complete set of apartments. Also, we shall always view a triangle building as a discrete system of apartments; if not, then we use the term *abstract* triangle building.

Recall that there is a *distance map* d on Δ defined as follows: if $x, y \in \Delta$, then there is an apartment – which we can denote without loss of generality by $\mathscr{A}$ – containing both x and y. By definition, $d(x, y) = d_{\mathscr{A}}(x, y)$ (see 1.1). A *germ of quarters* in Δ is an equivalence class in the set of quarters of Δ with respect to the equivalence relation: Q_1 is equivalent with Q_2 if $Q_1 \cap Q_2$ is a truncated quarter (see Tits [9]).

1.2.2. DEFINITIONS AND PROPERTIES. (1) One deduces that (Δ, d) is a metric space (see [9]).

(2) Let $x, y \in \Delta$. Then the closed line segment joining x to y is denoted by $[x, y]$. If B is an apartment containing x and y, then B also contains $[x, y]$. A *convex* set E is defined as usual (E is convex if for any x, y in E, $[x, y] \subset E$). If E is convex and can be written as the union of chambers or faces of chambers, then E is called a *chamber convex* set.

(3) Two chambers are called *adjacent* if they meet in a panel. A *gallery* joining two points x and y is a chain $(C_0, C_1, \ldots, C_m)$ of chambers such that consecutive chambers are adjacent and $x \in C_0$ and $y \in C_m$. m is called the *length* of the gallery. A gallery joining x and y with minimal length is called a gallery *stretched* between x and y (see [7]).

(4) The next two propositions can be found in [9]:

PROPOSITION (1.2.2), *Any germ of quarters contains a quarter with source any vertex of* Δ (see [9, Prop. 5]).

PROPOSITION (1.2.3). *Any subset of* Δ *which has the 'metric structure' of a subset of an apartment, is contained in an apartment* (see [9, Th. 1]).

A direct consequence of the last proposition is the next

COROLLARY (1.2.4). *Any convex set of* Δ *which can be embedded in* $\mathscr{A}$, *is contained in an apartment.*

1.3. *The Building at Infinity*

Two pennels p and q are *parallel* if they are on bounded distance from one another, i.e. if the sets $\{d(x, q) | x \in p\}$, and $\{d(y, p) | y \in q\}$ are bounded (where $d(x, q) = \inf\{d(x, y) | y \in q\}$ similar for $d(y, p)$). This relation is clearly an equivalence relation and the class of a pennel p w.r.t. that relation is denoted by $c(p)$.

Now define the 1-dimensional simplicial chamber complex $(\Delta_\infty, \mathrm{Ch}(\Delta_\infty))$ as follows:

Δ_∞ = parallel classes of pennels.

$\mathrm{Ch}(\Delta_\infty) = \{\{c(p), c(q)\} \mid$ there exist representatives p' and q' of $c(p)$ and $c(q)$ resp. such that $p \cup q$ is the boundary of a quarter$\}$.

PROPOSITION (1.3). $(\Delta_\infty, \mathrm{Ch}(\Delta_\infty))$ *is the rank* 2 *building corresponding to a projective plane. If B is an apartment of* Δ, *then the six germs of quarters of B define six elements of* $\mathrm{Ch}(\Delta_\infty)$ *which define a unique apartment* B_∞ *of* Δ_∞. *The map* $B \to B_\infty$ *is a bijection from the set of apartments of* Δ *to the set of apartments of* Δ_∞. *The 'trace at infinity' of a wall L of* Δ *is a pair* $\{M, Q\}$ *in* Δ_∞ *which is not a chamber, i.e.* $\{M, Q\}$ *is a wall (a non-incident point-line pair).*

NOTATION. From now on, Δ always denotes a triangle building. The building at infinity is denoted by Δ_∞ and one of the mutually dual projective geometries of Δ_∞ by PG(Δ) (cf. 4.2.1). We also write:

$$\begin{aligned}
\mathrm{Ch}(\Delta) &= \text{set of chambers of } \Delta \\
\mathrm{Pa}(\Delta) &= \text{set of panels of } \Delta \\
\mathrm{Ve}(\Delta) &= \text{set of vertices of } \Delta \\
\mathrm{Qu}(\Delta) &= \text{set of quarters of } \Delta \\
\mathrm{Pe}(\Delta) &= \text{set of pannels of } \Delta
\end{aligned}$$

2. Valuations on a PTR

2.1. *Planar Ternary Rings*

This section is a rewrite of Chapter V in [3].

2.1.1. Let R be a set not containing the symbol ∞ and let T be a ternary operation on R. Then we call (R, T) a *ternary ring*. Moreover, we call (R, T) a *planar ternary ring*, or *PTR* for short, if (O), (A), (B), (C), (D) and (E) below hold for all a, b, c, d in R:

(O) $0, 1 \in R$.

(A) $T(a,0,c) = T(0,b,c) = c$.

(B) $T(a,1,0) = T(1,a,0) = a$.

(C) If $a \neq c$, then there is a unique $x \in R$ such that $T(x,a,b) = T(x,c,d)$.

(D) There is a unique $x \in R$ such that $T(a,b,x) = c$.

(E) If $a \neq c$, then there is a unique $(x,y) \in R^2$ such that $T(a,x,y) = b$ and $T(c,x,y) = d$.

THEOREM (2.1.1). *If* (R, T) *is a PTR then the structure* PG(R, T) *defined as follows is a projective plane. The points of* PG(R, T) *are the ordered pairs* (x, y) *where* $x, y, \in R$ *together with elements of the form* (x) *where* $x \in R$ *and* (∞). *Lines are represented by ordered pairs* $[m, k]$ *where* $m, k \in R$ *together with elements of the form* $[m]$, *where* $m \in R$ *and* $[\infty]$. *Incidence is defined in the following manner:*

$$(x,y) \text{ is on } [m,k] \Leftrightarrow T(m,x,y) = k,$$

$$(x,y) \text{ is on } [k] \Leftrightarrow x = k,$$

$$(x) \text{ is on } [m,k] \Leftrightarrow x = m,$$

$$(x) \text{ is on } [\infty] \text{ for all } x \in R \text{ and } (\infty) \text{ is on } [k] \text{ for all } k \in R.$$

$$(\infty) \text{ is on } [\infty].$$

(see [3, Th. 5.2])

Note that we introduced the notation PG(R, T). Denote also $O = (0,0)$, $X = (0)$, $Y = (\infty)$ and $E = (1,1)$, then (O, X, Y, E) is a non-degenerate quadrangle in PG(R, T). We call (R, T) a *coordinatizing PTR of* PG(R, T) *with respect to* (O, X, Y, E). By [3, Th. 5.1], any projective plane – up to isomorphism – can be coordinatized by a PTR with respect to any non-degenerate quadrangle in the above way. However, distinct quadrangles may give rise to non-isomorphic PTR's. We now introduce some further notation:

Let $V = (P(V), L(V), \mathrm{I})$ be a projective plane with $P(V)$ the 'point set' $L(V)$ the 'line set' and 'I' the incidence relation. If P, Q are distinct points, then we denote by PQ the unique line incident with both P and Q. If M, L are distinct lines, then we denote by $L \cap M$ the unique point incident with both L and M. (Any line is viewed as the set of points incident with it.)

2.1.2. Let (R, T) be a PTR, then one defines a product in R by

$$a \cdot b = T(a, b, 0)$$

and a sum

$$a + b = T(1, a, b)$$

Also recall the following:

(1) (R, T) is *linear* if $T(a, b, c) = (a \cdot b) + c$.
(2) (R, T) is a *quasifield* if (R, T) is linear, $(R, +)$ is a group and the left distributive law holds in $(R, +, \cdot)$. Note that $(R, +)$ is Abelian in that case.
(3) (R, T) is a *nearfield* if (R, T) is a quasifield and $(R, \cdot)$ is a group.
(4) (R, T) is a *division ring* if (R, T) is a quasifield and also the right distributive law holds in $(R, +, \cdot)$.
(5) (R, T) is a *skewfield* if (R, T) is both a nearfield and a division ring.
(6) (R, T) is a *field* if (R, T) is a skewfield and $(R, \cdot)$ is Abelian.

2.2. *PTR's with Valuation v*

Let (R, T) be a PTR.. If $v: R^2 \to Z \cup \{\infty\}$ is a map satisfying (d1), (d2), (d3) and (d4) below, then we call (R, T, v) a *PTR with valuation* or briefly a *V-PTR*. We extend the order relation and the addition in Z (as usual: $\infty > z$; $z + \infty = \infty + z = \infty + \infty = \infty\ \forall z \in Z$) to $Z \cup \{\infty\}$ and then we have:

(d1) $v(a, b) = \infty \Leftrightarrow a = b$.
(d2) If $v(a, b) < v(b, c)$, then $v(a, b)$.

(d3) Suppose $T(a_1,b_1,c_1) = T(a_1,b_2,c_2)$ and $T(a_2,b_1,c_1) = T(a_2,b_2,c_3)$, then

$$v(a_1,a_2) + v(b_1,b_2) = v(c_2,c_3).$$

d(4) v is onto.

We usually write $v(x)$ instead of $v(x,0)$.

The next proposition was proved in [10]:

PROPOSITION (2.2). *Any V-PTR (R,T,v) has the following properties:*

(v1) *v is onto.*
(v2) $v(a,b) = v(b,a)$.
(v3) $v(a,c) \geqslant \inf\{v(a,b),\ v(b,c)\}$ *and if* $v(a,c) \neq v(b,c)$, *equality holds.*
(v4) $v(a,b) = \infty \Leftrightarrow a = b$.
(v5) $v(1) = 0$; $v(0) = \infty$.

For (v6) *through* (v11) *we suppose* $T(a_i,b_i,c_i) = d_i$, $i = 1, 2$.

(v6) *If* $a_1 = a_2$ *and* $b_1 = b_2$, *then* $v(c_1,c_2) = v(d_1,d_2)$.
(v7) *If* $a_1 = a_2$ *and* $c_1 = c_2$, *then* $v(b_1,b_2) + v(a_1) = v(d_1,d_2)$.
(v8) *If* $a_1 = a_2$ *and* $d_1 = d_2$, *then* $v(b_1,b_2) + v(a_1) = v(c_1,c_2)$.
(v9) *If* $b_1 = b_2$ *and* $c_1 = c_2$, *then* $v(a_1,a_2) + v(b_1) = v(d_1,d_2)$.
(v10) *If* $b_1 = b_2$ *and* $d_1 = d_2$, *then* $v(a_1,a_2) + v(b_1) = v(c_1,c_2)$.
(v11) *If* $c_1 = c_2$ *and* $d_1 = d_2$, *then*
$v(a_1,a_2) + v(b_1) = v(b_1,b_2) + v(a_2)$
and
$v(a_1,a_2) + v(b_2) = v(b_1,b_2) + v(a_1)$
and in particular
$v(a_1) + v(b_1) = v(a_2) + v(b_2)$.
(v12) *If* $T(a,b,c) = d$, *then* $v(a) + v(b) = v(c,d)$.

For (v13) *through* (v16) *we suppose*

$$T(a_1,b_1,c_1) = d_1 = T(a_1,b_2,c_2),$$

$T(a_2,b_1,c_1) = d_2$ *and* $T(a_3,b_3,c_3) = d_3$.

(v13) *If* $a_2 = a_3$; $b_2 = b_3$ *and* $c_2 = c_3$, *then*
$v(a_1,a_2) + v(b_1,b_2) = v(d_2,d_3)$.
(v14) *if* $a_2 = a_3$; $b_2 = b_3$ *and* $d_2 = d_3$, *then*
$v(a_1,a_2) + v(b_1,b_2) = v(c_2,c_3)$.
(v15) *if* $a_2 = a_3$; $c_2 = c_3$ *and* $d_2 = d_3$, *then*
$v(a_1,a_2) + v(b_1,b_2) = v(b_2,b_3) + v(a_2)$
and
$v(a_1,a_2) + v(b_1,b_3) = v(b_2,b_3) + v(a_1)$.

(v16) *If* $b_2 = b_3$; $c_2 = c_3$ *and* $d_2 = d_3$, *then*
$v(a_1, a_2) + v(b_1, b_2) = v(a_2, a_3) + v(b_2)$
and
$v(a_1, a_3) + v(b_1, b_2) = v(a_2, a_3) + v(b_1)$.

REMARK (2.2). If (R, T) is a quasifield, then (d3) can be replaced by:

(Q) $\quad v(a_1 \cdot b - a_2 \cdot b) = v(a_1 - a_2) + v(b)$

If (R, T) is a division ring, then (d3) can be replaced by:

DR $\quad v(a \cdot b) = v(a) + v(b)$

(see [10, §1.2])

EXAMPLES (2.2). (1) Q_p with the usual p-adic valuation.

(2) The field $K((t))$ of Laurent series over the field K. The valuation v for non-zero elements is defined by:

$$v(\Sigma a_l t^l) = \inf \{l \in Z \mid a_l \neq 0\}.$$

Given a field K and a finite automorphism group G of $K((t)), +, \cdot$ generated by $S = \{s_j \mid j \in J\}$, one defines the 'norm map'

$$n \colon K((t)) \to N \colon f(t) \mapsto \prod_{g \in G} f(t)^g.$$

Choose a map $\Phi \colon N \to G$ arbitrarily but such that Laurent series with same valuation go to the same automorphism. Then we denote the corresponding (André) quasifield $K((t)), +, \odot$ (where $f_1 \odot f_2 = f_1 \cdot f_2^{\Phi(n(f_1))}$, see [3, p. 187]) by $K(\Phi, S)$ (as in [10]). Particular examples are:

(3) Suppose K is a field with finite characteristic. Set $S = \{s\}$, where

$$s \colon K((t)) \to K(t)) \colon f(t) \to f\left(\frac{1}{1-t}\right).$$

Then $K(\Phi, s)$ is a V-PTR.

(4) Suppose K has a non-trivial root of identity, say $e^m = 1$. Set $S = \{s\}$, with

$$s \colon K((t)) \to K((t)) \colon f(t) \to f(et).$$

Then again $K(\Phi, s)$ is a V-PTR.

(5) If $\alpha^* = \{\alpha_j^* \colon K \to K \mid j \in J\}$ generates a finite automorphism group of K, then set

$$\alpha = \{\alpha_j \colon K((t)) \to K((t)) \colon \Sigma a_l t^l \to \Sigma \alpha_j^*(a_l) t^l \mid j \in J\}$$

Again $K(\Phi, \alpha)$ is a V-PTR.

(6) Suppose $K = \mathrm{GF}(q^2)$, q odd and $\alpha^*: K \to K: x \to x^q$. Let α be the automorphism of $K((t))$ induced by α^* as above. Now we choose Φ such that $\Phi(f) = \mathrm{id}$ if f is a square in $\mathrm{GF}(q)((t))$ and $\Phi(f) = \alpha$ otherwise (one can indeed check that $N \subseteq \mathrm{GF}(q)((t))$). Now $K(\Phi, \alpha)$ is a nearfield with valuation and we denote it by $\mathrm{GF}((q^2))$.

Our last class of examples is a distinct type:

(7) Let (R, T) be a quasifield. Then we can define the set of formal Laurent series $R((t))$ with coefficients in R. Addition and multiplication are defined exactly in the same way as for fields. It is a trivial verification that $R((t))$ is a quasifield. Moreover, if (R, T) is a division ring, then $R((t))$ is as well. One checks $R((t))$ is a V-PTR (v as above).

2.3. *Positive Valuated Ternary Rings*

Let (S, T) be a ternary ring (not necessarily planar), and $v: S \times S \to \mathbb{N} \cup \{\infty\}$ be a map such that for all a, b, c, d in S:

(PO) $0, 1 \in S$.

(PA) $T(a, 0, C) = T(0, b, c) = c$.

(PB) $T(a, 1, 0) = T(1, a, 0) = a$.

(PC) Suppose $v(b, d) \geqslant v(a, c)$, then there exists $x \in S$ such that $(T(x, a, b) = T(x, c, d)$.

(PD) There exists $x \in S$ such that $T(a, b, x) = c$.

(PE) Suppose $v(b, d) \geqslant v(a, c)$, then there is a couple (x, y) such that $T(a, x, y) = b$ and $T(c, x, y) = d$.

(Pd1) $v(a, b) = \infty, \quad a = b$.

(Pd2) If $v(a, b) < v(b, c)$, then $v(a, c) = v(a, b)$.

(Pd3) Suppose $T(a_1, b_1, c_1) = T(a_1, b_2, c_2)$ and $T(a_2, b_1, c_1) = T(a_2, b_2, c_3)$, then $v(a_1, a_2) + v(b_1, b_2) = v(c_2, c_3)$.

(Pd4) v is onto.

Then we call (S, T, v) a *positive valuated ternary ring* or briefly a *PV-TR* Again we write $v(x)$ for $v(x, 0)$, and then we have:

PROPOSITION (2.3). *If* (S, T, v) *is a PV-TR, then* (v1) *through* (v16) *hold. Moreover,* x, x, (x, y) *in resp* (PC), (PD), (PE) *are unique.*

For a proof, see [10, §4.3] and [11, §3.4.1]

EXAMPLES (2.3). (1) Let (R, T, v) be a V-PTR and define $R^+ = \{r \in R \mid v(r) \geqslant 0\}$, then one can check that (R^+, T, v) is a PV-TR.

(2) Suppose (R, T) is a PTR. Define $S = R[[t]] = \{\Sigma_{i=0}^{\infty} a_i t^i \mid a_i \in R\}$ and

extend T to S by the rule: the coefficient of t^n of $T(\Sigma a_l t^l, \Sigma b_l t^l, \Sigma c_l t^l)$ is:

$T(a_n, b_0, T(a_{n-1}, b_1, T(a_{n-2}, b_2, T(\ldots T(a_2, b_{n-2}, T(a_1, b_{n-1}, T(a_0, b_n, c_n)))\ldots)$.

Define $v\colon S \times S \to \mathbb{N} \cup \{\infty\}$ by:

$$v\left(\sum a_l t^l, \sum b_l t^l\right) = \begin{cases} \inf\{\mathrm{n} \mid \mathrm{a}_n \neq \mathrm{b}_n\}, & \text{if } \sum a_l t^l \neq \sum b_l t^l \\ \infty, & \text{otherwise} \end{cases}$$

(S, T, v) is a PV-TR (see [10, §7.3]).

2.4. *Complete Valuations*

2.4.1. Let (R, T, v) be a V-PTR. We can define a map $\delta\colon R \times R \to \mathbb{N}$ by

$$\delta(a, b) = 2^{-v(a,b)}.$$

(R, δ) is a metric space sine by (v3), $\delta(a, c) \leqslant \sup\{\delta(a, b), \delta(b, c)\}$. If (R, δ) is a complete metric space, then we say that v is *complete* and that (R, T, v) is *a complete V-PTR*, briefly a *CV-PTR*. For instance all the examples in (2.2) are complete. For a non-complete example, let K be a field and $K(t) \subset K((t))$ the field of rational functions in one variable. The valuation on $K((t))$ in (2.2) restricts to a valuation on $K(t)$ which is not complete.

PROPOSITION (2.4.1). *Suppose* (a_n), (b_n), (c_n), (d_n), $n \in \mathbb{N}$ *are sequences in a CV-PTR* (R, T, v). *Suppose that* $T(a_n, b_n, c_n) = d_n$ *for all* n. *If three of the above sequences are Cauchy, then also the fourth (granted that* (a_n) *or* (b_n) *does not converge to* 0) *does. If* a, b, c, d *are their resp limits, then*

$$T(a, b, c) = d.$$

Proof. Suppose, e.g., (b_n), (c_n), (d_n) are Cauchy. $b \neq 0$ implies that for $n \in \mathbb{N}$ large enough, $v(b_n)$ is constant (namely $v(b_n) = v(b)$). Define for all $n \in \mathbb{N}$ x_n:

(1) $\qquad T(a_n, b_n, x_n) = d.$

By (v6), we have $v(c_n, x_n) = v(d, d_n)$, hence (x_n) converges to c. Define (y_n) as

(2) $\qquad T(a_n, y_n, c) = d$

for all n for which this is well defined. Note that y_n is not well defined only for a finite number of n's, otherwise c would be equal to d (after all, there are infinitely many a_n's zero and thus infinitely many c_n's equal to d_n's). By (v12), $v(a_n) = v(c_n, d_n) - v(b_n)$, and so (a_n) would converge to 0, hence the result would follow. Hence we can assume that (y_n) is an infinite sequence and, by the above argument, also that $c \neq d$. By (1), (2), (v8) and (v12), we have:

$$v(b_n, y_n) = v(x_n, c) + v(b_n) - v(x_n, d).$$

For n large enough, $v(x_n, d)$ is constant (and finite). Hence, since (x_n) converges to c, (y_n) converges to b. Since $b \neq 0$, we can define a as $T(a, b, c) = d$. This implies together with (2) and (vii) that $v(a, a_n) = v(y_n, b) + v(a) - v(y_n)$. Hence (a_n) is Cauchy and converges to a. QED

2.4.2. In a similar way, we can define *complete positive valuated ternary rings*, or briefly *CPV-TR's*. Proposition (2.4.1) is also true for CPV-TR's.

EXAMPLES (2.4.2). (1) If (R, T, v) is complete, then so is (R^+, T, v).

(2) For any PTR(R, T), $(R[[t]], T, v)$ as defined in Examples (2.3) is complete.

REMARK (2.4.2) Proposition (2.4.1) remains true for V-PTR's and PV-TR's. Moreover, if three of the sequences converge, so does the fourth.

3. Triangle buildings defined by V-PTR's

This section is an abstract of [10]. The reader should refer to that paper for proofs and detailed information.

3.1. *The Geometries W_n*

The way a triangle building is defined by a V-PTR is via geometries W_n, $n \in \mathbb{N}$, which we define now.

Let (R, T, v) be a V-PTR. Let

$$R^+ = \{x \in R \mid v(x) \geqslant 0\}, \qquad R^- = \{x \in R \mid v(x) < 0\}$$

$$w(x, y) = v(x, y) - v(x) - v(y).$$

The relation E_n, with

$$xE_n y \quad \text{if} \begin{cases} (x, y) \in R^+ \times R^+ \text{ and } v(x, y) \geqslant n & \text{for } n > 0 \\ (x, y) \in R^- \times R^- \text{ and } w(x, y) \geqslant n & \text{for } n > 0 \\ (x, y) \in R \times R \quad \text{for } n = 0 \end{cases}$$

is an equivalence relation (see [10, §2.3]). We denote the quotient sets by $R_n = R/E_n$, $R_n^+ = R^+/E_n^+$, $R_n^- = R^-/E_n^-$ $(n \in \mathbb{N})$ where $E_n^+ = E_n/R^+ \times R^+$, $E_n^- = E_n/R^- \times R^-$. For $n = 0$, $W_0 = (P(W_0), L(W_0), \mathrm{I})$ is defined as the degenerate geometry $P(W_0) = L(W_0) = \{(0,0)\} = \{[0,0]\}$ and $(0,0)\,\mathrm{I}\,[0,0]$.

Now let n > 0. We define $W_n = (P(W_n), L(W_n), \mathrm{I})$:

(i) The point set $P(W_n) = R_n^+ \times R_n^+ \cup R_n^+ \times R_n^- \cup R_n^- \times R_n^-$. A point is denoted by round brackets, e.g. (x, y).

(ii) The line set $L(W_n) = R_n^+ \times R_n^+ \cup R_n^- \times R_n^+ \cup R_n^- \times R_n^-$. A line is denoted by square brackets, e.g. $[m, k]$.

NOTATION. (1) If $r \in R_n^+$, then we write r^+. Similarly, if $r \in R_n^-$, we write r^-.

Writing just r leaves the possibility open, if no hypotheses were made before. (2) If $r \in R_n$, then $\hat{r}$ denotes a representative in R of r. It will always be clear whether $\hat{r}$ is unique, variable, arbitrary, etc.

(iii) Incidence is defined as follows:

(I1) $(x^+, y^+)\,\mathrm{I}\,[m^+, k^+]$ if there exist $\hat{x}$, $\hat{y}$, $\hat{m}$, $\hat{k}$ such that

$$T(\hat{m}, \hat{x}, \hat{y}) = \hat{k}$$

(I2) $(x^+, y^+)\,\mathrm{I}\,[m^-, k^+]$ if there exist $\hat{x}$, $\hat{y}$, $\hat{m}$, $\hat{k}$ and $b \in R$ such that:

$$T(\hat{m}, \hat{k}, 0) = b \quad \text{and} \quad T(\hat{m}, \hat{x}, \hat{y}) = b$$

(I3) $(x^+, y^-)\,\mathrm{I}\,[m^+, k^+]$ if there exist $\hat{x}$, $\hat{y}$, $\hat{m}$, $\hat{k}$ and $a \in R$ such that:

$$T(\hat{x}, \hat{y}, a) = 0 \quad \text{and} \quad T(\hat{m}, \hat{y}, a) = \hat{k}$$

(I4) $(x^+, y^-)\,\mathrm{I}\,[m^-, k^-]$ if there exist $\hat{x}$, $\hat{y}$, $\hat{m}$, $\hat{k}$ and a, $b \in R$ such that:

$$T(\hat{x}, \hat{y}, a) = 0 \;\; T(b, \hat{k}, 0) = \hat{m} \quad \text{and} \quad T(b, \hat{y}, a) = \hat{m}$$

(I5) $(x^-, y^-)\,\mathrm{I}\,[m^-, k^+]$ if there exist $\hat{x}, \hat{y}, \hat{m}, \hat{k}$ and $a, b \in R$ such that:

$$T(\hat{x}, a, \hat{y}) = 0, \quad T(\hat{m}, \hat{k}, 0) = b \quad \text{and} \quad T(\hat{m}, a, \hat{y}) = b$$

(I6) $(x^-, y^-)\,\mathrm{I}\,[m^-, k^-]$ if there exist $\hat{x}$, $\hat{y}$, $\hat{m}$, $\hat{k}$ and a, $b \in R$ such that:

$$T(\hat{x}, a, \hat{y}) = 0, \quad T(b, \hat{k}, 0) = \hat{m} \quad \text{and} \quad T(b, a, \hat{y}) = \hat{m}$$

Symbolically, we write this definition as:

(I) $\quad P\,\mathrm{I}\,L$ if there exist $\hat{P}, \hat{L}$ such that $\hat{P}\,\mathrm{I}\,\hat{L}$.

EXAMPLES (3.1). (1) $n = 1$. W_1 is a projective plane coordinatized by the PTR (R_1^+, T_1), where $T_1(a, b, c) = d$ if there exist $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ such that $T(\hat{a}, \hat{b}, \hat{c}) = \hat{d}$. T_1 is well defined. (See [10, §2.7.2].)

(2) Suppose R is a skewfield, then W_n is the ring geometry over R_n^+. For example, $R = \mathrm{GF}(q)((t))$ or $\mathrm{GF}(q)(t)$ with the usual valuation, then

$$R_n^+ \cong \mathrm{GF}(q)[t]/t^n = 0.$$

If $R = Q_p$ with the p-adic valuation, then $R_n^+ = Z/p^n Z$ (see [10, §2.7.3]).

3.2. *Properties of W_n*

In fact, W_n is an n-uniform Hjelmslev plane (see [11]). But we need a little more than that!

3.2.1. We define the maps

$$\prod_j^n : R_n \to R_j : r_n \to r_j = \hat{r}_n / E_j (j \leqslant n)$$

where $\hat{r}_n$ is arbitrary. Π_j^n is well defined ([10, §3.5.5]) and can be extended to $P(W_n)$ and $L(W_n)$ via the 'coordinates'. It turns out that Π_j^n preserves incidence and is a surjective morphism ([10, §3.5.5]).

3.2.2. We define the '*partial valuation map*' u in W_n as follows: Let $P, Q \in P(W_n)$, L, $M \in L(W_n)$. Then we define:

$$u(P,Q) = \sup\left\{ j \in \mathbb{N} \mid \prod_j^n(P) = \prod_j^n(Q), 0 \leqslant j \leqslant n \right\},$$

$$u(L,M) = \sup\left\{ j \in \mathbb{N} \mid \prod_j^n(L) = \prod_j^n(M), 0 \leqslant j \leqslant n \right\},$$

$$u(P,L) = u(L,P) = \sup\left\{ j \in \mathbb{N} \mid \prod_j^n(P) \, \mathrm{I} \prod_j^n(L), 0 \leqslant j \leqslant n \right\}.$$

3.2.3. W_n now has the following three basic properties (PS), (RP) and (ND):

(PS) For any $P \in P(W_n)$ and any $k \in \{0, 1, \ldots, n-1\}$, we have:

$$\left(\prod_{k+1}^n\right)^{-1}\left(\prod_{k+1}^n(P)\right) \subsetneqq \left(\prod_j^n\right)^{-1}\left(\left(\prod_j^n(P)\right)\right)$$

Similarly for lines.

Note that $(\Pi_k^n)^{-1}$ determines a partition $P_k(W_n)$ (resp. $L_k(W_n)$) of $P(W_n)$ (resp. $L(W_n)$).

(RP) Let $P, Q \in P(W_n)$ and L, $M \in L(W_n)$. Again we view any line as the set of points incident with it. Let $k \leqslant \inf\{u(Q,L), u(L,P), u(P,M)\}$. Then

(i) $L \cap M \neq \emptyset$ and dually.

(ii) $u(QM) \geqslant k \Leftrightarrow u(P,Q) + u(L,M) \geqslant k$.

(ND) W_1 contains a non-degenerate quadrangle.

With a slightly different notation, this is proved in [10, §3.2.2] for (PS); [10, §3.5] for (RP) and [10, §2.7.2] for (ND). Note that we proved on our way:

PROPOSITION (3.2.3). *Let $P \in P(W_n)$; $L \in L(W_n)$. Then $P \mathrm{I} L$ is equivalent to each of the following two conditions:*

(i) *For any representative $\hat{P}$ of P, there exists $\hat{L}$ such that $\hat{P} \mathrm{I} \hat{L}$*

(ii) *For any representative $\hat{L}$ of L, there exists $\hat{P}$ such that $\hat{P} \mathrm{I} \hat{L}$*

(see [10, §3.5.2 and §3.5.3])

3.3. *Definition of* Δ

3.3.1. One can see easily that a line is completely determined by the set of points incident with it. Also the dual holds. For the next definition, we identify a line (resp. a point) with the set of points (resp. lines) incident with it. We define:

$$\tilde{B}_n^j = \{(\Pi_j^n)^{-1}(\Pi_j^n(P)) \cap L \mid P \in P(W_n), L \in L(W_n), P \mathrm{I} L\}$$
$$\bar{B}_n^{n-j} = \{(\Pi_j^n)^{-1}(\Pi_j^n(L)) \cap P \mid P \in P(W_n), L \in L(W_n), P \mathrm{I} L\};$$

for example,

$$\tilde{B}_n^n \cong P_n(V_n) \cong \bar{B}_n^n$$
$$\tilde{B}_n^0 \cong L_n(V_n) \cong \bar{B}_n^0.$$

There is a natural bijective correspondence between $\tilde{B}_n^k$ and $\bar{B}_n^k$ as follows: If $b_p \in \tilde{B}_n^k$, then there is a unique $b_1 \in \bar{B}_n^k$ such that (b_p, b_1, I) is a generalized digon and b_1 is maximal with that property, in other words if (b_p, b_1', I) is also a generalized digon with $b_1' \subset L(W_n)$, then $b_1' \subset b_1$ (see [10, §5.1.9]). We identify b_p and b_1 and denote it by b. The set of all such b's is denoted by B_n^k. We define the maps Π_{n-1}^p and Π_{n-1}^1 on B_n^k as:

$$\left.\begin{aligned} \Pi_{n-1}^p(b) &= \Pi_{n-1}^n(b_p) \varepsilon \tilde{B}_{n-1}^k \cong B_{n-1}^k \\ \Pi_{n-1}^1(b) &= \Pi_{n-1}^n(b_1) \in \bar{B}_{n-1}^{k-1} \cong B_{n-1}^{k-1} \end{aligned}\right\} \quad 0 < k < n.$$

Note that $b_p \subset b_p' \Leftrightarrow b_1' \subset b_1$.

3.3.2. We now define a simplicial chamber complex Δ together with a type map 'typ' on the set of vertices $\mathrm{Ve}(\Delta)$ of Δ. In what follows, $\mathrm{Pa}(\Delta)$ denotes the set of 1-simplices of Δ and $\mathrm{Ch}(\Delta)$ the set of chambers of Δ.

(i) $\mathrm{Ve}(\Delta)$ = union of all possible B_n^k over k and n.

(ii) typ: $\mathrm{Ve}(\Delta) \to Z/3Z: b \in B_n^k \to n + k \pmod 3$)

(iii) $\{b, b'\} \in \mathrm{Pa}(\Delta)$ if (1) $b'_p \subset b_p$ for $b \in B_n^k$ and $b' \in B_n^{k+1}$

or (2) $b' = \Pi_{n-1}^p(b)$ for $b \in B_n^k$

or (3) $b' = \Pi_{n-1}^1(b)$ for $b \in B_n^k$

(iv) $\{b, b', b''\} \in \mathrm{Ch}(\Delta)$ if $\{\{b, b'\}, \{b', b''\}, \{b'', b\}\} \subset \mathrm{Pa}(\Delta)$.

In [10], we prove that Δ is a triangle building.

3.4. *Case of a PV-TR*

In case we are dealing with a positive valuated ternary ring (S, T), the definition of W_n is slightly different. The equivalence relation E_n^+ is still well defined, and so is the quotient set $S_n = S/E_n^+$.

For $n = 0$, W_0 is the same as above. Suppose now $n > 0$. Let $S^0 = \{r \in S \mid v(r) > 0\}$, then also $S_n^0 = S^0/E_n^+$ is well defined.

$W_n = (P(W_n), L(W_n), \mathrm{I})$ is defined as follows:

(i) $P(W_n) = \{(w, y) \in S_n \times S_n\} \cup \{(x)_y \mid (x, y) \in S_n \times S_n^0\}$
$\cup \{(\infty_x)_y \mid (x, y) \in S_n^0 \times S_n^0\}$.

(ii) $L(W_n) = \{[m, k] \in S_n \times S_n\} \cup \{[k]_m \mid (m, k) \in S_n^0 \times S_n\}$
$\cup \{[\infty_k]_m \mid (m, k) \in S_n^0 \times S_n^0\}$.

(iii) With the same notation as in (3.1), incidence is defined as:

(PI1) $(x, y)\, \mathrm{I}\, [m, k]$ if $T(\hat{m}, \hat{x}, \hat{y}) = \hat{k}$ for some $\hat{m}, \hat{x}, \hat{y}, \hat{k}$

(PI2) $(x, y)\, \mathrm{I}\, [k]_m$ if $T(\hat{m}, \hat{y}, \hat{x}) = \hat{k}$ for some $\hat{m}, \hat{x}, \hat{y}, \hat{k}$

PI3 $(x)_y\, \mathrm{I}\, [m, k]$ if $T(\hat{k}, \hat{y}, \hat{x}) = \hat{m}$ for some $\hat{m}, \hat{x}, \hat{y}, \hat{k}$

(PI4) $(x)_y\, \mathrm{I}\, [\infty_k]_m$ if $T(\hat{m}, \hat{x}, \hat{y}) = \hat{k}$ for some $\hat{m}, \hat{x}, \hat{y}, \hat{k}$

(PI5) $(\infty_x)_y\, \mathrm{I}\, [k]_m$ if $T(\hat{k}, \hat{y}, \hat{x}) = \hat{m}$ for some $\hat{m}, \hat{x}, \hat{y}, \hat{k}$

(PI6) $(\infty_x)_y\, \mathrm{I}\, [\infty_k]_m$ if $T(\hat{k}, \hat{x}, \hat{y}) = \hat{m}$ for some $\hat{m}, \hat{x}, \hat{y}, \hat{k}$.

With that definition, we can prove again the properties (PS), (RP) and (ND), for suitable defined Π_n^k and u. Since Δ is defined only by using the geometries W_n and their properties, (S, T) implies the existence of a triangle building Δ. If (R, T, v) is a skewfield, then both ways (via R; via $S = R^+$) give rise to the same geometries W_n and hence to the same building Δ ([10, §4.4]).

3.5. EXAMPLES. (1) The buildings obtained from skewfields with discrete valuation are all classical.

(2) The buildings obtained from the André quasifields $K(\Phi, s)$ and $K(\Phi, \alpha)$ are non-classical and some (if not all) have no non-Desarguesian residues.

(3) The buildings obtained from the formal power series $R((t))$, with R a quasifield, are non-classical and have non-Desarguesian residues.

(4) The buildings obtained from the André nearfields $\mathrm{GF}((q^2))$ are non-classical and have non-Desarguesian residues.

(5) The buildings obtained from $(R[[t]], T)$ have residue planes coordinatized by (R, T).

(6) The buildings obtained from R^+, where (R, T, v) is a V-PTR, are in general not necessarily isomorphic to those obtained from R itself.

4. V-PTR'S DEFINED BY A TRIANGLE BUILDING

The aim of this section is to show:

THEOREM (4). *Any coordinatizing PTR of* $\mathrm{PG}(\Delta)$ *is a V-PTR.*

4.1. *Lemmas*

4.1.1. *Chamber convex sets*

(For general definition, see [1, §2.4].) The *chamber convex closure of two vertices a and b is by definition:*

$$\mathrm{cl}(a, b) = \bigcap \{Q \in \mathrm{Qu}(\Delta) \mid a, b \in Q\}$$

and is consequently a chamber convex set since any quarter is. It is the smallest chamber convex set containing a, b. We also have ([1, §2.4.4]):

LEMMA (4.1.1). *Suppose* $\mathrm{cl}(a, b)$ *contains at least one chamber, then it is the union of all chambers of all galleries stretched between a and b.*

By [7, §2.19(iv)]; $\mathrm{cl}(a, b)$ lies in each apartment containing a and b and hence $\mathrm{cl}(a, b)$ is either a non-degenerate parallelogram or an interval. (See Figure 2.)

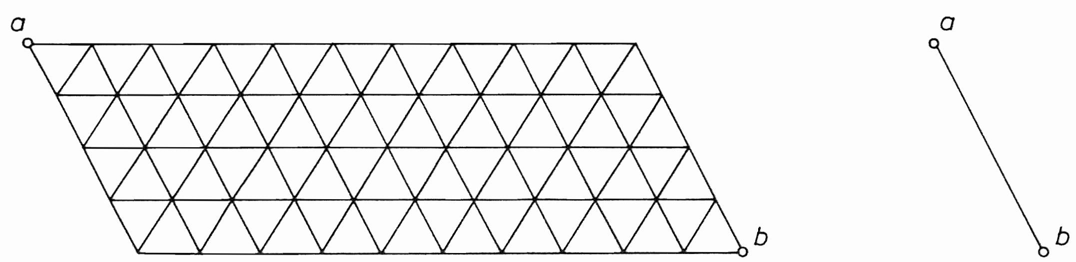

Fig. 2.

4.1.2. *The intersection of two apartments*

In general, all we can say about the intersection of two apartments A and B is that it is convex. However, if we know the mutual position of their traces at infinity A_∞ resp. B_∞, we have (see[11, §2.1.2]):

Let $A_\infty = \{x_i \mid i \,(\mathrm{mod}\ 6)$ and $x_i \mathrm{I}\, x_{i+1}\}$,

(i) $A_\infty \cap B = \emptyset$, then $A \cap B = \emptyset$ or $A \cap B$ is a bounded chamber convex set, i.e. a 6-, 5-, 4- or 3-gon bounded by intervals lying on pennels.

(ii) $A_\infty \cap B_\infty = \{x_1\}$, then $A \cap B$ is empty or a half strip which contains a pennel with trace at infinity x_1.

(iii) $A_\infty \cap B_\infty = \{x_1, x_4\}$, then $A \cap B$ is empty or a strip bounded by two parallel walls having $\{x_1, x_4\}$ as trace at infinity.

(iv) $A_\infty \cap B_\infty = \{x_1, x_2\} \Leftrightarrow A \cap B$ is a truncated quarter having $\{x_1, x_2\}$ as trace at infinity.

(v) $A_\infty \cap B_\infty = \{x_1, x_2, x_3\} \Leftrightarrow A \cap B$ is a double quarter having $\{x_1, x_2, x_3\}$ as trace at infinity.

(vi) $A_\infty \cap B_\infty = \{x_1, x_2, x_3, x_4\} \Leftrightarrow A \cap B$ is also a half apartment bounded by a wall having $\{x_1, x_4\}$ as trace at infinity and containing the three germs of quarters $\{x_1, x_2\}$, $\{x_2, x_3\}$ and $\{x_3, x_4\}$.

(vii) $A_\infty = B_\infty \Leftrightarrow A = B$.

4.1.3. *Equivalent definition of* Δ_∞

Just as the plane at infinity of a 3-dimensional affine space can be defined in two different but equivalent ways ((i) via parallel classes of lines and planes; (ii) via vector lines and vector planes after choosing an arbitrary origin), we can also define the spherical building at infinity of Δ in a second way. The next proposition follows mainly from Proposition (1.2.2):

PROPOSITION (4.1.3). *Choose an arbitrary vertex s in the triangle building Δ over $J = \{1, 2, 3\}$ and suppose without loss of generality* $\mathrm{typ}(s) = 3$. *We define an incidence structure* $(P(\Delta_\infty), L(\Delta_\infty), \mathrm{I})$ *as follows:*

(1) *$P(\Delta_\infty)$ is the set of all pennels p with source s such that* $\mathrm{typ}(s_p) = 1$, *where s_p is the vertex on p adjacent to s.*

(2) *$L(\Delta_\infty)$ is the set of all pennels l with source s such that* $\mathrm{typ}(s_l) = 2$, *where s_l is the vertex on p adjacent to s.*

Elements of $P(\Delta_\infty)$ are called point-pennels and elements of $L(\Delta_\infty)$ are called line-pennels.

(3) *A point-pennel p is incident with a line-pennel l if $p \cup l$ is the boundary of a quarter (with source s).*

$(P(\Delta_\infty), L(\Delta_\infty), \mathrm{I})$ *is a projective plane canonically isomorphic to* PG(Δ) *or its dual.*

LEMMA (4.1.4). *Let, with the same notation as in Proposition* (4.1.3), p_1 *and* p_2 *be two point-pennels. Suppose* $p_1 \cap p_2 = \{s\}$, *then* $p_1 \cup p_2$ *is the boundary of a double quarter and thus lies in an apartment.*

Proof. Let l be the line-pennel incident with p_1 and p_2. Let Q_1 resp. Q_2 be the quarters with topological border $l \cup p_1$, resp. $l \cup p_2$. Suppose s' is a vertex in both Q_1 and Q_2, but not on l, then $\mathrm{cl}(s, s')$ contains a vertex on p_1 (and also p_2) adjacent to s. So the intersection of p_1 and p_2 contains more than only s, a contradiction, hence $Q_1 \cap Q_2 = 1$. One can check (though it is not trivial) that $Q_1 \cup Q_2$ is convex, and hence by Corollary (1.2.4), $Q_1 \cup Q_2$ is a double quarter. QED

LEMMA (4.1.5). *Let* (O, X, Y, E) *be a non-degenerate quadrangle in* PG(Δ) *and let* A_O *resp.* A_X, A_Y, A_E *be the apartments of* Δ *determined by* $\{X, Y, E\}$ *resp.* $\{O, Y, E\}$, $\{O, X, E\}$, $\{O, X, Y\}$. *Then* (O, X, Y, E) *determines a unique vertex* $s = s(O, X, Y, E)$. *Moreover*:

$$A_O \cap A_X \cap A_Y \cap A_E = s,$$

and

$$A_O \cap A_X \cap A_Y = e$$

$$A_O \cap A_X \cap A_E = v$$

$$A_O \cap A_Y \cap A_E = x$$

$$A_X \cap A_Y \cap A_E = 0$$

are pennels with source $s(O, X, Y, E)$ *and with trace at infinity resp.* E, Y, X, O.

Proof. By 4.1.2(v) $A_O \cap A_X$ is a double quarter D_{YE} bounded by pennels y and e with same source $s \in \mathrm{Ve}(\Delta)$. Suppose x and o are pennels with source s and trace at infinity X, resp. O. Then $x \cap o = \{s\}$ since $x \cap D_{YE} = o \cap D_{YE} = \{s\}$. By Lemma (4.1.4) there is a double quarter D_{OX} bounded by x and o. Define D_{XY}, D_{XE}, D_{OY} and D_{OE} resp. by the double quarters bounded by $x \cup y$; $x \cup e$; $o \cup y$ and $o \cup e$ and lying in A_O or A_X. They all have source s. One can check that $D_{OX} \cup D_{XE} \cup D_{OE}$ is convex and hence an apartment, which is by uniqueness A_Y. Similar $D_{OX} \cup D_{OY} \cup D_{XY} = A_E$. Putting $s(O, X, Y, E) = s$, the result follows. QED.

4.2. *The Geometries* V_n, $n \in \mathbb{N}$

4.2.1. Let (R, T) be a coordinatizing planar ternary ring of PG(Δ) with respect to the non-degenerate ordered quadrangle (O, X, Y, E). Let $\bar{X} = OX$,

$\bar{Y} = OY$, $L_\infty = XY$. Let $s = s(O, X, Y, E)$. From now on, all pennels and (double) quarters have source s, if not stated explicitly otherwise. So let $o, x, y, e, \bar{x}, \bar{y}\, l_\infty$ be the pennels with trace at infinity resp. O, X, Y, etc. Also, from now on use the notation of Proposition (4.1.3) and we suppose without loss of generality that $(P(\Delta_\infty), L(\Delta_\infty), \mathrm{I})$ is isomorphic to $\mathrm{PG}(\Delta)$, as opposed to its dual.

4.2.2. Define the incidence structure $V_n = (P(V_n), L(V_n), \mathrm{I})$ as follows:

(i) The point set $P(V_n) = \{P^n \in \mathrm{Ve}(\Delta) \mid P^n \in p \in P(\Delta_\infty)$ and $d(P^n, s) = n\}$

(ii) The line set $L(V_n) = \{L^n \in \mathrm{Ve}(\Delta) \mid L^n \in l \in L(\Delta_\infty)$ and $d(L^n, s) = n\}$

(iii) $P^n \,\mathrm{I}\, L^n$ if there is a quarter Q containing P^n and L^n.

4.2.3. REMARK. (1) For $n = 0$, V_n is the trivial geometry isomorphic to W_n of the previous section.

(2) For $n = 1$, V_n is nothing other than the residue $R(s)$, thus a projective plane.

4.3. *Properties of* V_n

4.3.1. *The maps* $\Pi^n_j: V_n \to V_j$

NOTATION. Elements of V_n (points or lines) are always denoted by a capital letter and a superscript n. Small letters without a superscript stand for elements of $(P(\Delta_\infty), L(\Delta_\infty), \mathrm{I})$ (except $s \in \mathrm{Ve}(\Delta)$, numbers and elements of R). Capital letters without a superscript stand for lines and points of $\mathrm{PG}(\Delta)$. If a vertex $P^n \in P(V_n)$ is on a point-pennel p, then we say that p represents P^n and that P^n is the n-trace of p. Similarly for lines.

Now fix $j \in \mathbb{N}$ and let $n \geqslant j$. Let $P^n \in P(V_n)$, then there is a unique point P^j such that $P^j \in [s, P^n]$. Similarly for lines. We denote this map $P^n \to P^j$ by Π^n_j. Let Π_j be the union over $n \geqslant j$ of all Π^n_j and extend Π_j to $\mathrm{PG}(\Delta)$ by $\Pi_j(P) = P^j$ if the point-pennel p with trace at infinity P represents P^j. Similarly for lines. Clearly Π_j preserves incidence and is onto.

4.3.2. *The partial valuation map u*

Let $P^n, Q^n \in P(V_n)$; $L^n, M^n \in L(V_n)$, then we define: :

(1) $u(p^n, Q^n) = \sup\{j \in \mathbb{N} \mid \Pi_j(P^n) = \Pi_j(Q^n),\ 0 \leqslant j \leqslant n\}$,

(2) $u(L^n, M^n) = \sup\ \{j \in \mathbb{N} \mid \Pi_j(L^n) = \Pi_j(M^n), 0 \leqslant j \leqslant n\}$,

(3) $u(P^n, L^n) = u(L^n, P^n) = \sup\{j \in \mathbb{N} \mid \Pi_j(P^n)\ \mathrm{I}\ \Pi_j(L^n),\ 0 \leqslant j \leqslant n\}$.

Similar definitions for elements of $\mathrm{PG}(\Delta)$, with $\sup \mathbb{N} = \infty$.

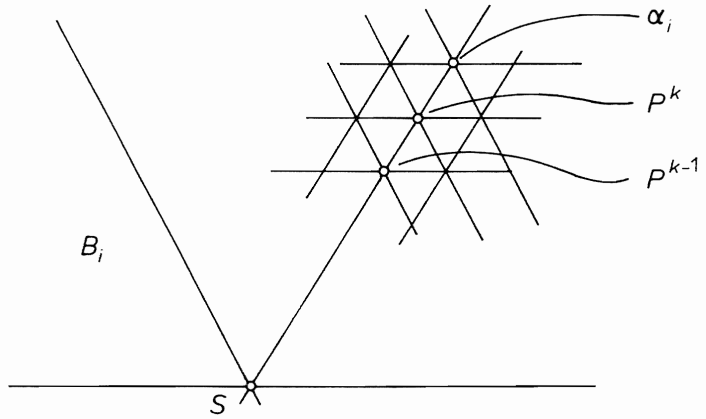

Fig. 3.

For $j \leqslant n$ (and both fixed), the inverse images of Π_j^n define a partition of each of the sets $P(V_n)$ and $L(V_n)$. Note that these partitions also can be obtained by the equivalence relations: P^n is equivalent to Q^n iff $u(P^n, Q^n) \geqslant j$ for points, and similarly for lines. This follows directly from the definitions.

We now prove the three basic properties (PS), (RP), (ND) stated in (3.2.3)

4.3.3. *Proof of* (PS)

(PS) *For any* $P^n \in P(V_n)$ *and any* $k \in \{0, 1, \ldots, n-1\}$, *we have:*

$$(\Pi_{k+1}^n)^{-1}(\Pi_{k+1}(P^n)) \subsetneqq (\Pi_k^n)^{-1}(\Pi_k(P^n)).$$

Similarly for lines.

The inclusion is trivial. To show proper inclusion, it suffices to prove that for any $k \in \mathbb{N}$ and any point P^k, there are at least two distinct points P_1^{k+1} and P_2^{k+1} such that $\Pi_k(P_i^{k+1}) = P^k$, $i = 1, 2$. So let $P^k \in P(V_k)$ and consider the residue $R(P^k)$. $R(P^k)$ is a projective plane. Let $\Pi_{k-1}(P^k) = P^{k-1}$. Without loss of generality, we can assume that P^{k-1} is a line in $R(P^k)$. Let α_1, α_2 be two points of $R(P^k)$, not incident with $P^{k-1}(\alpha_1, \alpha_2 \in \mathrm{Ve}(\Delta))$. Let B_i be an apartment containing s and $\{\alpha_i, P^k\}$, $i = 1, 2$. Since α_i is not adjacent to P^{k-1}, α_i is on a pennel in B_i, $i = 1, 2$. (See Figure 3). That is so because $P^{k-1} \in \mathrm{cl}(s, P^k)$ and hence P^{k-1} lies in both B_1 and B_2. Putting $\alpha_i = P_i^{k+1}$, $i = 1, 2$, the result follows. QED

4.3.4. *Proof of* (RP)

(RP) *Let* P^n, $Q^n \in P(V_n)$; L^n, $M^n \in L(V_n)$.

Let $k \leqslant \inf\{u(Q^n, L^n), u(L^n, P^n), u(P^n, M^n)\}$

(i) *there is a point incident with both* L^n *and* M^n (*and dual*)

(ii) $u(Q^n, M^n) \geqslant k \Leftrightarrow u(P^n, Q^n) + u(L^n, M^n) \geqslant k$.

(i) Let l and m be line-pennels representing resp. L^n and M^n. The unique point-pennel p incident with both l and m represents a point P^n which is incident with both L^n and M^n.

(ii) By mapping down all points and lines onto V_k, one can see that it suffices to prove the property for $k = n$.

LEMMA (4.3.4). *Let l be an arbitrary representative of a line L^n and suppose $P^n \mathrm{I} L^n$, with $P^n \in P(V_n)$. Then there is a point-pennel p which represents P^n such that $p \mathrm{I} l$ in $(P(\Delta_\infty), L(\Delta_\infty), \mathrm{I})$.*

Proof. Let B be an apartment through a quarter Q containing P^n and L^n. Let K^n be the line of V_n in B incident with P^n. Then $P^n \in \mathrm{cl}(L^n, K^n)$. (See Figure 4). Let k be the pennel in B representing K^n. l and k satisfy the conditions of Lemma (4.1.4). So let B' be an apartment containing k and l and a point-pennel p incident with both k and l in $(P(\Delta_\infty), L(\Delta_\infty), \mathrm{I})$. Since B' contains L^n, and M^n, it contains P^n, so p represents P^n. QED.

LEMMA (4.3.5) *Suppose $P^n \mathrm{I} L^n; P^n \mathrm{I} M^n$ and $Q^n \mathrm{I} L^n$. If $u(L^n, M^n) = 0$, then $Q^n \mathrm{I} M^n$ iff $P^n = Q^n$.*

Proof. If $P^n = Q^n$, then $Q^n \mathrm{I} M^n$.
Suppose now $Q^n \mathrm{I} M^n$. $u(L^n, M^n) = 0$ implies that any two pennels l and m representing L^n, resp. M^n only meet in s. As in the proof of Lemma (4.3.4) $Q^n \in \mathrm{cl}(L^n, M^n)$, moreover, the parallelogram (Q^n, L^n, s, M^n) coincides with the parallelogram (P^n, L^n, s, M^n), hence $Q^n = P^n$. QED

LEMMA (4.3.6). *Let $P^n \in P(V_n)$; $L^n \in L(V_n)$. If $d(P^n, L^n) \leqslant n$, then $P^n \mathrm{I} L^n$.*

Proof. Suppose $P^n \not\mathrm{I} L^n$. Let $j = u(P^n, L^n) < n$ and $\Pi_j(P^n) = P^j$; $\Pi_j(L^n) = L^j$. Choose l representing L^n arbitrary, then l represents L^j as well. Let q be a point-pennel such that $q \cup l$ is a wall (e.g. q is in an arbitrary apartment through l). Let l_0 be the line-pennel such that $q \mathrm{I} l_0 \mathrm{I} p$ and let p_0 be the point-pennel determined by $l_0 \mathrm{I} p_0 \mathrm{I} l$. Then $\{l, p_0, l_0, q\}$ determines a half apartment H bounded by the wall $q \cup l$ (just as in Lemmas (4.1.4) and (4.1.5)). Let L_0^j be the j-trace of l_0, then $L_0^j \mathrm{I} P^j \mathrm{I} L^j$. But in H we have $L_0^j \mathrm{I} P_0^j \mathrm{I} L^j$, where P_0^j is the j-trace of P_0. Since $u(L_0^j, L^j) = 0$ and by Lemma (4.3.5), $P_0^j = P^j$. But $u(P^n, L^n) = j$, so $p \cap p_0 = [s, P^j]$. (See Figure 5.) Let B be an arbitrary apartment containing H. Let m and q_0 be the pennels of B such that $q \mathrm{I} m \mathrm{I} q_0 \mathrm{I} l$. Let M be the wall (not necessarily through s) in B through L_j and parallel to $l_0 \cup q_0$. So M contains also P^j. Since $p \mathrm{I} l$, the triangle $\{p, p_0, q_0\}$ is non-degenerate in $(P(\Delta_\infty), L(\Delta_\infty), \mathrm{I})$ and hence it determines a unique apartment B' in Δ (but q, p_0, q_0 are not necessarily part of B'!). By 4.1.2(vi), $B \cap B'$ is a half apartment H' bounded by a wall M'

parallel to $l_0 \cup q_0$, so M' is parallel to M. By Lemma (4.1.4) and the fact that an apartment is convex, Δ has the following property (APP):

(APP) *The set of vertices of an apartment of Δ determined by the non-degenerate triangle $\{A, B, C\}$ of* PG(Δ) *is exactly the set of vertices α for which the pennels a_α, b_α, c_α with source α and trace at infinity resp. A, B, C meet pairwise in nothing more than α.*

Hence, one can easily see that $M = M'$. In B', we now have the situation of Figure 6:

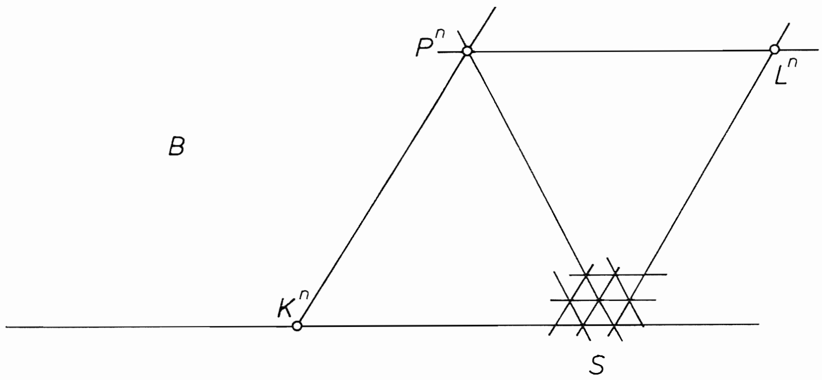

Fig. 4.

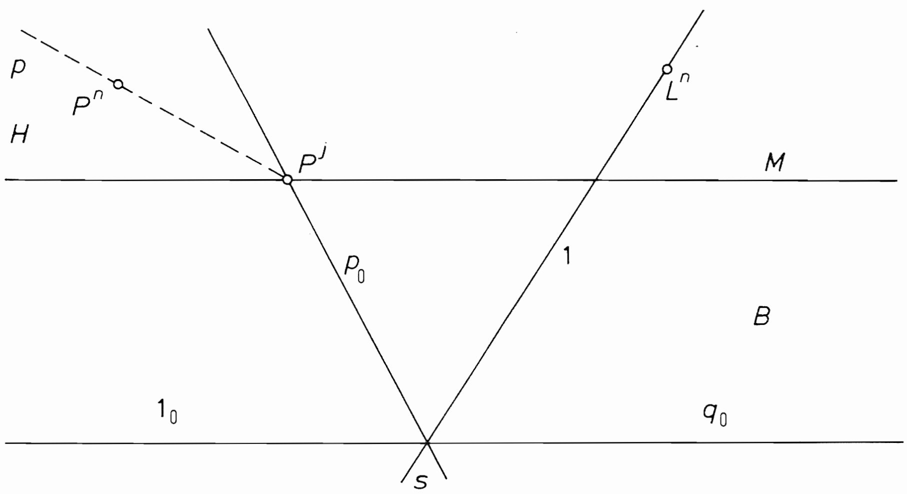

Fig. 5.

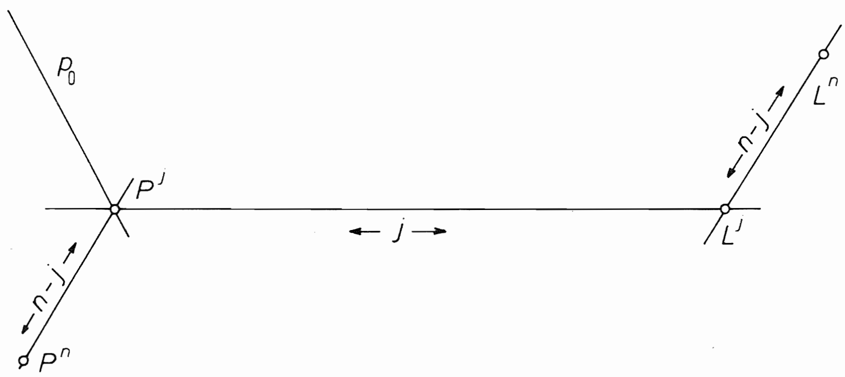

Fig. 6.

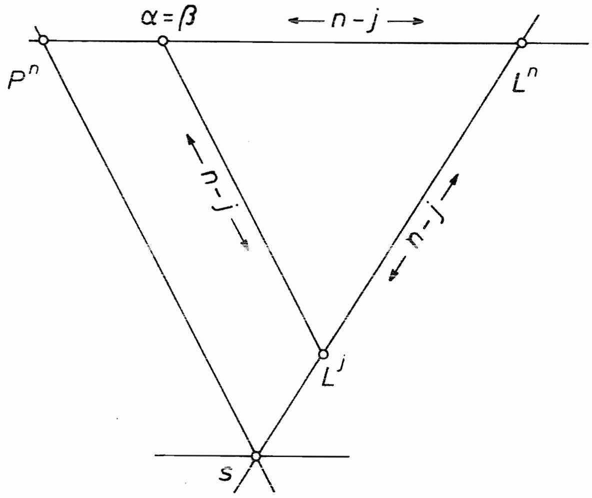

Fig. 7.

Clearly, $d(P^n, L^n) = \sqrt{n^2 + 3(n-j)^2} > n$. QED

COROLLARY (4.3.7). *P^n I L^n in V_n iff $d(P^n, L^n) = n$ in Δ*

Proof. If P^n I L^n, then $d(P^n, L^n) = n$ (clearly from the picture) and if $d(P^n, L^n) = n$, then P^n I L^n by (4.3.6). QED

We now finish off the proof of (RP)(ii). Since $k = n$, we have Q^n I L^n I P^n I M^n. If $L^n = M^n$, then there is nothing to prove, so suppose $u(L^n, M^n) = j < n$. Let $L^j = \Pi_j(L^n) = \Pi_j(M^n)$. By Lemma (4.1.4), Lemma (4.3.5) and Corollary (4.3.7), there is a unique vertex α with the properties

$$d(L^j, \alpha) = d(L^n, \alpha) = d(M^n, \alpha) = n - j.$$

On the other hand, any quarter containing P^n and L^n (or M^n) contains L_j, hence it contains $\text{cl}(L^j, P^n)$ which contains a vertex β at distance j from P^n and at distance $n-j$ from M^n, L^n *and* L^j (see Figure 7). Hence $\alpha = \beta$.

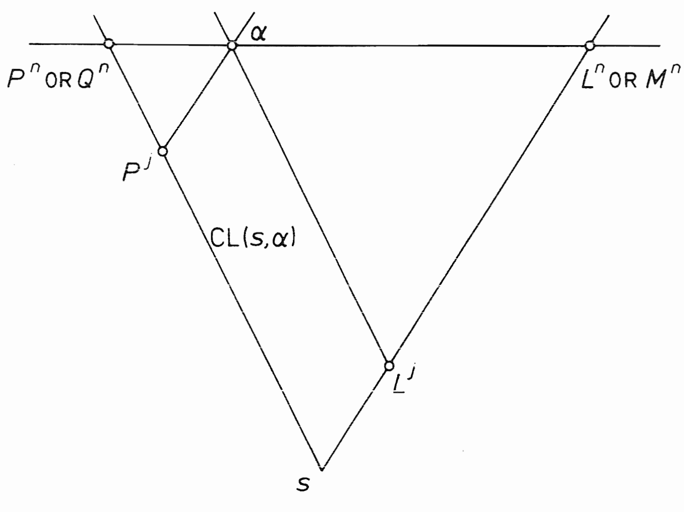

Fig. 8.

(1) Suppose first Q^n I M^n, then, as above, any quarter containing M^n (or L^n) and Q^n contains α. So any quarter containing L^n (or M^n) and P^n (or Q^n) contains $\mathrm{cl}(s, \alpha)$ which contains a vertex P^j at distance $n - j$ from s on $[s, P^n]$ and $[s, Q^n]$. Hence $[s, P^j] \subseteq [s, P^n] \cap [s, Q^n]$ and thus:

$$u(P^n, Q^n) \geqslant n - j = n - u(L^n, M^n).$$

(See Figure 8.)

(2) Suppose $u(P^n, Q^n) \geqslant n - j$, then all quarters containing Q^n and L^n, (or P^n and L^n) contain $P^j = \Pi_j(P^n) = \Pi_j(Q^n)$. Hence they all contain the set $\mathrm{cl}(P^j, L^n)$, which obviously contains α. Note that $d(P^j, \alpha) = j$, and so $d(Q^n, \alpha) = j$. Hence

$$d(Q^n, M^n) \leqslant d(Q^n, \alpha) + d(M^n, \alpha) = j + (n - j) = n.$$

By Lemma (4.3.6), Q^n I M^n. QED.

We now write down a few consequences of this property:

COROLLARY (4.3.8). *Let $P^n, Q^n \in P(V_n)$ and $L^n, M^n \in L(V_n)$. Suppose that*

$$u(Q^n, M^n) < \inf\{u(Q^n, L^n), u(L^n, P^n), u(P^n, M^n)\}$$

then we have

$$u(Q^n, M^n) = u(P^n, Q^n) + u(L^n, M^n).$$

Proof. Apply (RP) for k resp. equal to $u(Q^n, M^n)$ and $u(Q^n, M^n) + 1$.

QED

COROLLARY (4.3.9). *Let P, Q be points of* PG(Δ) *and L, M be lines of* PG(Δ). *Then* (RP) *holds for P, Q, L, M.*

Proof. Apply (RP) for $n > \sup\{u(Q, M), u(Q, L), u(L, P), u(P, M), u(P, Q) + u(L, M)\}$ QED

COROLLARY (4.3.10). *Let P^n, $Q^n \in P(V_n)$ and L^n, $M^n \in L(V_n)$ and Q^n I L^n I P^n I M_n. Then $u(Q^n, M^n) = u(P^n, Q^n) + u(L^n, M^n)$. This is also true in* PG(Δ).

Proof. Follows directly from Corollaries (4.3.8) and (4.3.9). QED

COROLLARY (4.3.11). *Let $P^n \in P(V_n)$ and $L^n \in L(V_n)$ be such that $u(P^n, L^n) = 0$. Suppose P^n is incident with two lines L_1^n and L_2^n. Then there are unique points P_i^n incident with both L^n and L_i^n, $i = 1, 2$. Moreover, we have:*

$$u(P_1^n, P_2^n) = u(L_1^n, L_2^n) = u(P_1^n, L_2^n) = u(P_2^n, L_1^n).$$

This is also true for points and lines in PG(Δ).

Proof. Let P_i^n and Q_i^n be incident with both L^n and L_i^n. Since $P^n \mathrm{I} L_i^n \mathrm{I} P_i^n \mathrm{I} L^n$, we have by Corollary (4.3.10), $0 = u(P^n, L^n) = u(P^n, P_i^n) + u(L^n, L_i^n)$, hence both last numbers are 0. Since $P_i^n \mathrm{I} L^n \mathrm{I} Q_i^n \mathrm{I} L_i^n \mathrm{I} P_i^n$, we have $u(P_i^n, Q_i^n) = n$, so P_i^n is unique $(i = 1,2)$. Now we have: $L_1^n \mathrm{I} P_1^n \mathrm{I} L^n \mathrm{I} P_2^n$, so

$$(1) \qquad u(L_1^n, P_2^n) = u(L^n, L_1^n) + u(P_1^n, P_2^n) = u(P_1^n, P_2^n)$$

and $L_1^n \mathrm{I} P^n \mathrm{I} L_2^n \mathrm{I} P_2^n$, so

$$(2) \qquad u(L_1^n, P_2^n) = u(L_1^n, L_2^n) + u(P^n, P_2^n) = u(L_1^n, L_2^n).$$

Combining (1) and (2) and by symmetry, the result follows. QED

4.3.5. *Proof of* (ND)

(ND) V_1 *contains a non-degenerate quadrangle.*

Proof. $V_1 = R(s)$ is a non-degenerate projective plane. QED

THEOREM (4.4). *The inverse limit* V *of* $(V_n)_{n \in \mathbb{N}}$ *with respect to the maps* Π_{n-1}^n *is isomorphic to* $(P(\Delta_\infty), L(\Delta_\infty), \mathrm{I}) \cong \mathrm{PG}(\Delta)$.

Proof. Define the map $f: (P(\Delta_\infty), L(\Delta_\infty), \mathrm{I}) \to V$ by $f(a) = (A^n)_{n \in \mathbb{N}}$, where A^n is the n-trace of the pennel a.

(1) f is well defined since $\Pi_{n-1}^n(A^n) = A^{n-1}$

(2) f is injective since two distinct pennels have distinct n-traces for n large enough.

(3) Let $(A^n)_{n \in \mathbb{N}}$ be an element of V. Then the union of all $[s, A^n]$ over n is convex and lies by Corollary (1.2.4) in an apartment. Therefore, it is a pennel itself. So $f(a) = (A^n)_{n \in \mathbb{N}}$ and f is onto.

(4) If $p \mathrm{I} l$, then $f(p) \mathrm{I} f(l)$ by definition.

If $f(p) \mathrm{I} f(l)$, then let s_n be the (chamber) convex closure of $\{s, p^n, L^n\}$; Clearly, $S_n \subset S_{n+1}$ and so the union Q over n of all S_n is convex. It clearly can be embedded in $\mathscr{A}$, so Q is a quarter which is obviously bounded by $p \cup l$. QED

4.5. *The Valuation Map v*

4.5.1. Suppose PG(Δ) is coordinatized by the PTR (R, T) with respect to the non-degenerate quadrangle (O, X, Y, E). By Lemma (4.1.5), (O^n, X^n, Y^n, E^n) is a non-degenerate quadrangle in V_n, where O^n, X^n, etc., denote the n-trace of the pennel corresponding resp. to O, X, etc. Define:

$$R^+ = \{r \in R \mid \Pi_1((O,r)) \neq Y^1\}, \qquad R^- = R - R^+.$$

We now define the valuation map $v: R^2 \to Z \cup \{\infty\}$ as:

$$v(r,s) = \begin{cases} u((0,r),(0,s)) & \textit{if } (r,s) \in R^+ \times R^+ \\ -u((\infty),(0,s)) & \textit{if } (r,s) \in R^+ \times R^- \\ -u((\infty),(0,r)) & \textit{if } (r,s) \in R^- \times R^+ \\ u((0,r),(0,s)) - u((\infty),(0,r)) - u((\infty),(0,s)) & \textit{if } (r,s) \in R^- \times R^- \end{cases}$$

We denote $v(0,r) = v(r,0)$ by $v(r)$. Clearly $v(r) \geqslant 0 \Leftrightarrow r \in R^+$.

PROPOSITION (4.5.1). *Let $r, s \in R$; $[m,k]$ and $[m',k']$ be lines in* PG(Δ) *(a,b) and (a',b') be points in* PG(Δ) *such that m, m', k, k', a, a', b, $b' \in R^+$, then we have*:

$$(1)\ v(r,s) = \begin{cases} u((r,0),(s,0)) & \textit{if } (r,s) \in R^+ \times R^+ \\ -u((0),(r,0)) & \textit{if } (r,s) \in R^- \times R^+ \\ -u((0),(s,0)) & \textit{if } (r,s) \in R^+ \times R^- \\ u((r,0),(s,0)) - u((0),(r,0)) - u((0),(s,0)) & \textit{if } (r,s) \in R^- \times R^- \end{cases}$$

$$(1)'\ v(r,s) = \begin{cases} u((r),(s)) & \textit{if } (r,s) \in R^+ \times R^+ \\ -u((\infty),(r)) & \textit{if } (r,s) \in R^- \times R^+ \\ -u((\infty),(s)) & \textit{if } (r,s) \in R^+ \times R^- \\ u((r),(s)) - u((\infty),(r)) - u((\infty),(s)) & \textit{if } (r,s) \in R^- \times R^-. \end{cases}$$

(2) $u((a,b),(a',b')) = \inf\{v(a,a'), v(b,b')\}$.

(2)′ $u([m,k],[m',k']) = \inf\{v(m,m'), v(k,k')\}$.

(3) $u((a,b),[m,k]) = v(k, T(m,a,b))$.

Proof. (1) follows from Corollary (4.3.11) applied twice: let $P = (0,r)$, $Q = (0,s)$ and $J = (1)$, then we have:

$$u((0,r),(0,s)) = u(JP, JP) = u((r,0),(s,0))$$

since $u(J,\bar{X}) = 0$ (after all, (O^1, X^1, Y^1, E^1) is non-degenerate in V_1).

(1)′ Similarly, but use L_∞ and $B = (1,0)$ instead of $\bar{X}$ and J resp.

(2) By Corollary (4.3.11), $u(XP, XP') = v(b,b')$ and $u(YP, YP') = v(a,a')$, where now $P = (a,b)$ and $P' = (a',b')$. Now $u(XP, YP) = 0$ (using $XP \mathrel{I} P \mathrel{I} YP \mathrel{I} Y$ and Corollary (4.3.10)), so the intersection of $\Pi_n(XP)$ and $\Pi_n(YP)$ is uniquely determined for all n. Similarly for XP'. Choose for n first inf $\{v(a,a'), v(b,b')\}$, then $\Pi_n(XP) = \Pi_n(XP')$ and $\Pi_n(YP) = \Pi_n(YP')$ and hence $\Pi_n(P) = \Pi_n(P')$, Secondly, choose $n = \inf\{v(a,a'), v(b,b')\} + 1$. Then $\Pi_n(P) = \Pi_n(P')$ implies $\Pi_n(XP) = \Pi_n(XP')$ and $\Pi_n(YP) = \Pi_n(YP')$, since $u(X,P) = u(Y,P) = u(X,P') = u(Y,P') = 0$; a contradiction.

(2)′ Dual to (2).

(3) By Corollary (4.3.10), we have:

$$u((a,b),[m,k]) = u([m,T(m,a,b)],[m,k])$$

By (2)′, this equals $v(k, T(m,a,b))$. QED

PROPOSITION (4.5.2). *Let P_1, P_2, P_3 be three points of* PG(Δ)), *then $u(P_1,P_2) < u(P_2,P_3)$ implies $u(P_1,P_3) = u(P_1,P_2)$.*

Proof. Obvious by mapping down the points onto V_n for suitable n. Note that the proposition also holds for lines. QED

PROPOSITION (4.5.3). *Let P_1, P_2, P_3, P_4 be four points of* PG(Δ), *then $u(P_1,P_2) + u(P_3,P_4) < u(P_1,P_3) + u(P_2,P_4)$ implies $u(P_1,P_2) + u(P_3,P_4) = u(P_1,P_4) + u(P_2,P_3)$. Similarly for four lines.*

Proof. Suppose

$$u(P_1,P_2) + u(P_3,P_4) \begin{cases} < u(P_1,P_3) + u(P_2,P_4) \\ < u(P_1,P_4) + u(P_2,P_3). \end{cases}$$

Without loss of generality, we can assume that $u(P_1,P_2) < u(P_1,P_3)$. By Proposition (4.5.2), this implies $u(P_1,P_2) = u(P_2,P_3)$. Hence $u(P_3,P_4) < u(P_1,P_4)$, which implies $u(P_1,P_3) = u(P_3,P_4)$. Hence $u(P_1,P_2) < u(P_2,P_4)$ which implies $u(P_1,P_4) = u(P_1,P_2)$. Hence $u(P_3,P_4) < u(P_2,P_3)$, which implies $u(P_2,P_4) = u(P_3,P_4)$. So we have

$$u(P_1,P_2) + u(P_3,P_4) \begin{cases} > 2.u(P_1,P_2) \\ > 2.u(P_3,P_4); \end{cases}$$

a contradiction, hence the two smallest sums have always to be equal. QED

4.5.4. We now prove the four axioms of valuations (see (2.2)).

(*) *v satisfies* (d1)

By definition. QED.

(**) *v satisfies* (d2)

Suppose $v(a,b) < v(b,c)$. We then have to show $v(a,c) = v(a,b)$. There are six possible cases (a,b,c in R^+ or R^- such that $v(a,b) < v(b,c)$):

(I) $a,b,c \in R^+$. This is a direct consequence of the definition and Proposition (4.5.2).

(II) $a \in R^-$; $b,c \in R^+$. We have $v(a,c) = v(a) = v(a,b)$ by definition.

(III) $a \in R^+; b,c \in R^-$. We have $v(a,b) = v(b) < v(b,c)$. So

$$-u((\infty),(0,b)) < u((0,b),(0,c)) - u((\infty),(0,b)) - u((\infty),(0,c)).$$

Hence

$$u((\infty),(0,b)) < u((0,b),(0,c))$$

and by Proposition (4.5.2)

$$u((\infty),(0,b)) = u((\infty),(0,c)).$$

We conclude

$$v(a,c) = v(c) = v(b) = v(a,b).$$

(IV) $a, c \in R^-; b \in R^+$. We have $v(a,b) = v(a) < v(c) = v(b,c)$, so

$$u((\infty),(0,c)) < u((\infty),(0,a)$$

and by Proposition (4.5.2)

$$u((\infty),(0,c)) = u((0,a),(0,c)).$$

Hence

$$v(a,c) = -u((\infty),(0,a)) = v(a) = v(a,b).$$

(V) $a, b \in R^-; c \in R^+$. We have $v(a,b) < v(b,c) = v(b)$. So

$$u((0,a),(0,b)) - u((\infty),(0,a)) < 0.$$

Again by Proposition (4.5.2)

$$u((0,a),(0,b)) = u((\infty),(0,b)).$$

Hence

$$v(a,b) = -u((\infty),(0,a)) = v(a) = v(a,c).$$

(VI) $a, b, c \in R^-. v(a,b) < v(b,c)$ is equivalent to

$$u((0,a),(0,b)) + u((\infty),(0,c)) < u((0,b),(0,c)) + u((\infty),(0,a)).$$

By Proposition (4.5.3)

$$u((0,a),(0,b)) + u((\infty),(0.c)) = u((0,a),(0,c)) + u((\infty),(0,b))$$

hence

$$v(a,b) = v(a,c). \qquad \text{QED}$$

(***) *v satisfies* (d3)

Suppose

$$T(a_1,b_1,c_1) = T(a_1,b_2,c_2) = d_1,$$
$$T(a_2,b_1,c_1) = T(a_2,b_2,c_3) = d_2.$$

(I) Suppose the valuation of each appearing element is positive (or 0). Let

$P_1 = (b_1, c_1)$, $P_2 = (b_2, c_2)$, $P_3 = (b_2, c_3)$, $L_1 = [a_1, d_1]$, $L_2 = [a_2, d_2]$. By Corollary (4.3.11),

$$u(L_1, L_2) = v(a_1, a_2) \quad \text{and} \quad u(P_1, P_2) = v(b_1, b_2).$$

By Proposition (4.5.1(3)),

$$u(P_2, L_2) = v(d_2, T(a_2, b_2, c_2)) = v(c_2, \bar{T}(b_2, a_2, d_2)) = v(c_2, c_3),$$

where $\bar{T}(a,b,c) = d \Leftrightarrow T(b,a,d) = c$. The result follows from $P_2 \mathrm{I} L_1 \mathrm{I} P_1 \mathrm{I} L_2 \mathrm{I} P_3$ and Corollary (4.3.10).

(II) Suppose $\inf\{v(a_1), v(a_2), v(b_1), \ldots, v(d_2)\} = k < 0$. Denote the n-trace of an element of PG(Δ) by the same letter with a superscript n. We coordinatize the apartment A with trace at infinity $\{O, X, Y\}$ by barycentric coordinates with respect to the quadrangle $(s; O^1, X^1, Y^1)$. This is well defined since A is a Euclidean plane (a copy of $\mathscr{A}$ in (1.1)). Consider the vertex s^* with coordinates $(\frac{1}{3} + k, \frac{1}{3}, \frac{1}{3} - k)$ (see Figure 9).

We recoordinatize PG(Δ) with respect to a new quadrangle (O, X, Y, E^*), where E^* is such that $s^* = S(O, X, Y, E^*)$ (this is always possible!) Denote by (R^*, T^*) the new PTR and by v^* resp. u^* the induced valuation map on (R^*, T^*), resp. the partial valuation map induced on PG(Δ). There is a bijective map $p \to p^*$ which assigns to a pennel p with source s, the pennel p^* with source s^* and same trace at infinity; this induces three bijective maps $b_\infty, b_{\bar{X}}, b_{\bar{Y}}: R \to R^*$ such that:

$$T(m, a, b) = k \Leftrightarrow T^*(b_\infty(m), b_{\bar{X}}(a), b_{\bar{Y}}(b)) = b_{\bar{Y}}(k),$$

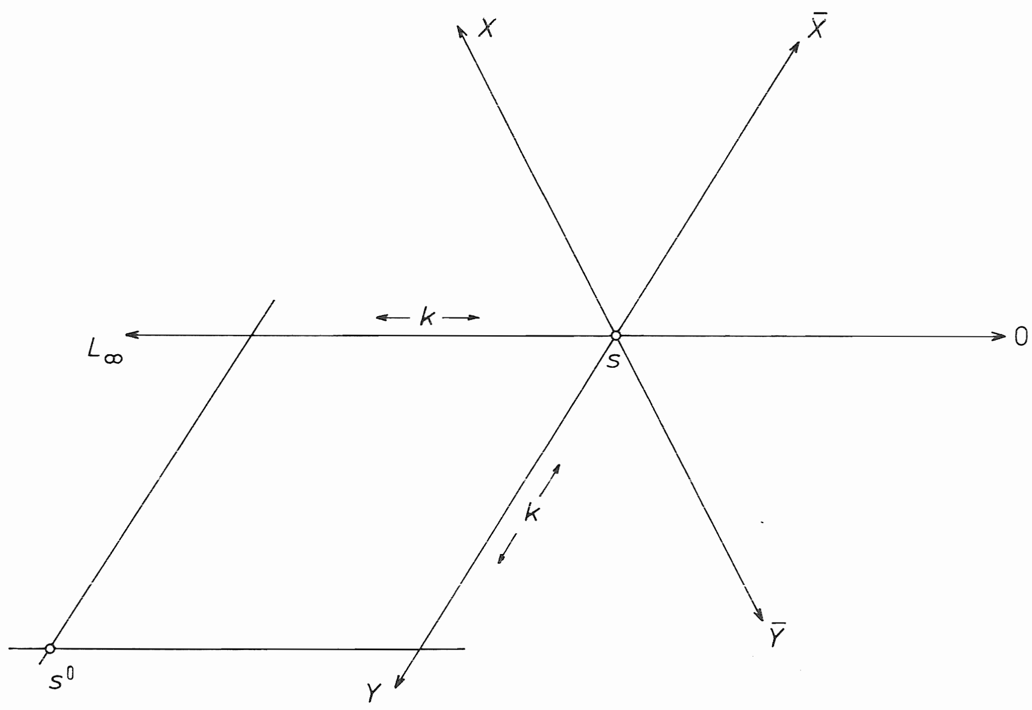

Fig. 9.

$[b_\infty(m), b_{\bar{Y}}(k)]$ and $(b_{\bar{X}}(a), b_{\bar{Y}}(b))$ are the new coordinates of $[m,k]$, resp. (a,b). If R is a division ring, then $(b_\infty, b_{\bar{X}}, b_{\bar{Y}})$ is an isotopism and R and R^* are isotopic (see [3, p. 177]).

Let P be a point of PG(Δ) on the line $\bar{Y}$ of PG(Δ), let A again be the apartment of Δ determined by the triangle $\{O, X, Y\}$, then the apartment B_P determined by $\{O, P, X\}$ meets A in a half apartment H bounded by a wall M parallel to $X \cup \bar{Y}$ (X and $\bar{Y}$ are here considered as elements of $P(\Delta_\infty)$ or $L(\Delta_\infty)$). We can do that for any point incident with $\bar{Y}$ in PG(A). So we obtain a set of apartments $(B_P \mid P \text{ I } \bar{Y}\}$. All these apartments meet pairwise in half apartments bounded by walls parallel to $X \cup \bar{Y}$. Hence we can factor out $X \cup \bar{Y}$. We denote the acquired set by $T_{X,\bar{Y}}$ (see Figure 10) and the corresponding quotient map by $\psi: \bigcup \{B_P \mid P \text{ I } \bar{Y}\} \to T_{X,\bar{Y}}$. We define a distance map d' on the set of elements of $T_{X,\bar{Y}}$, which have as inverse image a wall of Δ, by:

$$d'(\psi(M), \psi(M')) = \frac{2\sqrt{3}}{3} \cdot d(M, M').$$

This implies: $d'(\psi(M), \psi(M')) = 1$ iff M and M' are neighbouring walls. By the property (APP), we conclude that *if P and Q are incident with $\bar{Y}$, and B_{PQ} is the apartment determined by $\{P, Q, X\}$, then*

$$u(P, Q) = d'(\psi(s), \psi(B_{PQ})).$$

(In fact, $T_{X,\bar{Y}}$ is nothing other than the tree $(I(M^\infty), \{f^\infty \mid f(M)^\infty = M^\infty\})$ with $M^\infty = \{X, \bar{Y}\}$, introduced by Tits in [9, Prop. 4].)

Each point incident with $\bar{Y}$ determines a direction in $T_{X,\bar{Y}}$ (an 'end' of the tree). Now $d'(\psi(s), \psi(s^*)) = 2\,|\,k\,|$ and $\psi(s^*)$ lies in the direction of Y with respect to $\psi(s)$. To prove (d3), there are six cases to consider. We draw a picture in each of the cases.

Let $r, s \in R$, $b_{\bar{Y}}(r) = r^*$, $b_{\bar{Y}}(s) = s^*$, $P = (0, r)$ and $Q = (0, s)$ (old coordinates).

(I) $r, s \in R^+$. Then $v(r,s) = u(P,Q)$. Clearly $r^*, s^* \in (R^*)^+$ and $v^*(r^*, s^*) = v(r,s) + 2|k|$ (see Figure 11).

(II) $r \in R^+$, $2k \leqslant v(s) < 0$. Then $v(r,s) = -u(Y,Q)$. Clearly $r^*, s^* \in (R^*)^+$ and also $v^*(r^*, s^*) = u^*(P,Q) = d'(\psi(s^*), \psi(B_{PQ}))$. On the other hand, we have $-u(Y,Q) = -(2|k| - d'(\psi(s^*), \psi(B_{PQ})))$. Hence $v^*(r^*, s^*) = v(r,s) + 2|k|$ (Figure 12).

(III) $r \in R^+$, $v(s) < -2|k|$. Then $v(r,s) = -u(Y,Q)$. Clearly $r^* \in (R^*)^+$, $s^* \in (R^*)^-$ and thus $v^*(r^*, s^*) = -u^*(Y,Q) = -d'(\psi(s^*), \psi(B_{PQ}))$. On the other hand, we have $-u(Y,Q) = -(2|k| + d'(\psi(s^*), \psi(B_{PQ})))$. Hence $v^*(r^*, s^*) = v(r,s) + 2|k|$ (Figure 13).

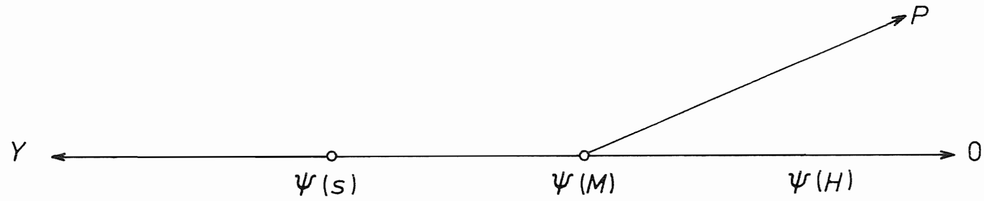

Fig. 10.

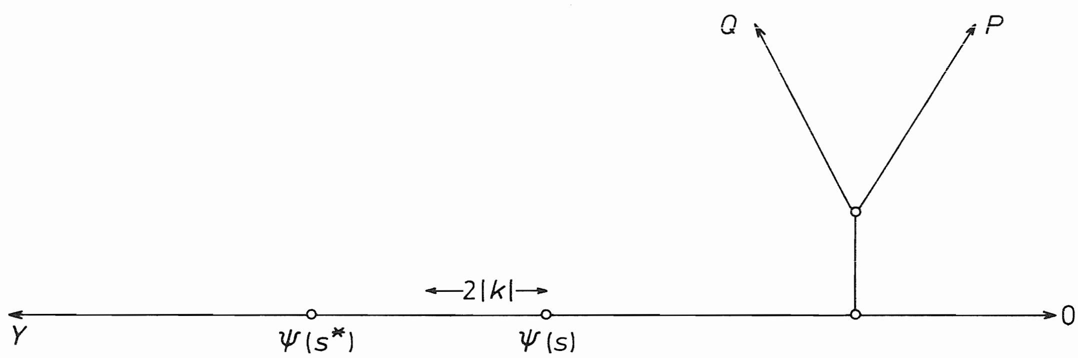

Fig. 11.

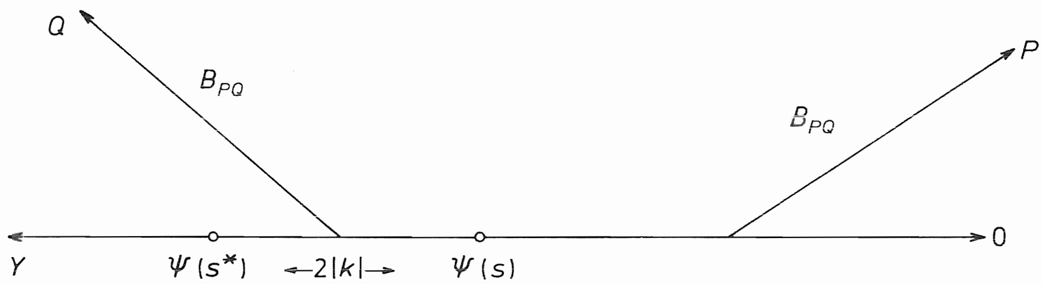

Fig. 12.

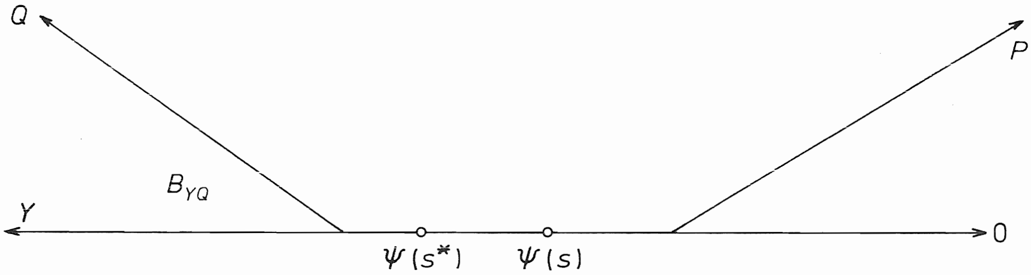

Fig. 13.

(IV) $2k \leqslant v(r) = v(s) < 0$. Then $v(r, s) = u(P, Q) - u(Y, P) - u(Y, Q)$. Clearly $r^*, s^* \in (R^*)^+$ and $v^*(r^*, s^*) = d'(\psi(s^*), \xi) = d'(\psi(s^*), \eta) + d'(\eta, \xi) = d'(\eta, \xi) - d'(\psi(s^*), \eta) + 2|k|$, where ξ and η are defined as on Figure 14. On the other hand, $v(r, s) = u(P, Q) - 2.u(Y, P) = d'(\eta, \xi) - d'(\psi(s), \eta)$. So again $v^*(r^*, s^*) = v(r, s) + 2|k|$ (see Figure 14).

(V) $v(r) < v(s) < 0$. Then $v(r) = v(r, s)$ and by (II) and (III), $v^*(r^*) = v(r) + 2|k| < v(s) + 2|k| \leqslant = v^*(s^*)$ and hence $v^*(r^*, s^*) = v^*(r^*)$ (after all, v^* satisfies (d2)), so again we have $v^*(r^*, s^*) = v(r, s) + 2|k|$.

(VI) $v(r) = v(s) < 2k$. By a similar argument as above (in (I) through (IV)), we again conclude $v^*(r^*, s^*) = v(r, s) + 2|k|$ (see Figure 15).

So, for all $r, s \in R$, we have

$$v^*(r^*, s^*) = v(r, s) + 2|k|. \qquad (*)$$

Similarly, one can show

$$v^*(b_\infty(r), b_\infty(s)) = v(r, s) + |k| \qquad (*)$$

$$v^*(b_{\bar{X}}(r), b_{\bar{X}}(s)) = v(r, s) + |k|. \qquad (*)$$

Now we have:

$$T^*(b_\infty(a_1), b_{\bar{X}}(b_1), b_{\bar{Y}}(c_1)) = b_{\bar{Y}}(d_1)$$
$$= T^*(b_\infty(a_1), b_{\bar{X}}(b_2), b_{\bar{Y}}(c_2))$$
$$T^*(b_\infty(a_2), b_{\bar{X}}(b_1), b_{\bar{Y}}(c_1)) = b_{\bar{Y}}(d_2)$$
$$= T^*(b_\infty(a_2), b_{\bar{X}}(b_2), b_{\bar{Y}}(c_3)).$$

All elements have positive valuation and so (d3) holds:

$$v^*(b_\infty(a_1), b_\infty(a_2)) + v^*(b_{\bar{X}}(b_1), b_{\bar{X}}(b_2))$$
$$= v^*(b_{\bar{Y}}(c_2), b_{\bar{Y}}(c_3)).$$

By (*), the result follows. QED.

REMARK(4.5.5.). Using the same technique, one can show in general that, if s^* has barycentric coordinates (k_0, l_0, m_0), then

$$v^*(b_\infty(r), b_\infty(s)) = v(r, s) + m_0 - l_0$$
$$v^*(b_{\bar{X}}(r), b_{\bar{X}}(s)) = v(r, s) + l_0 - k_0$$
$$v^*(b_{\bar{Y}}(r), b_{\bar{Y}}(s)) = v(r, s) + m_0 - k_0$$

All properties of v are invariant under these transformations.

(****) *v satisfies* (d4)

The following proof, using the above result differs from that in [11, §2.6.4]. We know $v(1) = 0$ since $u(A, 0) = 0$ and $v(0) \infty$. For $k \in Z$, take in

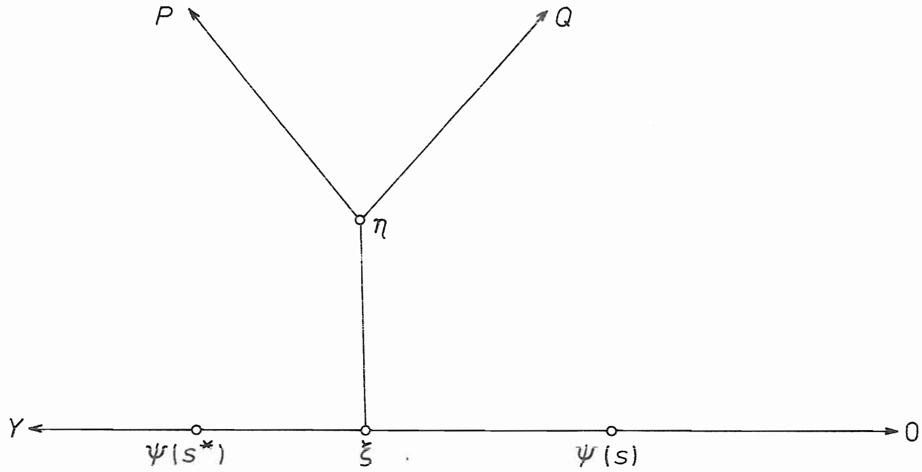

Fig. 14.

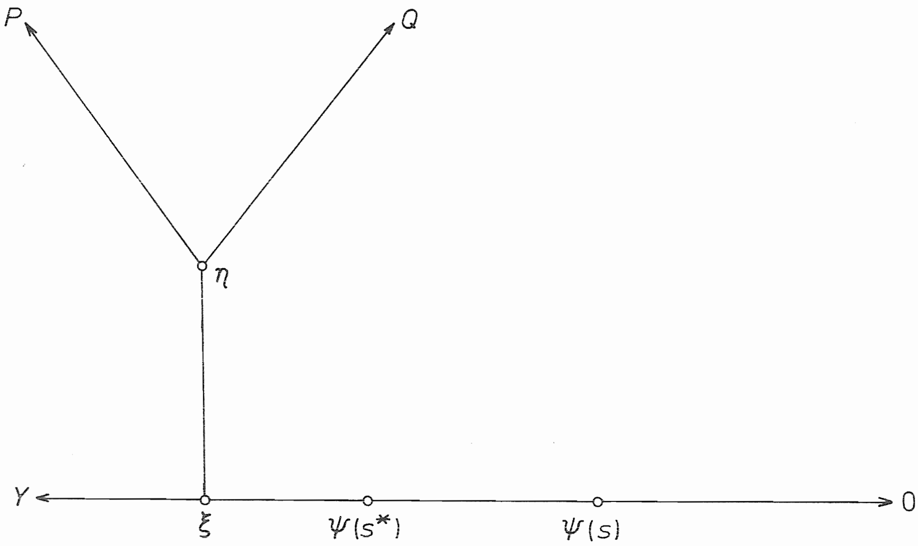

Fig. 15.

the apartment A the vertex s^* with barycentric coordinates the 3-tuple $(\frac{1}{3} + k, \frac{1}{3}, \frac{1}{3} - k)$. With the same notation as in (***), we have $v(r) = v^*(b_\infty(r)) + k$. So choosing r such that $b_\infty(r)$ is the 1 in (R^*, T^*), the result follows. QED

This completes the proof of Theorem (4). In Section 5 we shall show that (R, T, v) is complete.

REMARK (4.5.6) Suppose (O', X', Y', E') is another non-degenerate quadrangle of PG(Δ). Denote the corresponding V-PTR by (R', T', v').

(1) It might happen that $s(O, X, Y, E) = s(O', X', Y', E')$ (e.g. if (O, X, Y, E) is a permutation of (O', X', Y', E')), but nevertheless, this induces no general connection between (R, T, v) and $(R', T' v')$. The following example shows that the position of $s(O', X', Y', E)$ with respect to $s(O, X, Y, E)$ has no

influence on the PTR's. Let Δ be the triangle building corresponding to a 'proper' André-quasifield $K(\Phi, s)$ of (2.2). Then we can assume that R is precisely $K(\Phi, s)$ (by PTRΔPTR) of Section 5). With the notation of (4.3.2), we choose O' such that $\infty \neq u(O', XY) \geqslant 1$; E' is the image of E under the elation of PG(Δ) with axis XY (see [3, Ch. IV, §4]) mapping O to O'; $X' = X$ and $Y' = Y$. In view of (APP) one has $s(O, X, Y, E) \neq s(O', X, Y, E')$ and if we denote by (R'', T'') the PTR corresponding to (X, Y, O, E), then:

$$s(O,X,Y,E) = s(X,Y,O,E) \text{ and } (R,T) \not\cong (R'',T''),$$
$$s(O,X,Y,E) \neq s(O',X,Y,E') \text{ and } (R.T) \cong (R',T').$$

In the last case one can even show that v is compatible with v' (see also below).

(2) Suppose $(R, T) \cong (R', T')$. Then v is not necessarily compatible with v', even if $s(O, X, Y, E) = s(O', X', Y', E')$. But it seems reasonable to conjecture that v is compatible with $v' \Leftrightarrow$ the collineation of PG(Δ) corresponding to the isomorphism $(R, T) \to (R', T')$ extends to an automorphism of Δ.

5. Proof of the Main Theorem

In Section 4, we showed that PG(Δ) can be coordinatized by a V-PTR for any triangle building Δ and in [10], it was shown that a V-PTR gives rise to a triangle building. We now show that, in case we have a CV-PTR, these operations are mutual inverse. In detail:

(ΔPTRΔ) *If Δ is any triangle building, then any coordinatizing PTR of* PG(Δ) *gives rise to Δ itself.*

(PTRΔPTR) *If (R, T) is any CV−PTR, then* PG(Δ), *where Δ is the triangle building defined by (R, T), can be coordinatized by (R, T).*

5.1. *Proof of* (ΔPTRΔ)

Let Δ be a triangle building and (R, T) a coordinatizing V-PTR of PG(Δ) with respect to the non-degenerate quadrangle (O, X, Y, E). Let $s = a(O, X, Y, E)$ and V_n the geometry derived from Δ and s as in Section 4 (for all $n \in \mathbb{N}$). Then PG(Δ) $\cong V = \varprojlim V_n$ (see (4.4)). Let Γ be the triangle building obtained from (R, T) by using the method of Section 3. Let W_n denote the geometries derived from (R, T) as in Section 3. Let R^+ and R^- be as usual. If $r \in R^+$, then we write r^+; if $r \in R^-$, then we write r^-. Let $R_\infty = R \cup \{\infty\}$ and $R^-_\infty = R^- \cup \{\infty\}$.

DEFINITION (5.1.1). We define the incidence structure $W =$

$(P(W), L(W), \mathrm{I})$ as follows:

(i) $P(W) = R^+ \times R^+ \cup R^+ \times R^-_\infty \cup R^-_\infty \times R^-_\infty$
(ii) $L(W) = R^+ \times R^+ \cup R^-_\infty \times R^+ \cup R^-_\infty \times R^-_\infty$

As usual, points are denoted by round brackets and lines by square brackets. Incidence is defined as follows:

(iii) $(W1)$ $(x^+, y^+)\,\mathrm{I}\,[m^+, k^+]$ if $T(m, x, y) = k$
$(x^+, y^+)\,\mathrm{I}\,[m^-, k^+]$ if $T(m, k, 0) = b$ and $T(m, x, y) = b$ for some $b \in R$.
$(x^+, y^-)\,\mathrm{I}\,[m^+, k^+]'$ if $T(x, y, a) = 0$ and $T(m, y, a) = k$ for some $a \in R$.
$(x^+, y^-)\,\mathrm{I}\,[m^-, k^-]$ if $T(x, y, a) = 0$, $T(b, k, 0) = m$ and $T(b, y, a) = m$ for some $a, b \in R$.
$(x^-, y^-)\,\mathrm{I}\,[m^-, k^+]$ if $T(x, a, y) = 0$, $T(m, k, 0) = b$ and $T(m, a, y) = b$ for some $a, b \in R$.
$(x^-, y^-)\,\mathrm{I}\,[m^-, k^-]$ if $T(x, a, y) = 0$, $T(b, k, 0) = m$ and $T(b, a, y) = m$ for some $a, b \in R$.

$(W2)$ $(x, \infty)\,\mathrm{I}\,[m^-, k^-]$ if $T(x, k, 0) = m$.
$(\infty, y^-)\,\mathrm{I}\,[m^-, k^+]$ if $T(m, k, 0) = y$.
$(x^-, y^-)\,\mathrm{I}\,[\infty, k]$ if $T(x, k, y) = 0$.
$(x^+, y^-)\,\mathrm{I}\,m^-, \infty]$ if $T(x, y, m) = 0$.

$(W3)$ $(x, \infty)\,\mathrm{I}\,[x, k^+]$ for all $x \in R_\infty$ and all $k \in R^+$
$(\infty, x^-)\,\mathrm{I}\,[x^-, k^-]$ for all $x, k \in R^-_\infty$.
$(k^+, x^+)\,\mathrm{I}\,[\infty, k^+]$ for all $x, k \in R^+$.
$(x^+, k^-)\,\mathrm{I}\,[\infty, k^-]$ for all $x \in R^+$ and all $k \in R^-_\infty$.
$(x^-, k^-)\,\mathrm{I}\,[k^-, \infty]$ for all $x, k \in R^-_\infty$.

$(W4)$ $(0, \infty)\,\mathrm{I}\,[m^-, \infty]$ for all $m \in R^-_\infty$.
$(\infty, x^-)\,\mathrm{I}\,[\infty, 0]$ for all $x \in R^-_\infty$.

PROPOSITION (5.1.2). *W is isomorphic to* $\mathrm{PG}(R, T) \cong \mathrm{PG}(\Delta)$.

Proof. We establish the coordinate transformation formulas g: $\mathrm{PG}(R, T) \to W$, h: $W \to \mathrm{PG}(R, T)$ and we show that they are mutually inverse and also that they preserve incidence.

(1) *Definition of g*

$g: \mathrm{PG}(R, T) \to W$: $(x^+, y^+) \to (x^+, y^+)$
$(x^-, y^+) \to (z^+, x^-)$ where $T(z, x, y) = 0$
$(x^+, y^-) \to (z^-, y^-)$ where $T(z, x, y) = 0$
$(0, y^-) \to (\infty, y^-)$ if $x \neq 0$
$(x, y^-) \to (z^-, y^-)$ where $T(z, x, y) = 0$ if $v(y) < v(x)$

$$
\begin{array}{ll}
(x^-, y^-) \to (z^+, x^-) & \text{where } T(z, x, y) = 0 \\
 & \quad \text{if } v(y) \geqslant v(x) \\
(x) \to (x, \infty) & \text{for all } x \in R_\infty \\
[m^+, k^+] \to [m^+, k^+] & \\
[m^-, k^+] \to [m^-, l^+] & \text{where } T(m, l, 0) = k \\
[m^+, k^-] \to [k^-, l^-] & \text{where } T(m, l, 0) = k \\
 & \quad \text{if } m \neq 0 \\
[0, k^-] \to [k^-, \infty] & \\
[m^-, k^-] \to (k^-, l^-] & \text{where } T(m, l, 0) = k \\
 & \quad \text{if } v(k) < v(m) \\
[m^-, k^-] \to [m^-, l^+] & \text{where } T(m, l, 0) = k \\
 & \quad \text{if } v(k) \geqslant v(m) \\
[k] \to [\infty, k] & \text{for all } k \in R_\infty.
\end{array}
$$

(2) *Definition of h*

$$
\begin{array}{lll}
h\colon W \to \mathrm{PG}(R, T)\colon & (x^+, y^+) \to (x^+, y^+) & \\
 & (x^+, y^-) \to (y^- z) & \text{where } T(x, y, z) = 0 \\
 & (x^-, y^-) \to (z, y^-) & \text{where } T(x, z, y) = 0 \\
 & (x, \infty) + (x) & \text{for all } x \in R_\infty \\
 & (\infty, x^-) \to (0, x^-) & \\
 & [m^+, k^+] \to [m^+, k^+] & \\
 & [m^-, k^+] \to [m^-, l] & \text{where } T(m, k, 0) = l \\
 & [m^-, k^-] \to [l, m^-] & \text{where } T(l, k, 0) = m \\
 & [\infty, k] \to [k] & \text{for all } k \in R_\infty \\
 & [k^-, \infty] \to [0, k^-]. &
\end{array}
$$

(3) *g and h are mutually inverse.* This is a long and tiresome, though elementary work. We restrict ourselves to two examples.

(i) $(g \circ h)((x^+, y^-)) = (x^+, y^-)$
$h((x^+, y^-)) = (y^-, z)$ with $T(x, y, z) = 0$. There are two possibilities:
(a) if $v(z) \geqslant 0$, then $g((y^-, z)) = (z'^+, y^-)$ with $T(z', y, z) = 0$, hence $x = z'$;
(b) if $v(z) < 0$, then $v(z) = v(y) + v(x) \geqslant v(y)$ and hence $g((y^-, z)) = (z'^+, y^-)$ with $T(z', y, z) = 0$ and again we have $x = z'$.

(ii) $(h \circ g)([m^-, k^-]) = [m^-, k^-]$
(a) if $v(k) < v(m)$, then $g([m^-, k^-]) = [k^-, l^-]$ with $T(m, l, 0) = k$ and $h([k^-, l^-]) = [l', k^-]$ with $T(l', l, 0) = k$, hence $l' = m$;
(b) if $v(k) \geqslant v(m)$, then $g([m^-, k^-]) = [m^-, l^+]$ with $T(m, l, 0) = k$ and $h([m^-, l^+]) = [m^-, l']$ with $T(m, l, 0) = l'$, hence $l' = k$.

(4) *g preserves incidence.* This is again a long but elementary job. We give

one example. In PG(R, T), (m^-) I $[m^-, k^+]$ we have:

$$g((m^-)) = (m^-, \infty)$$
$$g([m^-, k^+]) = [m^-, l^+] \text{ with } T(m, l, 0) = k$$

By (W3), the result follows.

(5) *h preserves incidence.* Again, we restrict ourselves to an example for the same reason as above.

Suppose in W, (x^-, y^-) I $[m^-, k^-]$, then there are $a, b \in R$ such that

$$(*) \qquad T(x, a, y) = 0, \; T(b, k, 0) = m, \; T(b, a, y) = m.$$

But then $h((x^-, y^-)) = (a, y^-)$ and $h([m^-, k^-]) = [b, m^-]$. by $(*)$, the result follows. That completes the proof of the proposition. QED

PROPOSITION (5.1.3). *Let* $x, y, a, b, m, k, p, q \in R$. *Recall that for* $r, s \in R$, $w(r, s) = v(r, s) - v(r) - v(s)$. *If* $x \neq 0$ (*resp.* $a \neq 0$), *then let* z (*resp.* c) *be defined by* $T(z, x, y) = 0$ (*resp.* $T(c, a, b) = 0$). *Then in* PG(R, T), *we have:*

$$\begin{aligned}
&u((x^+, y^+), (a^+, b^+)) &&= \inf\{v(x, a), v(y, b)\}\\
&u((x^-, y^+), (a^-, b^+)) &&= \inf\{v(z, c), w(x, a)\}\\
&u((x^+, y^-), (a^+, b^-)) &&= \inf\{w(z, c), w(y, b)\}\\
&u((0, y^-), (0, b^-)) &&= w(y, b)\\
&u((x^-, y^-), (a^-, b^-)) &&= \inf\{w(z, c), w(y, b)\}\\
& &&\quad \textit{if } v(y) < v(x) \text{ and } v(b) < v(a)\\
&u((x^-, y^-), (a^-, b^-)) &&= \inf\{v(z, c), w(x, a)\}\\
& &&\quad \textit{if } v(y) \geqslant v(x) \text{ and } v(b) \geqslant v(a)\\
&u((x^+), (a^+)) &&= v(x, a)\\
&u((x^-), (a^-)) &&= w(x, a)\\
&u((x^-, y^+), (a^-, b^-)) &&= \inf\{v(z, c), w(x, a)\} \quad \textit{if } v(b) \geqslant v(a)\\
&u((x^-, y^+), (a^+)) &&= \inf\{v(z, a), |v(x)|\}\\
&u((x^+, y^-), (0, b^-)) &&= \inf\{w(y, b), |v(z)|\}\\
&u((x^+, y^-), (a^-, b^-)) &&= \inf\{w(z, c), w(y, b)\} \quad \textit{if } v(b) < v(a)\\
&u((x^+, y^-), (a^-)) &&= \inf\{w(z, a), |v(y)|\}\\
&u((x^+, y^-), (\infty)) &&= \inf\{|v(z)|, |v(y)|\}\\
&u((0, y^-), (a^-, b^-)) &&= \inf\{|v(c)|, w(y, b)\} \quad \textit{if } v(b) < v(a)\\
&u((0, y^-), (a^-)) &&= \inf\{|v(a)|, |v(y)|\}\\
&u((0, y^-), (\infty)) &&= |v(y)|\\
&u((x^-, y^-), (a^-)) &&= \inf\{w(z, a), |v(y)|\} \quad \textit{if } v(y) < v(x)\\
&u((x^-, y^-), (\infty)) &&= \inf\{|v(z)|, |v(y)|\} \quad \textit{if } v(y) < v(x)\\
&u((x^-, y^-), (a^+)) &&= \inf\{v(z, a), |v(x)|\} \quad \textit{if } v(y) \geqslant v(x)\\
&u((x^-), (\infty)) &&= |v(x)|.
\end{aligned}$$

In all other cases $u(P, Q) = 0$ *for* P, Q *points of* PG(R, T).

Let l and r be defined as $T(m,l,0)=k$, resp. $T(p,r,0)=q$ if $m \neq 0$, resp. $p \neq 0$. Then we have:

$$
\begin{aligned}
&u([m^+,k^+],[p^+,q^+]) &&= \inf\{v(m,p),v(k,q)\}\\
&u([m^-,k^+].[p^-,q^+]) &&= \inf\{w(m,p),v(l,r)\}\\
&u([m^+,k^-],[p^+,q^-]) &&= \inf\{w(k,q),w(l,r)\} \quad \textit{if } m \neq \text{ and } p \neq 0\\
&u([0,k^-],[0,q^-]) &&= w(k,q)\\
&u([m^-,k^-],[p^-,q^-]) &&= \inf\{w(k,q),w(l,r)\}\\
& && \qquad \textit{if } v(k)<v(m) \quad \text{and} \quad v(q)<v(p)\\
&u([m^-,k^-],[p^-,q^-]) &&= \inf\{w(m,p),v(l,r)\}\\
& && \qquad \textit{if } v(k)\geqslant v(m) \quad \text{and} \quad v(q)\geqslant v(p)\\
&u([k^+],[q^+]) &&= v(k,q)\\
&u([k^-),(q^-]) &&= w(k,q)\\
&u([m^-,k^+],[p^-,q^-]) &&= \inf\{w(m,p),v(l,r)\} \quad \textit{if } v(q)\geqslant v(p)\\
&u([m^-,k^+],[q^+]) &&= \inf\{v(l,q),|v(m)|\}\\
&u([m^+,k^-],[0,q^-]) &&= \inf\{w(k,q),|v(l)|\}\\
&u([m^+,k^-],[p^-,q^-]) &&= \inf\{w(k,q),w(l,r)\} \quad \textit{if } v(q)<v(p)\\
&u([m^+,k^-],[q^-]) &&= \inf\{w(l,q),|v(k)|\}\\
&u([m^+,k^-],[\infty]) &&= \inf\{|v(l)|,|v(k)|\}\\
&u([0,k^-],[p^-,q^-]) &&= \inf\{w(k,q),|v(r)|\} \quad \textit{if } v(q)<v(p)\\
&u([0,k^-],[q^-]) &&= \inf\{|v(q)|,|v(k)|\}\\
&u([0,k^-],[\infty]) &&= |v(k)|\\
&u([m^-,k^-],[q^-]) &&= \inf\{|v(k)|,w(l,q)\} \quad \textit{if } v(k)<v(m)\\
&u([m^-,k^-],[\infty]) &&= \inf\{|v(k)|,|v(l)|\} \quad \textit{if } v(k)<v(m)\\
&u([m^-,k^-],[q^-]) &&= \inf\{|v(m)|,v(l,q)\} \quad \textit{if } v(k)\geqslant v(m)\\
&u([k^-],[\infty]) &&= |v(k)|.
\end{aligned}
$$

In all other cases $u(L,M)=0$ for L,M lines of PG(R,T).

Proof. Again, this is a long case-by-case proof without difficulties worth mentioning. We give two typical examples:

(1) Suppose $P=(x^+,y^-)$ and $Q=(a^-,b^-)$ with $v(b)<v(a)$. Since $u((x,0),X)=0$, we have $u(L_\infty,[x])=0$ by Corollary (4.3.11). Since $u((0,y),Y)>0$, we have $u(L_\infty,[0,y])>0$ by Corollary (4.3.11). Hence $u([x],[0,y])=0$. Since $\Pi_1([0,y])$ is incident with $Y^1, u([0,y],Y)>0$. Now Y I $[x]$ I (x,y) I $[0,y]$, so by Corollary (4.3.10)

$$u(Y,(x,y))+u([x],[0,y])=u(Y,[0,y])>0,$$

hence $u(Y,(x,y))>0$. And since $Y^1 \not\mathrm{I}\, X^{-1}, u((x,y),\bar{X})>0$.

Note that the intersection of $[z,0]$ and $[0,y]$ is (x,y). By Corollary (4.3.11),

$$u([z,0],[0,y])=u((0,0),(0))=0.$$

On the other hand, (a,b) is incident with $[a]$ and $[0,b]$. Mapping down onto V_n for $n=|v(b)|$, we see that $u((a,b),Y)>0$ (because $u((a,b),L_\infty)>0$, and if $u((a,b),Y)$ were 0, then $[a]$ would have to be mapped down onto L^n_∞ contradicting $v(b)<v(a)$). Similarly, we conclude $u((a,b),\bar{X})>0$ and by Corollary (4.3.11), $u([c,0],[0,b])=0$. Also (a,b) is the intersection of $[c,0]$ and $[0,b]$. Now $u(P,0)=u(P,X)=u(Q,0)=u(Q,X)=0$, so by mapping down onto V_n for $n=\inf\{w(z,c),w(y,b)\}$, resp. $\inf\{w(z,c),w(y,b)\}+1$ we easily see that $u(P,Q)=\inf\{w(z,c),w(y,b)\}$ (after all, $w(z,c)=u([z,0],[c,0])$ and $w(y,b)=u([0,y],[0,b])$).

(2) $P=(x^-,y^-)$ with $v(y)\geqslant v(x)$ and $Q=(a^-,b^-)$ with $v(b)<v(a)$. Since (x,y) is the intersection of $[z,0]$ and $[x]$, and $v(z)\geqslant 0$, we have $u((x,y),Y)=0$. On the other hand, we know that $u((a,b),Y)>0$ (see (1)). Hence, by Proposition (4.5.2), $u((x,y),(a,b))=0$. QED

The next proposition is the key to the proof of (ΔPTRΔ).

PROPOSITION (5.1.4). *V_n is isomorphic to W_n for all $n\in\mathbb{N}$.*

Proof. We establish maps $\kappa\colon V_n\to W_n$ and $\lambda\colon W_n\to V_n$ and we show that they preserve incidence and that they are mutually inverse.

(1) *Definition of* $\kappa\colon V_n\to W_n$. Let P^n be a point of V_n and P any point of PG(Δ) such that $P^n=\Pi_n(P)$. Let $g(P)=(x,y)$, then define $\kappa(P^n)=(x,y)/E_n$, where ∞/E_n denotes the class of E_n containing all $r\in R$ for which $v(r)\leqslant -n$. Similar definition for the image of lines.

(2) *κ is well defined.* Let Q be another point such that $P^n=\Pi_n(Q)$. Let $g(Q)=(a,b)$. We must show that $(a,b)/E_n=(x,y)/E_n$. Again, this needs a long case-by-case proof full of similar arguments. We give an example. Let $P=(x^-,y^+)$, then $g(P)=(z^+,x^-)$ where $T(z,x,y)=0$. Since $n>0$ ($n=0$ is a trivial case), there are only a few possibilities for the 'shape' of the coordinates of Q (in view of Proposition (5.1.3)):

(i) $Q=(a^-,b^+)$ and $n\leqslant\inf\{v(z,c),w(x,a)\}$, where $T(c,a,b)=0$. But then $g(Q)=(c^+,a^-)$ and since $(z,x)/E_n=(c,a)/E_n$, the result follows.
(ii) $Q=(a^-,b^-)$ with $v(b)\geqslant v(a)$. Again let $T(c,a,b)=0$. Then $g(Q)=(c,a)$ with $n\leqslant\inf\{v(z,c),w(x,a)\}$. So the result follows.
(iii) $Q=(a^+)$ and $n\leqslant\inf\{v(z,a),|v(x)|\}$. $g(Q)=a,\infty)$ and again, the result follows easily.

Similarly for all other cases. We conclude that κ is well defined.

(3) *κ preserves incidence.* Let $P^n\,\mathrm{I}\,L^n$, then there are P and L in PG(Δ) such that $P\,\mathrm{I}\,L$ and $\Pi_n(P)=P^n$; $\Pi_n(L)=L^n$. Hence $g(P)\,\mathrm{I}\,g(L)$. Note that we always can choose P and L such that $g(P)$ and $g(L)$ does not contain an

∞-symbol (the proof is left to the reader since it is quite simple and uninformative).

From the definitions of incidence follows $\kappa(P^n)\,\mathrm{I}\,\kappa(L^n)$.

(4) *Definition of* λ: $W_n \to V_n$. Let $(x, y) \in P(W_n)$. Take any representative $(\hat{x}, \hat{y})$. This can be viewed as a point in W. Then by definition $\lambda((x, y)) = \Pi_n(h(\hat{x}, \hat{y}))$. Similarly for lines.

(5) λ *is well defined.* Again we prove that for one example. Let $x \in R_n^+$ and $y \in R_n^-$. Let $(\hat{x}^+, \hat{y}^-)$ and $(\hat{a}^+, \hat{b}^-)$ be two representatives of (x, y). Then $v(\hat{x}, \hat{a}) \geqslant n$ and $w(\hat{y}, \hat{b}) \geqslant n$. We have:

$$h((\hat{x}, \hat{y})) = (\hat{y}, \hat{z}) \quad \text{with} \quad T(\hat{x}, \hat{y}, \hat{z}) = 0$$
$$h((\hat{a}, \hat{b})) = (\hat{b}, \hat{c}) \quad \text{with} \quad T(\hat{a}, \hat{b}, \hat{c}) = 0$$

Note that $v(\hat{z}) \geqslant v(\hat{y})$ and $v(\hat{c}) \geqslant v(\hat{b})$ by (v12). So by Proposition (5.1.3) $u((\hat{y}, \hat{z}), (\hat{b}, \hat{c})) = \inf\{v(\hat{x}, \hat{a}), w(\hat{y}, \hat{b})\} \geqslant n$ (this is independent of the signs of $v(\hat{z})$ and $v(\hat{c})$!). Hence, $\lambda((x, y))$ is well defined.

(6) λ *preserves incidence.* This follows immediately from Definitions (5.1.1.)(iii)(W1) and (3.1)(iii).

Denote that identity map in any set F as id_F.

(7) $\lambda \circ \kappa = \mathrm{id}_{V_n}$. Let $P^n \in P(V_n)$. Let P in PG(Δ) be such that $\Pi_n(P) = P^n$ and P is incident with no coordinate axis (always possible: see [10, Lemma (6.1.1.)]). Then $g(P)$ contains no ∞ and hence $g(P)$ is a representative of $\kappa(P^n)$. Now $P = h(g(P))$ and $\Pi_n(\mathrm{P}) = \mathrm{P}^n$. So $(\lambda \circ \kappa)(P^n) = P^n$. Similarly for lines.

(8) $\mathrm{k} \circ \lambda = \mathrm{id}_{W_n}$. Reverse the reasoning of the proof of (7). QED

PROPOSITION (5.1.5). Δ *is isomorphic to* Γ, *the triangle building arising from* (R, T, v).

Proof. We establish $\rho: \Delta \to \Gamma$ and $\tau: \Gamma \to \Delta$ and show they are mutually inverse. Note that in order to prove that ρ and τ are morphisms, it suffices to show that they preserve adjacency of vertices.

(1) *Definition of* $\rho: \Delta \to \Gamma$. Denote for vertices α and β, $d^*(\alpha, \beta) = \inf\{n \in \mathbb{N} | n$ is the number of panels of a sequence of panels joining α to $\beta\}$. Clearly $d^*(\alpha, \beta) \geqslant d(\alpha, \beta)$. Now, let $\alpha \in \mathrm{Ve}(\Delta)$, then any apartment containing α and s, contains some. Let α be any vertex of Δ. The parallelogram $\mathrm{cl}(s, \alpha)$ has sides of length i and j where $i + j = n = d^*(s, \alpha)$; let $P^i(\alpha)$ and $L^i(\alpha)$ be the two angular points of $\mathrm{cl}(s, \alpha)$ with $P^i(\alpha) \in P(V_i)$ and $L^j(\alpha) \in L(V_j)$. Now let $P^n \in P(V_n)$ and $L^n \in L(V_n)$ be arbitrary but such that $\alpha \in [P^n, L^n]$. We now conceive the above points and lines of V_h as points and lines of $W_h, h = i, j, n$ (this is possible by Proposition (5.1.4)) and define $\rho(\alpha)$ to be the vertex of Γ corresponding to $(\Pi_i^n)^{-1}(P^i(\alpha)) \cap L^n$. Note that $\rho(\alpha) \in B_n^i$ (notation of 3.3.1) and that $\rho(\alpha)$ also corresponds to

$(\Pi_j^n)^{-1}(L^j(\alpha)) \cap P^n$, where P^n is viewed as the set of lines incident with P^n in W_n.

(2) *ρ is well defined.* Clearly, the definition is independent of the choice of P^n. Now, let M^n be such that $\alpha \in [P^n, M^n]$. Then $\Pi_j(L^n) = \Pi_j(M^n)$. The result follows from [10, Lemma (5.1.8)]. (After all, $\Pi_j(L^n) = L^j(\alpha) = \Pi_j(M^n)$)

(3) *ρ preserves adjacency.* Let $\{\alpha_1, \alpha_2\} \in Pa(\Delta)$. It can have three distinct positions w.r.t. s:

(i) $P^i(\alpha_1) \neq P^{i+1}(\alpha_2)$ and $L^j(\alpha_2) \neq L^{j-1}(\alpha_2)$, $i + j = n$. Then, considering an apartment through s and $\{\alpha_1, \alpha_2\}$ (see Figure 16), $\rho(\alpha_1)$ corresponds to $(\Pi_i^n)^{-1}(P^i(\alpha_1)) \cap L^n$ and $\rho(\alpha_2)$ corresponds to $(\Pi_{i+1}^n)^{-1}(P^{i+1}(\alpha_2)) \cap L^n$, where $\Pi_i(P^{i+1}(\alpha_2)) = P^i(\alpha_1)$. By Definition (3.3.1), $\{\rho(\alpha_1), \rho(\alpha_2)\} \in Pa(\Gamma)$.

(ii) $P^i(\alpha_1) = P^i(\alpha_2)$ and $L^j(\alpha_1) \neq L^{j-1}(\alpha_2)$, $i + j = n$. Then, as above, $\rho(\alpha_1)$ corresponds to $S_1 = (\Pi_i^n)^{-1}(P^i(\alpha_1)) \cap L^n$ and $\rho(\alpha_2)$ corresponds to $S_2 = (\Pi_i^{n-1})^{-1}(P^i(\alpha_2)) \cap L^{n-1}$, with $\Pi_{n-1}(L^n) = L^{n-1}$. Hence $\Pi_{n-1}(S_1) = S_2$, so by (3.3.1), $\rho(\alpha_1)$ is adjacent to $\rho(\alpha_2)$.

(iii) $P^i(\alpha_1) \neq P^{i+1}(\alpha_2)$ and $L^j(\alpha_1) = L^j(\alpha_2)$. Dual to (ii).

(4) *Definition of* $\tau: \Gamma \to \Delta$. Let $b \in B_n^j$. Then $b_p \in \tilde{B}_n^j$ and $b_l \in \bar{B}_n^j$. Let $P^n \in b_p$ and $L^n \in b_1$. Define $\tau(b)$ as the vertex in Δ on $[P^n, L^n]$ on distance j from L^n and distance $n - j$ from P^n (again we identified V_n and W_n by Proposition (5.1.4)).

(5) *τ is well defined.* First note that $P^n \mathrm{I} L^n$, so there is at least one vertex $\tau(b)$ satisfying the conditions in (4).

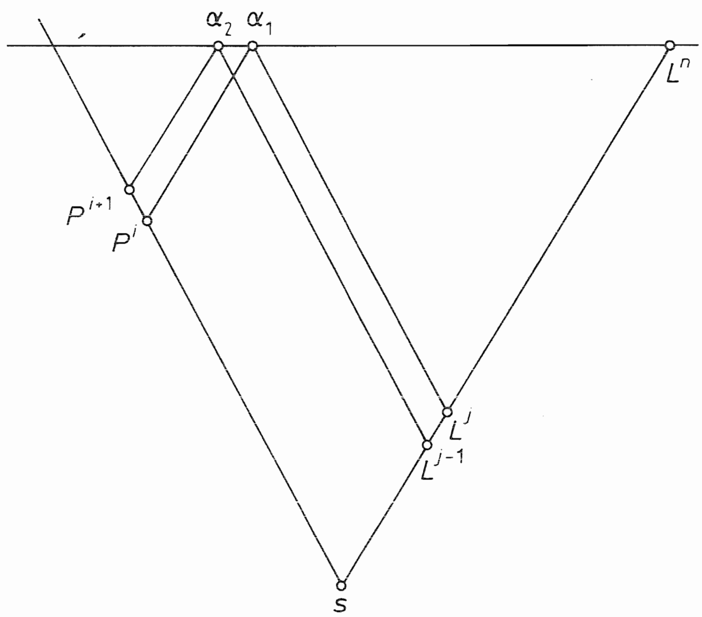

Fig. 16.

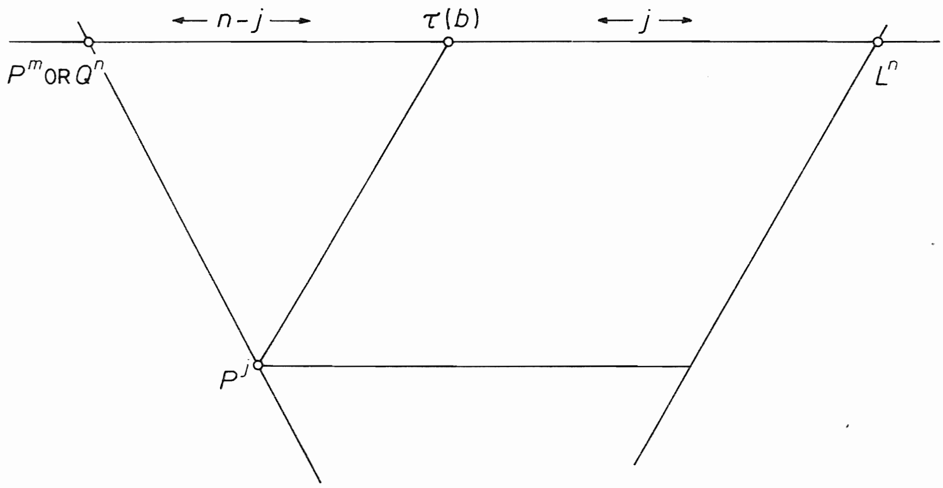

Fig. 17.

Let $Q^n \in b_p$ be distinct from P^n. Then $u(P^n, Q^n) \geq j$. Let $P^j = \Pi_j(P^n)$ and let A (resp. B) be an apartment through s, L^n and P^n (resp. Q^n), then the intersection of A and B contains P^n and L^j, and hence it contains $\mathrm{cl}(P^j, L^n)$ which contains $\tau(b)$ (see Figure 17). Hence $\tau(b)$ is on distance $n-j$ from Q^n. By the dual argument, we can also take another $M^n \in b_l$. We conclude that τ is well defined.

(6) *τ preserves adjacency*. Again, there are three cases from which two are mutually dual.

(i) Suppose $b \in B_n^j$, $c \in B_n^{j+1}$ and $c_p \subset b^p$, then $b_1 \subset c_1$. Choose $L^n \in b_l$, $P^n \in c_p$. By definition, τ(b) lies on $[P^n, L^n]$ on distance $j+1$ from L^n, while $\tau(c)$ is on the same $[P^n, L^n]$ on distance $j+1$ from L^n. Hence $\tau(b)$ and $\tau(c)$ are adjacent.

(ii) Suppose $b \in B_n^j$ and $c \in B_{n-1}^j$ with $\Pi_{n-1}^p(b) = c$. Choose $P^n \in b_p$ and $L^n \in b_l$ arbitrary, then by [10, Lemma (6.1.9)(i)], $\Pi_{n-1}(P^n) \in c_p$ and $\Pi_{n-1}(L^n) \in c_l$. So $\tau(c)$ lies in any quarter Q (with top s!) through P^n and L^n. Moreover, $\tau(c)$ lies on distance j from $\Pi_{n-1}(L^n)$ and distance $n-j-1$ from $\Pi_{n-1}(P^n)$. There is only one vertex in Q which satisfies these conditions and that vertex is adjacent to $\tau(b)$ (see Figure 18).

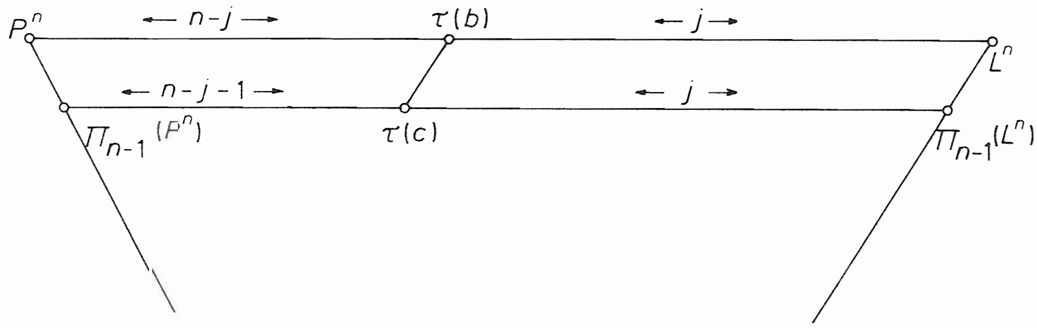

Fig. 18.

(iii) $b \in B_n^j$ and $c \in B_{n-1}^{j-1}$ with $\Pi_{n-1}^l(b) = c$ is dual to case (ii).

(7) $\tau \circ \rho = \mathrm{id}_\Delta$ and $\rho \circ \tau = \mathrm{id}_\Gamma$. This is obvious (by definition). QED

This concludes the proof of (ΔPTRΔ).

5.2. *Proof of* (PTRΔPTR)

Let (R, T) be a CV-PTR and let $\mathrm{PG}(R, T)$ be the corresponding projective plane. Let for all $n \in \mathbb{N}$, W_n be the geometry derived from (R, T), as in Section 3. Let Δ be the triangle building obtained from $(W_n)_{n \in \mathbb{N}}$ as in Section 3. Let $\mathrm{PG}(\Delta)$ be the 'plane at infinity' of Δ (at this point, we do not know whether $\mathrm{PG}(\Delta)$ is isomorphic to $\mathrm{PG}(R, T)$ or not!). For all $n \in \mathbb{N}$, let V_n be the geometry obtained from Δ by the procedure in Section 4, taking for s the unique element of B_0^0. Note that, at this point, we also do not know whether V_n is isomorphic to W_n or not. We prove the assertion by a sequence of propositions.

PROPOSITION (5.2.1). *V_n is isomorphic to W_n, for all $n \in \mathbb{N}$.*

Proof. Note that Δ is a combinatorial triangle building. By an elementary inductive argument, one can see that, if $P^n \in P(W_n)$, then $\{P^n, P^{n-1}, P^{n-2}, \ldots, P^1, P^0 = s\}$ is part of a pennel $(P^j = \Pi_j(P^n), j \leqslant n)$. Hence, $d(P^n, s) = n$. In this manner, we can view P^n as a point of V_n (after switching the names point and line in V_n, if necessary) because the type of P^1 is fixed and equal to $2(\mathrm{mod}\, 3)$ for all $P^n \in P(W_n)$. By another, but similar, inductive argument we can see that if $b_n \in B_n^j$, $j \neq n$, then $\{b_n, b_{n-1}, \ldots, b_1, b_0 = s\}$ (where $b_k = \Pi_k^p(b_{k+1})$ for $k < n$) is part of $\mathrm{cl}(b_n, s)$, but for $j \neq 0$, it cannot be part of a pennel (since the types do not suite, and for $j = 0$, b_1 has the wrong type in order that b_n should correspond to a point in V_n. So vertices of Δ corresponding to points of V_n can only correspond to points of W_n and hence there is a bijective correspondence between $P(W_n)$ and $P(V_n)$. Similarly for lines.

Suppose now $P^n \mathrel{\mathrm{I}} L^n$ in W_n. Then $((\Pi_k^n)^{-1}(\Pi_k(P^n)) \cap L^n)_{k \in J_n}$ is a sequence of adjacent vertices joining P^n to L^n, hence $d(P^n, L^n) \leqslant n$, so $P^n \mathrel{\mathrm{I}} L^n$ in V_n by Lemma (4.3.6).

Suppose $P^n \mathrel{\mathrm{I}} L^n$ in V_n. Consider $[P^n, L^n] = P^n, b_1, \ldots, b_{n-1}, L^n)$. Now, the only way to go from $P^n \in B_n^n$ to $L^n \in B_n^0$ in n steps is via B_n^k. Hence $b_k \in B_n^{n-k}$. Since b_k is adjacent to b_{k+1}, we have:

$$P^n \in (b_1)_p \subset (b_2)_p \subset \cdots \subset (b_{n-1})_p \subset L^n$$

Hence $P^n \mathrel{\mathrm{I}} L^n$ in W_n. QED

COROLLARY (5.2.2). *Let* V *be the inverse limit of* $(V_n)_{n\in\mathbb{N}}$ *w.r.t.* Π_n, *let* W *be the inverse limit of* $(W_n)_{n\in\mathbb{N}}$ *w.r.t.* Π_n, *then* V *is isomorphic to* W. *Moreover,* W *is isomorphic to* $\mathrm{PG}(\Delta)$.

Proof. This follows directly from Proposition (5.2.1) and Theorem (4.4).

PROPOSITION (5.2.3). *Let* $R_{\mathbb{N}}$ *be the inverse limit of* $(R_n)_{n\in\mathbb{N}}$ *(*R_n *as in* (3.1)*) w.r.t.* Π_n. *Then the map*

$$\mu: R_\infty \to R_{\mathbb{N}}: r \to \begin{cases} (r/E_n)_{n\in\mathbb{N}} & \text{if } r \neq \infty \\ (r_n/E_n)_{n\in\mathbb{N}} & r_n \text{ such that } v(r_n) \leqslant -n \end{cases}$$

is a bijection.

Proof. (1) μ *is onto.* Let us denote elements of $R_{\mathbb{N}}$ without the subscript '$n\in\mathbb{N}$'. Suppose $(r_n)\in R_{\mathbb{N}}$ and let $\hat{r}_n$ be arbitrary. Then $v(\hat{r}_n)$ has fixed sign. (i) Suppose $v(\hat{r}_n)\geqslant 0$ for all $n\in\mathbb{N}$. Then clearly $(\hat{r}_n)$ is a Cauchy sequence and if $\hat{r}$ is its limit, then $v(\hat{r},\hat{r}_n)\geqslant n$ for all $n\in\mathbb{N}$. Hence $\mu(\hat{r})=(r_n)$. (ii) Suppose $v(\hat{r}_n)<0$ for all $n\in\mathbb{N}$. If $v(\hat{r}_n)\leqslant -n$ for all n, then $\mu(\infty)=(r_n)$. So suppose $v(\hat{r}_k)>-k$ for some $k\in\mathbb{N}$. Then $v(\hat{r}_n)=v(\hat{r}_k)$, for all $n\geqslant k$ (see [10, §2.3]). Hence we have

$$v(\hat{r}_n,\hat{r}_m)=w(\hat{r}_n,\hat{r}_m)+2v(\hat{r}_k)\geqslant n-2k, \quad m>n\geqslant k.$$

So $(\hat{r}_n)$ is Cauchy! Again by [10, §2.3], one checks that $w(\hat{r},\hat{r}_n)\geqslant n$ for all $n\in\mathbb{N}$, where $\hat{r}$ is the limit of $(\hat{r}_n)$. Hence $\mu(\hat{r})=(r_n)$.

(2) μ *is one-to-one.* Suppose $r,s\in R_\infty$ and $r\neq s$. If $r=\infty$, then clearly $\mu(r)\neq\mu(s)$. Also, if $v(r)$ and $v(s)$ have distinct signs, then clearly $\mu(r)\neq\mu(s)$. If $v(r), v(s)\geqslant 0$, then $r/E_n\neq s/E_n$ for $n>v(r,s)$. If $v(r), v(s)<0$, then $r/E_n\neq s/E_n$ for $n>w(r,s)$. QED

PROPOSITION (5.2.4). $\mathrm{PG}(\Delta)$ *is isomorphic to* $\mathrm{PG}(R,T)$, *i.e.* $\mathrm{PG}(\Delta)$ *can be coordinatized by* (R,T).

Proof. In view of Proposition (5.1.2) and Corollary (5.2.2), it suffices to show that W can be coordinatized as in Definition (5.1.1).

The coordinates can be viewed as inverse limits of points or lines in W_n, up to the bijection μ (by Proposition (5.2.3)). We only have to check that the definition of incidence is correct.

(1) Suppose (P^n) and (L^n) are resp. a point and a line in W, with $P^n \mathrm{I} L^n$, for all $n\in\mathbb{N}$. Let $P^n=(x_n,y_n)$, $L=[m_n,k_n]$. Also, let $x,y,m,k\in R_\infty$ be such that $\mu(x)=(x_n)$, etc. The remainder of (1) is a long but easy case-by-case study. We give three typical examples:

(i) $x, y, m, k\in\bar{R}$. By Proposition (3.2.3), there are, for all $n\in\mathbb{N}$, $\hat{x}_n$, $\hat{y}_n$, a_n, $b\in R$ such that

(1) $\qquad T(\hat{x}_n,a_n,\hat{y}_n)=0$

(2) $T(b,k,0) = m$

(3) $T(b,a_n,\hat{y}_n) = m.$

Note that $(\hat{x}_n)$, resp. $(\hat{y}_n)$ converges to x, resp. y. Now (a_n) converges to $a \in R$ by Proposition (2.4.1), and we have by Equations (1), (2), (3):

$$T(x, a, y) = 0$$
$$T(b, k, 0) = m$$
$$T(b, a, y) = m.$$

Hence (x,y) I $[m,k]$ in W.

(ii) $x \in R^+$, y, $m \in R^-$, $k = \infty$. Again by Proposition (3.2.3), there are a, b_n, $\hat{m}_n$ and $\hat{k}_n \in R$ such that

(*1) $T(x, y, a) = 0$

(*2) $T(b_n, \hat{k}_n, 0) = \hat{m}_n$

(*3) $T(b_n, y, a) = \hat{m}_n$

By Proposition (2.4.1) and (*3), (b_n) converges to $b \in R$ and

(*4) $T(b, y, a) = m.$

Suppose $b \neq 0$, then by (*2), $v(\hat{m}_n) = v(\hat{k}_n) + v(b_n) \leqslant -n + v(b_n)$, hence $(\hat{m}_n)$ diverges to ∞; a contradiction. So $b = 0$ and by (*4), $a = m$. Hence by (*1) $T(x, y, m) = 0$. So by the incidence condition (W2), (x, y) I $[m, \infty]$.

(iii) $x = k = \infty$; y, $m \in R^-$. Again there are $\hat{x}_n$, $\hat{y}_n$, a_n, b_n such that for arbitrary $\hat{k}_n$:

(**1) $T(\hat{x}_n, a_n, \hat{y}_n) = 0$

(**2) $T(b_n, \hat{k}_n, 0) = m$

(**3) $T(b_n, a_n, \hat{y}_n) = m$

By (**2), $v(b_n) = v(m) - v(\hat{k}_n) \geqslant n + v(m)$, hence (b_n) converges to 0. By (**1), $v(a_n) = v(\hat{y}_n) - v(\hat{x}_n) \geqslant n + v(\hat{y}_n)$,
hence (a_n) converges to 0. By Proposition (2.4.1) and (**3), $T(0,0,y) = m$, so $y = m$ and by the incidence condition (W3), (∞, y) I $[y, \infty]$.

(2) Suppose that for $P = (x,y)$, $L = [m,k]$, one of the conditions (W1) through (W4) is satisfied (x, y, m, $k \in R_\infty$). Again, there are several cases. Let us examine two examples.

(i) If x, y, m, $k \in R$, then $(x,y)/E_n$ I $[m,k]/E_n$ by definition.

(ii) Suppose $x = \infty$; $y = m \in R^-$; $k \in R^-$, so (∞, y) I $[y,k]$. Define $b \in R$ as:

(1*) $T(b,k,0) = y$

Let $n \in \mathbb{N}$ and let $y_n \neq y$ be such that $w(y, y_n) \geqslant n - v(k)$ (this can always be arranged) and define a_n as:

(2*) $\quad T(b, a_n, y_n) = y$

a_n is well defined since $b = 0$ implies $y = 0$ (by (1*)), contradicting the fact that $v(y) < 0$. Note that also $a_n \neq 0$, otherwise $y_n = y$. So we can define x_n by

(3*) $\quad T(x_n, a_n, y_n) = 0.$

Now

$$\begin{aligned} v(x_n) &= v(y_n) - v(a_n) = v(y_n) - v(y, y_n) + v(b) \\ &= v(y_n) - v(y, y_n) + v(y) - v(k) \\ &= -v(k) - w(y, y_n) \leqslant -n. \end{aligned}$$

Hence, $(x_n, y)/E_n = (\infty, y)/E_n$ and by (1*), (2*), (3*),

$$(\infty, y)/E_n \, \mathrm{I} \, [y, k]/E_n. \qquad \text{QED}$$

REMARK (5.2.5) (W1) justifies our symbolic definition in (3.1).

This concludes the proof of (PTRΔPTR).

5.3. *End of the Proofs of Theorem* (I) *and the Main Theorem*

We can interpret Proposition (5.2.4) and Corollary (5.2.2) as follows: let Δ be the triangle building arising from an arbitrary V-PTR (R, T). Then PG(Δ) can be coordinatized by $R_{\mathbb{N}} = \varprojlim R_n$. But $R_{\mathbb{N}}$ is clearly complete (cf. Proposition (5.2.3)). Since every triangle building arises from some V-PTR (by (ΔPTRΔ)), any coordinatizing V-PTR of PG(Δ) is a CV-PTR. This completes the proof of Theorem (I) in the Introduction, and the Main Theorem now follows directly from (ΔPTRΔ), (PTRΔPTR) and Theorem (I). QED

5.4. *Complete Positive Valuated Ternary Rings*

THEOREM (5.4). *Let (S, T_+) be a CPV-TR. Let Δ be the triangle building derived from (S, T_+) as explained in Section* 3.4, *then* PG(Δ) *can be coordinated by a CV-PTR (R, T) with the property*:

$$(R^+, T) \cong (S, T_+)$$

If $(S, +, \cdot)$ is a ring, then (R, T) is a skewfield, namely the quotientfield of S.

Proof. Let W be the inverse limit of $(W_n)_{n \in \mathbb{N}}$. We again have that PG(Δ) is isomorphic to W. We coordinatize PG(Δ) w.r.t. the quadrangle

$$(((0,0), (0)_0, (\infty_0)_0, (1,1))/E_n^+)_{n \in \mathbb{N}}$$

Let $s \in S$, then $(P^n)_{n \in \mathbb{N}} = ((0, s/E_n^+))_{n \in \mathbb{N}}$ is a point of W and hence it defines a point $(0, \Xi(s))$ of PG(Δ), with $\Xi(s) \in R$. Now $v(\Xi(s)) \geqslant 0$ since $P^1 \neq Y^1$. Conversely, any point $P = (0, r)$ of PG(Δ) incident with the y-axis, defines a point $(0, r_n)$ in W_n incident with the line $[0]/E_n^+$. Taking any $\hat{r}_n$ for all $n \in \mathbb{N}$, one can see that $(\hat{r}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and thus it defines a unique element $¶(r) \in S$. Clearly $¶(\Xi(s)) = s$ and $\Xi(¶(r)) = r$.

Now let a, b, c, $d \in S$. $T_+(a, b, c) = d$ is equivalent with $(b, c)/E_n^+$ I $[a, d]/E_n^+$. (Use Proposition (2.4.1) to prove that.) But that is exactly equivalent to $T(\Xi(a), \Xi(b), \Xi(c)) = \Xi(d)$.

The second assertion follows from [10, §4.4]. QED

REMARKS. (1) The map $(S, T_+, v_+) \to (R, T, v)$ is an injective map (v_+ is the valuation map in S) from the set of all CPV-TR's to the set of all CV-PTR's.

(2) If (S, T_+) has all properties of a quasifield, except for the inverse for the multiplication, then (R, T) is *not* necessarily a quasifield!

(3) If (R, T, v) is a V-PTR, then there is a surjective map $(R, T, v) \to \Delta \to (R', T', v')$ from the set of all V-PTR's to the set of all CV-PTR's which maps (R, T, v) to its completion (R', T', v') with respect to v. (R, T, v) can be embedded in (R', T', v').

(4) If (S, T_+, v_+) is a PV-TR, then there is a surjective map $(S, T_+, v_+) \to \Delta \to (R, T, v)$ from the set of all PV-TR's to the set of all CPV-TR's, which maps (S, T_+, v_+) to the positive valuated part of a complete V-PTR. (S, T_+, v_+) can be embedded in both (R, T, v) and (R^+, T, v).

EXAMPLES

$$Z_p < Q_p; K[[t]]^+ < K[[t]] < K((t))$$
$$Q < Q_p; K[[t]]^+ < K(t) < K((t)), \quad (K \text{ a field}).$$

6. What about projective valuations?

In this section, we examine the connection between Tits' projective valuation and the valuation v of the present paper. In 6.1 we start with a brief summary of Tits' results [9].

6.1. *Recapitulation*

6.1.1. Suppose $I = Z/2Z$; $\mathbb{R}$ is the real line with the usual metric $|\cdot|$. Let $R_I = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the root system A_I; $\mathbf{e}_I = (\mathbf{1})$, $\mathbf{e}_2 = (-\mathbf{1})$; r_z is the reflexion about the point (z). Then $\text{sp}\{r_0, r_1\}$ is an affine Weyl group W and can be written as $\{r_0, 1I_{\mathbb{R}}\} \ltimes T$, where T is the group of translations generated by $2 \cdot \mathbf{e}_1$. The reflexions belonging to W are exactly those about integer points. These points are therefore called *walls*, but at the same time *vertices*. A *chamber* is a closed interval $[(z), (z+1)]$, $z \in Z$. A closed half line bounded by a vertex is a *quarter*. Since it is also bounded by a wall, quarters are also called *half apartments*. There is a type map typ: $Z \to I$: $z \to z \pmod 2$. Note that W is type-preserving and acts sharply transitively on the set of chambers (see [9]).

6.1.2. Discrete systems of apartments (for the definition we again refer to Tits [9]) having the above structure as apartment correspond to buildings of affine type $\tilde{A}_1$ (diagram $\circ \overset{(\infty)}{\text{——}} \circ$), which are in fact trees without finite endpoints. The 'building' at infinity (defined as in 1.3) of such a building is just a set without structure whose elements are called *ends* and which is denoted by End(T). An end is contained in an apartment if a representative does. Two ends p_1 and p_2 are contained in a unique apartment $A(p_1, p_2)$. For any three (pairwise distinct) ends p_1, p_2, p_3, there is a unique 'cross-point' $\kappa(p_1, p_2, p_3)$, namely the intersection of the apartments $A(p_1, p_2)$, $A(p_2, p_3)$, $A(p_3, p_1)$ (this intersection consists of one vertex since T is a tree). Let (p_1, p_2, p_3, p_4) be a fourtuple (p_1, p_2, p_3, p_4 pairwise distinct ends). We map $\mathbb{R}$ into $A(p_1, p_2)$ as follows: O is mapped to $\kappa(p_1, p_2, p_3)$; $[o, \infty]$ is mapped onto $A(p_1, p_2) \cap A(p_2, p_3)$ and $Z \subset \mathbb{R}$ is mapped onto the set of vertices of $A(p_1, p_2)$. By definition, $\beta(p_1, p_2; p_3, p_4) \in \mathbb{R}$ denotes the inverse image of the vertex $\kappa(p_1, p_2, p_4) \in A(p_1, p_2)$. If d_T denotes the distance in T induced by the axioms, then $|\beta(p_1, p_2; p_3, p_4)| = d_T(\kappa(p_1, p_2, p_3), \kappa(p_1, p_2, p_4))$. Tits ([9]) shows that T is uniquely determined by β: T can be recovered from the set End(T) and the function β. Moreover Tits proves:

PROPOSITION (6.1.2). *Let T_∞ be a set with at least three elements and β be a map from the set of 4-tuples of pairwise distinct elements of T_∞, to the set of integers Z. Then (T_∞, β) defines a tree T (with a discrete set of vertices) $\Leftrightarrow$ β satisfies* (VP1), (VP2), (VP3) *and* (VP4), *stated below:*

Let a, b, c, d, e be five pairwise distinct elements of T_∞.

(VP1) $\beta(a, b; c, d) = \beta(c, d; a, b) = -\beta(a, b; d, c)$.

(VP2) *If $\beta(a, b; c, d) = k > 0$, then $\beta(a, b; c, d) = k$ and $\beta(a, c; b, d) = 0$.*

(VP3) $\beta(a, b; c, e) = \beta(a, b; c, d) + \beta(a, b; d, e)$.

(VP4) *β is non-degenerate, i.e. $1 \in \operatorname{Im} \beta$.*

In this formulation, we exclude trees with only one 'crosspoint'.

We can extend β to 4-tuples containing exactly two equal elements as follows:

$$\beta(a,b;c,c) = \beta(c,c;a,b) = 0$$
$$\beta(a,b;c,b) = \beta(b,a;b,c) = +\infty$$
$$\beta(a,b;b,c) = \beta(b,a;c,b) = -\infty.$$

With that definition, (VP1) through (VP4) still hold (for (VP3) with the additional condition that the sum must be defined).

A map β defined on 4-tuples over a set T_∞, satisfying (VP1) through (VP4), is called a *projective valuation*.

The reasons why we want $1 \in \operatorname{Im}\beta$ are:

(1) We do not want to call a map which maps everything to 0 a projective valuation. Otherwise Proposition (6.1.4) has to be restated.
(2) We do not want to call a map β which maps anything to a multiple of an integer $p \neq 1, -1$ a projective valuation (then $(1/p)\beta$ would also be a projective valuation defining the same tree up to vertices adjacent with exactly two other vertices). Otherwise Proposition (6.1.4), Theorem (6.2) and Remark (6.2.2) have to be adjusted, e.g. a building would induce infinitely many projective valuations instead of exactly one (see Proposition (6.1.4)).

6.1.3. Let T_∞ be a set with projective valuation β. Let a, b, $c \in T_\infty$ be pairwise distinct. Denote by $S(a, b, c)$ the set $\{x \in T_\infty \mid \beta(a, b; c, x) > 0\}$. A *meteor* is a family $M = \{S(a, b_n, c_n) \mid n \in \mathbb{N}$ and $b_{n+1}, c_{n+1} \in S(a, b_n, c_n)$ for all $n\}$. M is briefly denoted by $\{S(a, b_n, c_n)\}_{n\in\mathbb{N}}$.

LEMMA (6.1.3). *Let $M = \{S(a, b_n, c_n)\}_{n\in\mathbb{N}}$ be a meteor, then we have:*

$$S(a, b_{n+1}, c_{n+1}) \subsetneqq S(a, b_n, c_n)$$

Proof. (1) $S(a, b_{n+1}, c_{n+1}) \neq S(a, b_n, c_n)$ since $c_{n+1} \in S(a, b_n, c_n) - S(a, b_{n+1}, c_{n+1})$.

(2) Suppose $x \in S(a, b_{n+1}, c_{n+1})$, then

$$k_1 = \beta(a, b_{n+1}; c_{n+1}, x) > 0.$$

We also have by definition:

$$k_2 = \beta(a, b_n; c_n, b_{n+1}) > 0$$
$$k_3 = \beta(a, b_n; c_n, c_{n+1}) > 0.$$

We have to show that $\beta(a, b_n; c_n, x) > 0$. By (VP3), we have:

$$(*) \qquad \beta(a, b_n; c_n, x) = k_2 + \beta(a, b_n; b_{n+1}, x).$$

If $k = \beta(a,b_n;b_{n+1},x) \geqslant 0$, the result follows, so suppose $k < 0$. Then by (VP1) and (VP2), $\beta(a,b_{n+1};x,b_n) = -k > 0$. Hence by (VP3):

$$0 < -k + k_1 = \beta(a,b_{n+1};c_{n+1},b_n) = \beta(a,b_n;c_{n+1},b_{n+1}).$$

Adding k_3, we have (using (VP3)):

$$-k + k_1 + k_3 = \beta(a,b_n;c_n,b_{n+1}) = k_2.$$

So $k_2 + k = k_1 + k_3 > 0$, and by (*), the result follows. QED

PROPOSITION (6.1.4). *Let V be a projective plane. Let L be a line of V and G the group of projectivities of L into itself. Then the set of triangle buildings* Δ *for which V is isomorphic to* PG(Δ), *is in bijective correspondence with the set of projective valuations on L invariant under the action of G and for which the intersection of any meteor is non-empty.*

Without the last condition, this can be found in [9, §9]. We have added that condition on the meteors to avoid triangle buildings with a non-maximal set of apartments (see below).

In the next paragraph, we study the correspondence between valuation and projective valuation.

6.2. *Equivalence Theorem*

THEOREM (6.2). *Let* (R, T) *be a PTR and* $\mathrm{PG}(R,T)$ *the corresponding projective plane. Let* $\bar{Y}$ *be the y-axis of* $\mathrm{PG}(R,T)$ *and* β *a map from the set of all 4-tuples over* $\bar{Y}$ *with at most two equal elements to the set* $Z \cup \{+\infty, -\infty\}$. *Suppose we also have a map* $v: R^2 \to Z \cup \{+\infty\}$. *Suppose:*

(vβ1) $v(a,b) = \beta(\infty,0;1,a) + \beta(\infty,a;0,b)$ *if* $a \neq 0$

(vβ2) $v(0,b) = \beta(\infty,0;1,b)$

(βv1) $\beta(a,b;c,d) = v(a,c) - v(a,d) + v(b,d) - v(b,c)$

(βv2) $\beta(\infty,b;c,d) = (c,d;\infty,\mathrm{b}) = (b,\infty;d,c) = (d,c;b,\infty) = v(b,d) - v(b,c)$,

where r in the argument of β *stands for* $(0,r)$ *and* ∞ *for* (∞). *We then have:*

(1) (R,T,v) *satisfies* (d1) *and* (d2) $\Leftrightarrow (\bar{Y},\beta)$ *satisfies* (VP1), (VP2) *and* (VP3).

(2) (R,T,v) *satisfies* (d1), (d2) *and* (d4) $\Leftrightarrow (\bar{Y},\beta)$ *satisfies* (VP1), (VP2), (VP3) *and* (VP4).

(3) (R,T,v) *satisfies* (d1), (d2), (d3) *and* (d4) $\Leftrightarrow (\bar{Y},\beta)$ *satisfies* (VP1), (VP2), (VP3), (VP4) *and* β *is invariant under the action of G, the group of projectivities of* $\bar{Y}$ *into itself.*

(4) (R, T, v) *is a CV-PTR* $\Leftrightarrow \beta$ *is a projective valuation on* $\bar{Y}$, *invariant*

under the action of G and with the property that the intersection of any meteor is non-empty.

Proof. The direct algebraic proof of this is straightforward, but long. (1) and (2) are nearly trivial. To show one direction of (3), the point is to choose carefully the projectivities and to write down the algebraic equation of them in terms of T. To show the converse, the point is that any projectivity can be written as the juxtaposition of certain 'special' projectivities $g_1 \circ g_2^{-1}$ where $\mathrm{Im}(g_1) = \mathrm{Im}(g_2)$ can be any line; the center of g_2 can be any point; but the center of g_1 can be chosen freely. A detailed proof is written down in [11, §4].

We restrict ourselves to the proof of (4), granted (3). The proof of (4) is not contained in [11].

(i) Let $(R, T), v)$ be a CV-PTR. By (3), β is a projective valuation invariant under the action of G.

Let $M = \{S(x, y_n, z_n)\}_{n \in \mathbb{N}}$ be a meteor. Since G is 3-transitive, there is a $g \in G$ which maps M onto a meteor $M' = \{S(\infty, a_n, b_n)\}$ with $a_0 = 0$ and $b_0 = 1$. So $g(X) = \infty$; $g(y_n) = b_n$. Clearly, the intersection of M is non-empty iff the intersection of M' is non-empty.

Since $a_1, b_1 \in S(\infty, 0, 1)$, we have:

$$\beta(\infty, 0; 1, a_1) = v(a_1) > 0$$
$$\beta(\infty, 0; 1, b_1) = v(b_1) > 0$$

Hence,

$$(^\circ) \qquad v(a_0, a_1) > v(a_0, b_0) = v(1) = 0.$$

We now prove by induction that $v(a_n, b_n) \geqslant n$.

(I) For $n = 0$, this is trivial (see $(^\circ)$).

(II) Let n be arbitrary. Since $a_n, b_n \in S(\infty, a_{n-1}, b_{n-1})$, we have:

$$(*) \qquad v(a_{n-1}, a_n) > v(a_{n-1}, b_{n-1}) \quad \text{and} \quad v(a_{n-1}, b_n) > v(a_{n-1}, b_{n-1}).$$

Hence, by (v3),

$$v(a_n, b_n) \geqslant \inf\{v(a_{n-1}, a_n), v(a_{n-1}, b_n)\} > v(a_{n-1}, b_{n-1}) \geqslant n - 1.$$

So $v(a_n, b_n) \geqslant n$ and hence (a_n) and (b_n) are 'asymptotic' sequences. By (*), $v(a_{n-1}, a_n) > v(a_{n-1}, b_{n-1}) > n - 1$, so (a_n) is Cauchy and converges to some $a \in R$. It is easy to see that $v(a, a_n) > v(a_n, b_n)$ for all $n \in \mathbb{N}$. So we have $\beta(\infty, a_n; b_n, a) = v(a, a_n) - v(a_n, b_n) > 0$, for all $n \in \mathbb{N}$. Hence a is in the intersection of M', and so $g^{-1}(a)$ is in the intersection of M.

(ii) Let (x_n) be a Cauchy sequence. Then, from a certain number $k \in \mathbb{N}$ on, $v(x_n)$ has constant sign. Suppose first this sign is positive. Let $(\hat{a}_n)$ be a subsequence such that $v(a_n, a_{n+1}) \geqslant n+1$. Let $y_n \in R$ be such that $v(y_n) = n$ and let $b_n = T(1, y_n, a_n)$. By (v12), we have:

$$v(a_n, b_n) = n < n + 1 \leqslant v(a_n, a_{n+1}).$$

Hence $\beta(\infty, a_n; b_n, a_{n+1}) > 0$, so $a_{n+1} \in S(\infty, a_n, b_n)$. Define d_n by:

$$(\dagger) \qquad T(1, d_n, a_{n+1}) = a_n$$

then $v(d_n) = v(a_n, a_{n+1}) \geqslant n+1$, hence $v(d_n, y_{n+1}) \geqslant n+1$. Since $b_{n+1} = T(1, y_{n+1}, a_{n+1})$, we have by (vT) and (†):

$$v(a_n, b_{n+1}) = v(y_{n+1}, d_n) > n = v(b_n, a_n).$$

Hence $\beta(\infty, a_n; b_n, b_{n+1}) > 0$, so $b_{n+1} \in S(\infty, a_n, b_n)$. We conclude that the set $M = \{S(\infty, a_n, b_n)\}_{n \in \mathbb{N}}$ is a meteor. If $a \in R$ is in its intersection, then $v(a, a_n) > v(a_n, b_n) = n$ for all $n \in \mathbb{N}$. So a is the limit of (a_n) and hence of (x_n).

Suppose now, $v(x_n)$ has constant negative sign from a certain number $k \in \mathbb{N}$ on. Note that $v(x_n)$ itself is constant after a while, so if we define y_n as $T(x_n, y_n, 0) = 1$ for $n \geqslant k$, then by (v11),

$$v(y_n, y_{n+1}) = v(x_n, x_{n+1}) - v(x_n) - v(x_{n+1})$$

and hence (y_n) is Cauchy and has a positive constant valuation for n large. By the first part of (ii), (y_n) converges to $y \in R - \{0\}$. Defining $x \in R$ by $T(x, y, 0) = 1$, we see that $v(x, x_n) = v(y, y_n) + v(x_n) - v(y)$, and hence (x_n) converges to x. QED

REMARK (6.2.1). From this proof it follows that the intersection of a meteor in $(\bar{Y}, \beta)$, where β is a projective valuation invariant under the action of G, contains at most one element (since a Cauchy sequence has at most one limit). However, this is a general property and the proof runs as follows: as in the proof of Lemma (6.1.3), one shows that, if $x \in S(a, b_n, c_n)$, then $\beta(a, b_n; c_{n-1}, x) > 1$, and by induction, this becomes $\beta(a, b_n; c_0, x) > n$. So if $y \in S(a, b_n, c_n)$, then also $\beta(a, b_n; c_0, y) > n$. From this, one deduces $\beta(a, x; c_0, y) > n$. This last equality is true for all $n \in \mathbb{N}$ if x and y are in the intersection of $M = \{S(a, b_n, c_n)\}_{n \in \mathbb{N}}$. But that is not possible for pairwise distinct a, c_0, x, y. Now, clearly $x, y \neq c_0, a$ and also $c_0 \neq a$ by definition, hence $x = y$.

REMARK (6.2.2). The transformation formulas (vβ) and (βv) have a nice geometric interpretation.

Consider again – with the notation of (4.5.4) – the tree $T_{X,\bar{Y}}$. Recall that in this tree, an end corresponds to a point on the y-axis. In fact $T_{X,\bar{Y}}$ is the rank 2 affine building (or the discrete system of apartments) corresponding to the projective line $\bar{Y}$. Let us display (vβ) in two cases:

(1) $v(a) = \beta(\infty, 0; 1, a)$. Note that $\kappa(\infty, 0, 1) = \psi(s)$ and that $d' = d_{T_{X,\bar{Y}}}$, Figure 19 tells us that the formula is right!

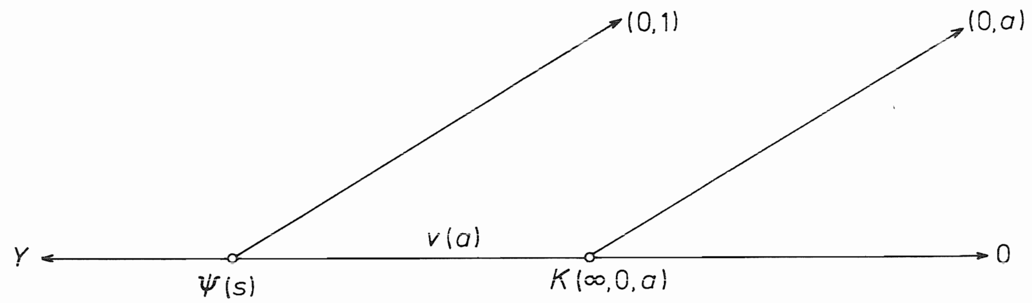

Fig. 19.

(2) $v(a) = v(b) < 0$ (see Figure 20).

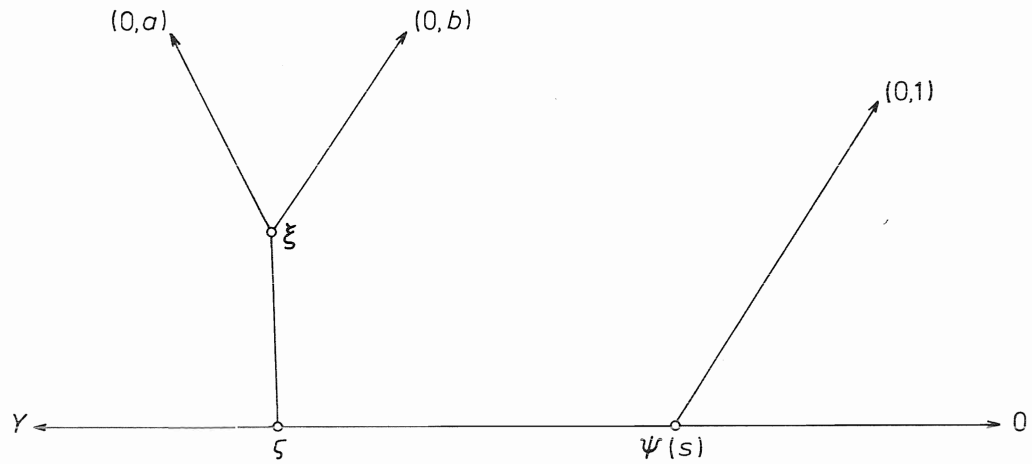

Fig. 20.

We denote $\kappa(\infty, 0, a) = \kappa(\infty, a, 0)$ by ζ and $\kappa(\infty, a, b)$ by ξ. Then we have:

$$\begin{aligned}\beta(\infty, 0; 1, a) + \beta(\infty, a; 0, b) &= -d'(\psi(s), \zeta) + d'(\zeta, \xi) \\ &= d'(\psi(s), \zeta) + d'(\zeta, \xi) - 2d'(\psi(s), \zeta) \\ &= d'(\psi(s), \xi) - d'(\psi(s), \zeta) - d'(\psi(s), \zeta) \\ &= w(a, b) + v(a) + v(b) = v(a, b)!\end{aligned}$$

Note also that (vβ) can be derived from (βv) and conversely.

There is an algebraic connection between (vβ1) and (vβ2). If we apply (vβ1) and (VIP3) on $v(0, b)$ formally, then we get:

$$\begin{aligned}v(0, b) &= \beta(\infty, 0; 1, 0) + \beta(\infty, 0; 0, b) \\ &= \beta(\infty, 0; 1, b)\end{aligned}$$

This is nothing other than (vβ2).

REMARK (6.2.3). Let (R, T, v) be a V-PTR, but not complete. Let Δ be the corresponding triangle building and (R', T', v') the competion of (R, T, v) with respect to v, coordinatizing PG(Δ). Let PG(R, T) be the subplane of PG(Δ) coordinatized by (R, T). Take away the apartments of Δ which do not correspond to apartments in PG(R, T). We then obtain a triangle building Γ whose set of apartments is not maximal of course and whose structure at infinity (similar defined as the spherical building at infinity for triangle buildings with a maximal set of apartments, cf. 1.3) is isomorphic to the spherical rank 2 building corresponding to PG(R, T). The fact that Γ is a system of apartments is due to the fact that, at finite distance, Γ looks exactly like Δ: all we did was remove apartments which are limits of sequences of apartments that stay. Such buildings are called 'symmetric discrete systems of apartments' in [9]. So, in fact, any V-PTR, complete or not, can be seen as a coordinatizing PTR of the projective plane corresponding to the building at infinity of a system of apartments. In fact one could even show that *there is a bijective correspondence between the class of symmetric discrete systems of apartments of type* $\tilde{A}_2$ *and the class of projective planes coordinatized by a V-PTR.*

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