Polar Kangaroos of Type E_6

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Abstract

An automorphism of a point-line geometry is called a *kangaroo* if its displacement spectrum has a gap; that is, at least one certain distance smaller than the diameter of the geometry cannot occur between a point and its image. In this paper we consider kangaroos in the exceptional long root subgroup geometry of type E_6 over an arbitrary field and classify them.

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1 Introduction

The notion of "kangaroo collineation", or briefly, a *kangaroo*, arose in connection with the classification of domestic automorphisms, in particular in spherical buildings of exceptional type E_7, see [7], a *domestic* automorphism being one that does not map a chamber to an opposite. Then a *k-kangaroo* is a collineation of a point-line geometry (related to a spherical building) having a gap in its displacement spectrum, namely, a point and its image are never at distance k, see [8]. These collineations show up again in the classification of so-called *uniclass* automorphism, that is, automorphisms of a spherical building the displacement spectrum on the chambers of which is contained in a single (possibly twisted) conjugacy class of the corresponding Weyl group (which is a finite Coxeter group). The general philosophy seems to be that kangaroo collineations have a large fix structure and their centralizer is very often a maximal subgroup of almost simple type of the corresponding Chevally group. The examples in the exceptional cases so far are all intimately related to the Tits indices [15] appearing in the Freudenthal-Tits Magic Square. For example, with the notation that shall be explained below, the 1-kangaroos of $\mathsf{E}_{6,1}(\mathbb{K})$ are precisely the nontrivial fixators of a quaternion or octonion Veronesean (related to the Tits index ${}^1\mathsf{E}^{28}_{6,2}$) [7]; the 3-kangaroos of $\mathsf{E}_{6,\{1,6\}}(\mathbb{K})$ are precisely the polarities fixing a split building of type F_4 and relate to the Tits index ${}^2\mathsf{E}^{2}_{6,4}$ [19]. Also, the $\{0,2\}$-kangaroos of $\mathsf{E}_{7,7}(\mathbb{K})$ are the fixators of a certain subcomplex isomorphic to a building of type F_4 and relate to the Tits index $\mathsf{E}^{9}_{7,4}$ [7].

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In [8] a link is explained between uniclass automorphisms and kangaroos of the corresponding long root subgroup geometries. The latter are point-line geometries naturally related to the polar node of the Dynkin diagram and where points at distance 2 from each other come in two flavours, denoted distance 2 and distance $2'$. In the present paper the aforementioned 1-kangaroos in $\mathsf{E}_{6,1}(\mathbb{K})$ are characterized as the only $\{1,2'\}$-kangaroo collineations of $\mathsf{E}_{6,2}(\mathbb{K})$. On top, we also classify the more general 1-kangaroos of $\mathsf{E}_{6,2}(\mathbb{K})$ which admit at least one fixed point and find that they are the nontrivial fixators of (large) subbuildings of type G_2 (generalized hexagon, more precisely, *triality hexagon* and *mixed hexagons*) and relate to the Tits index ${}^1\mathsf{E}_{6,2}^{16}$ and the maximal subgroups of types ${}^3\mathsf{D}_4$, ${}^6\mathsf{D}_4$ or mixed type G_2 of groups of type E_6.

Also the central elations are kangaroos; in the present paper, we first characterize those in buildings of type E_6 as the only collineations being 2-kangaroos of $\mathsf{E}_{6,1}(\mathbb{K})$. However, the natural homes of such elations being the long root subgroup geometries, it is natural to look for a characterization as a kangaroo in $\mathsf{E}_{6,2}(\mathbb{K})$. We show that a collineation of $\mathsf{E}_{6,2}(\mathbb{K})$ is a central elation if, and only if, it is a $\{2,2'\}$-kangaroo. We actually conjecture that this is true for all long root subgroup geometries.

In summary, informally stated (for precise statements, see Section 2.5), we prove in the present paper that

Main Result—(a) *A nontrivial collineation of the long root subgroup geometry of type* E_6 *is a* $\{1,2'\}$*-kangaroo if and only its fixed point structure in* $\mathsf{E}_{6,1}(\mathbb{K})$ *is a naturally included quaternion or octonion Veronesean.*
(b) *A nontrivial collineation of the long root subgroup geometry of type* E_6 *with at least one fixed point is a* 1*-kangaroo if and only its fixed point structure is a fully embedded Moufang hexagon of type* ${}^3\mathsf{D}_4$*,* ${}^6\mathsf{D}_4$*, or of mixed type.*
(c) *A nontrivial collineation of the long root subgroup geometry of type* E_6 *is a* $\{2,2'\}$*-kangaroo if and only it is a long root elation.*

It is worthwhile to note that, although we conjecture that similar characterizations hold for collineations of the other spherical buildings of exceptional type, those of type E_6 play a somewhat distinguished role because one does not have to take into account the possibility that the collineation maps everything to an opposite (a so-called *anisotropic* collineation), since in type E_6 such an anisotropic automorphism is never type-preserving.

Outline of the paper—In the next section we introduce the geometries that we will be working with; this is mainly the parapolar space $\mathsf{E}_{6,1}(\mathbb{K})$, the so-called *minuscule geometry of type* E_6 *over the field* $\mathbb{K}$. There is no central place in the literature where one can find all basic properties of this geometry. Most geometric properties are already proved by Tits [14]; for a more group theoretic approach see Aschbachter [1].

In the second part of Section 2 we introduce root elations and some geometric substructures of $\mathsf{E}_{6,1}(\mathbb{K})$ which allow us to state our main results in a more precise way than in the introduction above.

Section 3 is devoted to the proofs of the main results. Most of the assertions involving the polar $\{1,2'\}$-kangaroos follows from [7] once we show that a $\{1,2'\}$-kangaroo in $\mathsf{E}_{6,2}(\mathbb{K})$ is equivalent to a 1-kangaroo of $\mathsf{E}_{6,1}(\mathbb{K})$. So the emphasis is on the polar $\{2,2'\}$-kangaroos and the polar 1-kangaroos with a fixed point.

We end this introduction by noting that we are not aware of the existence of a polar 1-kangaroo which has no fixed point and which is not a polar $2'$-kangaroo. This is still open.

2 Preliminaries and statement of the Main Result

Throughout, we will work with incidence structures called *partial linear spaces*. In this subsection, we introduce the general definitions we will need.

2.1 Point-line geometries

Definition 2.1. A *point-line geometry* is a pair $\Delta = (\mathscr{P}, \mathscr{L})$ with $\mathscr{P}$ a set and $\mathscr{L}$ a set of subsets of $\mathscr{P}$. The elements of $\mathscr{P}$ are called *points*, the members of $\mathscr{L}$ are called *lines*. If $p \in \mathscr{P}$ and $L \in \mathscr{L}$ with $p \in L$, we say that the point p *lies on* the line L, and the line L *contains* the point p, or *goes through* p. If two (not necessarily distinct) points p and q are contained in a common line, they are called *collinear*, denoted $p \perp q$. If they are not contained in a common line, we say that they are *noncollinear*. For any point p and any subset $P \subset \mathscr{P}$, we denote

$$p^{\perp} := \{q \in \mathscr{P} \mid q \perp p\} \text{ and } P^{\perp} := \bigcap_{p \in P} p^{\perp}.$$

A *partial linear space* is a point-line geometry in which every line contains at least three points, and where there is a unique line through every pair of distinct collinear points p and q. That line is then denoted with pq.

Example 2.2. Let V be a vector space of dimension at least 3. Let $\mathscr{P}$ be the set of 1-spaces of V, and let $\mathscr{L}$ be the set of 2-spaces of V, each of them regarded as the set of 1-spaces it contains. Then $(\mathscr{P}, \mathscr{L})$ is called a *projective space (of dimension* $\mathsf{dim}V - 1$*) and denoted by* $\mathsf{PG}(V)$*, or* $\mathsf{PG}(n, \mathbb{K})$ *if* V *is defined over the field* $\mathbb{K}$ *and had dimension* $n+1$.

Definition 2.3. Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a partial linear space.

- (i) A *path of length* n in Δ from point x to point y is a sequence $(p_0, p_1, \dots, p_{n-1}, p_n)$, with $(p_0, p_n) = (x, y)$, of points of Δ such that $p_{i-1} \perp p_i$ for all $i \in \{1, \dots, n-1\}$. It is called a *geodesic* when there exist no paths of Δ from x to y of length strictly smaller than n, in which case *the distance* between x and y in Δ is defined to be n, notation $\delta_{\Delta}(x, y) = n$, or $\delta(p, q)$ if no confusion is possible.
- (ii) The partial linear space Δ is called *connected* when for any two points x and y, there is a path (of finite length) from x to y. If moreover the set $\{\delta_{\Delta}(x, y) \mid x, y \in \mathscr{P}\}$ has a supremum in $\mathbb{N}$, this supremum is called the *diameter* of Δ.
- (iii) A subset S of $\mathscr{P}$ is called a *subspace* of Δ when every line L of $\mathscr{L}$ that contains at least two points of S, is contained in S. A subspace that intersects every line in at least a point, is called a *hyperplane*; it is *proper* if it does not coincide with $\mathscr{P}$. A subspace is called *convex* if it contains all points on every geodesic that

connects any two points in S. We usually regard subspaces of Δ in the obvious way as subgeometries of Δ.

(*iv*) A subspace S in which all points are collinear, or equivalently, for which $S \subseteq S^\perp$, is called a *singular* subspace. If S is moreover not contained in any other singular subspace, it is called a *maximal* singular subspace. If it is contained in at least one other singular subspace, but all such singular subspaces are maximal, then we call it *submaximal*. A singular subspace is called *projective* if, as a subgeometry, it is a projective space (cf. Example 2.2). Note that every singular subspace is trivially convex.

(*v*) For a subset P of $\mathscr{P}$, the *subspace generated by* P is denoted $\langle P \rangle_\Delta$ and is defined to be the intersection of all subspaces containing P. The *convex hull of* P is denoted by $\mathsf{conv}_\Delta(P)$ and is defined to be the intersection of all convex subspaces that contain P. A subspace generated by three mutually collinear points, not on a common line, is called a *plane*. Note that, in general, this is not necessarily a singular subspace; however we will only deal with geometries satisfying Axiom (GS) (see below), which implies that subspaces generated by pairwise collinear points are singular; in particular planes will be singular subspaces.

2.2 Polar and parapolar spaces

We recall the definition of a polar space, and gather some basic properties. We take the viewpoint of Buekenhout–Shult [4]. All results in this section are well known, the standard reference being [3]. Since we are only interested in polar spaces of finite rank, we include this in our definition.

Definition 2.4. A *polar space* is a point-line geometry Γ in which for every point p the set $p^\perp$ is a hyperplane, and each nested family of singular subspaces is finite and had size $r+1$ at least 3. The polar space is *nondegenerate* if $p^\perp$ is always a proper hyperplane. The integer r is the *rank* of the polar space.

One shows that a polar space Γ is partial linear, and that each singular subspace is a projective space, see [4]. The maximal singular subspaces of a polar space of rank r have dimension $r-1$. Two singular subspaces are called Γ-*opposite* if no point of either of them is collinear to all points of the other.

Example 2.5. Let $\mathbb{K}$ be a field, n an integer at least 3, and let $\mathscr{H}$ be a hyperbolic quadric in $\mathsf{PG}(2n-1,\mathbb{K})$, that is, a quadric with standard equation $X_{-1}X_1 + X_{-2}X_2 + \cdots + X_{-n}X_n = 0$. The the points and lines on $\mathscr{H}$ define a point-line geometry that is a polar space of rank n and that we will denote by $\mathsf{D}_{n,1}(\mathbb{K})$. We call it a *hyperbolic* polar space. It has the peculiar property that every submaximal singular subspace is contained in exactly two maximal singular subspaces. Also, intersecting in a subspace of even codimension defines an equivalence relation on the set of maximal singular subspaces.

We also recall the definition of a parapolar space and introduce the ones that we are concerned with in this paper,

Definition 2.6. A *parapolar space* Δ is a connected point-line geometry, which is not a polar space, and for which every pair $\{p, q\}$ of points with $|p^\perp \cap q^\perp| \geq 2$ is contained in a convex subspaces isomorphic to a nondegenerate polar space. Any such convex subspace is called a *symp* of Δ (which is short for *symplecton*).

A pair of points p and q is called *special* if $|p^\perp \cap q^\perp| = 1$. A pair of noncollinear points p and q is called symplectic if $|p^\perp \cap q^\perp| \geq 2$. In this case, $\mathsf{conv}_\Delta(\{p, q\}$ is a nondegenerate polar space. A parapolar space is called *strong* when it contains no pair of special points.

Remark 2.7. The definition of parapolar space immediately implies that it is a partial linear space. Also, parapolar spaces are so-called *gamma spaces*, that is, they satisfy the following axiom, which is sometimes superfluously added in the definition.

(GS) Every point is collinear to zero, one or all points of any line.

In the present paper, we will only be concerned with parapolar spaces all symps of which have the same rank $r \geq 3$. We say that the parapolar space has *(constant* or *uniform symplectic) rank* r. If $r \geq 3$, then all singular subspaces are projective.

Example 2.8. Let $\mathscr{H}$ be a hyperbolic quadric in $\mathsf{PG}(2n-1, \mathbb{K})$ as in Example 2.5, with $n \geq 5$. Let Υ_1 and Υ_2 be the two natural systems of maximal singular subspaces. Let Ξ be the set of singular subspaces of dimension $r-3$ and set $L(W) = \{U \in \Upsilon_1 \mid W \subseteq U\}$ for each $W \in \Xi$. Then the point-line geometry with point set Υ_1 and line set $\{L(W) \mid W \in \Xi\}$ is a strong parapolar space with diameter $\lfloor \frac{n}{2} \rfloor$ and rank 4. We denote it by $\mathsf{D}_{n,n}(\mathbb{K})$.

Definition 2.9. Let Γ be a nondegenerate polar or parapolar space of rank r and let U be a singular subspace of Γ of dimension at most $r-3$. We define $\mathsf{Res}_\Gamma(U)$ to be the point-line geometry $(\mathscr{P}, \mathscr{L})$ with

$$\mathscr{P} := \{\text{singular subspaces } K \text{ of } \Gamma \text{ with } N \subset K \text{ and } \mathsf{codim}_K(N) = 1\},$$
$$\mathscr{L} := \{\text{singular subspaces } L \text{ of } \Gamma \text{ with } N \subset L \text{ and } \mathsf{codim}_L(N) = 2\},$$

where any element of $\mathscr{L}$ is identified with the set of elements of $\mathscr{P}$ contained in it.

If U is a point, then we say that $\mathsf{Res}_\Gamma(U)$ is a *point residual.*

Lemma 2.10 ([10])**.** *Let Γ be a possibly degenerate polar space of rank r and let U be a singular subspace of Γ of dimension d at most $r-3$. Then the point-line geometry $\mathsf{Res}_\Gamma(U)$ is a polar space of rank $r-1-d$, which is nondegenerate if and only if $U^{\perp\perp} = U$. If Γ is a parapolar space of rank r and U is nonempty and has dimension $d \leq r-3$, then $\mathsf{Res}_\Gamma(U)$ is a strong parapolar space of rank $r-1-d$. If $d \geq 1$, then $\mathsf{Res}_\Gamma(U)$ has diameter* 2.

A lot of background information about parapolar spaces is provided in the standard reference [12].

2.3 Parapolar spaces of type E_6

Let $\mathbb{K}$ be a field. We now introduce the parapolar spaces $\mathsf{E}_{6,1}(\mathbb{K})$ and $\mathsf{E}_{6,2}(\mathbb{K})$. They are related to the building of type E_6 over the field $\mathbb{K}$, but we will not need that relationship; instead we define these geometries by one of their characterizations in the literature, see Theorem 15.4.3 in [12].

Definition 2.11. A parapolar space $\Delta = (X, \mathscr{L})$ is denoted by $\mathsf{E}_{6,1}(\mathbb{K})$ and called *of type* $\mathsf{E}_{6,1}$ if it satisfies the following conditions.

(i) Two different points are either collinear or symplectic. In other words, Δ is strong and has diameter 2.

(ii) The symps are hyperbolic polar spaces of rank 5 isomorphic to $\mathsf{D}_{5,1}(\mathbb{K})$.

(iii) If a point p is collinear with at least one point of a symp ξ not containing p, then $p^\perp \cap \xi$ is a maximal singular subspace of ξ.

(iv) Two different symps with at least two common points, have a maximal singular subspace in common.

Note that these axioms are not entirely independent, but we are not concerned about that.

Before we define $\mathsf{E}_{6,2}(\mathbb{K})$, we note that $\mathsf{E}_{6,1}(\mathbb{K})$ has maximal singular subspaces of dimension 5, and that two such subspaces intersect in at most a plane. Let Ω be the set of maximal subspaces of dimension 5 of $\mathsf{E}_{6,1}(\mathbb{K})$ and let Π be the set of (projective) planes of $\mathsf{E}_{6,1}(\mathbb{K})$.

Definition 2.12. With the above notation, define for $\pi \in \Pi$ the set $L(\pi) = \{W \in \Omega \mid \pi \subseteq W\}$. Then the point-line geometry $(\Omega, \{L(\pi) \mid \pi \in \Pi\}$ is denoted by $\mathsf{E}_{6,2}(\mathbb{K})$.

Proposition 2.13. *The point-line geometry* $\mathsf{E}_{6,2}(\mathbb{K})$ *is parapolar space of rank* 4 *and diameter* 3*. It is not strong.*

In fact, $\mathsf{E}_{6,2}(\mathbb{K})$ is the long root subgroup geometry of the building of type E_6 over $\mathbb{K}$, and hence the Lie incidence geometry related to the so-called polar node of the E_6 diagram.

It follows from the previous proposition that in $\mathsf{E}_{6,2}(\mathbb{K})$ two distinct points are either collinear, symplectic, special or at distance 3 from each other. Special pairs of points p, q are also said to be at distance $2' = \delta(p, q)$. In Lemma 2.21 below, we make the connection with the mutual positions of the two 5-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$ corresponding to two points of $\mathsf{E}_{6,2}(\mathbb{K})$.

Definition 2.14. (i) Let θ be a collineation of $\mathsf{E}_{6,2}(\mathbb{K})$, and let, as above, Ω be the point set of $\mathsf{E}_{6,2}(\mathbb{K})$. Then the *displacement* of θ is the set $\{\delta(p, p^\theta) \mid p \in \Omega\}$.

(ii) Let K be a nonempty subset of $\{0, 1, 2, 2', 3\}$. An automorphism of $\mathsf{E}_{6,2}(\mathbb{K})$ is called a *K-kangaroo collineation*, or briefly a *K-kangaroo*, or even briefer a *kangaroo*, if its displacement does not contain any member of K.

Similar definitions hold in $\mathsf{E}_{6,1}(\mathbb{K})$, where the possible distances of points are $0, 1$ and 2. In order to facilitate expressing that we mean a kangaroo in $\mathsf{E}_{6,2}(\mathbb{K})$, and not in $\mathsf{E}_{6,1}(\mathbb{K})$, we sometimes add the adjective *polar*. Hence we talk about *polar (K-)kangaroos.*

2.4 Root elations

A *root elation* in $\mathsf{E}_6(\mathbb{K})$ is an automorphism that fixes all chambers having a panel in a given half apartment. It is well known that all nontrivial elations of a spherical building of rank at least 3 with simply laced diagram are all conjugate to one another. In $\mathsf{E}_{6,2}(\mathbb{K})$, a root elation is sometimes called a *central elation* as it fixes all points collinear or symplectic to a given point, the *centre*, and it stabilizes all lines containing a point collinear to the centre.

Lemma 2.15. *Let θ be a root elation of $\mathsf{E}_{6,1}(\mathbb{K})$. Then there is a unique 5-space U such that every pointwise fixed 5-space is adjacent or equal to U.*

2.5 Main Results

Before we can state our Main Result, we need to describe some substructures of $\mathsf{E}_{6,1}(\mathbb{K})$.

Definition 2.16. (*i*) An *ovoid* in a polar space is a set of points intersecting each maximal singular subspace in precisely one point.

(*ii*) A *special ovoid* in $\mathsf{E}_{6,1}(\mathbb{K})$ is a set of pairwise noncollinear points, not all contained in one common symp, intersecting each symp in either the empty set, a singleton, or an ovoid.

(*iii*) A point-line geometry $\Gamma = (X, \mathscr{L})$ is a *generalized hexagon* if the graph on $X \cup \mathscr{L}$ with a point and line adjacent if the point belongs to the line, and no further adjacencies, has girth 12 and diameter 6.

(*iv*) A *simple spread* of $\mathsf{PG}(2n+1, \mathbb{K})$ is a set of n-dimensional subspaces partitioning the point set.

(*v*) Two planes of $\mathsf{E}_{6,1}(\mathbb{K})$ are called *opposite*, if no pair of points taken from distinct planes is collinear; *semi-opposite* if each point of each plane is collinear to exactly one point of the other plane; *collinear* if each point of each plane is collinear to each point of the other plane.

(*vi*) A set $\mathscr{S}$ of planes of $\mathsf{E}_{6,1}(\mathbb{K})$ is called a *semi-spread* if each pair of members of $\mathscr{S}$ is either opposite, semi-opposite or collinear, and the members of $\mathscr{S}$ in any maximal singular subspace W of dimension 5 containing at least one member of $\mathscr{S}$ form a spread of W.

We can now formulate our Main Result.

Main Result 1. *Let θ be a nontrivial 1-kangaroo of $\mathsf{E}_{6,2}(\mathbb{K})$.*

(*i*) *If θ fixes at least one point of $\mathsf{E}_{6,2}(\mathbb{K})$, then its fix structure is a semi-spread of $\mathsf{E}_{6,1}(\mathbb{K})$.*

(*ii*) *If θ is also a $2'$-kangaroo, then its fix structure is a special ovoid of $\mathsf{E}_{6,1}(\mathbb{K})$.*

Conversely, each elementwise stabilizer in $\mathsf{E}_{6,1}(\mathbb{K})$ of a semi-spread is a 1-kangaroo of $\mathsf{E}_{6,2}(\mathbb{K})$ which fixes at least one plane of $\mathsf{E}_{6,1}(\mathbb{K})$. Each elementwise stabilizer in $\mathsf{E}_{6,1}(\mathbb{K})$ of a special ovoid is a $\{1, 2'\}$-kangaroo of $\mathsf{E}_{6,2}(\mathbb{K})$.

Moreover, we show:

Proposition 2.17. (*i*) *A semi-spread of $\mathsf{E}_{6,1}(\mathbb{K})$ which is the fix structure of a nontrivial collineation defines a Moufang generalized hexagon in $\mathsf{E}_{6,2}(\mathbb{K})$ either of type ${}^3\mathsf{D}_4$ or ${}^6\mathsf{D}_4$, or of mixed type.*

(*ii*) *The points of a special ovoid $\mathscr{O}$ which is the fix structure of a nontrivial collineation of $\mathsf{E}_{6,1}(\mathbb{K})$ and the symps containing at least two points of $\mathscr{O}$ define a Moufang projective plane either over a quaternion or octonion division algebra over $\mathbb{K}$, or over an inseparable quadratic field extension of degree 4 of $\mathbb{K}$.*

Concerning existence we show:

Proposition 2.18. (*i*) *The isomorphism classes of cubic extensions of $\mathbb{K}$ are in natural one-to-one correspondence to the projective equivalence classes of semi-spreads of $\mathsf{E}_{6,1}(\mathbb{K})$ fixed by a nontrivial collineation.*

(*ii*) *The isomorphism classes of quaternion and octonion division algebras over $\mathbb{K}$, including inseparable quadratic field extensions of degree 4, are in natural one-to-one correspondence to the projective equivalence classes of special ovoids of $\mathbb{K}$ fixed by a nontrivial collineation.*

We also characterize root elations by means of kangaroos.

Main Result 2. *An automorphism of $\mathsf{E}_6(\mathbb{K})$ is a root elation if and only if it is a $\{2, 2'\}$ kangaroo of the corresponding geometry $\mathsf{E}_{6,2}(\mathbb{K})$ if and only if it is a 2-kangaroo of the corresponding geometry $\mathsf{E}_{6,1}(\mathbb{K})$.*

2.6 Trivia about $\mathsf{E}_{6,1}(\mathbb{K})$

In this section we collect a number of well-known properties of the geometry $\Delta = (\mathscr{P}, \mathscr{L})$ of type $\mathsf{E}_{6,1}$. To have everything at one place, we also include the axioms. Most results are proved in [14], the others follow directly from (*ii*) below and a straight forward argument in the associated polar space $\mathsf{D}_{5,1}(\mathbb{K})$.

Lemma 2.19. (*i*) *Δ is a strong parapolar space with diameter 2.*

(*ii*) *The point residual at any point is isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$.*

(*iii*) *All symps of Δ are isomorphic to $\mathsf{D}_{5,1}(\mathbb{K})$.*

(*iv*) *All singular subspaces of dimension $d \geq 2$ of Δ are isomorphic to $\mathsf{PG}(d, \mathbb{K})$.*

(*v*) *The maximal singular subspaces of Δ have dimension 4 and 5. They are referred to as the 4-spaces and 5-spaces, respectively.*

(*vi*) *Each singular subspace U of dimension 4 is contained in a unique maximal singular subspace. If the latter is a 5-space, then U is referred to as a $4'$-space.*

(*vii*) *If a point p is not contained in a symp ξ, but is collinear to at least one point of ξ, then it is collinear to all points of a $4'$-space U of ξ. The space spanned by p and U is a 5-space.*

(*viii*) *For each symp ξ, its maximal singular subspaces contained in some other symp are 4-spaces and form one natural class of generators, and its maximal singular subspaces contained in a 5-space of Δ are $4'$-spaces and form the other class.*

(*ix*) *Two distinct symps intersect in either a point, or a 4-space.*

(*x*) *For each point $p \in \mathscr{P}$ there exists a symp ξ disjoint from $p^\perp$.*

(*xi*) *A singular 5-space that intersects a given symp in at least a plane, intersects the symp in a $4'$-space. There exist 5-spaces intersecting a given symp in precisely a given line.*

(*xii*) *Each singular* 3*-space is contained in a unique maximal singular* 4*-space and a unique singular* 5*-space.*
(*xiii*) *If two collinear points are collinear to respective* 3*-spaces of a* 5*-space, then these* 3*-spaces have a plane in common.*

The following properties deserve a separate mention. They are about the possible positions of two subspaces of Δ. Proofs can be found in [14].

Lemma 2.20. *For a point p and* 5*-space U of Δ, we have either*

(*i*) $p \in U$,
(*ii*) $p^\perp \cap U$ *is a* 3*-space or*
(*iii*) $p^\perp \cap U$ *is a point.*

With the notation of Lemma 2.20, we call p *far* from U if $p^\perp \cap U$ is a unique point and *close* if it is a 3-space.

Lemma 2.21. *For two* 5*-spaces U, V, we have either*

(0) $U = V$,
(1) $U \cap V$ *is a plane (and every point of $U \setminus V$ is close to V),*
(2) $U \cap V$ *is a point (and there exists a unique $4'$-space H of U with the property that a point $x \in U \setminus V$ is close to V if and only if $x \in H$),*
($2'$) *there is a unique* 5*-space W, such that $\alpha := U \cap W$ and $\beta := W \cap V$ are both planes (and every point of $U \setminus \alpha$ is far from V; each such point is collinear with a point of β), or*
(3) *U is opposite V, which means that for $p \in U$, there is a unique $q \in V$ with $p \perp q$ (so every point of U is far from V).*

The numbers represent the distance between U and V, where $2'$ corresponds to being special.

We call two 5-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$ *symplectic, special* or *opposite* if the corresponding points of $\mathsf{E}_{6,2}(\mathbb{K})$ are symplectic, special or opposite, respectively. If two 5-space intersect in a plane, we call them *adjacent* (as "collinear" would be confusing).

We also have the following basic results.

Lemma 2.22. *Let π be a plane and U a singular* 5*-space. If each point of π is collinear to a unique point of U, then there exists a unique singular* 5*-space V containing π and not opposite U; it is special to U and the plane $\alpha \subseteq U$ collinear to a plane of V coincides precisely with the set of points of U collinear to some point of π.*

Proof. We sketch the proof. Pick three points p_1, p_2, p_3 of π in a triangle and let $q_1, q_2, q_3 \in U$ be the respective collinear point. The symps defined by p_1, q_2 (which also contains p_2, q_1) and p_1, q_3, respectively, intersect by Lemma 2.19(*ix*) in a 4-space A (as they already share the line p_1q_1). The set $\{p_2, p_3, q_2, q_3\}^\perp \cap A$ is by a dimension argument nonempty and hence contains at least one point x. Then the 3-space generated by x and π is, by Lemma 2.19(*xii*), contained in a unique 5-space V. Since all of q_1, q_2, q_3 are collinear to x, the latter is not contained in π. Hence each of q_1, q_2, q_3 is collinear to at least two points of V and hence to a 3-space. Using Lemma 2.19($xiii$) one deduces that there exists a plane α in V collinear to q_1, q_2, q_3. It follows that V and U are special; now the assertions are clear. $\square$

The next lemma follows from the second part of Proposition 4.4 of [5].

Lemma 2.23. *Given two opposite* 5*-space* U *and* W*, and given any point* p *on any line* L *intersecting both* U *and* W*, there exists a* 5*-space containing* p *and nontrivially intersecting each line connecting a point of* U *with a point of* W*.*

The set of points of $\mathsf{E}_{6,2}(\mathbb{K})$ corresponding to the set of 5-spaces arising from U and W as in the previous lemma is called an *imaginary line*.

3 Proofs

3.1 Root elations

We first prove Main Result 2. We proceed in two steps.

Proposition 3.1. *A nontrivial type preserving automorphism* θ *of* $\mathsf{E}_6(\mathbb{K})$ *is a root elation if, and only if, it does not map any point of* $\mathsf{E}_{6,1}(\mathbb{K})$ *to a point at distance* 2 *if, and only if, no symp intersects its image in a unique point.*

Proof. We first notice that, by Theorem 8(2) of [9], θ being a root elation is equivalent to mapping no incident (point,symp)-pair of $\mathsf{E}_{6,1}(\mathbb{K})$ to an opposite.

Now assume that a type preserving automorphism θ maps a (point,symp)-pair (p, ξ) of $\mathsf{E}_{6,1}(\mathbb{K})$ to an opposite. Then, since no point of ξ^θ is collinear to p, the point p^θ is at distance 2 of p.

Conversely assume that θ maps a point p of $\mathsf{E}_{6,1}(\mathbb{K})$ to a point p^θ at distance 2 of p. Set $\xi := \xi(p, p^\theta)$. We claim that there exists a symp ζ containing p, not adjacent to ξ, such that ζ^θ is not adjacent to ξ. Indeed, let ξ' be the preimage of ξ under the action of θ. We select a symp ζ through p which is, as a point of $\mathsf{Res}(p)$ opposite both ξ and ξ' (also both as points of $\mathsf{Res}(p)$; this choice is possible since $\mathsf{Res}(p)$ is a polar space). Then ζ is not adjacent to ξ and ζ^θ is not adjacent to $\xi'^\theta = \xi$. The claim follows.

Now p is opposite ζ^θ as a point of ζ^θ collinear to p yields a $4'$-space of ζ^θ collinear to p and hence yields a 3-space of ζ^θ collinear to both p and p^θ. This implies that ζ^θ and ξ are adjacent, a contradiction. Similarly, p^θ is opposite ζ. Hence (p, ζ) is opposite (p^θ, ζ^θ) and the first equivalence of the assertion is proved. The second one is the dual of the first one. $\square$

Our next aim is to show the following proposition.

Proposition 3.2. *A nontrivial type preserving automorphism* θ *of* $\mathsf{E}_6(\mathbb{K})$ *is a root elation if and only if it is a polar* $\{2, 2'\}$*-kangaroo.*

We proceed with a few lemmas.

Lemma 3.3. *A nontrivial automorphism θ of $\mathsf{E}_6(\mathbb{K})$ which is a 2-kangaroo in $\mathsf{E}_{6,1}(\mathbb{K})$, is a polar $\{2,2'\}$-kangaroo.*

Proof. Suppose that θ does not map any point of $\mathsf{E}_{6,1}(\mathbb{K})$ to a point at distance 2 and let W be a singular 5-space of $\mathsf{E}_{6,1}(\mathbb{K})$. Suppose first that $\delta(W, W^\theta) = 2'$. Let $\pi \subseteq W$ and $\pi' \subseteq W^\theta$ be the unique planes contained in a common 5-space. Since, by Lemma 2.21(2′), every point of $W \setminus \pi$ is collinear to a unique point of W^θ, which on top lies in π', our assumption implies that all points of $W \setminus \pi$ are mapped into π', a contradiction.

Now suppose that $W \cap W^\theta = \{p\}$, with p a point of $\mathsf{E}_{6,1}(\mathbb{K})$. Then Lemma 2.21(2) implies that, for a certain hyperplane H, every point of $U \setminus H$ is mapped onto p, a contradiction. The lemma is proved. □

From now on we assume that θ is a polar $\{2,2'\}$-kangaroo. We argue in $\mathsf{E}_{6,1}(\mathbb{K})$ because we want to show that θ is a 2-kangaroo of $\mathsf{E}_{6,1}(\mathbb{K})$. For clearness's sake we repeat our convention in each lemma.

Lemma 3.4. *If a polar $\{2,2'\}$-kangaroo of $\mathsf{E}_{6,1}(\mathbb{K})$ maps a 5-space onto an adjacent one, then the intersection has a line in common with its image.*

Proof. Let U be a 5-space mapped onto an adjacent one and set $\pi = U \cap U^\theta$. Select an arbitrary 5-space $W \notin \{U, U^\theta\}$ containing π. Suppose for a contradiction that $|\pi \cap \pi^\theta| \leq 1$. Since $\pi \cup \pi^\theta \subseteq U^\theta$, we see that W and W^θ are adjacent and their intersection contains a line L disjoint from $\pi \cap \pi^\theta$. The line L is also disjoint from U^θ as $W \cap U^\theta = \pi$ and $W^\theta \cap U^\theta = \pi^\theta$. But L is collinear to $\pi \cap \pi^\theta$, yielding a singular 6-space, a contradiction. This proves the assertion. □

Lemma 3.5. *If a polar $\{2,2'\}$-kangaroo θ of $\mathsf{E}_{6,1}(\mathbb{K})$ maps a plane π onto a plane π^θ intersecting π in a line L, and such that all points of π are collinear to all points of π^θ, then each point of L is fixed.*

Proof. Suppose first, for a contradiction, that a line $M \subseteq \pi$ is mapped onto a disjoint line $M^\theta \subseteq \pi^\theta$. Let U be the unique 5-space containing $\pi \cup \pi^\theta$. Let W be a 5-space containing M but disjoint from M^θ. Then W^θ is adjacent to W and hence intersects W in some plane α. Since W and W^θ both meet U in planes, and they contain respective disjoint lines of U, their intersection has at most one point in common with U. Hence there exists a line $K \subseteq W \cap W^\theta$ disjoint from U. But K is collinear to $M \cup M^\theta$, yielding a 5-space intersecting U in exactly the 3-space spanned by π and π^θ, contradicting Lemma 2.21.

Now assume for a contradiction that L is not fixed and let $x \in L$ be such that $x^\theta \in \pi^\theta \setminus L$. Let M be a line in π containing x but not $x^{\theta^{-1}}$. Then M and M^θ are disjoint, contradicting the first paragraph. Hence L is stabilized. If some point $x \in L$ were not fixed, then the same argument leads to the same contradiction to the first paragraph. This shows the assertion completely. □

Lemma 3.6. *If a polar $\{2,2'\}$-kangaroo θ of $\mathsf{E}_{6,1}(\mathbb{K})$ maps a 5-space U onto an adjacent one, then the intersection plane π is pointwise fixed.*

Proof. We first show that π is stabilized, suppose for a contradiction it is not. Then $\pi^\theta \cap \pi$ is a line L by Lemma 3.4. By Lemma 3.5, the line L is fixed pointwise. Let $p \in \pi \setminus L$ be arbitrary. Let α^θ be any plane in U^θ containing p and disjoint from $(\pi \cup \pi^\theta) \setminus \{p\}$. Then $\alpha \cap \pi = \emptyset$. Let W be any 5-space containing α and distinct from U. Since $p \in U^\theta$ is collinear to α, the 5-spaces W and W^θ are not opposite. They are clearly not equal, so they are adjacent by assumption. But then both U and W^θ have planes in common with both W and U^θ. Since W and U^θ are clearly special, the uniqueness in Lemma 2.21(*iii*) yields $U = W^\theta$, clearly a contradiction, so U does not contain α^θ.

Hence π is stabilized. Now suppose, for a contradiction, that some point $p \in \pi$ is not fixed. Consider a plane β in U intersecting π in just p. Select a 5-space $W \neq U$ containing β (and automatically symplectic to U^θ). Then W and W^θ are adjacent (since they are not equal and not opposite), yielding by the first part of the proof, a plane in the intersection which is disjoint from and collinear to the solid generated by β and p^θ, a contradiction. □

Lemma 3.7. *A polar $\{2, 2'\}$-kangaroo θ of $\mathsf{E}_{6,1}(\mathbb{K})$ does not map any symp ξ to a symp ξ^θ intersecting ξ in a unique point.*

Proof. Let ξ be any symp and suppose for a contradiction that it intersects ξ^θ in the unique point p. Select a 5-space U intersecting ξ in a $4'$-space $V \subseteq W$. Both p and V^θ are contained in ξ^θ, consequently $p^\perp \cap V^\theta$ is at least a 3-space, implying that U and U^θ are not opposite. Since by our assumption on ξ and ξ^θ they are not equal either, they are adjacent. So $U \cap U^\theta$ is, by Lemma 3.6, a pointwise fixed plane which intersects V in at least a pointwise fixed line L, implying $L \subseteq \xi \cap \xi^\theta$, a contradiction. □

Now the last assertion of Proposition 3.1 combined with the previous lemma proves Proposition 3.2.

3.2 The fix structure of polar kangaroos

Standing Hypothesis. Throughout, let θ be a nontrivial 1-kangaroo of $\mathsf{E}_{6,2}(\mathbb{K})$. However, we argue in $\mathsf{E}_{6,1}(\mathbb{K})$, where the singular 5-spaces are the points of $\mathsf{E}_{6,2}(\mathbb{K})$. Hence we consider θ as a collineation of $\mathsf{E}_{6,1}(\mathbb{K})$ where for a 5-space U, the intersection $U \cap U^\theta$ is never a plane.

We collect some properties of such a 1-kangaroo of $\mathsf{E}_{6,2}(\mathbb{K})$. We use the notation of the precious section and we set $\mathsf{E}_{6,1}(\mathbb{K}) = (X, \mathscr{L})$.

Lemma 3.8. *If $\pi \in \Pi$ is a plane with the property that $\pi \cap \pi^\theta$ contains a line, then $\pi = \pi^\theta$.*

Proof. Suppose for a contradiction that $\mathscr{L} \ni L \subseteq \pi \cap \pi^\theta$. If $\pi \neq \pi^\theta$, choose a 5-space U such that $\pi \subseteq U$, but $\pi^\theta \not\subseteq U$. Then $L \subseteq \pi^\theta \subseteq U^\theta$, so $L \subseteq U \cap U^\theta$ while $U \neq U^\theta$. Lemma 2.21 implies that $U \cap U^\theta$ is a plane, contradicting the fact that θ is a polar 1-kangaroo. Hence $\pi = \pi^\theta$ and the lemma is proved. □

Lemma 3.9. *No line is stabilized by θ.*

Proof. Suppose for a contradiction that θ stabilizes the line L. By Lemma 3.8, every plane through L is fixed, so every 3-space through L is fixed. A plane π with $L \not\subseteq \pi$ in such a fixed 3-space is also fixed, as $\pi \cap \pi^\theta$ is at least a line. As all planes in such a 3-space are fixed, all points are fixed. Through connectivity, everything is fixed pointwise and so θ is the identity, a contradiction. □

Lemma 3.10. *No fixed singular 5-space contains a fixed point.*

Proof. Suppose for a contradiction that the singular 5-space U contains a fixed point $x \in X$. If there is another fixed point $y = y^\theta$ in U, we have a fixed line xy, contradicting Lemma 3.9. Let $p \in U \setminus \{x\}$ be arbitrary. Since the line px is not fixed, x, p, p^θ span a plane π. Since $xp^\theta \subseteq \pi \cap \pi^\theta$, we deduce from Lemma 3.8 that π is stabilized. Considering a point $p' \in U \setminus \pi$, we obtain a second fixed plane π' spanned by x, p', p'^θ, which intersects π in just $\{x\}$ by Lemma 3.9. Then θ fixed the 4-space Σ spanned by π and π'. Now consider a point $p'' \in U \setminus \Sigma$, then we again obtain a fixed plane π'' containing x and p''. However, π'' has to intersect Σ in precisely a line, which is then also fixed, contradicting Lemma 3.9. □

Lemma 3.11. *The fixed planes in a fixed singular* 5*-space* U *define a simple spread of* U*.*

Proof. For a point $p \in U$, we have $p^{\theta^2} \notin pp^\theta$, as we would otherwise have a fixed line. The plane spanned by $p, p^\theta, p^{\theta^2}$ is fixed as otherwise it intersects its image in the line $p^\theta p^{\theta^2}$, which contradicts Lemma 3.8. Varying p, we obtain a set of fixed planes covering the point set of U. No two such fixed planes intersect in a point or a line, as either would itself be fixed, contradicting Lemmas 3.9 and 3.10. □

Lemma 3.12. *A* 5*-space* U *that has a fixed plane* $\pi \subseteq U$ *is itself fixed.*

Proof. As $\pi \subseteq U \cap U^\theta$, this follows immediately from the definition of a 1-kangaroo. □

We can now prove (i) of the Main Result.

Theorem 3.13. *If* θ *fixes at least one singular* 5*-space, then the planes fixed by* θ *form a semi-spread of* $\mathsf{E}_{6,1}(\mathbb{K})$*.*

Proof. Let α and β be two fixed planes. Let U be an arbitrary singular 5-space containing α, which is fixed by Lemma 3.12. Then $\beta \cap U$ is neither a line nor a point, by Lemmas 3.9 and 3.10. If $\beta \subseteq U$, then α and β are collinear. Hence we may assume that $\beta \cap U = \emptyset$.

Suppose first that each point of β is collinear to exactly a point of U. Then by Lemma 2.22, these points form a plane β' of U, which is fixed, and so β and β' are fixed planes which are semi-opposite. If $\alpha \neq \beta'$, then α and β are opposite.

Hence we may assume that some point $p \in \beta$ is collinear to a 3-space $\Sigma \subseteq U$. Then p^θ, which is distinct from p by Lemma 3.10 combined with Lemma 3.12, is collinear to the 3-space Σ^θ. Since U has dimension 5, the intersection $\Sigma \cap \Sigma^\theta$ contains a line L. Then L and pp^θ are contained in a singular 3-space, which is on its turn contained in a unique singular 5-space W by Lemma 2.19. Then $W \cap U$ is a plane π, by Lemma 2.21. Hence pp^θ is collinear to π, a plane of Σ and of Σ^θ. Now p^{θ^2} is not contained in pp^θ by Lemma 3.9, but is collinear to Σ^{θ^2}, which intersects by the same token as previously Σ^θ in a plane π', and hence π in at least a line. If $\pi \cap \pi'$ is exactly a line, then the set of points of U collinear to β is that line, and must be fixed, contradicting Lemma 3.9. Hence $\pi = \pi'$ and β is collinear to π, which is also fixed by θ. If $\alpha \neq \pi$, then β is semi-opposite α as each point of β is collinear to a 3-space of U which contains π and hence intersects α in a unique point. □

In the proof of Lemma 3.15 below we need the existence of a plane in a fixed 5-space that is mapped onto a disjoint plane. We can prove this existence in a slightly more general situation, and that is what we will do in the next lemma.

Lemma 3.14. *Let $\mathscr{S}$ be a simple spread of $\mathsf{PG}(2n+1,\mathbb{K})$ elementwise fixed by a nontrivial collineation σ. Then*

(*i*) *no point of $\mathsf{PG}(2n+1,\mathbb{K})$ is fixed by σ, and*
(*ii*) *there exists a subspace of dimension n mapped onto a disjoint subspace.*

Proof. Suppose for a contradiction that p is a fixed point of σ. Let q be an arbitrary point not contained in the member Σ_p of $\mathscr{S}$ containing p, and let r be a third point on the line pq. Let Σ_q and Σ_r be the members of $\mathscr{S}$ containing q and r, respectively. Then the subspace spanned by p and Σ_r is fixed by σ and intersects Σ_q precisely in q. Since also Σ_q is fixed, we deduce $q^\sigma = q$. Hence all points outside Σ_p are fixed, and this is enough to conclude that σ is the identity, contradicting the nontriviality of σ. This proves (*i*).

Now suppose for a contradiction that every subspace Σ of $\mathsf{PG}(2n+1,\mathbb{K})$ of dimension n is mapped onto a nondisjoint subspace. Then, with the terminology of [13], σ is domestic and by Theorem 2.1 in [13] it pointwise fixes a subspace of dimension n. This contradicts (*i*) and proves (*ii*). □

Lemma 3.15. *If θ fixes at least one singular 5-space, then it is not a $2'$-kangaroo.*

Proof. Let U be a fixed 5-space. Combining Lemma 3.11 with Lemma 3.14 we find a plane π disjoint from π^θ. A singular 5-space V through π gets mapped to a 5-space V^θ through π^θ at distance $2'$. Indeed, if V met V^θ in some point p, then p would be collinear to all points of U and not belong to it, contradicting Lemma 2.20. □

Corollary 3.16. *A non-trivial polar $\{1,2'\}$-kangaroo of $\mathsf{E}_{6,2}(\mathbb{K})$ is also a 0-kangaroo.*

Revised Standing Hypothesis. Now suppose there are no 5-spaces mapped to distance $2'$, so by the previous corollary, there are no fixed 5-spaces either. Then θ is a polar $\{0,1,2'\}$-kangaroo (of $\mathsf{E}_{6,2}(\mathbb{K})$). We assume this throughout as standing hypothesis.

Lemma 3.17. *A line L in a symp ξ with $L^\theta \subseteq \xi$ and $L \cap L^\theta = \emptyset$, is ξ-opposite L^θ.*

Proof. If L and L^θ are contained in a common 3-space V, then the image of any 5-space U containing V shares L^θ with U, hence by Lemma 2.21 is either adjacent or equal to U, a contradiction. Now suppose that not all points of L are collinear to L^θ, but there is a unique point $x \in L$ collinear to all points of L^θ. Referring to Lemma 2.19, consider a 5-space U through L, such that $U \cap \xi = L$. We may choose U such that it does not contain $x^{\theta^{-1}}$. Then U is disjoint from L^θ, but $x \in U$ is collinear to all points of $L^\theta \subseteq U^\theta$, implying that U^θ is not opposite U (cf. Lemma 2.21). Consequently $U \cap U^\theta$ is a point p. Picking non-collinear points $u \in L$ and $v \in L^\theta$, we see by convexity of ξ that $p \in \xi$. But then $p \in \xi \cap U = L$. The only point of L collinear with all points of L^θ is however x. But as $x^{\theta^{-1}} \notin U$, we conclude $x \notin U^\theta$ and this contradiction shows the lemma. $\square$

Lemma 3.18. *Suppose that θ has no fixed points in X. Then no point $p \in X$ is mapped onto a collinear one.*

Proof. Suppose for a contradiction that there is a point $p \in \xi$ such that $p \perp p^\theta$. Then $p^\theta \perp p^{\theta^2}$. If $p \perp p^{\theta^2}$, then the plane spanned by $p, p^\theta\, p^{\theta^2}$ has the line $p^\theta p^{\theta^2}$ in common with its image, hence is fixed by Lemma 3.8. Then Lemma 3.12 yields fixed singular 5-spaces, a contradiction. So p is not collinear to p^{θ^2}.

Now let π be a plane containing pp^θ. Select a line M in π through p, but distinct from pp^θ. By Lemma 3.8 combined with Lemma 3.12, the plane $\pi^{\theta-1}$ intersects π in just p. Hence we can select a singular 5-space U intersecting π in M and $\pi^{\theta^{-1}}$ in $\{p\}$. Since U^θ contains p^θ, which is collinear to every point of $M \subseteq U$, the singular 5-spaces Y and U^θ are not opposite, hence they share exactly one point x. Since x is not fixed, the lines $L := px$ and $L^\theta = p^\theta x^\theta$ are disjoint.

If $x^\theta \perp p$, then L and L^θ are contained in a common 3-space, and hence also in a common symp, contradicting Lemma 3.17. If x^θ is not collinear to p, then the lines L and L^θ are contained in the symp defined by the convex closure of p and x^θ, again contradicting Lemma 3.17 as x is collinear to all points of L^θ. $\square$

We can also say something if θ maps no points to collinear ones.

Lemma 3.19. *If θ does not map any point to a collinear one, then it fixes at least one point.*

Proof. Suppose for a contradiction that θ does not fix any point. Let $x \in X$ be arbitrary. Then x^θ and x are contained in a unique symp ξ. Let U be a singular 5-space containing x and intersecting ξ in a $4'$-space. Then U^θ contains x^θ, which is collinear to the points of a 3-space of U. Hence $U \cap U^\theta$ is a point p. Then, since $p^\theta \in U^\theta$ and thus $p \perp p^\theta$, we see that p is fixed. $\square$

The last two lemmas immediately imply:

Corollary 3.20. *There is at least one fixed point for θ, and dually, at least one fixed symp.*

Now we determine the fix structure in a fixed symp.

Proposition 3.21. *A point p in a fixed symp ξ is either fixed or mapped to a noncollinear point.*

Proof. Suppose for a contradiction that there is a point $p \in \xi$ such that $p \perp p^\theta$. Repeating the first paragraph of the proof of Lemma 3.18, we deduce that p is not collinear to p^{θ^2}. Now consider a $4'$-space $U \in \xi$ through pp^θ. Select a point $q \in U$ not on pp^θ, and not collinear to p^{θ^2}. We claim that the line qp^θ is disjoint from its image. Indeed, if $x \in qp^\theta \cap q^\theta p^{\theta^2}$, then

(*i*) either $x = p^\theta$, but then $q^\theta \in p^\theta p^{\theta^2}$ implying the contradiction $q \in pp^\theta$,
(*ii*) or $x \in qp^\theta \setminus \{p^\theta\}$, and then, as $x \perp p^{\theta^2}$, also $q \perp p^{\theta^2}$, again a contradiction to the choice of q.

The claim is proved. Now Lemma 3.17 implies that the line $L = qp^\theta \subseteq \xi$ is ξ-opposite $L^\theta = q^\theta p^{\theta^2}$, which is ridiculous as the point p^θ is collinear to all points of $q^\theta p^{\theta^2}$. This final contradiction proves the lemma. □

Corollary 3.22. *The fixed points in a fixed symp form an ovoid.*

Proof. Suppose V is a 4- or a $4'$-space of ξ. Then $V \cap V^\theta$ contains a point $p \perp p^\theta$, as $\{p, p^\theta\} \subseteq V^\theta$. Hence p is fixed by Proposition 3.21. Hence every maximal singular subspace contains at least one fixed point. If it contained at least two fixed points, then it fixes a line, contradicting Lemma 3.9. The assertion is proved. □

Theorem 3.23. *The fixed points of θ form a special ovoid.*

Proof. Let ξ be an arbitrary symp. If ξ contains at least two fixed points, then these are not collinear by Lemma 3.9 and hence ξ is fixed. Consequently Corollary 3.22 implies that the fixed point of θ in ξ form an ovoid of ξ. Now the dual of Corollary 3.22 yields several fixed symps through each fixed point of ξ, yielding on their turn again many fixed points outside ξ. Hence the set of fixed points is a special ovoid. □

3.3 Collineations of $\mathsf{E}_{6,1}(\mathbb{K})$ fixing a special ovoid

Proposition 5.11 of [7] implies that a collineation pointwise fixing a special ovoid is a 1-kangaroo in $\mathsf{E}_{6,1}(\mathbb{K})$. So it suffices to show that such a 1-kangaroo is automatically a polar $\{1, 2'\}$-kangaroo.

Proposition 3.24. *Let θ be a 1-kangaroo in $\mathsf{E}_{6,1}(\mathbb{K})$ with at least one fixed point. Then θ is a polar $\{1, 2'\}$-kangaroo (in $\mathsf{E}_{6,2}(\mathbb{K})$).*

Proof. Theorem 5.1 of [7] asserts that the fix structure of θ is a special ovoid. Suppose now that a singular 5-space U is mapped onto an adjacent one. Then each point of $U \cap U^\theta$ is mapped inside U^θ, hence is fixed (as it is not mapped onto a collinear point by assumption of the 1-kangaroo). This contradicts the fact that a special ovoid does not contain collinear points.

Now assume that a singular 5-space U is mapped onto a special one. Select a point $p \in U$ mapped into the unique plane β of U^θ collinear to a plane α of U. Then the symp ξ through p and p^θ is fixed (this is Lemma 5.3 of [7]). Clearly ξ contains α and so $U \cap \xi$ is a $4'$-space. Consequently $U^\theta \cap \xi$ is also a $4'$-space, necessarily meeting U in at least one point as $4'$-spaces cannot be opposite in a symp. This contradicts U and U^θ being special. □

3.4 Collineations of $\mathsf{E}_{6,1}(\mathbb{K})$ fixing a semi-spread

This case is a bit more involved. We begin with proving some properties of collineations of $\mathsf{PG}(5,\mathbb{K})$ fixing a plane spread $\mathscr{S}$ of $\mathsf{PG}(5,\mathbb{K})$ elementwise.

Lemma 3.25. *Let $\mathscr{S}$ be a plane spread of $\mathsf{PG}(5,\mathbb{K})$ and let θ be a nontrivial collineation stabilizing each member of $\mathscr{S}$. Then θ does not fix any point, any line, any 3-space, any hyperplane and any plane not belonging to $\mathscr{S}$. Also, if θ maps some line L to an intersecting line, then the plane spanned by L and L^θ belongs to $\mathscr{S}$. Dually, if θ maps some 3-space S to a 3-space S^θ with $S \cap S^\theta = \alpha$ a plane, then $\alpha \in \mathscr{S}$. Finally, θ does not map any plane π to a plane π^θ satisfying $\langle \pi, \pi^\theta \rangle$ is 3-dimensional.*

Proof. By Lemma 3.14, θ does not fix any point. Let L be any line and pick $x \in L$. Since x is not fixed, x and x^θ are contained in the same member π of $\mathscr{S}$. Hence, if L is fixed, then $L \subseteq \pi \in \mathscr{S}$. Let $\alpha \in \mathscr{S} \setminus \{\pi\}$ be arbitrary. Then $A = \langle L, \alpha \rangle$ is a fixed 4-space, which intersects each plane $\beta \in \mathscr{S} \setminus \{\alpha\}$ in a unique line L_β. For distinct $\beta, \beta' \in \mathscr{S} \setminus \{\alpha\}$ contained in U, the 3-space B spanned by L_β and $L_{\beta'}$ is stabilized and intersects any member of $\mathscr{S}$ in U containing a point of $A \setminus B$ in a unique point, which is hence fixed. This contradicts Lemma 3.14.

Suppose now for a contradiction that θ fixes a 3-space S. Then any member of $\mathscr{S}$ containing a point outside S intersects S in either a fixed point, or a fixed line, a contradiction. Similarly, θ does not fix a hyperplane either. Clearly, θ does not fix any plane not contained in $\mathscr{S}$ (as otherwise the intersection with a non-disjoint member of $\mathscr{S}$ is also fixed).

Now suppose that for some line L we have $L \cap L^\theta$ is a point x. Then x is not fixed and hence the unique member π of $\mathscr{S}$ containing x also contains x^θ and $x^{\theta^{-1}}$, which, together with x, thus generate the plane π.

Now let π be a plane with $\mathsf{dim}\langle \pi, \pi^\theta \rangle = 3$. Then the line $L = \pi \cap \pi^\theta$ is mapped onto a line in π^θ. Since $L^\theta \neq L$ by the first part of the proof, $\langle L, L^\theta \rangle = \pi^\theta$ is fixed by the previous paragraph, which implies $\pi = \pi^\theta$, a contradiction.

Finally, let S be a solid such that $S \cap S^\theta = \alpha$ is a plane. Then both α and α^θ are contained in S^θ, leading to $\alpha = \alpha^\theta$ by the previous paragraph. Since θ does not fix any plane not belonging to $\mathscr{S}$, we conclude $\alpha \in \mathscr{S}$.

All assertions are proved. $\square$

Proposition 3.26. *Let $\mathscr{S}$ be a semi-spread of $\mathsf{E}_{6,1}(\mathbb{K})$. Let $\mathscr{W}$ be the set of singular 5-spaces containing at least two members of $\mathscr{S}$. Then $\Gamma := (\mathscr{W}, \mathscr{S})$ is a generalized hexagon fully embedded in $\mathsf{E}_{6,2}(\mathbb{K})$. Moreover, the hexagon is a Moufang hexagon.*

Proof. Let $\pi \in \mathscr{S}$ and $W \in \mathscr{W}$. We show that either $\pi \subseteq W$, or there exists a unique $U \in \mathscr{W}$ such that $\pi \subseteq U$ and $U \cap W \in \mathscr{S}$, or there exist unique $U, U' \in \mathscr{W}$ such that $\pi \subseteq U$, $U \cap U' \in \mathscr{S}$ and $U' \cap W \in \mathscr{S}$.

We may assume that π is not contained in W, and that π is not collinear to any member of $\mathscr{S}$ contained in W (as otherwise the assertion in the previous paragraph is obvious). Hence π is opposite or semi-opposite each member of $\mathscr{S}$ in W. This implies that each point of π is far from W, and so π is semi-opposite a unique member $\alpha \in \mathscr{S}$ contained in W. The first assertion now follows from Lemma 2.22 and the definition of semi-spread (which implies the fullness of the embedding).

Let $U \in \mathscr{W}$ be arbitrary and let $W \in \mathscr{W}$ be opposite. Let π be a member of $\mathscr{S}$ collinear to respective members π_U and π_W of $\mathscr{S}$ contained in U and W, respectively. Then π_U and π_W are semi-opposite, hence for each point $p \in \langle \pi, \pi_U \rangle$ there is a line L_p through p containing a point of π_W. Let V be a 5-space corresponding to an arbitrary point of the imaginary line of $\mathsf{E}_{6,2}(\mathbb{K})$ defined by U and W. Then there is a unique plane π_V intersecting all lines L_p from above. Since all these lines are collinear to π, the plane π_V is collinear to π and hence the 5-space X generated by π and π_V belongs to $\mathscr{W}$. Since π_V is semi-opposite π_W, it belongs to $\mathscr{S}$. It follows that $V \in \mathscr{W}$. Hence we have shown, with self-explaining terminology, that $\mathscr{W}$ is closed under taking hyperbolic lines. hence $\mathscr{W}$ is the union of hyperbolic lines containing W. It follows that central elations with centre W preserve Γ. It is now routine to see that the group of central elations of $\mathsf{E}_{6,2}(\mathbb{K})$ with centre W induces a group of central elations in Γ acting transitively on the members of $\mathscr{W}$ through π, except for the unique member of $\mathscr{W}$ through π adjacent to W. It now follows from [11] (see also Theorem 6.3.9 of [18]) that Γ is a Moufang hexagon and the lemma is proved. $\square$

Lemma 3.27. *Suppose $\mathscr{S}$ is a semi-spread of $\mathsf{E}_{6,1}(\mathbb{K})$ and θ is a collineation pointwise fixing each member of $\mathscr{S}$. Then θ is the identity.*

Proof. Let p be any point of $\mathsf{E}_{6,1}(\mathbb{K})$. Let U be a 5-space containing some member of $\mathscr{S}$. If $p \in U$, then p is fixed. If p is close to U, then, since the 4-space A spanned by p and $p^\perp \cap U$ is unique with respect to containing $p^\perp \cap U$,and the latter is fixed, also A is stabilized. Hence $p^\theta \perp p$. Now suppose p is far from U and let $\pi \in \mathscr{S}$ be the unique plane of U belonging to $\mathscr{S}$ containing a point x of $p^\perp$. Let y be any point of $\pi \setminus \{x\}$ and pick $z \in \pi \setminus xy$. Then p is close to the 5-space spanned by z and $z^\perp \cap \xi(p, y)$ and is hence mapped to a collinear point by the foregoing.

Consequently θ is a root elation by Proposition 3.1. But this contradicts Lemma 2.15, as the latter clearly implies that no two opposite 5-spaces are pointwise fixed. $\square$

Lemma 3.28. *Suppose $\mathscr{S}$ is a semi-spread of $\mathsf{E}_{6,1}(\mathbb{K})$. Then every point of $\mathsf{E}_{6,1}(\mathbb{K})$ that is not contained in a member of $\mathscr{S}$ is close to some 5-space containing a member of $\mathscr{S}$.*

Proof. Suppose some point p is far from a 5-space W containing members of $\mathscr{S}$. Then p is collinear to a unique point $x \in W$; let π be the unique member of $\mathscr{S}$ containing x. Then in the point residual at x, which is isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$, a straight forward argument in the associated polar space $\mathsf{D}_{5,1}(\mathbb{K})$ shows that there is a unique 5-space through π close to p. This proves the lemma. $\square$

Proposition 3.29. *Suppose θ is a collineation of $\mathsf{E}_{6,1}(\mathbb{K})$ stabilizing each member of a semi-spread $\mathscr{S}$. Then θ is a polar 1-kangaroo.*

Proof. We first claim that there are no fixed points. Indeed, suppose for a contradiction that there is a fixed point p. If p belongs to a 5-space U containing at least one member of $\mathscr{S}$, then by Lemma 3.14, U is pointwise fixed. By connectivity, all planes of $\mathscr{S}$ are then pointwise fixed. Lemma 3.27 implies that θ is the identity.

Now let the fixed point p not be contained in any member of $\mathscr{S}$. then, by Lemma 3.28, p is close to some 5-space U containing a member of $\mathscr{S}$. Then $p^\perp \cap U$ is a 3-space S, which must be fixed. By Lemma 3.25, θ induced the identity in U, and by connectivity and Lemma 3.14, θ pointwise fixes each member of $\mathscr{S}$, again leading to the identity by Lemma 3.27. The claim follows.

Now we claim that a point p which is mapped onto a collinear point, is contained in a member of $\mathscr{S}$. Indeed, suppose for a contradiction that some point p not contained in any member of $\mathscr{S}$ is mapped onto a collinear point. By Lemma 3.28, p is close to some 5-space U containing a spread of members of $\mathscr{S}$. Set $S := p^\perp \cap U$. By Lemma 2.19($xiii$), S^θ has a plane α in common with S. By Lemma 3.25, the plane α belongs to $\mathscr{S}$. Hence the unique 5-space containing the 3-space spanned by p and α contains a spread of planes of $\mathscr{S}$, and so p belongs to a member of $\mathscr{S}$.

We now show that the intersection of a 5-space and its image is never a plane. Suppose for a contradiction that for some 5-space U we have $U \cap U^\theta$ is a plane π. Pick $p \in \pi$. Then the foregoing implies that $L := \langle p, p^\theta \rangle$ is a line contained in some member $\alpha = \langle p^{\theta^{-1}}, p, p^\theta \rangle$ of $\mathscr{S}$. It follows that $p^{\theta^{-1}} \notin U^\theta$ (as otherwise $\alpha \subseteq U^\theta$ and θ would fix U^θ). Pick $p' \in \pi \setminus \{p\}$. Then there is a unique (fixed) 5-space W containing the 3-space spanned by α and p'. It follows that W contains $\langle p, p^\theta, p', p'^\theta \rangle$, which is 3-dimensional (as the lines $\langle p, p^\theta \rangle$ and $\langle p', p'^\theta \rangle$ are contained in disjoint planes of W). This implies $W = U^\theta$, a contradiction.

Hence θ is a polar 1-kangaroo. □

Main Result 1 is proved.

Now we proceed to the proofs of Proposition 2.17 and Proposition 2.18. The second assertions are every time proved in [7]. So we concentrate on the semi-spreads.

3.5 Proof of Proposition 2.17

Let θ be a collineation of $\mathsf{E}_{6,2}(\mathbb{K})$ pointwise fixing the Moufang generalized hexagon Γ corresponding to a semi-spread $\mathscr{S}$ of $\mathsf{E}_{6,1}(\mathbb{K})$.

Since the lines of Γ are pointwise fixed, θ is inherited from a linear map of the underlying vector space of any embedding of $\mathsf{E}_{6,2}(\mathbb{K})$ into projective space (for instance the Weyl embedding [2]). The residue of a fixed point is the plane Grassmannian of $\mathsf{PG}(5,\mathbb{K})$; hence θ induces in the corresponding 5-space U of $\mathsf{E}_{6,1}(\mathbb{K})$ a linear collineation elementwise fixing a spread $\mathscr{S}_U$. Since, by Lemma 3.25 and Lemma 3.27, no point and no line of U is fixed, we can choose the basis in U such that the matrix of θ restricted to U is

$$M_\theta = \begin{pmatrix} 0 & 0 & a & & & \\ 1 & 0 & b & & & \\ 0 & 1 & c & & & \\ & & & 0 & 0 & a' \\ & & & 1 & 0 & b' \\ & & & 0 & 1 & c' \end{pmatrix},$$

with $a, a', b, b', c, c' \in \mathbb{K}$. It is a routine calculation to express for $p = (1, 0, 0, 1, 0, 0)$, that p^{θ^3} is contained in the plane $\langle p, p^\theta, p^{\theta^2}\rangle$. We obtain $a = a'$, $b = b'$ and $c = c'$. Moreover, expressing that no point is fixed results in the polynomial $x^3 - cx^2 - bx - a$ being irreducible over $\mathbb{K}$.

We now claim that $\mathscr{S}_U$ is regular, that is, given any three distinct members of $\mathscr{S}_U$ and a line L intersecting each of these members, the members of $\mathscr{S}_U$ intersecting L only depends on the three given members. Indeed, we may take $\alpha_1 = (1, 0, 0, 0, 0, 0)^{\langle\theta\rangle}$, $\alpha_2 = (0, 0, 0, 1, 0, 0)^{\langle\theta\rangle}$ and $\alpha_3 = (1, 0, 0, 1, 0, 0)^{\langle\theta\rangle}$ as three distinct members of $\mathscr{S}_U$.

A generic point of α_1 is given by $p = (x_1, x_2, x_3, 0, 0, 0)$, $x_1, x_2, x_3 \in \mathbb{K}$. One calculates that a generic point of the line L_p through p intersecting α_2 and α_3 nontrivially is given by $p_k = (x_1, x_2, x_3, kx_1, kx_2, kx_3)$, with $k \in \mathbb{K} \cup \{\infty\}$, where $k = \infty$ corresponds to the point $(0, 0, 0, x_1, x_2, x_3)$ as usual in projective geometry. Then it is clear from the matrix M_θ above, taking into account $a = a'$, $b = b'$ and $c = c'$, that the points in the orbit of p_k under the action of $\langle\theta\rangle$ have coordinates of the form $(y_2, y_2, y_3, ky_1, ky_2, ky_3)$, hence belong to the plane spanned by the points $(1, 0, 0, k, 0, 0)$, $(0, 1, 0, 0, k, 0)$ and $(0, 0, 1, 0, 0, k)$, which does not depend on p, but only on k. Varying $k \in \mathbb{K} \cup \{\infty\}$, the set of such planes only depends on α_1, α_2 and α_3. The claim is proved.

Now it follows from the main result of [6] that $\mathbb{K}$ is contained in the centre of the skew field $\mathbb{L}$ determined by the spread $\mathscr{S}$. Obviously, $\mathbb{L}$ has dimension 3 over $\mathbb{K}$, and so $\mathbb{L}$ is commutative. It follows from the classification of Moufang hexagons in [17] that Γ is a Moufang hexagon either of type ${}^3\mathsf{D}_4$ or ${}^6\mathsf{D}_4$ (more precisely, with Tits indices ${}^3\mathsf{D}^2_{4,2}$ or ${}^3\mathsf{D}^2_{4,2}$, respectively), or of mixed type (if $\mathsf{char}\mathbb{K} = 3$ and $a = b = 0$).

The proof of Proposition 2.17 is complete.

3.6 Proof of Proposition 2.18

In this subsection we will use some building-theoretic arguments, in particular, Tits' extension theorem 4.16 of [16]. We refer to [16] for all notions and background, in particular apartments, chambers, etc.

Now let $\mathbb{K}$ and $\mathbb{L}$ be fields such that $[\mathbb{L} : \mathbb{K}] = 3$, so $\mathbb{L}$ is a third degree extension of $\mathbb{K}$, say with associated irreducible (over $\mathbb{K}$) polynomial $x^3 - cx^2 - bx - a$. Then using the matrix M_θ above (with again $(a, b, c) = (a', b', c')$) we construct a regular spread $\mathscr{S}$ of a 5-space $U \cong \mathsf{PG}(5, \mathbb{K})$ of $\mathsf{E}_{6,1}(\mathbb{K})$, elementwise fixed by the corresponding (linear) collineation θ_U of U. Pick two members α_1 and α_2 of $\mathscr{S}$ and let Σ_U be an apartment of U containing α_1 and α_2. Let U' be a 5-space opposite U. Let C_i^U, $i = 1, 2$, be arbitrary opposite chamber in Σ_U containing α_i, and set $C_i := C_i^U \cup \{U\}$. Let C_2' be the projection of C_1 onto U'. Then, by Proposition 3.29 in [16], the chambers C_2 and C_2' are opposite and hence define a unique apartment Σ, which also contains C_1 as it is the projection of C_2' onto U (by 3.28 of [16]). Now set $D_i^U = (C_i^U)^{\theta_U}$, $i = 1, 2$, define $D_i = D_i^U \cup \{U\}$ and let D_2' be the projection of D_1 onto U'. Let Υ be the apartment containing the opposite chambers D_2 and D_2'. Extend θ with an isomorphism $\Sigma \to \Upsilon$ in the obvious way (fixing U and U').

Let $E_2(C_1)$ be the union of all rank 2 residues defined by C_1. We claim that there is an adjacency preserving mapping $E_2(C_2) \cup \Sigma \to E_2(D_1) \cup \Upsilon$ that extends the action of θ. Indeed, the only rank 2 residue of C_1 where θ is not defined is the one of type $\{2,4\}$, denoted by $R_{2,4}$. Let $L \in C_1$ be the element of type 3 (hence L is a line of $\mathsf{E}_{6,1}(\mathbb{K})$ contained in α_1) and let S be the element of type 5 of C_1 (hence S is a 4-space intersecting U in a 3-space containing α_1). Then $R_{2,4}$ is a projective plane consisting of the 5-spaces containing L and intersecting S in a 3-space, and the planes in S through L, with natural incidence. The elements of $R_{2,4}$ on which θ is already defined are the planes through L contained in the 3-space $S \cap U$, the two 5-spaces U, U_1 of Σ through α_1 (which are fixed) and the unique 5-space U^* of Σ containing L, intersecting S in a 3-space, but not containing α_1. Let $R'_{2,4}$ be the residue of type $\{2,4\}$ of the chamber D_1. Then $R_{2,4}$ and $R'_{2,4}$ share the rank 1 residue at α_1 and we can define θ on that residue as the identity. It follows that θ now defines a partial map of projective planes $R_{2,4} \to R'_{2,4}$ which, in geometric terms (calling the 5-spaces *points* and the planes *lines* in these residues), consists of the union of a linear mapping on a line pencil stabilizing exactly one (common) line, the identity on that line, and a corresponding pair of points not on that line. There exists now a unique isomorphism $R_{2,4} \to R'_{2,4}$ extending this partial map. This can be seen by suitable coordinatization as follows. First we introduce coordinates in $R_{2,4}$: Let $(1,0,0)$ correspond to the vertex of the said line pencil. Let $(0,1,0)$ be the point corresponding to U_1 and $(0,0,1)$ to U^*. Since $R'_{2,4}$ shares a rank 1 residue, we can choose the same coordinates $(*,*,0)$ for points corresponding to 5-spaces through α_1, and we let $(0,0,1)$ correspond to the image of U^* , The line pencil through $(1,0,0)$ can be given binary coordinates by assigning the coordinate $[a,b]$ to the line with equation $ay + bz = 0$ (calling the coordinates (x,y,z)). Then the action of θ on that pencil is given by a matrix

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix},$$

since the line $[0,1]$ is fixed and the line of $R_{2,4}$ generated by $(1,0,0)$ ad $(0,0,1)$ is by construction mapped onto the line of $R'_{2,4}$ generated by the same base points. Hence the mapping with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}$$

extends θ and defines an isomorphism from $R_{2,4}$ to $R'_{2,4}$.

Hence we obtain an adjacency preserving map from $E_2(C_1) \cup \Sigma$ to $E_2(D_1) \cup \Upsilon$. Applying Tits' Extension Theorem 4.16 of [16], we can extend this map to an automorphism of $\mathsf{E}_6(\mathbb{K})$, which by the uniqueness in Theorem 4.16 of [16], coincides with θ over its entire domain (in particular, over U).

Denoting that extension still with θ, Proposition 2.18 now follows from the following proposition.

Proposition 3.30. *With the above set-up, the planes of $\mathsf{E}_{6,1}(\mathbb{K})$ fixed by θ form a semi-spread of $\mathsf{E}_{6,1}(K)$.*

Proof. Let π be a fixed plane. We first claim that every plane of $\mathscr{S}$ is either opposite, semi-opposite, collinear or equal to π. Indeed, we consider the following possibilities.

(i) *Each point $p \in \pi$ is collinear to a unique point p_U of U.* This case easily leads to π being semi-opposite the unique member of $\mathscr{S}$ consisting of the points p_U if p ranges over π. Obviously, π is opposite each other member of $\mathscr{S}$.

(ii) *The intersection $\pi \cap U$ is nontrivial.* Since θ does not have any fixed points or fixed lines in U, this implies that $\pi \subseteq U$, which leads to $\pi \in \mathscr{S}$.

(iii) *Some point $p \in \pi$ is close to U and $\pi \cap U = \emptyset$.* Set $S = p^\perp \cap U$. Since S is not fixed and p^θ is collinear to p, the intersection $S \cap S^\theta$ is a plane α (use Lemma 2.19($xiii$)), which belongs to $\mathscr{S}$ by Lemma 3.25. We claim that the line $L := \langle p, p^\theta \rangle$ is not fixed. Indeed, if it were, then, by considering a symp through a point of $S \setminus S^\theta$ and p^θ, we see that the set $\{x^\perp \cap U \mid x \in L\}$ defines a unique 4-space of U, which must be fixed, contradicting Lemma 3.25. The claim follows. But now one sees that all points of π are collinear to all points of α. It follows that π is collinear to α and semi-opposite every other members of $\mathscr{S}$.

This shows our claim that every plane of $\mathscr{S}$ is either opposite, semi-opposite, collinear or equal to π.

Next we claim that the fixed planes in each 5-space W containing at least one fixed plane π form a spread isomorphic to $\mathscr{S}$ (and with conjugate action of θ). Indeed, first suppose W is opposite U and π is opposite α_1. Since all 5-spaces through α_1 are fixed, it follows from projecting that also all 5-spaces through π are fixed. As W is opposite U, the claim follows again by projecting. In particular, all 5-spaces opposite U through fixed planes in U' distinct from the unique plane in U' semi-opposite α_1, satisfy our claim. Interchanging the roles of U and U', we deduce that all 5-spaces opposite U' through members of $\mathscr{S}$ satisfy our claim. Since the 5-spaces through members of $\mathscr{S}$ not opposite U' are opposite 5-spaces for which the claim already holds, we see that the claim holds for all 5-spaces containing a member of $\mathscr{S}$. Let U_1' be the 5-space of Σ opposite U_1. It is easy to see that U_1' is fixed (it belongs to $\Sigma \cap \Upsilon$), and so we can interchange the roles of (U, U') and (U_1, U_1'). The same thing can be said about (U_2, U_2'), where U_2 is the unique 5-space in Σ containing α_2 and distinct from U, and U_2' is the opposite 5-space in Σ.

Now suppose that π is opposite some member α of $\mathscr{S}$. Then by projecting the residues of π and α onto each other, the claim follows for all 5-spaces containing π. If π is collinear to some member of $\mathscr{S}$, then it is opposite some fixed plane of either U_1 or U_2 (or both). The claim again follows for all 5-spaces through π.

It now follows from this claim that each 5-space through any fixed plane plays the same role as U. Hence, by the first part of the proof, two arbitrary fixed planes are either opposite, semi-opposite, collinear or equal. The proof of the proposition is complete. $\square$

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